

Algebraic Structures of Characteristics in Involutive Systems of Non-Linear Partial Differential Equations

By

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Introduction

An involutive system of partial differential equations is, roughly speaking, such a system that its general solution can be obtained by solving successively equations of Cauchy-Kowalevsky's type. It was E. Cartan who introduced the notion of involution for exterior differential systems. On this subject one has the Cartan-Kähler theorem and the prolongation theorem due to E. Cartan, M. Kuranishi and M. Matsuda (cf. Cartan [2, 4]; Kuranishi [11-12]; Matsuda [13-14]). M. Kuranishi constructed the process of standard prolongation, which was applied to infinite Lie groups by himself. It was also applied to the equivalence problem of G-structures by V.W. Guillemin, I.M. Singer and S. Sternberg. In this course they clarified the algebraic structures of involutive systems. By their results M. Kuranishi gave a clear definition of involutive systems. His standard prolongation was improved by M. Matsuda, who combined it with the classical method of prolongation due to Lagrange and Jacobi.

On the other hand, it is well-known that, in the classical and modern theory of partial differential equations, consideration of characteristics in various senses leads us to fruitful results. In our subject the two concepts of Cauchy characteristics and Monge ones will be particularly important. Effectiveness of the former was shown by E. Cartan for general systems ([2-4]). However, that of the latter seems to have been shown only for special systems (cf. Cartan [4], Chap. IV, Part III).

The principal aim of this paper is to investigate algebraic structures

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of the above two characteristics of general non-linear involutive systems. In our investigation, a principal role will be played by the introduced concept "characteristic module". It is to be emphasized that we deal with non-linear involutive systems with several unknown functions and that we discuss also the case when the characters of higher order do not necessarily vanish. Besides, our investigation enables us to establish the method of integration for involutive systems whose characters of order more than one vanish, generalizing the one obtained recently by the present author ([9-10]).

All notions such as functions and manifolds occurring in this paper are assumed to be in the real or complex analytic category though our argument can be carried out in the C^∞ category except when the Cartan-Kähler existence theorem is applied.

Our main results are summarized as follows. Let \mathcal{O} be a system of partial differential equations of order l in a fibered manifold $(\mathcal{M}, \mathcal{N}, \rho)$. It is defined to be a locally finitely generated subsheaf of ideals in the sheaf of rings of germs of analytic functions on the space of l -jets of cross-sections of $(\mathcal{M}, \mathcal{N}, \rho)$. Suppose that \mathcal{O} is involutive. We introduce the characteristic module M of \mathcal{O} by generalizing the characteristic ideal defined for systems with a single unknown function (cf. Kakié [10]); The latter is, roughly speaking, the ideal in a polynomial ring generated by the characteristic polynomials of all differential equations in \mathcal{O} . The module M is a homogeneous submodule of that Noetherian graded module L over a polynomial ring which is canonically associated with $(\mathcal{M}, \mathcal{N}, \rho)$. The characteristic module M admits a reduced primary decomposition in $L: M = \bigcap_{j=1}^{\nu} Q_j$ where each Q_j is a primary homogeneous submodule of L . For a homogeneous submodule N of L , we denote by $\mu(M; L)$ the multiplicity of N with respect to L ; $\mu(M; L)$ is defined to be the positive integer obtained from the coefficient of the highest degree r of the Hilbert characteristic polynomial of the graded module L/N by multiplying it by $r!$. Let s_1, s_2, \dots, s_n be the Cartan's characters of \mathcal{O} of order $1, 2, \dots, n$ respectively ($n = \dim \mathcal{N}$). One of the fundamental results is the following: Assume that \mathcal{O} is involutive and that $s_p > 0, s_{p+1} = \dots = s_n = 0$. Let \mathfrak{P}_j be the associated prime ideal of $Q_j (1 \leq j \leq \nu)$. Then the greatest projective dimension of the homogeneous prime ideals \mathfrak{P}_j is equal to $p-1$.

Moreover the following is valid:

$$s_p = \sum_{\text{proj dim } \mathfrak{P}_j = p-1} \mu(Q_j; L)$$

(Theorem III in § 5). The primary decomposition of M enables us to investigate algebraic structures of Monge characteristics of \mathcal{O} . Corresponding to each \mathfrak{P}_j , a Monge characteristic system of \mathcal{O} is defined (§§ 6-7). Furthermore we can give a sufficient condition in order that the differential system associated with \mathcal{O} admits Cauchy-Cartan characteristics (Theorem IV). Suppose that $s_2 = \dots = s_n = 0$. In this case, we can clarify more completely algebraic structures of Monge characteristics. In fact, considering the problem of constructing from \mathcal{O} suitable involutive systems for integration, we can show effectiveness of Monge characteristics (Theorems V and VI in § 9). It is a classical problem to find a process of reducing the solution of the Cauchy problem for \mathcal{O} with a given initial value to the integration of a completely integrable Pfaffian system. As an application of our theory, we can solve this problem for \mathcal{O} with $s_2 = \dots = s_n = 0$ (cf. Kakié [9-10]).

This paper is divided into three parts. Part I (§§ 1-3) is devoted to the investigation on modules associated with involutive subspaces. Part II (§§ 4-7) is concerned with involutive systems of partial differential equations. The concept of characteristic module is introduced and algebraic structures of Monge characteristics is studied. In part III (§§ 8-10), we deal with only involutive systems of which characters of order more than one vanish.

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Part I. Modules Associated with Involutive Subspaces

§ 1. Primary Decomposition

Let K be a field of characteristic zero, and R be the polynomial

ring over K in n variables. The ring R is a graded one with usual direct sum decomposition: $R = \sum_{q=0}^{\infty} R_q$. Let $L = \sum_{q=0}^{\infty} L_q$ be a finitely generated graded R -module. Obviously L is a Noetherian module; it satisfies the maximal condition concerning its submodules. Let $M = \sum_{q=0}^{\infty} M_q$ be a homogeneous R -submodule of L . The submodule M admits a reduced primary decomposition in $L: M = \bigcap_{j=1}^{\nu} Q_j$ where the modules Q_j are homogeneous primary submodules of L (cf. Bourbaki [1], Chap. IV). Let Q_j be \mathfrak{P}_j -primary in L . For any element ξ in R , the homomorphism h_{ξ} in the R -module L/Q_j defined by $h_{\xi}(z) = \xi z$ is injective or locally nilpotent (presque nilpotent). The ideal \mathfrak{P}_j is the set of all elements $\xi \in R$ such that the homomorphism h_{ξ} is locally nilpotent. Since M is a homogeneous submodule, the ideals \mathfrak{P}_j are necessarily homogeneous. The reduced primary decomposition is unique in the sense that the number ν of the submodules Q_j and the set $\{\mathfrak{P}_1, \dots, \mathfrak{P}_{\nu}\}$ are uniquely determined only by M . Let $\text{Ass}(N)$ denote the set of prime ideals associated with an R -module N . Then $\text{Ass}(L/M) = \{\mathfrak{P}_1, \dots, \mathfrak{P}_{\nu}\}$. Moreover if \mathfrak{P}_j is a minimal element in the set $\text{Ass}(L/M)$, the corresponding submodule Q_j is uniquely determined by M .

Let $P(N, x)$ denote the characteristic polynomial of Hilbert of N , where $N = \sum_{q=0}^{\infty} N_q$ is a finitely generated graded R -module. (cf. Serre [17], Chap. II; Zeriski-Samuel [18], Chap. VII, § 12). The number $P(N, q)$ indicates $\dim N_q$ for every sufficiently large integer q .

Let r be the degree of $P(L/M, x)$. The polynomial $P(L/M, x)$ can be written in the following form:

$$P(L/M, x) = a_0 \binom{x}{r} + a_1 \binom{x}{r-1} + \dots + a_r,$$

where $\binom{x}{s} = \frac{x(x-1)\cdots(x-s+1)}{s!}$ and the coefficients a_0, \dots, a_r are integers. The coefficient a_0 is usually called the *multiplicity of M with respect to L over K* . We shall denote it by $\mu(M; L)$; $\mu(M; L) = a_0$. It is a positive integer except the case when $M_q = L_q$ for large q .

The degree of $P(L/M, x)$ is closely related to the projective dimensions of the homogeneous ideals in $\text{Ass}(L/M)$. In fact, the following is valid (Zariski-Samuel [18], p. 235, Theorem 42').

Lemma 1. *The degree of the characteristic polynomial $P(L/M, x)$ is equal to the greatest projective dimension of the associated prime ideals $\text{Ass}(L/M)$ of L/M .*

Let \mathcal{G}_p be the set of all p -dimensional subspaces of the vector space R_1 of dimension n over K ($0 \leq p \leq n$). \mathcal{G}_p forms a $(n-p)p$ -dimensional algebraic variety with Plücker coordinates. This is usually called Grassmann manifold when K is the field of real or complex numbers.

The multiplicity $\mu(M; L)$ possesses a geometrical interpretation. In fact, one has the following

Lemma 2. *Let r be the degree of $P(L/M, x)$. Then the following are valid:*

(i) *Let ρ be the smallest integer p for which there is an element $F_p \in \mathcal{G}_p$ such that $(M + F_p L) \cap L_q$ coincides with L_q for all large q . Then r is equal to $\rho - 1$.*

(ii) *For any element $F_r \in \mathcal{G}_r$ outside its proper algebraic subvariety, the characteristic polynomial of the graded R -module $L/(M + F_r L)$ is of degree 0 and $\mu(M; L)$ is equal to $\mu(M + F_r L; L)$. Moreover*

$$\mu(M; L) = \min \{ \dim L_q / (M + F_r L) \cap L_q; F_r \in \mathcal{G}_r \}$$

for sufficiently large q .

Proof. We prove this lemma by dividing into five steps. In this proof, we denote $\mu(M; L)$ simply by $\mu(M)$. By polynomials of negative degrees we mean zero.

(a) Let F_p be an arbitrary element of \mathcal{G}_p ($0 \leq p \leq n$). Then the characteristic polynomial of $L/(M + F_p L)$ is of degree $\geq r - p$. If F_r is an element of \mathcal{G}_r such that the characteristic polynomial of $L/(M + F_r L)$ is exactly of degree 0, then $\mu(M)$ is equal to or less than $\mu(M + F_r L)$.

Proof. We write $N = L/M$. Let us consider the homomorphism θ_η from N into itself defined by $\theta_\eta(y) = \eta y$, where $\eta \in R_1$. Writing $S = \ker \theta_\eta$ and $T = N/\theta_\eta(N)$, we have the exact sequence of modules

$$0 \rightarrow S \xrightarrow{i} N \xrightarrow{\theta_\eta} N \xrightarrow{j} T \rightarrow 0,$$

where i and j are natural homomorphisms. This gives rise to the exact sequence of vector spaces over K

$$0 \rightarrow S_q \xrightarrow{i} N_q \xrightarrow{\theta_q} N_{q+1} \xrightarrow{j} T_{q+1} \rightarrow 0,$$

where $N = \sum_{q=0}^{\infty} N_q$, $S = \sum_{q=0}^{\infty} S_q$, $T = \sum_{q=0}^{\infty} T_q$ (direct sum decompositions of graded modules). Obviously

$$T_{q+1} = N_{q+1}/\theta_q(N_q) = L_{q+1}/(M + \eta L) \cap L_{q+1}.$$

Hence, applying the well-known lemma (Zariski-Samuel [18], p. 233, Lemma), we get from the above exact sequence the following equalities:

$$(1) \quad P(N, q+1) - P(N, q) = P(L/(M + \eta L), q+1) - \dim S_q$$

(q : sufficiently large).

The term on the left side can be written in the form

$$\begin{aligned} P(N, q+1) - P(N, q) &= \mu(M) \binom{q}{r-1} + b_1 \binom{q}{r-2} + \cdots + b_{r-1} \\ &= \mu(M) \binom{q+1}{r-1} + c_1 \binom{q+1}{r-2} + \cdots + c_{r-1}. \end{aligned}$$

Therefore we conclude from (1) that the characteristic polynomial of $L/(M + \eta L)$ is of degree $\geq r-1$, and that if it is of degree $r-1$, $\mu(M) \leq \mu(M + \eta L)$. Proceeding step by step in just the same manner, we know that the characteristic polynomial of $L/\{M + (\eta_1, \dots, \eta_p)L\}$ is of degree $\geq r-p$ ($1 \leq p \leq n$), and that if the characteristic polynomial of $L/\{M + (\eta_1, \dots, \eta_r)L\}$ is exactly of degree 0, then that of $L/\{M + (\eta_1, \dots, \eta_p)L\}$ is of degree $r-p$ for each $p=1, 2, \dots, r$. In this case it also holds that

$$\mu(M) \leq \mu(M + \eta_1 L) \leq \cdots \leq \mu(M + (\eta_1, \dots, \eta_r)L),$$

where $\eta_p \in R_1$. this completes the proof of (a).

(b) There exists a sequence $F_0 \subset \cdots \subset F_r \subset F_{r+1}$ where $F_p \in \mathcal{G}_p$ such that the characteristic polynomial of $L/(M + F_p L)$ is exactly of degree $r-p$ ($0 \leq p \leq r+1$), and that $\mu(M) = \mu(M + F_1 L) = \cdots = \mu(M + F_r L)$.

Proof. We use the same notations as in the proof of (a). Suppose firstly that none of the ideals in $\text{Ass}(N) = \{\mathfrak{A}_1, \dots, \mathfrak{A}_\nu\}$ coincides with the maximal ideal $\mathfrak{X} = \sum_{q=1}^{\infty} R_q$ in R . We can choose an element $\eta_1 \in R_1$

not belonging to any ideal \mathfrak{P}_j . Then the homomorphism θ_{η_1} is injective (cf. Bourbaki [1], p.132, Corollaire 2). Hence $\ker \theta_{\eta_1} = 0$. Suppose secondly that one of the ideals \mathfrak{P}_j , say \mathfrak{P}_ν , coincides with \mathfrak{K} . Since $\mathfrak{K}^\sigma L \subset Q_\nu$ for some positive integer σ , it follows that $Q_\nu \cap L_q = L_q$ for large q . Therefore $M_q = \bigcap_{j=1}^{\nu-1} Q_j \cap L_q$ for large q . Let η_1 be an element in R_1 not belonging to $\bigcup_{j=1}^{\nu-1} \mathfrak{P}_j$. By the same reason as above, θ_{η_1} is injective in N_q , and hence $\ker \theta_{\eta_1} \cap N_q = 0$ for large q . Hence, in any case, we have the equality (1) in which $\eta = \eta_1$ and $\dim S_q = 0$. By the remark below (1), it follows that $P(L/(M + \eta_1 L), x)$ is of degree $r - 1$ and that $\mu(M + \eta_1 L) = \mu(M)$. Repeating the same argument successively, we can find $r + 1$ elements $\eta_1, \dots, \eta_{r+1}$ in R_1 such that the characteristic polynomial of $L/\{M + (\eta_1, \dots, \eta_p)L\}$ is of degree $r - p$ for each $p = 1, 2, \dots, r + 1$, and that $\mu(M) = \mu(M + \eta_1 L) = \dots = \mu(M + (\eta_1, \dots, \eta_r)L)$. As is readily seen, $\eta_1, \dots, \eta_{r+1}$ are necessarily linearly independent. Let F_p be the space spanned by η_1, \dots, η_p . Then the space F_p have the required property.

(c) There is an element F_{r+1} of \mathcal{G}_{r+1} such that

$$(2) \quad (M + F_{r+1}L) \cap L_q = L_q \text{ for large } q.$$

Proof. This is an immediate consequence of (b).

(d) Let $N' = \sum_{q=0}^{\infty} N'_q$ be a homogeneous submodule of L such that there is an element $\eta \in R_1$ for which $(N' + \eta L) \cap L_q = L_q$ for large q . Then $\dim L_q/N'_q \geq \dim L_{q+1}/N'_{q+1}$ for large q .

Proof. The assumption implies that the homomorphism defined by the multiplication by η from L_q/N'_q to L_{q+1}/N'_{q+1} is surjective for large q . The required result follows immediately from this.

(e) We are now in a position to complete the proof of Lemma 2. If there is an element $F_{d+1} \in \mathcal{G}_{d+1}$ such that $(M + F_{d+1}L) \cap L_q = L_q$ for large q , then $P(L/M, x)$ is of degree $\leq d$. In fact, recalling that $P(L/M, q) = \dim L_q/M_q$ for large q , we find no difficulty in showing that

$$P(L/M, q) \leq \text{Const.} \binom{q+d}{q} \text{ for large } q.$$

This inequality implies that $P(L/M, x)$ is of degree $\leq d$. It follows from this and (c) that the assertion (i) is valid.

Let us next prove (ii). Since (c) is valid, we can readily prove

that the set S of elements $F_r \in \mathcal{Q}_r$ for which there is no element $\eta \in \mathcal{R}_1$ such that (2) holds for the space F_{r+1} spanned by F_r and η forms a proper algebraic subvariety of \mathcal{Q}_r . We put

$$\delta_q(F_r) = \dim L_q / (M + F_r L) \cap L_q,$$

$$\delta_q = \min \{ \delta_q(F_r); F_r \in \mathcal{Q}_r \}.$$

The function $\delta_q(F_r)$ attains its minimum value outside a proper algebraic subvariety $D(q)$ of \mathcal{Q}_r . By virtue of (b), there exists $F_r \in \mathcal{Q}_r$ such that $\mu(M) = \mu(M + F_r L) = \delta_q(F_r)$ for large q . Hence we have $\delta_q \leq \mu(M)$ for large q . Let F_r be an element of \mathcal{Q}_r not belonging to its proper subvariety $S \cup D(q)$. Since the assumption (d) is satisfied for the module $M + F_r L$, we have $\delta_q(F_r) \geq \delta_{q+1}(F_r)$ if q is large enough. It follows that

$$(3) \quad \delta_q = \delta_q(F_r) \geq \delta_{q+1}(F_r) \geq \delta_{q+1}.$$

Thus the sequence $\{\delta_q\}$ of non-negative integers is a descending one when q varies sufficiently large integers. We conclude from this fact that $\delta_q = \delta_{q+1} = \delta_{q+2} = \dots$ for large q . Then it follows from (3) that for any F_r outside $S \cup D(q)$, $\delta_q(F_r) = \delta_{q+1}(F_r) = \delta_{q+2}(F_r) = \dots$. From this it follows that the characteristic polynomial of $M + F_r L$ is of degree 0 and $\mu(M + F_r L) = \delta_q(F_r) = \delta_q$ for such F_r . On the other hand, (a) asserts that $\mu(M) \leq \mu(M + F_r L) = \delta_q$ for such F_r . Combining this with the fact $\delta_q \leq \mu(M)$ shown already, we have $\mu(M) = \delta_q$ for large q . Thus we have all the required results in (ii). Q.E.D.

Let us now consider a reduced primary decomposition $M = \bigcap_{j=1}^r Q_j$ in L , Q_j being \mathfrak{P}_j -primary. If the projective dimension of \mathfrak{P}_k is equal to $\deg P(L/M, x)$, then \mathfrak{P}_k is a minimal element in the set $\{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$ (cf. Lemma 1), and hence the corresponding module Q_k is uniquely determined only by M by the uniqueness theorem of reduced primary decompositions: In particular, $\mu(Q_k; L)$ is also so. There exists a link between the multiplicity $\mu(M; L)$ and the multiplicities $\mu(Q_j; L)$. In fact, the following theorem is valid.

Theorem I. Let $M = \bigcap_{j=1}^{\nu} Q_j$ be a reduced primary decomposition of M in L , and let Q_j be \mathfrak{P}_j -primary. Let r denote the degree of the Hilbert characteristic polynomial $P(L/M, x)$. Then

$$\mu(M; L) = \sum_{\text{proj dim } \mathfrak{P}_j=r} \mu(Q_j; L).$$

To prove this theorem, we shall recall the following lemma.

Lemma 3. Let N_1 and N_2 be homogeneous submodules of L . Then

$$P(L/N_1 \cap N_2, x) + P(L/(N_1 + N_2), x) = P(L/N_1, x) + P(L/N_2, x).$$

Proof. Obviously we have the following two exact sequences of R -modules with natural homomorphisms:

$$0 \rightarrow (N_1 + N_2)/N_1 \rightarrow L/N_1 \rightarrow L/(N_1 + N_2) \rightarrow 0,$$

$$0 \rightarrow N_2/N_1 \cap N_2 \rightarrow L/N_1 \cap N_2 \rightarrow L/N_2 \rightarrow 0.$$

It follows from these that the following equalities are valid for all sufficiently large q :

$$P((N_1 + N_2)/N_1, q) - P(L/N_1, q) + P(L/(N_1 + N_2), q) = 0,$$

$$P(N_2/N_1 \cap N_2, q) - P(L/N_1 \cap N_2, q) + P(L/N_2, q) = 0.$$

$(N_1 + N_2)/N_1$ is isomorphic to $N_2/N_1 \cap N_2$, and hence we have the desired result. Q.E.D.

Let $\mathfrak{r}_L(M)$ be the ideal in R defined to be the set of all elements $\xi \in R$ such that $\xi^\sigma L \subset M$ for some integer σ . This ideal $\mathfrak{r}_L(M)$ coincides with the intersection of the prime ideals associated with L/M ; $\mathfrak{r}_L(M) = \bigcap_{j=1}^{\nu} \mathfrak{P}_j$. It is clear that $\mathfrak{r}_L(N_1 + N_2)$ contains $\mathfrak{r}_L(N_1) + \mathfrak{r}_L(N_2)$ for any two submodules N_1 and N_2 of L .

Proof of Theorem I: Applying Lemma 3 to $N_1 = Q_1$ and $N_2 = \bigcap_{j=2}^{\nu} Q_j$, we have

$$\begin{aligned}
 P(L/M, x) &= P(L/\bigcap_{j=1}^{\nu} Q_j, x) \\
 &= P(L/Q_1, x) + P(L/\bigcap_{j=2}^{\nu} Q_j, x) - P(L/N, x),
 \end{aligned}$$

where we write $N = Q_1 + \bigcap_{j=2}^{\nu} Q_j$. Let us prove that the prime ideals associated with L/N are of projective dimension less than r . Let \mathfrak{A} be any element of $\text{Ass}(L/N)$. Since $r_L(N)$ contains $\mathfrak{P}_1 + \bigcap_{j=2}^{\nu} \mathfrak{P}_j$, the ideal \mathfrak{A} also contains the latter ideal. It follows that \mathfrak{A} contains \mathfrak{P}_1 and \mathfrak{P}_k for some $k \neq 1$. If one of \mathfrak{P}_1 and \mathfrak{P}_k is of projective dimension $< r$, then it is obvious that \mathfrak{A} is also so. Assume that both ideals \mathfrak{P}_1 and \mathfrak{P}_k are of projective dimension r . Then $\mathfrak{P}_1 + \mathfrak{P}_k$ is distinct from \mathfrak{P}_1 , for $\mathfrak{P}_1 \neq \mathfrak{P}_k$. We conclude that $\mathfrak{A} \not\supseteq \mathfrak{P}_1$. Hence \mathfrak{A} is of projective dimension $< r = \text{proj dim } \mathfrak{P}_1$, for in the contrary case, both prime ideals would have the same dimension and one of the two ideals contains the other, and hence they would coincide. Thus we have the desired result. What we have just deduced implies, by virtue of Lemma 1, that $P(L/N, x)$ is of degree $< r$. Consequently the above formula indicates that the term of (the highest) degree r of $P(L/M, x)$ is equal to the sum of those of $P(L/Q_1, x)$ and $P(L/\bigcap_{j=2}^{\nu} Q_j, x)$; that is,

$$P(L/M, x) \equiv P(L/Q_1, x) + P(L/\bigcap_{j=2}^{\nu} Q_j, x) \pmod{1, x, \dots, x^{r-1}}.$$

Repeating the same argument for the modules $\bigcap_{j=k}^{\nu} Q_j$ ($2 \leq k \leq \nu$) successively, we obtain

$$P(L/M, x) \equiv \sum_{j=1}^{\nu} P(L/Q_j, x) \pmod{1, x, \dots, x^{r-1}}.$$

Since $P(L/Q_j, x)$ is of degree r if and only if \mathfrak{P}_j is of projective dimension r (Lemma 1), it follows that $\mu(M; L)$ is equal to the sum of $\mu(Q_k; L)$ where Q_k are those submodules the prime ideals associated with which are of projective dimension r . Q.E.D.

In concluding this section, we recall that the *exponent* of a \mathfrak{P} -primary submodule Q of L is, by definition, the smallest integer σ for which $\mathfrak{P}^{\sigma} L \subset Q$.

§ 2. Involutive Subspaces

Let E and F be vector spaces over K of dimensions m and n respectively. Let F^* denote the dual space to F and $S^q(F^*)$ denote the q -th symmetric power on F^* . The vector space $E \otimes S^l(F^*)$ can be considered as the subspace of the space $\{E \otimes S^{l-1}(F^*)\} \otimes F^* = \text{Hom}(F, E \otimes S^{l-1}(F^*))$ ($l \geq 1$). Let A be a subspace of $E \otimes S^l(F^*)$. The prolongation pA of A is defined by

$$pA = A \otimes F^* \cap E \otimes S^{l+1}(F^*).$$

We shall denote by $\mathcal{G}_k(F)$ the algebraic variety of all k -dimensional subspaces of F . For an element $F_k \in \mathcal{G}_k(F)$, $A(F_k)$ is defined to be the space of those elements in A which annihilate the subspace F_k . Let us put

$$g_k[A] = \min \{ \dim A(F_k); F_k \in \mathcal{G}_k(F) \} \quad (k=0, 1, \dots, n).$$

It is obvious that $g_0[A] \geq g_1[A] \geq \dots \geq g_n[A] = 0$. The space A is called an involutive subspace of $\{E \otimes S^{l-1}(F^*)\} \otimes F^*$ if and only if $\dim pA$ is equal to $\sum_{k=0}^n g_k[A]$ (cf. Kurashi [12], § 6). Throughout this and the next sections, by involutive subspaces we mean involutive subspaces of $\{E \otimes S^{l-1}(F^*)\} \otimes F^*$ except when otherwise is expressly stated.

Let R be the graded ring $\sum_{q=0}^{\infty} S^q(F)$ and R_q be $S^q(F)$ ($S^0(F)$ means K). The graded ring R is (isomorphic to) the polynomial ring over K in n variables. We shall denote by L the graded R -module $E^* \otimes R$:

$$L = \sum_{q=0}^{\infty} E^* \otimes S^q(F), L_q = E^* \otimes S^q(F).$$

This module L is finitely generated as an R -module, and hence it is a Noetherian R -module.

Let D be the annihilator of A in the space L_l dual to $E \otimes S^l(F^*)$. We shall use the following notation: Let Ξ and Z be subsets of R and L respectively. The set of all finite sums $\sum_i \xi_i z_i$ where $\xi_i \in \Xi, z_i \in Z$ is denoted by ΞL . Now it is not hard to see that the annihilator of $A(F_k)$ is $D + (F_k L) \cap L_l$. Therefore we have

$$\begin{aligned} \dim A(F_k) &= \dim L_l / \{D + (F_k L) \cap L_l\} \\ &= \dim \{L_l / (F_k L) \cap L_l\} / \{(D + (F_k L) \cap L_l) / (F_k L) \cap L_l\} \end{aligned}$$

$$= \binom{l+n-k-1}{l} - \dim D/D \cap F_k L,$$

where $\binom{p+q-1}{p}$ is the binomial coefficient $(p+q-1)!/p!q!$, which indicates the number of forms of degree p in q variables. Since

$$\binom{l+n}{l+1} = \binom{l+n-1}{l} + \binom{l+n-2}{l} + \cdots + \binom{l}{l},$$

it follows from the above equalities that

$$\sum_{k=0}^n g_k[A] = \binom{l+n}{l+1} - \sum_{k=0}^n \max \{ \dim D/D \cap F_k L; F_k \in \mathcal{G}_k(F) \}.$$

On the other hand, the annihilator of pA in L_{l+1} coincides with FD (see Kuranishi [12], § 6, Lemma 6.1), and $\dim L_{l+1} = \binom{l+n}{l+1}$, and hence we have

$$\dim pA = \binom{l+n}{l+1} - \dim FD.$$

Let N be the homogeneous submodule of L generated by D ; $N = \sum_{q=0}^{\infty} N_q$. Clearly $N_q = 0$ for $q = 0, 1, \dots, l-1$, $N_l = D$ and $N_{l+1} = FD$. Then it follows from the above argument that A is an involutive subspace if and only if

$$(4) \quad \dim N_{l+1} = \sum_{k=0}^n \max \{ \dim N_l/N_l \cap F_k L; F_k \in \mathcal{G}_k(F) \}.$$

Let N be an arbitrary homogeneous submodule of L such that it is generated by a subspace of L_l . It is clear that $\dim N_l/N_l \cap F_k L$ where $F_k \in \mathcal{G}_k(F)$ takes its maximum value outside a certain proper algebraic subvariety of $\mathcal{G}_k(F)$. Hence we find no difficulty in proving that there is a sequence $F_0 \subset F_1 \subset \cdots \subset F_n = F$ where $F_k \in \mathcal{G}_k(F)$ such that $\dim N_l/N_l \cap F_k L$ attains the maximum value ($k = 0, 1, \dots, n$). The submodule $F_k L$ of L is obviously a (F_k) -primary submodule of L , (F_k) being the ideal in R generated by F_k . Accordingly $F_{k+1}z \subset F_k L$ implies $z \in F_k L$, and hence we have

$$\{z \in N_l; F_{k+1}z \subset F_k N_l\} \subset N_l \cap F_k L.$$

The vector space $F_{k+1}N_l/F_k N_l$ is isomorphic to $\hat{\xi}_{k+1}N_l/\hat{\xi}_{k+1}N_l \cap F_k N_l$,

where ξ_{k+1} is an element of F_{k+1} not belonging to F_k , and the latter space has the same dimension as the space $N_l/\{z \in N_l; \xi_{k+1}z \in F_k N_l\}$. Hence we obtain the inequalities

$$(5) \quad \dim F_{k+1}N_l/F_k N_l \geq \dim N_l/N_l \cap F_k L \quad (k=0, 1, \dots, n-1).$$

Since $N_{l+1} = FN_l$, it follows that

$$(6) \quad \begin{aligned} \dim N_{l+1} &= \sum_{k=0}^{n-1} \dim F_{k+1}N_l/F_k N_l \\ &\geq \sum_{k=0}^n \dim N_l/N_l \cap F_k L \\ &= \sum_{k=0}^n \max\{\dim N_l/N_l \cap F_k' L; F_k' \in \mathcal{G}_k(F)\}. \end{aligned}$$

Accordingly we have the following

Proposition 1. *Let N be a homogeneous submodule of L generated by a subspace of L_l ; $N = \sum_{q=l}^{\infty} N_q$. Then*

$$\dim N_{l+1} \geq \sum_{k=0}^n \max\{\dim N_l/N_l \cap F_k L; F_k \in \mathcal{G}_k(F)\}.$$

The equality holds if and only if the subspace of $E \otimes S^l(F^)$ of which annihilator in L_l is the space N_l is an involutive subspace.*

Suppose that A is an involutive subspace. Let N be the submodule of L generated by the annihilator D of A . Then equality (4) holds. Let $F_0 \subset F_1 \subset \dots \subset F_n$ be a sequence of subspaces of F such that the function $\dim N_l/N_l \cap F_k' L$ on $\mathcal{G}_k(F)$ attains the maximum value at F_k ($0 \leq k \leq n$). Such a sequence $F_0 \subset F_1 \subset \dots \subset F_n$ actually exists as we already explained. Combining (6) with (5), we see that the equalities in (5) hold. The argument above (5) indicates that the equalities in (5) hold if and only if

$$\{z \in N_l; F_{k+1}z \subset F_k N_l\} = N_l \cap F_k L \quad (k=0, 1, \dots, n-1).$$

Conversely, we readily see that if these conditions are satisfied for some sequence $F_0 \subset F_1 \subset \dots \subset F_n$, then equality (4) holds, that is, A is an involutive subspace. These results can be summarized as follows.

Lemma 4. *Let A and N be as above. A is an involutive subspace if and only if there exists a sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ where $F_k \in \mathcal{G}_k(F)$ such that*

$$\dim F_{k+1}N_l/F_kN_l = \dim N_l/N_l \cap F_kL \quad (k=0, 1, \dots, n-1),$$

or such that

$$\{z \in N_l; F_{k+1}z \subset F_kN_l\} = N_l \cap F_kL \quad (k=0, 1, \dots, n-1).$$

In this case the latter conditions are satisfied for $F_0 \subset F_1 \subset \cdots \subset F_n$ if and only if the function $\dim N_l/N_l \cap F_kL$ on $\mathcal{G}_k(F)$ attains the maximum value at F_k ($0 \leq k \leq n$).

Using this lemma, we shall deduce the following

Proposition 2. *Let N be as in Proposition 1. If equality (4) holds, then the following equality holds for any integer $q \geq l$:*

$$\dim N_{q+1} = \sum_{k=0}^n \max \{ \dim N_q/N_q \cap F_kL; F_k \in \mathcal{G}_k(F) \}.$$

Proof. By Lemma 4, the desired equality holds if and only if there is a sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ where $F_k \in \mathcal{G}_k(F)$ such that

$$(7)_q \quad \{z \in N_q; F_{k+1}z \subset F_kN_q\} = N_q \cap F_kL \quad (k=0, 1, \dots, n-1).$$

The assumption means that $(7)_l$ holds for some sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ by Lemma 4. We shall prove that $(7)_q$ holds for the same sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ by induction on q . Suppose that $(7)_q$ holds. Let us prove $(7)_{q+1}$. To do so, it is sufficient to prove

$$(8) \quad W = \{z \in N_{q+1}; F_{k+1}z \subset F_kN_{q+1}\} \supset N_{q+1} \cap F_kL$$

for each fixed k (cf. the argument above (5)). We write $V_s = F_sN_q \cap F_kL$. We first prove that V_{s+1} is contained in V_k for $s \geq k$ (, and hence $V_{s+1} = V_k$ for $s \geq k$). To show this, it suffices to prove that for each $s \geq k$, V_{s+1} is contained in V_s . Let z be any element of V_{s+1} . When F_{s+1} is spanned by F_s and ξ_{s+1} , z can be written as follows: $z = \xi_{s+1}z_{s+1} + \hat{z}$ where $z_{s+1} \in N_q$, $\hat{z} \in F_sN_q$. Since $\xi_{s+1} \notin F_s$, z_{s+1} is necessarily an element of $F_sL \cap N_q = V_s$. It follows from the induction assumption $(7)_q$ that $\xi_{s+1}z_{s+1} \in F_sN_q$. Consequently $z = \xi_{s+1}z_{s+1} + \hat{z}$ belongs to F_sN_q . This implies that $z \in V_s$. Hence $V_{s+1} \subset V_s$ as desired. We now prove (8). Obviously $V_k \subset W$.

Therefore, by what we have just deduced, $V_s \subset W$ for all $s=0, 1, \dots, n$. In particular we have $V_n \subset W$. Since $N_{q+1} = F_n N_q$, $V_n = N_{q+1} \cap F_k L$. Combining these, we have (8) as desired. Q.E.D.

Remark. The above proof also implies the following: If $F_0 \subset F_1 \subset \dots \subset F_n$ where $F_k \in \mathcal{G}_k(F)$ is a sequence such that $(7)_l$ holds. Then $F_s N_q \cap F_k L = F_k N_q$ for $s \geq k$ and $q \geq l$.

Note. It is well-known that $\dim pA \leq \sum_{k=0}^n g_k[A]$, and that if A is an involutive subspace, then pA is also an involutive subspace of $\{E \otimes S^l(F^*)\} \otimes F^*$ (cf. Kuranishi [12]). Propositions 1 and 2 are equivalent to these facts respectively. The reason why we give our direct proofs is that the discussion in the course of the proofs is needed in the following investigation.

§ 3. Modules Associated with Involutive Subspaces

Let R and L be the same ring and R -module as in Section 2 respectively. Let N be a homogeneous submodule of L . We denote by \mathfrak{X} the maximal ideal in R ; $\mathfrak{X} = \sum_{q=1}^{\infty} R_q$.

Notation. We denote by $H(N)$ the smallest (homogeneous) submodule M of L which contains N and which possesses the property

$$(9) \quad \mathfrak{X}z \subset M \text{ implies } z \in M.$$

The submodule $H(N)$ is constructed from N as follows. Writing $M^{(0)} = N$, we define the submodules $M^{(s)}$ of L inductively by

$$M^{(s+1)} = \{z \in L; \mathfrak{X}z \subset M^{(s)}\} \quad (s=0, 1, 2, \dots).$$

It is obvious that $M^{(0)} \subset M^{(1)} \subset M^{(2)} \subset \dots$. Since L is Noetherian, there exists an integer s such that $M^{(s)} = M^{(s+1)} = \dots$. Then $H(N)$ is nothing else than $\bigcup_{i=0}^{\infty} M^{(i)} = M^{(s)}$. It is to be noted that $H(N) \cap L_q$ coincides with $N \cap L_q$ for all sufficiently large q . Conversely, if N and N' be two homogeneous submodules of L such that $N \cap L_q = N' \cap L_q$ for all large q , then $H(N) = H(N')$.

We can interpret the process of constructing $H(N)$ from N quite differently, applying the theory of primary decompositions: *A reduced*

primary decomposition of $H(N)$ in L is obtained from that of N by omitting the component which is \mathfrak{K} -primary (if it exists). When N is itself \mathfrak{K} -primary, by this statement we mean that $H(N) = L$. Let us prove this fact. Let Q be a \mathfrak{P} -primary submodule of L . We first prove that if $\mathfrak{P} \subsetneq \mathfrak{K}$, then $H(Q) = Q$. It is enough to show that Q has the property (9). There exists an element $\xi \in \mathfrak{K}$ not belonging to \mathfrak{P} . It follows from the very definition of primary modules that $\xi z \in Q$ implies $z \in Q$. This shows that $\mathfrak{K}z \subset Q$ implies $z \in Q$. Hence our desired result follows. We next prove that if $\mathfrak{P} = \mathfrak{K}$, then $H(Q) = L$. Since $\mathfrak{K}^\sigma L \subset Q$ for some positive integer σ and L is generated by L_0 , $Q \cap L_q = L_q$ for $q \geq \sigma$. From this and the definition of $H(Q)$ it follows that $H(Q) = L$. We now prove our original assertion. Let $N = \bigcap_{j=1}^p N_{(q_j)}$ be a reduced primary decomposition in L . It is not hard to show that $H(N) = \bigcap_{j=1}^p H(N_{(q_j)})$ (cf. the first process of constructing $H(N)$ from N). Hence our assertion follows from what we have just deduced.

Let A be a subspace of $E \otimes S^l(F^*)$. We associate with A an R -submodule of L as follows.

Definition. Let N be the submodule of L generated by the annihilator D of A in L_l . We define the (homogeneous) submodule $M(A)$ of L to be $H(N)$, and call it the *module associated with A* .

It is clear that the module associated with pA coincides with $M(A)$; In other words, the module associated with a space is unchanged under the prolongation. Moreover there never exists the maximal ideal \mathfrak{K} in $\text{Ass}(L/M(A))$ as we explained above. These are the reasons why we associate with A the module $M(A) = H(N)$ instead of N itself.

In the remaining part of this section, we devote ourselves to investigating the structure of the module $M(A)$ under the condition that A is an involutive subspace. A link between $M(A)$ and N (see the above definition) is given by the following lemma. We write: $M(A) = \sum_{q=0}^{\infty} M_q(A)$, $N = \sum_{q=0}^{\infty} N_q$ (direct sum decomposition of graded modules). Remark that $N_q = 0$ for $q = 0, 1, \dots, l-1$ and $N_l = D$.

Lemma 5. *Suppose that A is an involutive subspace. Then, under the above circumstances, $M_q(A) = N_q$ for all $q \geq l$.*

This lemma indicates that the elements to be added to N in order to obtain the module $M(A)$ (if they exist) are of degrees less than l . Of course, this is not valid in general when A is not an involutive subspace.

Proof. For brevity, we write $M = M(A)$, $M_q = M_q(A)$. It follows from the definition of $H(N)$ that there exists an integer s such that $M_q = N_q$ for $q \geq s$. Assume $s > l$. Let us prove $M_{s-1} = N_{s-1}$. Obviously $M_{s-1} \supset N_{s-1}$. Let F_k be an element of $\mathcal{G}_k(F)$ ($0 \leq k \leq n$). The natural homomorphism from $N_{s-1}/N_{s-1} \cap F_k L$ to $M_{s-1}/M_{s-1} \cap F_k L$ is obviously injective. Therefore we have

$$\dim N_{s-1}/N_{s-1} \cap F_k L \leq \dim M_{s-1}/M_{s-1} \cap F_k L \quad (0 \leq k \leq n).$$

It follows from this that

$$\begin{aligned} \sum_{k=0}^n \max \{ \dim N_{s-1}/N_{s-1} \cap F_k L; F_k \in \mathcal{G}_k(F) \} \\ \leq \sum_{k=0}^n \max \{ \dim M_{s-1}/M_{s-1} \cap F_k L; F_k \in \mathcal{G}_k(F) \}. \end{aligned}$$

By virtue of Propositions 1 and 2 in § 2, the number on the left side is equal to $\dim N_s$. Besides, applying Proposition 1 to the module generated by M_{s-1} , we know that the number on the right side is equal to or less than $\dim M_s$. Combining these facts with $\dim M_s = \dim N_s$, we conclude that the above inequality is actually the equality. Since each term on the left side is equal to or less than the corresponding term on the right side, as a consequence of the equality, we have $\dim M_{s-1} = \dim N_{s-1}$. Hence $M_{s-1} = N_{s-1}$. If $s-1 > l$, by the same reasoning, we have $M_{s-2} = N_{s-2}$. Proceed step by step, we can complete the proof of the fact that $M_q = N_q$ for all $q \geq l$. Q.E.D.

Let us calculate the degree of the characteristic polynomial of $L/M(A)$ and $\mu(M(A); L)$ for an involutive subspace A .

Lemma 6. *Suppose that A is an involutive subspace and that $g_{p-1}[A] > 0$, $g_p[A] = 0$. Then the greatest projective dimension of the prime ideals in $\text{Ass}(L/M(A))$ is equal to $p-1$. Furthermore $\mu(M(A); L)$ is equal to $g_{p-1}[A]$.*

Proof. We write: $M = M(A)$. The annihilator of $A(F_p)$ in L_l is $M_l + (F_p L) \cap L_l$ (cf. Lemma 5), and hence the assumption $g_p[A] = 0$ implies that $(M + F_p L) \cap L_l = L_l$ for any $F_p \in \mathcal{G}_p(F)$ outside a proper subvariety. Therefore we can prove without any difficulty that there is a sequence $F_0 \subset F_1 \subset \cdots \subset F_n$ where $F_k \in \mathcal{G}_k(F)$ for which the conditions in Lemma 4 in § 2 are satisfied and moreover $(M + F_p L) \cap L_l = L_l$. Then by Remark at the end of § 2 and by Lemma 5, we have

$$(10) \quad F_{p-1} L \cap F_k M_l = F_{p-1} M_l \quad \text{for } k \geq p-1.$$

Let us prove the equality

$$(11) \quad \dim M_{l+1}/M_{l+1} \cap F_{p-1} L = \binom{l+n-p}{l+1} + \dim M_l/M_l \cap F_{p-1} L.$$

Remarking Lemma 5, we have the equalities

$$\begin{aligned} \dim M_{l+1}/M_{l+1} \cap F_{p-1} L &= \sum_{k=p}^n \dim F_k M_l / \{F_{p-1} L \cap F_k M_l + F_{k-1} M_l\} \\ &= \sum_{k=p}^n \dim F_k M_l / F_{k-1} M_l \quad (\text{by (10)}) \\ &= \sum_{k=p}^n \dim M_l / M_l \cap F_{k-1} L \quad (\text{by the choice of } F_k). \end{aligned}$$

Obviously $(M + F_k L) \cap L_l = L_l$ for $k \geq p$, and hence

$$\dim M_l / M_l \cap F_k L = \dim L_l / (F_k L) \cap L_l = \binom{l+n-k-1}{l}.$$

Since $\sum_{k=p}^{n-1} \binom{l+n-k-1}{l} = \binom{l+n-p}{l+1}$, it follows that equality (11) is indeed valid.

Let us put

$$\begin{aligned} \delta_q(F_{p-1}) &= \dim L_q / (M + F_{p-1} L) \cap L_q \\ &= \binom{q+n-p}{q} - \dim M_q / M_q \cap F_{p-1} L, \end{aligned}$$

$$\delta_q = \min \{ \delta_q(F_{p-1}); F_{p-1} \in \mathcal{G}_{p-1}(F) \}.$$

Equality (11) means that $\delta_i(F_{p-1}) = \delta_{i+1}(F_{p-1})$. Since the condition in Lemma 4 is satisfied by our assumption,

$$\delta_i = \delta_i(F_{p-1}) = \delta_{i,1}(F_{p-1}) = \delta_{i+1}.$$

Since pA is an involutive subspace of $\{E \otimes S^l(F^*)\} \otimes F^*$, (cf. Note at the end of § 2), we can prove in just the same manner that $\delta_{i+1} = \delta_{i+2}$; and so on. Thus we have $\delta_i = \delta_{i,1} = \delta_{i+2} = \dots$.

Now the annihilator of $A(F_{p-1})$ in L_i is $D + (F_{p-1}L) \cap L_i = (M + F_{p-1}L) \cap L_i$, and hence $g_{p-1}[A] = \delta_i > 0$. The fact that δ_q is positive means that $(M + F_{p-1}L) \cap L_q \neq L_q$ for any $F_{p-1} \in \mathcal{G}_{p-1}(F)$. Hence what we have shown above implies that p is equal to the smallest integer k for which there is an element $F_k \in \mathcal{G}_k(F)$ such that $(M + F_kL) \cap L_q = L_q$. Applying (i) in Lemma 2 in § 1, we conclude that the degree of the characteristic module $P(L/M(A), x)$ is equal to $p-1$. This means that the greatest projective dimension of the prime ideals in $\text{Ass}(L/M(A))$ is equal to $p-1$ by Lemma 1 in § 1. Moreover applying (ii) of Lemma 2, we have $\mu(M(A); L) = \delta_q$ for large q . Combining this with the fact $g_{p-1}[A] = \delta_i = \delta_{i+1} = \dots$, we obtain $g_{p-1}[A] = \mu(M(A); L)$ as desired.

Q.E.D.

We note that under the circumstances in Lemma 6, the integer $g_p[A]$ is unchanged under the prolongation; $g_p[A] = g_p[pA]$. This is an immediate consequence of Lemma 6, for $M(A)$ is also the module associated with pA .

Applying Theorem I in the preceding section and using Lemma 6, we have the following fundamental result.

Theorem II. *Suppose that A is an involutive subspace of $\{E \otimes S^{l-1}(F^*)\} \otimes F^*$ and that $g_{p-1}[A] > 0$, $g_p[A] = 0$. Let $M(A) = \bigcap_{j=1}^v Q_j$ be a reduced primary decomposition of the module $M(A)$ associated with A in L , and let Q_j be \mathfrak{P}_j -primary in L .*

Then the greatest projective dimension of the prime ideals in $\text{Ass}(L/M(A)) = \{\mathfrak{P}_1, \dots, \mathfrak{P}_v\}$ is equal to $p-1$, and the following is valid:

$$g_{p-1}[A] = \sum_{\text{proj dim } \mathfrak{P}_j = p-1} \mu(Q_j; L).$$

Part II. Involutive Systems of Partial Differential Equations

§ 4. Criterion of Involution

Let $(\mathcal{M}, \mathcal{N}, \rho)$ be an analytic fibered manifold where ρ is the projection from \mathcal{M} onto \mathcal{N} , and $J^l(\mathcal{M}, \mathcal{N}, \rho)$ be the space of l -jets of cross-sections of $(\mathcal{M}, \mathcal{N}, \rho)$. We shall often denote the space $J^l(\mathcal{M}, \mathcal{N}, \rho)$ briefly by J^l . The sheaf of germs of analytic functions on an open set \mathcal{U} in J^l is denoted by $\mathcal{O}(\mathcal{U})$. A locally finitely generated subsheaf of ideals of the sheaf of rings $\mathcal{O}(\mathcal{U})$ is called a system of partial differential equations of order l in $(\mathcal{M}, \mathcal{N}, \rho)$. A point X of J^l is called an *integral* one of \mathcal{O} if every $\varphi \in \mathcal{O}_X$ vanishes at X , where \mathcal{O}_X denotes the ring of germs of sections of \mathcal{O} at X . The set of integral points of \mathcal{O} is denoted by $I\mathcal{O}$. An integral point X is called an *ordinary* one if $\mathcal{O} = 0$ gives a regular local equation of the variety $I\mathcal{O}$ around X . A cross-section f of $(\mathcal{M}, \mathcal{N}, \rho)$ over an open set V in \mathcal{N} is said to be a *solution of \mathcal{O}* if and only if the l -jet $j_a^l(f)$ of f at a is an integral point of \mathcal{O} for every a in V .

Let ρ_k^l be the projection from J^l onto J^k defined by $\rho_k^l(j_a^l(f)) = j_a^k(f)$ ($l \geq k \geq 0$) and ρ_{l-1}^l be $\rho \circ \rho_0^l$. For each point $X \in J^l$ and $\bar{X} = \rho_{l-1}^l X$, one has the injection i from $\mathcal{O}_{\bar{X}}(J^{l-1})$ into $\mathcal{O}_X(J^l)$ defined by $i\varphi = \varphi \circ \rho_0^l$. We identify $\mathcal{O}_{\bar{X}}(J^{l-1})$ with its image $i\mathcal{O}_{\bar{X}}(J^{l-1})$.

Let $(\rho_{l-1}^l)_*$ be the differential of the mapping ρ_{l-1}^l at X ; $(\rho_{l-1}^l)_*: T_X(J^l) \rightarrow T_{\bar{X}}(J^{l-1})$, where $T_X(J^l)$ denotes the tangent space to J^l at X . The kernel of $(\rho_{l-1}^l)_*$ is denoted by $Q_X(J^l)$. For brevity, we shall write $a = \rho_{l-1}^l X$ and $b = \rho_0^l X$.

Let $Q_b(\mathcal{M})$ be the kernel of the mapping $\rho_*: T_b(\mathcal{M}) \rightarrow T_a(\mathcal{N})$. The space $Q_X(J^l)$ can be identified with $Q_b(\mathcal{M}) \otimes S^l(T_a^*(\mathcal{N}))$. The subspace $C_X(\mathcal{O})$ of $Q_X(J^l)$ is defined by

$$C_X(\mathcal{O}) = \{\mathcal{X} \in Q_X(J^l); \mathcal{X}(\varphi) = 0 \text{ for every } \varphi \in \mathcal{O}_X\} \quad (X \in I\mathcal{O}).$$

This space is considered canonically as a subspace of $Q_{\bar{X}}(J^{l-1}) \otimes T_a^*(\mathcal{N})$. The notion of involutive subspaces of $Q_{\bar{X}}(J^{l-1}) \otimes T_a^*(\mathcal{N})$ can be introduced as we recalled in § 2 (See Kuranishi [12], § 6). The prolongation of such subspaces is also defined (see § 2). In particular, $pC_X(\mathcal{O})$ is defined by

$$pC_X(\mathcal{O}) = C_X(\mathcal{O}) \otimes T_a^*(\mathcal{N}) \cap Q_{\bar{X}}(J^{l-1}) \otimes S^2(T_a^*(\mathcal{N})).$$

The (total) prolongation of \mathcal{O} is defined as follows. Let φ be a function on an open set \mathcal{U} in J^l and ξ be a vector field on $\rho_{-1}^l \mathcal{U} \subset \mathcal{N}$. Let φ_ξ denote a function on $(\rho_{-1}^{l+1})^{-1} \mathcal{U}$ defined by

$$\varphi_\xi(j_a^{l+1}(f)) = \xi(\varphi(j_a^l(f))).$$

For each open set $\tilde{\mathcal{U}}$ in J^{l+1} , we denote by $\mathcal{G}_{\tilde{\mathcal{U}}}$ that ideal in the ring of analytic functions on $\tilde{\mathcal{U}}$ which is generated by all sections of the sheaf \mathcal{O} over $\rho_{-1}^{l+1} \tilde{\mathcal{U}}$ and all φ_ξ constructed from every section φ of \mathcal{O} over $\rho_{-1}^{l+1} \tilde{\mathcal{U}}$ and every vector field ξ on $\rho_{-1}^{l+1} \tilde{\mathcal{U}}$. Let $\rho_{\tilde{\mathcal{U}}\mathcal{V}}$ denote the restriction mapping from $\mathcal{G}_{\tilde{\mathcal{V}}}$ to $\mathcal{G}_{\tilde{\mathcal{U}}}$, where $\tilde{\mathcal{U}} \subset \tilde{\mathcal{V}}$. Then we have the presheaf $\{\mathcal{G}_{\tilde{\mathcal{U}}}, \rho_{\tilde{\mathcal{U}}\mathcal{V}}\}$ over J^{l+1} . The prolongation $p\mathcal{O}$ of \mathcal{O} is defined to be the sheaf over J^{l+1} associated with the presheaf $\{\mathcal{G}_{\tilde{\mathcal{U}}}, \rho_{\tilde{\mathcal{U}}\mathcal{V}}\}$.

Let \tilde{X} be a point of $(\rho_{-1}^{l+1})^{-1}X$. $(p\mathcal{O})_{\tilde{X}} \cap \mathcal{O}_X(J^l)$ is independent of the choice of such \tilde{X} . \mathcal{O} is said to be *p-closed at X* if and only if $(p\mathcal{O})_{\tilde{X}} \cap \mathcal{O}_X(J^l)$ is contained in \mathcal{O}_X ; When X is an ordinary integral point, \mathcal{O} is *p-closed at X* if and only if there exists a neighbourhood \mathcal{U} of X such that the mapping ρ_{-1}^{l+1} from $I(p\mathcal{O}) \cap (\rho_{-1}^{l+1})^{-1}\mathcal{U}$ to $I\mathcal{O} \cap \mathcal{U}$ is surjective (Matsuda [13]).

M. Kuranishi [12] gave a modern characterization of involutive systems of partial differential equations. Let us recall the criterion of involution given by M. Matsuda [13].

Theorem A. *\mathcal{O} is involutive at X_0 if and only if the following four conditions are satisfied:*

- (i) X_0 is an ordinary integral point of \mathcal{O} .
- (ii) $\dim pC_X(\mathcal{O})$ remains constant on a neighbourhood of X_0 in $I\mathcal{O}$.
- (iii) $C_{X_0}(\mathcal{O})$ is an involutive subspace of $Q_{\bar{X}_0}(J^{l-1}) \otimes T_{a_0}^*(\mathcal{N})$ ($\bar{X}_0 = \rho_{-1}^l X_0, a_0 = \rho_{-1}^l X_0$).
- (iv) \mathcal{O} is *p-closed at X_0* .

Note. If \mathcal{O} is involutive at X_0 , then \mathcal{O} is involutive at each point on a neighbourhood of X_0 in $I\mathcal{O}$.

Let $(x_1, \dots, x_n, y_1, \dots, y_m)$ be a coordinate system of \mathcal{M} on U (n

$= \dim \mathcal{N}$, $n + m = \dim \mathcal{M}$) such that there exists a coordinate system (x_1', \dots, x_n') of \mathcal{N} on ρU satisfying $x_i = x_i' \cdot \rho$ ($1 \leq i \leq n$). Then a coordinate system of J^l on $(\rho_0^l)^{-1}U$ is given by

$$(x_i, y_\alpha, p_\alpha^i, p_\alpha^{i_1 i_2}, \dots, p_\alpha^{i_1 \dots i_l}; 1 \leq \alpha \leq m, 1 \leq i, i_k \leq n),$$

where

$$p_\alpha^{i_1 \dots i_k} (j_\alpha^l(f)) = \frac{\partial^k f_\alpha}{\partial x_{i_1} \dots \partial x_{i_k}}(a), \quad f_\alpha \text{ being } y_\alpha\text{-coordinate of } f.$$

By the contact forms of orders up to and including l , we mean the following Pfaffian forms on $J^l(\mathcal{M}, \mathcal{N}, \rho)$:

$$\begin{cases} dy_\alpha - \sum_{i=1}^n p_\alpha^i dx_i & (1 \leq \alpha \leq m), \\ dp_\alpha^{i_1 \dots i_k} - \sum_{i=1}^n p_\alpha^{i_1 \dots i_k i} dx_i & \left(\begin{array}{l} 1 \leq \alpha \leq m, 1 \leq k < l \\ 1 \leq i_1, \dots, i_k \leq n \end{array} \right). \end{cases}$$

In the usual way, one can associate with \mathcal{O} the exterior differential system $\Sigma(\mathcal{O})$ defined on the domain of \mathcal{O} in J^l which is generated as a differential ideal by \mathcal{O} and the contact forms of orders up to and including l . Every solution of \mathcal{O} corresponds in a one-to-one manner to an n -dimensional integral manifold \mathfrak{M} of $\Sigma(\mathcal{O})$ such that $\dim(\rho^l_{-1})_* T(\mathfrak{M}) = n$.

The differential system $\Sigma(\mathcal{O})$ is said to be *involutive at X with respect to \mathcal{N}* if and only if the following two conditions (i) and (ii) are satisfied:

(i) There exists at least one n -dimensional integral element E_n of $\Sigma(\mathcal{O})$ and of origin X such that $\dim(\rho^l_{-1})_* E_n = n$,

(ii) Every such integral element E_n is an ordinary one in the sense that there exists a chain of integral elements $E_0 \subset E_1 \subset \dots \subset E_n$ where each E_k except the last E_n is a regular integral element of $\Sigma(\mathcal{O})$ and of dimension k (such a chain is called a regular chain). (cf. Cartan [4], Chap. V; Kuranishi [11], Definition I. 17; Kähler [8].)

The link between two concepts of involution is clarified by the following theorem (Kuranishi [12], Appendix; Matsuda [15]).

Theorem B. *\mathcal{O} is involutive at X if and only if $\Sigma(\mathcal{O})$ is involutive at X with respect to \mathcal{N} .*

We finally recall the following theorem due to E. Cartan and Y. Matsushima [16].

Theorem C. *If \mathcal{O} is involutive at X , then $p\mathcal{O}$ is involutive at each point on $I(p\mathcal{O}) \cap (\rho_i^{l+1})^{-1}X$.*

§ 5. The Characteristic Module

Let us denote by K the field of real or complex numbers according as our discussion is carried out in the real or complex analytic category. Let X be an integral point of \mathcal{O} . The spaces $Q_b(\mathcal{M})$ and $T_a(\mathcal{N})$ are vector spaces over K of dimensions m and n respectively (cf. § 4). We shall apply the theory developed in Part I in which we shall set $E = Q_b(\mathcal{M})$ and $F = T_a(\mathcal{N})$. We shall also use freely the same notations as in Part I.

Suppose that \mathcal{O} is involutive at X . We begin with recalling Cartan's characters of \mathcal{O} at X . From the space $C_X(\mathcal{O})$ we obtain the following decreasing sequence of integers (see § 2):

$$g_0[C_X(\mathcal{O})] \geq g_1[C_X(\mathcal{O})] \geq \dots \geq g_n[C_X(\mathcal{O})] = 0.$$

For brevity we write $g_k(X) = g_k[C_X(\mathcal{O})]$. One can prove the following inequalities (see Kuranishi [12], § 6):

$$g_{k-2}(X) - g_{k-1}(X) \geq g_{k-1}(X) - g_k(X) \quad (k=2, \dots, n).$$

Let us put

$$s_k(X) = g_{k-1}(X) - g_k(X) \quad (k=1, 2, \dots, n).$$

These numbers $s_k(X)$ are indeed those characters of the differential system $\Sigma(\mathcal{O})$ associated with \mathcal{O} which were introduced by E. Cartan [2] (also see Kähler [8], p. 55). The above inequalities imply that $s_1(X) \geq s_2(X) \geq \dots \geq s_n(X) \geq 0$. It should be noted that the inequalities of this type valid for Cartan's characters are always valid only in case of an exterior differential system generated as a differential ideal by differential forms of degree ≤ 1 . We call the number $s_k(X)$ *Cartan's character of order k of \mathcal{O} at X* . The characters $s_k(X)$ play an important role in the Cartan-Kähler existence theorem for the differential system $\Sigma(\mathcal{O})$ and they indi-

cate the numbers of arbitrary functions upon which depends the general integral manifold of dimension n (cf. Kähler [8], p. 55). In case of partial differential equations, however, it is more adequate to introduce the new numbers

$$\sigma_k(X) = s_k(X) - s_{k+1}(X) \quad (1 \leq k < n), \quad \sigma_n(X) = s_n(X)$$

as M. Kuranishi did in his lecture note [12]. In fact, it can be seen that the latter numbers $\sigma_k(X)$ indicate more precisely the numbers of arbitrary functions upon which depends the general solution of \mathcal{O} (Compare the existence proof in Cartan [2] or Kähler [8] with that in Kuranishi [12] or Matsuda [14]). Let $s_p(X)$ (resp. $\sigma_r(X)$) be the last non-vanishing number in $s_1(X), \dots, s_n(X)$ (resp. $\sigma_1(X), \dots, \sigma_n(X)$). Then it is clear that $p=r$ and $s_p(X) = \sigma_r(X)$. It is this character $s_p(X)$ which is of great importance in the following investigation. One reason why the character $s_p(X)$ is especially important consists in the fact that it is unchanged under the total prolongation; more precisely, the character of order p of $p\mathcal{O}$ at each point of $I(p\mathcal{O}) \cap (\rho_i^{l+1})^{-1}X$ is equal to $s_p(X)$ (cf. Lemma 6 in § 3).

The characters $s_k(X)$ remain constant on a connected open set $\mathcal{C}\mathcal{V}$ in $I\mathcal{O}$ at each point of which \mathcal{O} is involutive. In particular, they are constant on a neighbourhood in $I\mathcal{O}$ of a point at which \mathcal{O} is involutive. In fact, $C_X(\mathcal{O})$ is an involutive subspace of $\mathcal{Q}_X(J^{l-1}) \otimes T_a^*(\mathcal{N})$ if and only if $\dim pC_X(\mathcal{O}) = \sum_{k=0}^n g_k(X)$ (cf. § 2). Hence we conclude from Theorem A that $\dim pC_X(\mathcal{O})$ remains constant on $\mathcal{C}\mathcal{V}$ and it is equal to $\sum_{k=0}^n g_k(X)$ for each $X \in \mathcal{C}\mathcal{V}$. Since the functions $g_k(X)$ are upper semi-continuous and integral-valued, it follows that the functions $g_k(X)$ are constant on $\mathcal{C}\mathcal{V}$. Therefore the characters $s_k(X)$ are also so.

We now introduce the important concept "characteristic module". To indicate precisely the dependence on the point X , we write:

$$R_q(X) = S^q(T_a^*(\mathcal{N})), \quad R(X) = \sum_{q=0}^{\infty} R_q(X)$$

$$L_q(X) = \mathcal{Q}_b(\mathcal{M}) \otimes S^q(T_a^*(\mathcal{N})), \quad L(X) = \sum_{q=0}^{\infty} L_q(X).$$

Definition. We define the *characteristic module* $M(X)$ of \mathcal{O} at X to be the module associated with the space $C_X(\mathcal{O})$ in the sense of Section 3.

The module $M(X)$ is a homogeneous submodule of the $R(X)$ -module $L(X)$. It follows immediately from the definition that the characteristic module is unchanged under the prolongation in the sense that the characteristic module of $p\mathcal{O}$ at \tilde{X} coincides with that of \mathcal{O} at X , where \tilde{X} is any integral point of $p\mathcal{O}$ satisfying $\rho_i^{l+1}\tilde{X} = X$.

Suppose that $s_p(X) > 0$ and $s_{p+1}(X) = \dots = s_n(X) = 0$. Clearly this assumption means that $g_{p-1}(X) > 0$ and $g_p(X) = 0$. Moreover $s_p(X)$ is equal to $g_{p-1}(X)$. Hence applying Theorem II in § 3, we have the following fundamental result.

Theorem III. *Suppose that \mathcal{O} is involutive at X and that $s_p(X) > 0$, $s_{p+1}(X) = \dots = s_n(X) = 0$ ($1 \leq p \leq n$). Let $M(X) = \bigcap_{j=1}^{\nu} Q_j$ be a reduced primary decomposition of the characteristic module $M(X)$ of \mathcal{O} at X in $L(X)$ where Q_j is a \mathfrak{P}_j -primary homogeneous submodule of $L(X)$ ($1 \leq j \leq \nu$).*

Then the greatest projective dimension of the homogeneous prime ideals $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_\nu$ is equal to $p-1$ and the following equality is valid:

$$s_p(X) = \sum_{\text{proj dim } \mathfrak{P}_j = p-1} \mu(Q_j; L(X)),$$

where $\mu(Q_j; L(X))$ denotes the multiplicity of Q_j with respect to $L(X)$.

Remark. Theorem III is the complete generalization of our previous result ([10], Theorem 2).

§ 6. Characteristics

This section is devoted to the investigation on what is called characteristics in the sense of Monge or Cauchy. It is the primary decomposition of the characteristic module which enables us to clarify the structure of characteristics for general non-linear systems of partial differential equations.

Let X be an integral point of \mathcal{O} . Obviously it is also an integral point of the differential system $\Sigma(\mathcal{O})$ canonically associated with \mathcal{O} (see § 4). Let E_p be a p -dimensional integral element of $\Sigma(\mathcal{O})$ and of origin

X ($0 \leq p \leq n$). The space of such vectors $\Gamma \in T_X(J^l)$ that E_p and Γ span an integral element of $\mathcal{L}(\mathcal{O})$ is denoted by $H(E_p)$ and it is usually called the polar element. Since $\mathcal{L}(\mathcal{O})$ is generated by differential forms of degree ≤ 2 as an exterior algebra, a subspace E of $T_X(J^l)$ is an integral element of $\mathcal{L}(\mathcal{O})$ if and only if any linear element contained in E is an integral one and furthermore, for any pair $\{E_1, E_1'\}$ of one-dimensional subspaces of E , E_1 and E_1' span an integral element, in other words, $H(E_1)$ contains E_1' . Let E_1 be a linear element of origin X on which the contact forms of orders up to and including l vanish. E_1 is spanned by the vector Γ expressed in the following manner by using the local coordinate system of J^l around X (cf. § 4):

$$(12) \quad \Gamma = \sum_{i=1}^n \lambda_i \frac{d}{dx_i} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \zeta_\alpha^{i_1 \dots i_l} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}},$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^m p_\alpha^{i_1} \frac{\partial}{\partial y_\alpha} + \sum_{q=1}^{l-1} \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_q} p_\alpha^{i_1 \dots i_q} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_q}}.$$

The element E_1 is an integral one of $\mathcal{L}(\mathcal{O})$ if and only if

$$(13) \quad \sum_{i=1}^n \frac{dF}{dx_i} \lambda_i + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \frac{\partial F}{\partial p_\alpha^{i_1 \dots i_l}} \zeta_\alpha^{i_1 \dots i_l} = 0 \quad (F \in \mathcal{O}_X),$$

where functions of l -jets are assumed to represent their values at X . The vector $\hat{\Gamma}$ of $T_X(J^l)$ possessing the form

$$\hat{\Gamma} = \sum_{i=1}^n \hat{\lambda}_i \frac{d}{dx_i} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \hat{\zeta}_\alpha^{i_1 \dots i_l} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}}$$

belongs to $H(E_1)$ if and only if $\hat{\Gamma}$ spans a linear integral element and moreover

$$(14) \quad \sum_{i=1}^n (-\zeta_\alpha^{i_1 \dots i_{l-1} i} \hat{\lambda}_i + \hat{\zeta}_\alpha^{i_1 \dots i_{l-1} i} \lambda_i) = 0 \quad (1 \leq \alpha \leq m, 1 \leq i_1, \dots, i_{l-1} \leq n),$$

where the scalars $\zeta_\alpha^{i_1 \dots i_l}$ and the scalars $\hat{\zeta}_\alpha^{i_1 \dots i_l}$ are respectively supposed to be defined for all superscripts $\{i_1, \dots, i_l\}$ in such a manner that they are respectively symmetric with respect to the superscripts $\{i_1, \dots, i_l\}$. In fact, condition (14) means that the differential forms $\sum_{i=1}^n dp_\alpha^{i_1 \dots i_{l-1} i} \wedge dx_i$ vanishes on the element spanned by E_1 and $\hat{\Gamma}$.

Suppose that \mathcal{O} is involutive at X . Then, according to Theorem B in § 4, $\Sigma(\mathcal{O})$ is involutive at X with respect to \mathcal{N} , and hence there exists at least one n -dimensional integral element E_n of $\Sigma(\mathcal{O})$ and of origin X such that $(\rho_{-1}^l)_*E_n$ is of dimension n . Let E_n be an n -dimensional subspace of $T_X(J^l)$ on which the contact forms of orders up to and including l vanish. We know at once that the E_n is spanned by n vectors of the form

$$(15) \quad \Gamma^{(i)} = \frac{d}{dx_i} + \sum_{\alpha=1}^n \sum_{i_1 \leq \dots \leq i_l} \psi_\alpha^{i_1 \dots i_l i} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}} \quad (i=1, 2, \dots, n).$$

In order that E_n is an integral element of $\Sigma(\mathcal{O})$, it is necessary and sufficient that

$$(16) \quad \begin{cases} \frac{dF}{dx_i} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \psi_\alpha^{i_1 \dots i_l i} \frac{\partial F}{\partial p_\alpha^{i_1 \dots i_l}} = 0 & (i=1, 2, \dots, n; F \in \mathcal{O}_X), \\ \psi_\alpha^{i_1 \dots i_{l+1}} = \psi_\alpha^{k_1 \dots k_{l+1}} & \text{if } \{i_1, \dots, i_{l+1}\} = \{k_1, \dots, k_{l+1}\}. \end{cases}$$

Let \mathfrak{B} be one of the prime ideals in $\text{Ass}(L(X)/M(X))$, $M(X)$ being the characteristic module of \mathcal{O} at X . Recall that $\text{Ass}(L(X)/M(X))$ coincides with the set $\{\mathfrak{B}_1, \dots, \mathfrak{B}_v\}$ under the circumstances of Theorem III in the preceding section. We shall denote by $c(\mathfrak{B})$ the vector space of elements of degree 1 in \mathfrak{B} ; $c(\mathfrak{B}) = \mathfrak{B} \cap R_1(X)$. Of course $c(\mathfrak{B})$ may possibly vanish.

Definition. We define the subspace $B(\mathfrak{B})$ of $T_X(J^l)$ to be the space spanned by all those vectors each of which is contained in an n -dimensional integral element E_n of $\Sigma(\mathcal{O})$ of origin X such that $\dim(\rho_{-1}^l)_*E_n = n$ and is also contained in $(\rho_{-1}^l)_*^{-1}c(\mathfrak{B})$. We denote by $D(\mathfrak{B})$ the annihilator of $B(\mathfrak{B})$ in the dual space $T_X^*(J^l)$ to $T_X(J^l)$.

The space $B(P)$ is expressed as follows:

$$B(\mathfrak{B}) = \left\langle \sum_{i=1}^n \lambda_i \left(\frac{d}{dx_i} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \psi_\alpha^{i_1 \dots i_l i} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}} \right); \sum_{i=1}^n \lambda_i \frac{\partial}{\partial x_i} \in c(\mathfrak{B}), (16) \right\rangle,$$

where by $\langle \dots \rangle$ we mean the vector space over K spanned by the elements in it.

Let us investigate the structure of $B(\mathfrak{A})$ in detail. Let ξ be an element of $T_a(\mathcal{N})$ (cf. § 4). The linear mapping $\sigma(\xi)$ from $Q_X(J^l) \otimes T_a^*(\mathcal{N})$ to $Q_X(J^l)$ is defined to be the one obtained by extending linearly the mapping

$$\sigma(\xi)(w \otimes v^*) = v^*(\xi)w, \quad \text{where } w \in Q_X(J^l), \quad v^* \in T_a^*(\mathcal{N}).$$

This mapping is expressed as follows: When $\xi = \sum_{i=1}^n \lambda_i \partial / \partial x_i$,

$$\begin{aligned} \sigma(\xi) & \left(\sum_{i=1}^n \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \varphi_\alpha^{i_1 \dots i_l i} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}} \otimes dx_i \right) \\ & = \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \left(\sum_{i=1}^n \lambda_i \varphi_\alpha^{i_1 \dots i_l i} \right) \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}}. \end{aligned}$$

Since we have the expressions

$$\begin{aligned} C_X(\mathcal{O}) & = \left\{ \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \varphi_\alpha^{i_1 \dots i_l} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}}; \right. \\ & \quad \left. \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \varphi_\alpha^{i_1 \dots i_l} \frac{\partial F}{\partial p_\alpha^{i_1 \dots i_l}} = 0 \quad (F \in \mathcal{O}_X) \right\}, \end{aligned}$$

$$pC_X(\mathcal{O}) = \left\{ \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_{l+1}} \varphi_\alpha^{i_1 \dots i_{l+1}} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_{l+1}}}; \right.$$

$$(17) \dots \left. \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \varphi_\alpha^{i_1 \dots i_l i} \frac{\partial F}{\partial p_\alpha^{i_1 \dots i_l}} = 0 \quad (i=1, 2, \dots, n; F \in \mathcal{O}_X) \right\},$$

where the same convention as in (14) is supposed to be applied, we know that $\sigma(\xi)$ induces the mapping from $pC_X(\mathcal{O})$ to $C_X(\mathcal{O})$.

Definition. Corresponding to each $\mathfrak{A} \in \text{Ass}(L(X)/M(X))$, we introduce the following subspace of $C_X(\mathcal{O})$:

$$C(\mathfrak{A}) = \langle \sigma(\xi) pC_X(\mathcal{O}); \xi \in c(\mathfrak{A}) \rangle.$$

For simplicity, we write: $R=R(X)$, $L=L(X)$, $M=M(X)$. On account of Lemma 5 in § 3, the annihilator of $C_X(\mathcal{O})$ in L_l is M_l and that of $pC_X(\mathcal{O})$ in L_{l+1} coincides with M_{l+1} . Hence the dual spaces to $C_X(\mathcal{O})$ and $pC_X(\mathcal{O})$ are canonically identified with L_l/M_l and L_{l+1}/M_{l+1} respectively. Let $\sigma^*(\xi)$ be the adjoint mapping of $\sigma(\xi)$; $\sigma^*(\xi): L_l/M_l \rightarrow L_{l+1}/M_{l+1}$. It is easy to show that $\sigma^*(\xi)$ is given by $\sigma^*(\xi)\bar{z} = \xi\bar{z}$

($\mathfrak{z} \in L_i/M_i$), the multiplication being the one when L/M is regarded as a graded R -module. Moreover the kernel of $\sigma^*(\xi)$ coincides with the annihilator of the image of $\sigma(\xi)$. Therefore we know that the annihilator $\text{Ann } C(\mathfrak{P})$ of $C(\mathfrak{P})$ in L_i is the intersection of the spaces $\{z \in L_i; \xi z \in M_{i+1}\}$, where ξ ranges over all vectors in $c(\mathfrak{P})$. Consequently we have

$$(18) \quad \text{Ann } C(\mathfrak{P}) = \{z \in L_i; c(\mathfrak{P})z \subset M_{i+1}\}.$$

In case \mathfrak{P} is generated by linear forms, that is, by $c(\mathfrak{P})$, the conditions $c(\mathfrak{P})z \subset M_{i+1}$ and $\mathfrak{P}z \subset M$ are equivalent. Hence we have the following

Lemma 7. *The annihilator $\text{Ann } C(\mathfrak{P})$ of $C(\mathfrak{P})$ in L_i is given by (18). If the ideal \mathfrak{P} is generated by $c(\mathfrak{P})$, then*

$$\text{Ann } C(\mathfrak{P}) = \{z \in L_i; \mathfrak{P}z \subset M\}.$$

Using the space $C(\mathfrak{P})$, we can decompose $B(\mathfrak{P})$ as follows.

Proposition 3. *The space $B(\mathfrak{P})$ admits the following direct sum decomposition:*

$$B(\mathfrak{P}) = \{E_n \cap (\rho_{-1}^l)_*^{-1}c(\mathfrak{P})\} \oplus C(\mathfrak{P}),$$

where E_n is an n -dimensional integral element of $\Sigma(\mathcal{O})$ of origin X satisfying $\dim (\rho_{-1}^l)_*E_n = n$.

Proof. Suppose that E_n is spanned by (15). Let Γ be any vector of $B(\mathfrak{P})$. It is expressed as follows:

$$\Gamma = \sum_{i=1}^n \lambda_i \left(\frac{d}{dx_i} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \chi_\alpha^{i_1 \dots i_l i} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l i}} \right),$$

where $\sum_{i=1}^n \lambda_i \partial / \partial x_i \in c(\mathfrak{P})$ and the scalars $\chi_\alpha^{i_1 \dots i_l i}$ satisfy the same condition as the condition (16) satisfied by the $\psi_\alpha^{i_1 \dots i_l i}$. This vector possesses the expression $\Gamma = \Gamma_1 + \Gamma_2$ where $\Gamma_1 = \sum_{i=1}^n \lambda_i \Gamma^{(i)}$ and

$$\Gamma_2 = \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \left\{ \sum_{i=1}^n \lambda_i (\chi_\alpha^{i_1 \dots i_l i} - \psi_\alpha^{i_1 \dots i_l i}) \right\} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l i}}.$$

Obviously Γ_1 belongs to E_n . We assert that Γ_2 is actually a vector of $C(\mathfrak{B})$. In fact, the scalars $\varphi_\alpha^{i_1 \dots i_r} = \chi_\alpha^{i_1 \dots i_r} - \psi_\alpha^{i_1 \dots i_r}$ satisfy condition (17), and hence the vector

$$\tilde{\mathcal{X}} = \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_{r_1}} \varphi_\alpha^{i_1 \dots i_{r_1}} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_{r_1}}}$$

belongs to $pC_X(\mathcal{O})$. Putting $\xi = \sum_{i=1}^n \lambda_i \partial / \partial x_i \in c(\mathfrak{B})$, we have $\Gamma_2 = \sigma(\xi) \tilde{\mathcal{X}}$. It follows from the very definition that $\Gamma_2 \in C(\mathfrak{B})$. Thus the proof will be complete if we show that $C(\mathfrak{B})$ is a subspace of $B(\mathfrak{B})$. This fact is, however, proved without any difficulty. Q.E.D.

The remaining part of this section is devoted to the investigation on characteristics in the sense of Cauchy and Cartan. Let E_p be an integral element of origin E_0 . The space $H(E_p)$ is obviously a subspace of $H(E_0)$. The element E_p is called a *Cauchy-Cartan characteristic element* if and only if $H(E_p)$ coincides with $H(E_0)$ (cf. Cartan [2], Part VIII; Cartan [4], Chap. III; Goursat [7], § 66). As is well known, Cauchy-Cartan characteristic elements can be characterized as an integral element of the characteristic system in the sense of Cartan. The Cartan characteristic system can be always constructed in the well-known manner, and hence one can know, at least in theory, whether or not a given differential system admits Cauchy-Cartan characteristics. There exists, however, the more explicit condition in order that a differential system admits Cauchy-Cartan characteristics. In fact, in the case of Pfaffian systems, Cartan [2] himself established such a condition stated by means of Cartan's characters. Owing to our theory developed above, we can establish such a new condition for the differential system $\mathcal{L}(\mathcal{O})$. We have actually the following

Theorem IV. *Assume that the following two conditions are satisfied:*

- (i) \mathcal{O} is involutive at X .
- (ii) The characteristic module of \mathcal{O} at X itself is a \mathfrak{B} -primary submodule of $L = L(X)$ and its exponent is equal to 1.

Then the subspace $B(\mathfrak{B})$ of $T_X(J^r)$ is a r -dimensional integral

element of $\Sigma(\mathcal{O})$ such that $\dim(\rho_{-1}^l)_*B(\mathfrak{B}) = \gamma$, where $\gamma = \dim c(\mathfrak{B})$. Moreover $B(\mathfrak{B})$ is a Cauchy-Cartan characteristic element of $\Sigma(\mathcal{O})$ possessing the maximum dimension.

Proof. By virtue of Theorem B in § 4, condition (i) implies that there exists an n -dimensional integral element E_n of $\Sigma(\mathcal{O})$ and of origin X such that $(\rho_{-1}^l)_*E_n$ is of dimension n . In this proof, we denote by M the characteristic module of \mathcal{O} at X . We prove this theorem by dividing into four steps.

(a) $B(\mathfrak{B}) = E_n \cap (\rho_{-1}^l)_*^{-1}c(\mathfrak{B})$. In particular, $B(\mathfrak{B})$ is a γ -dimensional integral element.

In fact, on account of Proposition 3, it suffices to show that the space $C(\mathfrak{B})$ vanishes. Since the exponent of M is equal to 1 by assumption, Lemma 7 indicates that the annihilator of $C(\mathfrak{B})$ in L_i coincides with the total space L_i . This means that $C(\mathfrak{B})$ vanishes.

(b) $H(X) = E_n \oplus C_X(\mathcal{O})$ (direct sum decomposition).

Suppose E_n is spanned by the n vectors (15). Let Γ be any vector of $H(X)$ expressed by (12). It is obvious that Γ can be written in the following form:

$$\Gamma = \sum_{i=1}^n \lambda_i \Gamma^{(i)} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} (\zeta_\alpha^{i_1 \dots i_l} - \sum_{i=1}^n \lambda_i \psi_\alpha^{i_1 \dots i_l i}) \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}}.$$

The first term on the right hand side is obviously a vector of E_n . Since the ζ 's and the ψ 's satisfy (13) and (16) respectively, we conclude at once that the second term is a vector in $C_X(\mathcal{O})$. This completes the proof of (b).

(c) Let E_1 be a linear integral element of $\Sigma(\mathcal{O})$ of origin X . In order that E_1 is contained in $H(E_1')$ for any linear element E_1' contained in $C_X(\mathcal{O})$, in other words, $H(E_1)$ contains $C_X(\mathcal{O})$, it is necessary and sufficient that $(\rho_{-1}^l)_*E_1$ is contained in $c(\mathfrak{B})$.

In fact, suppose that E_1 be spanned by (12) and E_1' be spanned by

$$(19) \quad \mathcal{X} = \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \varphi_\alpha^{i_1 \dots i_l} \frac{\partial}{\partial p_\alpha^{i_1 \dots i_l}}$$

Then, E_1 is contained in $H(E_1')$ if and only if the following are satisfied (cf. (14)):

$$\sum_{i=1}^n \lambda_i \varphi_\alpha^{i_1 \cdots i_{i-1} i} = 0 \quad (1 \leq \alpha \leq m, 1 \leq i_1, \dots, i_{i-1} \leq n),$$

where the same convention as in (14) is assumed to be applied. Hence E_1 is contained in $H(E_1')$ for all E_1' contained in $C_X(\emptyset)$ if and only if

$$\sum_{i=1}^n \lambda_i \varphi_\alpha^{i_1 \cdots i_{i-1} i} \equiv 0 \pmod{\sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_i} \frac{\partial F}{\partial p_\alpha^{i_1 \cdots i_i}} \varphi_\alpha^{i_1 \cdots i_i}; F \in \emptyset_X},$$

where the φ 's are regarded as indeterminates. We write $\xi_i = \partial / \partial x_i$ for brevity. Recalling Lemma 5 in § 3, we find no difficulty in showing that the latter condition is equivalent to the condition

$$\sum_{i=1}^n \lambda_i dy_\alpha \otimes \xi_{i_1} \otimes \xi_{i_2} \cdots \otimes \xi_{i_{i-1}} \otimes \xi_{i_i} \in M_l \quad \left(\alpha = 1, 2, \dots, m, \right. \\ \left. 1 \leq i_1, \dots, i_{i-1} \leq n \right).$$

Bearing in mind that the elements $dy_\alpha \otimes \xi_{i_1} \otimes \xi_{i_2} \cdots \otimes \xi_{i_{i-1}} \otimes \xi_{i_i}$ span L_{l-1} , we find immediately that this can be stated as follows:

$$(20) \quad \left(\sum_{i=1}^n \lambda_i \xi_i \right) L_{l-1} \subset M_l.$$

When $M=L$, the theorem is obviously valid (In this case, \emptyset is actually completely integrable at X). In the contrary case, M_{l-1} is a proper subspace of L_{l-1} (cf. the very definition of the module associated with involutive subspaces in § 3). In other words, there is an element of L_{l-1} not belonging to M . Since M is a \mathfrak{P} -primary submodule of L with exponent 1, it follows that (20) holds if and only if $\sum_{i=1}^n \lambda_i \xi_i$ belongs to \mathfrak{P} , that is, to $c(\mathfrak{P})$. The latter condition means that $(\rho_{l-1}^l)_* E_1$ is contained in $c(\mathfrak{P})$. The required result has been Therefore proved.

(d) We now complete the proof of Theorem IV. It suffices to prove the following:

“A linear element of origin X is a Cauchy-Cartan characteristic element of $\mathcal{L}(\emptyset)$ if and only if it is contained in $B(\mathfrak{P})$.”

Proof. Sufficiency: Suppose that E_1 is a linear element contained in $B(\mathfrak{P})$. By virtue of (a), E_1 is contained in the element E_n . Let Γ' be any vector of $H(X)$. By (b), Γ' admits the decomposition $\Gamma' = \Gamma_1' + \Gamma_2'$ where $\Gamma_1' \in E_n$ and $\Gamma_2' \in C_X(\emptyset)$. Since E_1 and Γ_1' is contained in the same integral element E_n , $\Gamma_1' \in H(E_1)$. Let E_1' denote the integral element spanned by Γ_2' . By virtue of (a), $(\rho_{l-1}^l)_* E_1 \subset c(\mathfrak{P})$.

Hence (c) implies that $H(E_1') \supset E_1$, that is, $H(E_1) \supset E_1' \ni \Gamma_2'$. Combining what we have proved, we obtain $\Gamma' \in H(E_1)$. This implies that $H(E_1)$ coincides with $H(X)$, and hence E_1 is actually a Cauchy-Cartan characteristic element.

Necessity: Suppose that E_1 is a Cauchy-Cartan characteristic element, that is, $H(E_1) = H(X)$. Then, it is necessary that $(\rho_{-1}^l)_* E_1$ is of dimension 1. In fact, if the contrary is true, that is, if $(\rho_{-1}^l)_* E_1 = 0$, then E_1 is contained in $C_X(\emptyset)$ (cf. (b)). Assume E_1 is spanned by \mathcal{X} given by (19). The integral element E_n is obviously a subspace of $H(E_1)$, and hence any vector $\Gamma \in E_n$ belongs to $H(E_1)$. Therefore we have

$$\sum_{i=1}^n \lambda_i \varphi_\alpha^{i_1 \dots i_{i-1} i} = 0 \quad (1 \leq \alpha \leq m, 1 \leq i_1, \dots, i_{i-1} \leq n)$$

for any system $(\lambda_1, \dots, \lambda_n)$ of scalars (cf. (14)). This means that all the elements

$$(21) \quad \sum_{i=1}^n \lambda_i dy_\alpha \otimes \xi_{i_1} \otimes \dots \otimes \xi_{i_{i-1}} \otimes \xi_i \quad (1 \leq \alpha \leq m, 1 \leq i_1, \dots, i_{i-1} \leq n)$$

annihilate \mathcal{X} . Since $(\lambda_1, \dots, \lambda_n)$ is arbitrary, the elements (21) spans L_1 . Therefore \mathcal{X} must vanish. This is a contradiction, and hence $(\rho_{-1}^l)_* E_1 \neq 0$.

By what we have just proved, we can choose an $(n-1)$ -dimensional subspace E_{n-1} of E_n such that $(\rho_{-1}^l)_* E_{n-1}$ and $(\rho_{-1}^l)_* E_1$ span the total space $T_\alpha(\mathcal{N})$. The assumption $H(E_1) = H(X)$ implies that E_1 and E_{n-1} span an n -dimensional integral element E_n' such that $\dim (\rho_{-1}^l)_* E_n' = n$. On the other hand, since $C_X(\emptyset) \subset H(X) = H(E_1)$, (c) indicates that $(\rho_{-1}^l)_* E_1$ is contained in $c(\mathfrak{B})$. Consequently E_1 is contained in $E_n' \cap (\rho_{-1}^l)_*^{-1} c(\mathfrak{B})$. By the very definition of $B(\mathfrak{B})$, it follows that E_1 is contained in $B(\mathfrak{B})$. This completes the proof of necessity. Q.E.D.

§ 7. Monge Characteristic Systems

Let X_0 be an integral point of \emptyset . Suppose that \emptyset is involutive at X_0 . There exists a neighbourhood \mathcal{C} of X_0 in $I\emptyset$ such that \emptyset is involutive at each point on \mathcal{C} and that the characters $s_1(X), \dots, s_n(X)$ remains constant on \mathcal{C} (cf. §§ 4-5). Let $M(X)$ be the characteristic module of \emptyset at $X \in \mathcal{C}$ and let it admit the reduced primary decomposition in $L(X)$

$$M(X) = \bigcap_{j=1}^{\nu} Q_j(X) \quad (\nu = \nu(X)),$$

where each $Q_j(X)$ is a $\mathfrak{P}_j(X)$ -primary homogeneous submodule of $L(X)$, $\mathfrak{P}_j(X)$ being a homogeneous prime ideal in $R(X)$. We shall assume that the following regularity conditions (a), (b), and (c) are satisfied: (a) $\nu(X)$ remains constant on $\mathcal{C}\mathcal{V}$. (b) Each subspace $c(\mathfrak{P}_j(X)) = \mathfrak{P}_j \cdot (X) \cap R_1(X)$ of $T_a(\mathcal{N})$ ($a = \rho_{-1}^l X$) has the constant dimension γ_j on $\mathcal{C}\mathcal{V}$, and it depends analytically on $X \in \mathcal{C}\mathcal{V}$ in the sense that there exist γ_j vector fields $\xi_k(X) = \sum_{i=1}^n a_{ki}(X) \partial / \partial x_i$ ($1 \leq k \leq \gamma_j$) where the functions a_{ki} are analytic on $\mathcal{C}\mathcal{V}$ such that their germs at X span the space $c(\mathfrak{P}_j(X))$ for each X on $\mathcal{C}\mathcal{V}$. (c) The dimension of $C(\mathfrak{P}_j(X))$ remains constant on $\mathcal{C}\mathcal{V}$.

We shall denote by \mathfrak{P}_j the ideal $\mathfrak{P}_j(X)$ considered as a field which assigns to each $X \in \mathcal{C}\mathcal{V}$ the ideal $\mathfrak{P}_j(X)$. Let \mathfrak{P} be one of $\mathfrak{P}_1, \dots, \mathfrak{P}_\nu$ such that $c(\mathfrak{P}(X))$ is of positive dimension. For each $X \in \mathcal{C}\mathcal{V}$, \mathfrak{P} defines a subspace $B(\mathfrak{P}(X))$ of $T_X(J^l)$ and its annihilator $D(\mathfrak{P}(X))$ in $L_l(X)$ (see § 6). By Proposition 3 in § 6, $B(\mathfrak{P}(X))$ is the direct sum of two subspaces $E_n \cap (\rho_{-1}^l)_*^{-1} c(\mathfrak{P}(X))$ and $C(\mathfrak{P}(X))$. Therefore the space $B(\mathfrak{P}(X))$ has the constant dimension on $\mathcal{C}\mathcal{V}$. This fact enables us without any difficulty to prove that there exists an analytic Pfaffian system on $\mathcal{C}\mathcal{V}$ such that the germs of Pfaffian forms belonging to it at X span the space $D(\mathfrak{P}(X))$ for each $X \in \mathcal{C}\mathcal{V}$. We shall denote the Pfaffian system by $D(\mathfrak{P})$.

Definition. *The Monge characteristic system (of order l) of \mathcal{O} defined on $\mathcal{C}\mathcal{V}$ corresponding to \mathfrak{P} is defined to be the Pfaffian system $D(\mathfrak{P})$. We shall denote it by $\mathcal{A}^l(\mathfrak{P})$.*

Let us next define Monge characteristic systems of higher order. By virtue of Theorem C due to Cartan and Matsushima, the prolonged system $p^{q-l}\mathcal{O}$ is involutive at each point on $\mathcal{C}\mathcal{V}^q = I(p^{q-l}\mathcal{O}) \cap (\rho_l^q)^{-1}\mathcal{C}\mathcal{V}$ ($q \geq l$). Let X^q be a point on $\mathcal{C}\mathcal{V}^q$ such that $\rho_l^q X^q = X$. It follows immediately from the definition that the characteristic module of $p^{q-l}\mathcal{O}$ at X^q coincides with that of \mathcal{O} at X . Hence we obtain the Monge characteristic system (of order q) of $p^{q-l}\mathcal{O}$ defined on $\mathcal{C}\mathcal{V}^q$ corresponding to \mathfrak{P} .

Definition. The Monge characteristic system of order $q (\geq l)$ of \emptyset over $\mathcal{C}\mathcal{V}$ corresponding to \mathfrak{F} is defined to be the Monge characteristic system of $p^{q-l}\emptyset$ defined on $\mathcal{C}\mathcal{V}^q$ corresponding to \mathfrak{F} . We shall denote it by $\mathcal{A}^q(\mathfrak{F})$.

Let u be a function of q -jets defined on a neighbourhood of $X^q \in \mathcal{C}\mathcal{V}^q$.

Definition. u is called an *integral of $\mathcal{A}^q(\mathfrak{F})$ at X^q* if and only if

$$du \equiv 0 \pmod{\mathcal{A}^q(\mathfrak{F})} \text{ on a neighbourhood of } X^q \text{ in } \mathcal{C}\mathcal{V}^q.$$

A function u such that $du \neq 0$ at X^q is called a *relative integral of $\mathcal{A}^q(\mathfrak{F})$ at X^q* if and only if u vanishes at X^q and

$$du \equiv 0 \pmod{\mathcal{A}^q(\mathfrak{F})} \text{ on a neighbourhood of } X^q \text{ in } \mathcal{C}\mathcal{V}^q(u),$$

where $\mathcal{C}\mathcal{V}^q(u)$ denotes the set of the points of $\mathcal{C}\mathcal{V}^q$ at which u vanishes.

We find no difficulty in proving that a function of q -jets is an integral of $\mathcal{A}^q(\mathfrak{F})$ at X^q if and only if it is an integral of $\mathcal{A}^{q+1}(\mathfrak{F})$ at each point on $\mathcal{C}\mathcal{V}^{q+1} \cap (\rho_q^{q+1})^{-1}X^q$ (cf. Kakié [9], § 5). Therefore we may say that a function is, or is not, an integral of a characteristic system without specifying the order. We shall denote by $\mathcal{A}(\mathfrak{F})$ the Monge characteristic system corresponding to \mathfrak{F} without the specifying order. If a function of q -jets which cannot be regarded as a function of $(q-1)$ -jets is a (relative) integral of $\mathcal{A}^q(\mathfrak{F})$, we say that it is a (relative) integral of order q of $\mathcal{A}(\mathfrak{F})$.

Remark. E. Cartan carried out a very suggestive discussion concerning Monge characteristics ([4], Chap. IV, Part III). Our discussion clarifies, however, the structure of Monge characteristics of general non-linear involutive systems much more completely than the classical theory.

Part III. Involutive Systems Whose Characters of Order More Than One Vanish

§ 8. Construction of New Involutive Subspaces

The remaining sections are devoted to investigating involutive systems of which characters of order more than 1 vanish. Such systems are

similar to systems with one unknown function of two independent variables. Suppose that \mathcal{O} is involutive at X and its characters $s_2(X), \dots, s_n(X)$ vanish. By virtue of Theorem A in § 4, the space $C_X(\mathcal{O})$ is an involutive subspace of $Q_X(J^{l-1}) \otimes T_a^*(\mathcal{N})$. Moreover the latter assumption means that $g_1[C_X(\mathcal{O})] = 0$ (see § 5). In this section, we treat the algebraic problem of constructing new involutive subspaces contained in such an involutive one. Instead of dealing with $C_X(\mathcal{O})$ itself, we find it appropriate to consider any subspace of $Q_X(J^l)$. Throughout this section, by involutive subspaces we mean involutive subspaces of $Q_X(J^{l-1}) \otimes T_a^*(\mathcal{N})$. For brevity we shall write: $R = R(X), L = L(X)$ (see § 5).

The following lemma is an immediate consequence of the definition (see § 2).

Lemma 8. *Let A be a subspace of $Q_X(J^l)$ such that $g_1[A] = 0$. Then A is an involutive subspace if and only if $\dim pA = \dim A$.*

Suppose that A is an involutive subspace in $Q_X(J^l)$ satisfying $g_1[A] = 0$. Associated with A we have the submodule $M(A)$ of L (the module associated with A ; see § 3). Except when any confusion occurs, we shall write: $M = M(A)$. Let $M = \bigcap_{j=1}^v Q_j$ be a reduced primary decomposition of M in L where each Q_j is a \mathfrak{P}_j -primary homogeneous submodule of L . No \mathfrak{P}_j coincides with the maximal ideal \mathfrak{X} in R (see § 3). By Lemma 6 in § 3, the homogeneous prime ideals \mathfrak{P}_j in R are of projective dimension 0. The fact that \mathfrak{P}_j is generated by an $(n-1)$ -dimensional subspace of R_1 means that the zero of \mathfrak{P}_j is located in the $(n-1)$ -dimensional projective space $P^{n-1}(K)$ over K .

Let Z be a subset of L_i . We shall denote by $A[Z]$ the subspace of A whose annihilator in L_i is the space spanned by M_i and Z . In particular, $A[0] = M_i$ (Lemma 5 in § 3).

Lemma 9. *Assume that A is an involutive subspace satisfying $g_1[A] = 0$. Let z be an element of L_i not belonging to M_i . Then $A[z]$ is an involutive subspace if and only if the following two equivalent conditions are satisfied:*

(a) *There exist $n-1$ independent linear forms ξ_1, \dots, ξ_{n-1} in R such that $\xi_1 z, \dots, \xi_{n-1} z \in M_{i+1}$.*

(b) *There exists a prime ideal \mathfrak{P} belonging to $\text{Ass}(L/M)$ such that it is generated by $n-1$ independent linear forms and that $\mathfrak{P}z \subset M$.*

Proof. We write: $A' = A[z]$. Obviously $g_1[A'] = 0$. By Lemma 8, A' is an involutive subspace if and only if $\dim pA' = \dim A'$. Since $\dim pA = \dim A$, we readily find that A' is an involutive subspace if and only if

$$(22) \quad \dim \langle R_1 z, R_1 M_l \rangle / R_1 M_l = 1.$$

We remark that $R_1 z$ is not contained in $R_1 M_l = M_{l+1}$. In fact, if the contrary is valid, we have $R_1 z \subset M = \bigcap_{j=1}^{\nu} Q_j$. Clearly there is $\xi \in R_1$ not belonging to $\bigcup_{j=1}^{\nu} \mathfrak{P}_j$. Therefore, using the property of primary submodules, we can readily show that $z \in M$. This contradicts the assumption. Thus we have proved the above remark. Bearing in mind this fact, we find no difficulty in proving that condition (22) holds if and only if condition (a) holds, and hence A' is an involutive subspace if and only if (a) holds.

Let us prove that the conditions (a) and (b) are equivalent. Suppose that (a) valid. Let \mathfrak{P} be the prime ideal generated by ξ_1, \dots, ξ_{n-1} . Then $\mathfrak{P}z \subset M$. Since $z \notin M$, this implies that $\mathfrak{P} \subset r_L(M) = \bigcap_{j=1}^{\nu} \mathfrak{P}_j$ (cf. § 1). Hence there is a prime ideal \mathfrak{P}_k such that $\mathfrak{P} \subset \mathfrak{P}_k$. Remarking that both ideals have the same projective dimension 0, we have $\mathfrak{P} = \mathfrak{P}_k$. Thus (a) implies (b). Suppose conversely that (b) is valid. Let \mathfrak{P} be generated by ξ_1, \dots, ξ_{n-1} . Then it is obvious that for the ξ_i , the condition in (a) is satisfied. Thus (b) implies (a). Q.E.D.

Corresponding to each $\mathfrak{P}_\beta \in \text{Ass}(L/M)$, we have a subspace $C(\mathfrak{P}_\beta)$ of A (see § 6):

$$C(\mathfrak{P}_\beta) = \langle \sigma(\xi) pA; \xi \in c(\mathfrak{P}_\beta) \rangle.$$

(Except in this section, we denote by the same symbol $C(\mathfrak{P}_\beta)$ the space constructed from the space $A = C_X(\emptyset)$.) By Lemma 7 in § 6, the annihilator of $C(\mathfrak{P}_\beta)$ in L_l is $\{z \in L_l; c(\mathfrak{P}_\beta)z \subset M\}$. Suppose that \mathfrak{P}_β is generated by $n-1$ independent linear forms, that is, it is generated by

$c(\mathfrak{P}_\beta)$. Then

$$\text{Ann } C(\mathfrak{P}_\beta) = \{z \in L_i; \mathfrak{P}_\beta z \subset M\}.$$

We now modify Lemma 9 as follows.

Proposition 4. *Suppose that A is an involutive subspace satisfying $g_1[A]=0$. Let A' be a subspace of A and of codimension 1 in A . Then in order that A' is an involutive subspace, it is necessary and sufficient that there exists a prime ideal \mathfrak{P}_β in $\text{Ass}(L/M)$ of which zero is located in $P^{n-1}(K)$ such that A' contains $C(\mathfrak{P}_\beta)$.*

Proof. Assume A' is given by $A[z]$ ($z \in L_i$). By Lemma 9, A' is an involutive subspace if and only if there exists an ideal \mathfrak{P}_β in $\text{Ass}(L/M)$ of which zero is located in $P^{n-1}(K)$ such that $\mathfrak{P}_\beta z \subset M$. The remark made above indicates that $\mathfrak{P}_\beta z \subset M$ if and only if $z \in \text{Ann } C(\mathfrak{P}_\beta)$. The latter means that $A[z] \supset C(\mathfrak{P}_\beta)$. Consequently we have the desired result. Q.E.D.

Let us generalize Proposition 4.

Proposition 5. *Assume that A is an involutive subspace satisfying $g_1[A]=0$. Let \mathfrak{P}_β be an ideal in $\text{Ass}(L/M)$ whose zero is located in $P^{n-1}(K)$. Then any subspace A' of A containing $C(\mathfrak{P}_\beta)$ is an involutive one. Moreover the module $M(A')$ associated with A' admits the primary decomposition in L of the following form: $M(A') = N \cap \{\bigcap_{j \neq \beta} Q_j\}$ where N is either the total module or a \mathfrak{P}_β -primary submodule containing Q_β such that its multiplicity $\mu(N; L)$ is less than $\mu(Q_\beta; L)$ by the codimension of A' in A .*

Proof. Let s denote the codimension of A' in A . Suppose that $A' = A[z_1, \dots, z_s]$ where $z_1, \dots, z_s \in L_i$. On account of Lemma 7 in § 6, the assumption $A' \supset C(\mathfrak{P}_\beta)$ implies $\mathfrak{P}_\beta z_i \subset M$ ($1 \leq i \leq s$). Remarking that \mathfrak{P}_β is generated by $n-1$ independent linear forms, we conclude from this that

$$\dim \langle M_{i+1}, R_i z_i; 1 \leq i \leq s \rangle / M_{i+1} \leq s.$$

In this inequality, the equality is actually valid. In fact, suppose that the number on the left side is less than s . Then there is a linear combination $z = \sum_{i=1}^s c_i z_i$ with $c_i \in K$ such that $R_1 z \subset M_{l+1}$. This implies that $z \in M_l$ (recall the property of the module associated with an involutive subspace; see § 3). This is a contradiction to the choice of the elements z_i . Thus the equality holds. This equality implies that $\dim pA' = \dim A'$. Hence, by Lemma 8, A' is an involutive subspace.

We next prove the latter assertion. Let $M(A') = \bigcap_{k=1}^r N_{(k)}$ be a reduced primary decomposition where $N_{(k)}$ is an \mathfrak{A}_k -primary submodule of L . No \mathfrak{A}_k is the maximal ideal \mathfrak{X} in R . Since $g_1[A'] = 0$, on account of Lemma 6 in § 3, the homogeneous prime ideals \mathfrak{A}_k are of projective dimension 0. Obviously $M(A') \supset M$, and hence $N_{(k)} \supset \bigcap_{j=1}^{\nu} Q_j$ ($1 \leq k \leq r$). Taking the operation $\varkappa_L(\)$ (see § 1), we get $\mathfrak{A}_k \supset \bigcap_{j=1}^{\nu} \mathfrak{P}_j$. From this it follows that each \mathfrak{A}_k contains one of the ideals \mathfrak{P}_j . Since both ideals have the same dimension, they must coincide. We may assume without loss of generality that $\mathfrak{A}_k = \mathfrak{P}_k$ ($1 \leq k \leq r$). The prime ideal $\mathfrak{A}_k = \mathfrak{P}_k$ does not contain \mathfrak{P}_j for $j \neq k$. We prove that the facts $N_{(k)} \supset \bigcap_{j=1}^{\nu} Q_j$, $\mathfrak{A}_k \not\supset \mathfrak{P}_j$ for each $j \neq k$ implies that $N_{(k)} \supset Q_k$. In fact, let q be an integer in $\{1, \dots, \nu\}$ distinct from k . We first prove that $N_{(k)} \supset \bigcap_{j \neq q} Q_j$. To show this, it is sufficient to prove that any element z of $\bigcap_{j \neq q} Q_j$ not belonging to Q_q belongs to $N_{(k)}$. By assumption there is an element ξ of \mathfrak{P}_q not belonging to \mathfrak{A}_k . There is a positive integer s such that $\xi^s z \in Q_q$. Hence $\xi^s z \in \bigcap_{j=1}^{\nu} Q_j \subset N_{(k)}$. Since the homomorphism from $L/N_{(k)}$ into itself defined by $\xi \in \mathfrak{A}_k$ is an isomorphism (cf. The definition of primary submodules), we have $z \in N_{(k)}$. By the same reasoning, we have $N_{(k)} \supset \bigcap_{j \neq q, t} Q_j$ where $q \neq k$, $t \neq k$. Proceed step by step, we finally obtain $N_{(k)} \supset Q_k$.

Thus we have proved that $N_{(k)} \supset Q_k$ ($1 \leq k \leq r$).

To simplify the explanation, without loss of generality, we assume $\beta = \nu$. Then

$$\text{Ann } A' \subset \text{Ann } C(\mathfrak{P}_\nu) = \{z \in L; \mathfrak{P}_\nu z \subset M\}.$$

Since \mathfrak{P}_ν does not contain \mathfrak{P}_j for $j \neq \beta$, $\mathfrak{P}_\nu z \subset M = \bigcap_{j=1}^{\nu} Q_j$ implies that $z \in \bigcap_{j \neq \nu} Q_j$ (cf. the above argument). Hence $\text{Ann } A'$ is contained in

$\{\bigcap_{j \neq \nu} Q_j\} \cap L_\nu$. It follows that $M(A')$ is contained in $H(\bigcap_{j \neq \nu} Q_j) = \bigcap_{j=1}^{\nu-1} Q_j$ (see § 3). By the same reasoning as we have just done, we know that each \mathfrak{P}_j ($1 \leq j \leq \nu-1$) coincides with one of the ideals \mathfrak{U}_k and that if $\mathfrak{P}_j = \mathfrak{U}_k$, then $Q_j \supset N_{(k)}$. Combining this with the above result, we conclude that the number r is equal to either $\nu-1$ or ν , and that $N_{(k)} = Q_k$ ($1 \leq k \leq \nu-1$). When $r = \nu-1$, the proof has been already completed. When $r = \nu$, we have already proved that $\mathfrak{U}_\nu = \mathfrak{P}_\nu$ and $N_{(\nu)} \supset Q_\nu$. Therefore, in any case, we have the primary decomposition of $M(A')$ stated in the proposition. Moreover, applying Theorem II in § 3, we have

$$\sum_{j=1}^{\nu} \mu(Q_j; L) = \dim A, \quad \mu(N; L) + \sum_{j=1}^{\nu-1} \mu(Q_j; L) = \dim A'.$$

Hence $\mu(N; L) = \mu(Q_\nu; L) - (\dim A - \dim A')$. Thus we have proved all the required results. Q.E.D.

Proposition 6. *Suppose that A is an involutive subspace satisfying $g_1[A] = 0$. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ be s prime ideals in $\text{Ass}(L/M)$ such that their zeros are located in $P^{n-1}(K)$. Let A_k be any subspace of A containing $C(\mathfrak{P}_k)$ ($1 \leq k \leq s$). Then the space $\bigcap_{k=1}^s A_k$ is an involutive subspace whose codimension in A is equal to the sum of the codimensions in A of the s spaces A_k . Moreover the module associated with $\bigcap_{k=1}^s A_k$ admits the primary decomposition $\{\bigcap_{k=1}^s N_{(k)}\} \cap \{\bigcap_{j=s+1}^{\nu} Q_j\}$ where each $N_{(k)}$ is either the total module or a \mathfrak{P}_k -primary submodule of L .*

Proof. We write: $B_0 = A$, $B_k = \bigcap_{j=1}^k A_j$ ($1 \leq k \leq s$). By assumption, B_0 is an involutive subspace. By Proposition 5, B_1 is also an involutive subspace and its associated module $M(B_1)$ admits the primary decomposition $N_{(1)} \cap \{\bigcap_{j=2}^{\nu} Q_j\}$ where $N_{(1)}$ is either the total module or a \mathfrak{P}_1 -primary submodule. We note that B_2 is a subspace of the involutive subspace B_1 constructed from B_1 exactly in the same way as B_1 is constructed from A . In fact, the space $C(\mathfrak{P}_j)$ constructed from A contains the corresponding space constructed from $B_1 = A_1$ as is immediately seen. Therefore we can apply Proposition 5 again, and we conclude that B_2 is an involutive

subspace and $M(B_2)$ admits the primary decomposition $N_{(1)} \cap N_{(2)} \cap \{\bigcap_{j=3}^s Q_j\}$ where $N_{(2)}$ is either the total module or a \mathfrak{B}_2 -primary submodule. Proceeding step by step in the same way, we see that the subspaces B_k are involutive ones. In particular, $\bigcap_{j=1}^s A_j$ is indeed an involutive subspace.

Let us prove the assertion concerning the codimension of that space. We prove that $\dim B_k/B_{k+1}$ is equal to the codimension γ of A_{k+1} in A ($1 \leq k < s$). Suppose A_{k+1} is given by $A[z_1, \dots, z_\gamma]$ where $z_i \in L_i$. It follows from $C(\mathfrak{B}_{k+1}) \subset A_{k+1}$ that $\mathfrak{B}_{k+1} z_i \subset M$ ($1 \leq i \leq \gamma$). This implies $z_i \in \bigcap_{j \neq k+1} Q_j$ (cf. the argument in the proof of Proposition 5). Since $z_i \notin M$, $z_i \notin Q_{k-1}$. Hence $z_i \in \text{Ann } B_k = M(B_k) \cap L_i$ (cf. Lemma 5 in § 3), for $\text{Ann } B_k$ is contained in $Q_{k+1} \cap L_i$. From this it follows that $B_{k+1} = B_k \cap A_{k+1}$ is of codimension γ in B_k .

Using what we have just deduced, we can readily calculate the codimension of B_s , and we know that the assertion is indeed true.

Q.E.D.

In the above proof, if the exponents of Q_k ($1 \leq k \leq s$) in L are equal to 1, $\text{Ann } C(\mathfrak{B}_k)$ coincides with $\{\bigcap_{j \neq k} Q_j\} \cap L_i$. In fact, by the same argument as in the proof of Proposition 5, we can prove the following:

$$\{z \in L; \mathfrak{B}_k z \subset M\} = \bigcap_{j \neq k} Q_j.$$

Hence we have the desired result. Accordingly when $A_k = C(\mathfrak{B}_k)$ ($1 \leq k \leq s$), we readily see that the submodule $N_{(k)}$ must coincide with the total module (note that the modules associated with involutive subspaces coincides if and only if their homogeneous part of degree l coincides; see § 3). Hence we have the following

Corollary. *Under the circumstances of Proposition 6, suppose further that the exponents of Q_k ($1 \leq k \leq s$) in L is equal to 1. Then the module associated with the involutive subspace $\bigcap_{k=1}^s C(\mathfrak{B}_k)$ admits the reduced primary decomposition $\bigcap_{j=s+1}^s Q_j$.*

In concluding this section, we calculate the codimension of $C(\mathfrak{B}_\beta)$ in A .

Lemma 10. *Suppose that A is an involutive subspace such that $g_1[A]=0$. Let \mathfrak{A}_β be an ideal in $\text{Ass}(L/M)$ such that the zero of \mathfrak{A}_β is located in $P^{n-1}(K)$. If the exponent of Q_β in L is equal to 1, the codimension of $C(\mathfrak{A}_\beta)$ in A is equal to $\mu(Q_\beta; L)$.*

Proof. By Proposition 6 and its corollary, $C(\mathfrak{A}_\beta)$ is an involutive subspace and the module associated with it admits the reduced primary decomposition $\bigcap_{j \neq \beta} Q_j$. Since the prolongation of involutive subspaces are also involutive ones (cf. Note at the end of § 2), applying Lemma 8 successively, we have

$$\dim A = \dim p^i A, \quad \dim C(\mathfrak{A}_\beta) = \dim p^i C(\mathfrak{A}_\beta) \quad \text{for } i \geq 0.$$

By virtue of Lemma 5 in § 3 and Lemma 6.1 in Kuranishi's work [12], the annihilators of $p^i A$ and $p^i C(\mathfrak{A}_\beta)$ are M_{l+i} and $\{\bigcap_{j \neq \beta} Q_j\} \cap L_{l+i}$ respectively. Accordingly the codimension of $C(\mathfrak{A}_\beta)$ in A can be known by calculating the dimensions of the latter spaces.

Since the prime ideals \mathfrak{A}_j are distinct and of projective dimension 0, it is not difficult to show that the space $(\mathfrak{A}_\beta + \bigcap_{j \neq \beta} \mathfrak{A}_j^\sigma) \cap R_q$ coincides with the total space R_q for large q , where σ is a positive integer. Since $\mathfrak{A}_j^\sigma L \subset Q_j$ ($1 \leq j \leq \nu$) for large σ , we know at once that $(Q_\beta + \bigcap_{j \neq \beta} Q_j) \cap L_q = L_q$ for large q . Let us prove the vector space isomorphism

$$\left(\bigcap_{j \neq \beta} Q_j\right) \cap L_q / M_q \cong L_q / Q_\beta \cap L_q.$$

Consider the natural mapping τ from $(\bigcap_{j \neq \beta} Q_j) \cap L_q$ to the space on the right side. By the above fact, τ is actually surjective. Since the kernel of τ is obviously M_q , we get the desired isomorphism.

Combining what we have shown, we find that the codimension of $C(\mathfrak{A}_\beta)$ in A is equal to $\dim L_q / Q_\beta \cap L_q$ for large q . It follows from the definition of $\mu(Q_\beta; L)$ that the latter is equal to $\mu(Q_\beta; L)$ for large q , and hence we have the desired result. Q.E.D.

§ 9. Construction of New Involutive Systems

The problem considered in this section is to construct new involutive systems of which solutions are also those of a given one. One of the

essential problems to be solved in treating this problem is the algebraic one of constructing new involutive subspaces contained in a given one. In the preceding section, we have solved the latter problem only for involutive subspaces associated with involutive systems whose characters of order more than 1 vanish. In the general case, we find pretty different circumstances in solving this problem. It is on account of this situation that we shall deal with only such systems.

Suppose that \emptyset satisfies the following condition:

(H-1) \emptyset is involutive at X_0 and $s_1(X_0) > 0, s_2(X_0) = \dots = s_n(X_0) = 0$.

Then there is a neighbourhood $\mathcal{C}\mathcal{V}$ of X_0 in $I\emptyset$ such that the condition (H-1) is satisfied at each point on $\mathcal{C}\mathcal{V}$ and the character $s_1(X)$ remains constant on $\mathcal{C}\mathcal{V}$ (cf. §§ 4-5). Let the characteristic module $M(X)$ of \emptyset at $X \in \mathcal{C}\mathcal{V}$ admit the reduced primary decomposition in $L(X)$

$$M(X) = Q_1(X) \cap Q_2(X) \cap \dots \cap Q_\nu(X) \quad (\nu = \nu(X))$$

where $Q_j(X)$ is a $\mathfrak{P}_j(X)$ -primary homogeneous submodule of $L(X)$, $\mathfrak{P}_j(X)$ being a homogeneous prime ideal in $R(X)$. For simplicity we write: $\mu_j(X) = \mu(Q_j(X); L(X))$. In virtue of Theorem III in § 5, the ideals $\mathfrak{P}_j(X)$ are of projective dimension 0 and moreover $\sum_{j=1}^{\nu} \mu_j(X) = s_1(X)$. We shall furthermore assume the following condition:

(H-2) There exists a neighbourhood $\mathcal{C}\mathcal{V}$ of X_0 in $I\emptyset$ such that the following are satisfied: (a) The number $\nu(X)$ remains constant on $\mathcal{C}\mathcal{V}$ and the exponents of the submodules $Q_j(X)$ in $L(X)$ are equal to 1. (b) Each ideal $\mathfrak{P}_j(X)$ is generated by an $(n-1)$ -dimensional subspace of $R_1(X)$ and it is analytic in the sense that there exist $n-1$ vector fields $\xi_k(X) = \sum_{i=1}^n a_{ki}(X) \partial / \partial x_i$ ($k=1, 2, \dots, n-1$) where the functions $a_{ki}(X)$ are analytic on $\mathcal{C}\mathcal{V}$ such that the germs of the vector fields $\xi_k(X)$ at X generate the ideal $\mathfrak{P}_j(X)$ for each $X \in \mathcal{C}\mathcal{V}$.

Applying Lemma 10 at the end of § 8, we know that $\dim C(\mathfrak{P}_j)$ is equal to $g_0(X) - \mu_j(X)$ ($g_0(X) = \dim C_X(\emptyset)$). Since the functions $\mu_j(X)$ are lower semi-continuous and $s_1(X) = g_0(X)$ is constant on $\mathcal{C}\mathcal{V}$, the equality $\sum_{j=1}^{\nu} \mu_j(X) = s_1(X)$ implies that the numbers $\mu_j(X)$ remain constant on $\mathcal{C}\mathcal{V}$. Therefore condition (c) stated at the beginning of § 8 is satisfied, and hence we can define Monge characteristic systems $\mathcal{A}^h(\mathfrak{P}_j)$ which are actually analytic Pfaffian systems on $\mathcal{C}\mathcal{V}$ (see § 8). Hereafter

we shall denote briefly by s_1 and μ_j respectively the numbers $s_1(X)$ and $\mu_j(X)$ being constant on $\mathcal{C}\mathcal{V}$.

Let u_1, \dots, u_r be r functions defined on an open set \mathcal{U}_1 in J^{h_1}, \dots , on an open set \mathcal{U}_r in J^{h_r} respectively ($h_1, \dots, h_r \geq l$). We shall write: $h = \max\{h_i; 1 \leq i \leq r\}$, $\tilde{\mathcal{U}} = \bigcap_{i=1}^r (\rho_{h_i}^h)^{-1} \mathcal{U}_i$. Let (u_i) denote the subsheaf of ideals in the sheaf $\mathcal{O}(\mathcal{U}_i)$ of rings of germs of analytic functions on \mathcal{U}_i associated with the presheaf $\{\mathcal{F}_{\mathcal{U}'}; \mathcal{U}' \subset \mathcal{U}_i\}$ with natural restriction mappings, where $\mathcal{F}_{\mathcal{U}'}$ denotes the ideal generated by u_i in the ring of analytic functions on \mathcal{U}' . Let $\mathcal{H}_{\mathcal{U}^*}$ be that ideal in the ring of analytic functions on \mathcal{U}^* which is generated by all sections of the sheaves $p^{h-l}\mathcal{O}$, $p^{h-h_1}(u_1), \dots, p^{h-h_r}(u_r)$. We denote by $\mathcal{O}^h(u_1, \dots, u_r)$ the sheaf of ideals of $\mathcal{O}(\tilde{\mathcal{U}})$ associated with the presheaf $\{\mathcal{H}_{\mathcal{U}^*}; \mathcal{U}^* \subset \tilde{\mathcal{U}}\}$ with natural restriction mappings. $\mathcal{O}^h(u_1, \dots, u_r)$ is a system of partial differential equations of order h with domain $\tilde{\mathcal{U}}$ such that its solutions are also those of \mathcal{O} .

Let u be a function of q -jets defined on a neighbourhood of X^q in $J^q(q \geq l)$. We write:

$$\pi_q^* u = \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_q} \frac{\partial u}{\partial p_{\alpha}^{i_1 \dots i_q}}(X^q) dy_{\alpha} \otimes \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_q}},$$

which is an element of $L_q = \mathcal{O}_b(\mathcal{M}) \otimes S^q(T_a^*(\mathcal{N}))$. We say that u is *independent of \mathcal{O} at X^q* if and only if

$$\pi_q^* u \notin 0 \pmod{\pi_q^* F; F \in (p^{q-l}\mathcal{O})_{X^q}}.$$

When \mathcal{O} is involutive at $X = \rho_l^q X^q \in \mathcal{C}\mathcal{V}$, this condition is equivalent to the condition $\pi_q^* u \notin M_q(X)$ (cf. Lemma 5 in § 3).

We are now in a position to discuss the problem proposed at the beginning of this section.

Proposition 7. *Suppose that \mathcal{O} satisfies the conditions (H-1) and (H-2). Let u be a function of l -jets defined on a neighbourhood of X_0 in $I\mathcal{O}$ which is independent of \mathcal{O} at X_0 . Then $\mathcal{O}^l(u)$ is involutive at X_0 if and only if u is a relative integral of a characteristic system $\mathcal{A}^l(\mathfrak{F}_{\beta})$ at X_0 .*

In this case, the characteristic module of $\mathcal{O}^l(u)$ at X_0 admits the primary decomposition in $L(X_0) : N \cap \{\bigcap_{j \neq \beta} \mathcal{O}_j(X_0)\}$ where N is either

the total module or a $\mathfrak{P}_\beta(X_0)$ -primary submodule containing $\mathcal{Q}_\beta(X_0)$ such that $\mu(N; L(X_0)) = \mu_\beta - 1$.

Proof. Sufficiency: We prove that the four conditions (i)-(iv) of Theorem A in § 4 are satisfied for $\mathcal{O}^l(u)$. Since u is independent of \mathcal{O} at X_0 and u vanishes at X_0 , it is clear that X_0 is an ordinary integral point of $\mathcal{O}^l(u)$. Hence the condition (i) is satisfied for $\mathcal{O}^l(u)$. Let us prove that the condition (iii) for $\mathcal{O}^l(u)$ is satisfied. We call in this proof a vector field $\sum_{i=1}^n \lambda_i(X) \partial/\partial x_i$ which assigns to each point in a subset of $I\mathcal{O}$ a vector of $c(\mathfrak{P}_\beta(X))$ a characteristic vector field belonging to $c(\mathfrak{P}_\beta)$. Recalling the definition of characteristic systems and the argument in § 6, we find no difficulty in showing that u is a relative integral of $\mathcal{A}^l(\mathfrak{P}_\beta)$ at X_0 if and only if u vanishes at X_0 and there is a neighbourhood \mathcal{W} of X_0 in $I\mathcal{O}^l(u)$ such that

$$(23) \quad \left\{ \begin{array}{l} \text{for any characteristic vector field } \sum_{i=1}^n \lambda_i \partial/\partial x_i \text{ on } \mathcal{W} \\ \text{belonging to } c(\mathfrak{P}_\beta), \\ \sum_{i=1}^n \lambda_i \left(\frac{du}{dx_i} + \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_i} \phi_\alpha^{i_1 \dots i_i} \frac{\partial u}{\partial p_\alpha^{i_1 \dots i_i}} \right) = 0 \\ \text{in consequence of (16) on } \mathcal{W}. \end{array} \right.$$

We put: $z(X) = \pi_i^* u \in L_i(X)$. Choosing \mathcal{W} sufficiently small if necessary, we may assume that u is independent of \mathcal{O} at each point on \mathcal{W} . This means that

$$(24) \quad z(X) \in M(X) \quad (X \in \mathcal{W}).$$

On the other hand, (23) implies that

$$(25) \quad \left\{ \begin{array}{l} \text{for any characteristic vector field } \sum_{i=1}^n \lambda_i \partial/\partial x_i \text{ on } \mathcal{W} \\ \text{belonging to } c(\mathfrak{P}_\beta), \\ \sum_{i=1}^n \lambda_i \left(\sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_i} \phi_\alpha^{i_1 \dots i_i} \frac{\partial u}{\partial p_\alpha^{i_1 \dots i_i}} \right) = 0 \\ \text{in consequence of (17) on } \mathcal{W}. \end{array} \right.$$

Since $\mathfrak{P}_\beta(X)$ is generated by $c(\mathfrak{P}_\beta(X))$, it is readily seen that (25) means that

$$(26) \quad \mathfrak{P}_\beta(X) \ z(X) \subset M(X) \quad (X \in \mathcal{W}).$$

Applying Lemma 9 in § 8, we conclude from (24) and (26) that $C_X(\mathcal{O}^l(u))$ is an involutive subspace of $Q_X(J^{l-1}) \otimes T_a^*(\mathcal{N})$ for each $X \in \mathcal{W}$. Thus the condition (iii) is satisfied for $\mathcal{O}^l(u)$.

The condition (ii) is proved as follows. Obviously $g_1[C_X(\mathcal{O}^l(u))] = 0$. Therefore, in virtue of Lemma 8 in § 8, what we have just proved implies that the space $pC_X(\mathcal{O}^l(u))$ has the same dimension as $C_X(\mathcal{O}^l(u))$ for each $X \in \mathcal{W}$. Since u is independent of \mathcal{O} at $X \in \mathcal{W}$, the latter space has the dimension $\dim C_X(\mathcal{O}) - 1$, which is constant on \mathcal{W} . Combining these, we know that the condition (ii) is satisfied for $\mathcal{O}^l(u)$.

It remains to prove that $\mathcal{O}^l(u)$ is p -closed at X_0 . It suffices to show that the mapping ρ_i^{l+1} from $I\{p\mathcal{O}^l(u)\} \cap (\rho_i^{l+1})^{-1} \mathcal{W}$ to \mathcal{W} is surjective (§ 3), that is, for each $X \in \mathcal{W}$, the following system of linear equations in $p_\alpha^{i_1 \dots i_{l+1}}$ actually possesses solutions:

$$(27) \quad \begin{cases} \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \frac{\partial F}{\partial p_\alpha^{i_1 \dots i_l}} p_\alpha^{i_1 \dots i_l i} + \frac{dF}{dx_i} = 0 & \left\{ \begin{array}{l} i=1, 2, \dots, n, \\ F \in \mathcal{O}_X \end{array} \right\}, \\ \sum_{\alpha=1}^m \sum_{i_1 \leq \dots \leq i_l} \frac{\partial u}{\partial p_\alpha^{i_1 \dots i_l}} p_\alpha^{i_1 \dots i_l i} + \frac{du}{dx_i} = 0 & (i=1, 2, \dots, n), \end{cases}$$

the functions being supposed to denote their values at X . Since $c(\mathfrak{P}_\beta \cdot (X))$ is exactly the set of linear forms $\xi = \sum_{i=1}^n \lambda_i \partial / \partial x_i$ such that $\xi z(X) \in M(X)$, it follows from (23) that (27) has solutions. Thus we have proved that the four conditions of Theorem A are satisfied for $\mathcal{O}^l(u)$, and hence $\mathcal{O}^l(u)$ is involutive at X_0 .

Necessity: Suppose conversely that $\mathcal{O}^l(u)$ is involutive at X_0 . There is a neighbourhood \mathcal{W} of X_0 in $I\mathcal{O}^l(u)$ such that $\mathcal{O}^l(u)$ is involutive at each point X on \mathcal{W} and its characters remains constant. On account of Theorem A, $C_X(\mathcal{O}^l(u))$ is an involutive subspace of $Q_X(J^{l-1}) \otimes T_a^*(\mathcal{N})$ for each $X \in \mathcal{W}$. Since $g_1[C_X(\mathcal{O}^l(u))] = 0$, we can apply Proposition 4 in § 7; we know that there is a field of prime ideals $\mathfrak{P}_\beta(X)$ on \mathcal{W} such that $C_X(\mathcal{O}^l(u))$ contains $C(\mathfrak{P}_\beta(X))$ for each $X \in \mathcal{W}$. Bearing in mind the assumption (a) in (H-2) and recalling the definition of the latter space (cf. §§ 6 and 8), we conclude that (26) holds, and hence (25) holds. Now $\mathcal{O}^l(u)$ is p -closed at X_0 , and hence we may assume that the mapping ρ_i^{l+1} from $I\{p\mathcal{O}^l(u)\} \cap (\rho_i^{l+1})^{-1} \mathcal{W}$ to \mathcal{W} is surjective; that

is, system (27) actually admits solutions. Combining (25) with this fact, we conclude that (23) is valid. This means that u is a relative integral of $\Delta^l(\mathfrak{P}_\beta)$ at X_0 . The proof of Necessity is thus complete.

The characteristic module of $\Phi^l(u)$ at X_0 is nothing else than the module associated with the involutive subspace $C_X(\Phi^l(u))$, which is of codimension 1 in $C_X(\Phi)$. Therefore the last assertion follows immediately from Proposition 5 in § 8. Q.E.D.

Proposition 7 can be generalized as follows.

Theorem V. *Suppose that Φ satisfies the conditions (H-1) and (H-2). Let X_0^q be a point of $I(p^{q-l}\Phi) \cap (\rho_i^q)^{-1}X_0(q \geq l)$, and u be a function of q -jets defined on a neighbourhood of X_0^q which is independent of Φ at X_0^q . Then in order that $\Phi^q(u)$ is involutive at X_0^q , it is necessary and sufficient that u is a relative integral of a characteristic system $\Delta^q(\mathfrak{P}_\beta)$ at X_0^q .*

In this case, the characteristic module of $\Phi^q(u)$ at X_0^q admits the primary decomposition in $L(X_0):N \cap \{\bigcap_{j \neq \beta} Q_j(X_0)\}$ where N is either the total module or a $\mathfrak{P}_\beta(X_0)$ -primary submodule containing $Q_\beta(X_0)$ such that $\mu(N; L(X_0)) = \mu_\beta - 1$.

Proof. As we pointed out in § 7, the prolonged system $p^{q-l}\Phi$ is involutive at X_0^q and its characteristic module at X_0^q is nothing else but that of Φ at X_0 . Hence it is immediately seen that the conditions (H-1) and (H-2) are satisfied for $p^{q-l}\Phi$. Moreover, by the very definition, the characteristic systems of order q of Φ are those (of order q) of $p^{q-l}\Phi$. Applying Proposition 7, we have at once the desired result.

Q.E.D.

Let us next try to construct involutive systems by adjoining to Φ several equations. Let X^{q_1}, \dots, X^{q_r} be a point of J^{q_1}, \dots , a point of J^{q_r} respectively such that $\rho_i^{q_i} X^{q_i} = X_0$ and $\rho_{i_j}^{q_i} X^{q_i} = X^{q_j}$ for any pair $\{q_i, q_j\}$ with $q_i \geq q_j$ ($q_1, \dots, q_r \geq l$). Let \tilde{X}_0 denote such the point of J^q that $\rho_{q_i}^q \tilde{X}_0 = X^{q_i}$ ($1 \leq i \leq r$), where $q = \max\{q_i; 1 \leq i \leq r\}$. Let $S = \{u_1, \dots, u_r\}$ be a

system of r functions where each u_i is a function of q_i -jets on a neighbourhood of X^{q_i} . We may assume without loss of generality that the functions u_i are arranged in such a manner that $q_1 \leq q_2 \leq \dots \leq q_r$. We say that S is *independent of \mathcal{O} over X_0* if and only if u_{i+1} is independent of $\mathcal{O}^{q_i}(u_1, \dots, u_i)$ at $X^{q_{i+1}}$ for each $i=0, 1, \dots, r-1$.

Theorem VI. *Suppose that \mathcal{O} satisfies the conditions (H-1) and (H-2), and that $\tilde{X}_0 \in I(p^{q-1}\mathcal{O})$. Let $S_1 = \{u_1, \dots, u_{r_1}\}, \dots, S_s = \{u_{r_{s-1}+1}, \dots, u_r\}$ be s systems of functions where each u_i is a function of q_i -jets defined on a neighbourhood of X^{q_i} . If each system S_β is independent of \mathcal{O} over X_0 ($1 \leq \beta \leq s$) and if every $u_i \in S_\beta$ is a relative integral of $\mathcal{A}^{q_i}(\mathfrak{B}_\beta)$ at X^{q_i} , then $\mathcal{O}^q(u_1, \dots, u_r)$ is involutive at \tilde{X}_0 .*

If furthermore $r_1 = \mu_1, r_2 - r_1 = \mu_2, \dots, r - r_{s-1} = \mu_s$, then the characteristic module of $\mathcal{O}^q(u_1, \dots, u_r)$ at \tilde{X}_0 admits the reduced primary decomposition $\bigcap_{j=s+1}^r Q_j(X_0)$ in $L(X_0)$.

Proof. We may assume that q_1 is the smallest one of the numbers q_i . By Theorem V, $\mathcal{O}^{q_1}(u_1)$ is involutive at X^{q_1} , and its characteristic module at X^{q_1} admits the primary decomposition $N_1 \cap \{\bigcap_{j=2}^r Q_j(X_0)\}$ where N_1 is either the total module or a $\mathfrak{B}_1(X_0)$ -primary submodule such that $\mu(N_1; L(X_0)) = \mu_1 - 1$. We readily see that the conditions (H-1) and (H-2) are satisfied for $\mathcal{O}^{q_1}(u_1)$ at X^{q_1} . Let q_k be the smallest one of the numbers q_i except q_1 . For brevity we write $h = q_k$. When $u_k \in S_1$, it follows from the assumptions that u_k is independent of $\mathcal{O}^{q_1}(u_1)$ at X^h . Suppose $u_k \in S_\beta$ with $\beta \neq 1$. We set $z_k(X) = \pi_h^* u \in L_h(X)$. The assumption that u_k is a relative integral of $\mathcal{A}(\mathfrak{B}_\beta)$ independent of \mathcal{O} at X^h implies that $z_k(X_0) \notin M_1(X_0)$ and $\mathfrak{B}_\beta(X_0) z_k(X_0) \subset M(X_0)$ (cf. the discussion in the proof of Proposition 7, in particular, (24) and (26)). By the same reasoning as in the proof of Proposition 6 in § 8, we deduce from these that $z_k(X_0) \notin Q_\beta(X_0)$ and $z_k(X_0) \in \bigcap_{j \neq \beta} Q_j(X_0)$. It follows that $z_k(X_0)$ does not belong to the characteristic module of $\mathcal{O}^{q_1}(u_1)$ at X^{q_1} . This means that u_k is independent of $\mathcal{O}^{q_1}(u_1)$ at X^h . Thus, in any case, we can apply Theorem V. Hence we know that $\mathcal{O}^h(u_1, u_2)$ is involutive at X^h and its characteristic module at X^h is given by $N_1' \cap \{\bigcap_{j \neq 1} Q_j(X_0)\}$ or

$N_1 \cap N_\beta \cap \{ \bigcap_{j=1, \beta} Q_j(X_0) \}$ according as $u_k \in S_1$ or $u_k \in S_\beta$, where the N 's are primary submodules such that $\mu(N_1'; L(X_0)) = \mu_1 - 2$, $\mu(N_\beta; L(X_0)) = \mu_\beta - 1$. Proceeding step by step in the same way, we can complete the proof of the assertion that $\mathcal{O}^q(u_1, \dots, u_r)$ is involutive at \tilde{X}_0 . It also follows that its characteristic module at \tilde{X}_0 is given by $\{ \bigcap_{j=1}^s Q_j' \} \cap \{ \bigcap_{j=s+1}^r Q_j' \}$ where the Q_j' are primary submodules of $L(X_0)$ such that $\mu(Q_j'; L(X_0)) = \mu_j - (r_j - r_{j-1})$ ($1 \leq j \leq s$) where $r_s = r$, $r_0 = 0$.

Suppose $r_j - r_{j-1} = \mu_j$ ($1 \leq j \leq s$). Then $\mu(Q_j'; L(X_0)) = 0$, that is, the codimension of $Q_j' \cap L_h(X_0)$ in $L_h(X_0)$ equals zero for large h ; $Q_j' \cap L_h(X_0) = L_h(X_0)$ (cf. the definition of multiplicities in § 1). Since Q_j' is primary in $L(X_0)$, $H(Q_j') = Q_j'$ (see the beginning of § 3). Combining these, we conclude that the submodules Q_j' must be the total module. Consequently the last assertion is also valid. Q.E.D.

§ 10. The Method of Integration

Suppose that \mathcal{O} is involutive at X_0 and the characters $s_2(X_0), \dots, s_n(X_0)$ vanish. As usual, we identify a solution of \mathcal{O} with the corresponding n -dimensional integral manifold of $\Sigma(\mathcal{O})$ in the space J^n . By a p -dimensional manifold in J^n , we mean all along such a manifold \mathfrak{M}_p that $(\rho^l_{-1})_* T_X(\mathfrak{M}_p)$ is of dimension p for each point X on \mathfrak{M}_p . An integral curve \mathfrak{F} of $\Sigma(\mathcal{O})$ is said to be *non-characteristic* if and only if no spaces $c(\mathfrak{R}_j(X))$ contain $(\rho^l_{-1})_* T_X(\mathfrak{F})$ for each $X \in \mathfrak{F}$. By the Cartan-Kähler theorem (cf. Kähler [8], p. 26), there exists a unique analytic n -dimensional integral manifold of $\Sigma(\mathcal{O})$ passing through a given non-characteristic integral curve. Thus the following Cauchy's problem possesses a unique solution: "Find an n -dimensional integral manifold of $\Sigma(\mathcal{O})$ passing through a given non-characteristic integral curve." Throughout this section, by Cauchy's problem for \mathcal{O} we mean the one just explained. The problem to engage our attention is the following one arisen from the classical method of integration: To reduce the solution of Cauchy's problem for \mathcal{O} to the integration of a completely integrable Pfaffian system. (Refer to Goursat [6] and Forsyth [5] for the classical theory of integration. See also Kakié [9-10] for its recent development.)

Suppose $s_j(X_0) = 0$. Then \mathcal{O} is completely integrable at X_0 ; there

passes a unique n -dimensional integral manifold of $\Sigma(\mathcal{O})$ passing through a given integral point sufficiently near X_0 . This manifold can be obtained by integrating the system $\Sigma(\mathcal{O})$ which is completely integrable at X_0 . We shall exclude this trivial case in the following discussion.

The vanishing of $s_1(X_0)$ means that the characteristic module $M(X_0)$ of \mathcal{O} at X_0 coincides with the total module; $M(X_0)$ is a proper submodule of $L(X_0)$ if and only if $s_1(X_0)$ is positive. It is, however, the number ν of the components in the reduced primary decomposition of $M(X_0)$ which is of importance in treating our problem.

Suppose that \mathcal{O} satisfies the conditions (H-1) and (H-2). Let us distinguish the following two cases.

1^o) *The case when $\nu=1$:* By Theorem IV in § 6, the differential system $\Sigma(\mathcal{O})$ admits Cauchy-Cartan characteristics of dimension $n-1$. Moreover its Cartan characteristic system is given by the only one Monge characteristic system of order l of \mathcal{O} . Let \mathfrak{F} be a non-characteristic integral curve of $\Sigma(\mathcal{O})$. The one-parameter family of $(n-1)$ -dimensional Cauchy-Cartan characteristic manifolds each of which passes through a point of \mathfrak{F} generates an n -dimensional integral manifold \mathfrak{M}_n of $\Sigma(\mathcal{O})$ (cf. Cartan [4], Chap. III, Goursat [7], § 66). Since the Cartan characteristic system is completely integrable (Cartan [4], Chap. III), the manifold \mathfrak{M}_n is obtained by integrating a completely integrable Pfaffian system. This fact may be stated as follows:

"If $\nu=1$, then the solution of Cauchy's problem for \mathcal{O} can be reduced to the integration of a completely integrable Pfaffian system."

2^o) *The case when $\nu>1$:* It is this case that we must consider to establish a method of integration. Let \mathfrak{F} be a non-characteristic integral curve of $\Sigma(\mathcal{O})$. The curve \mathfrak{F} can be uniquely prolonged to a curve \mathfrak{F}^q in J^q in such a manner that \mathfrak{F}^q is an integral curve of $\Sigma(p^{q-1}\mathcal{O})$ and that $\rho_i^q \mathfrak{F}^q = \mathfrak{F}$ ($q \geq l$).

Let $X^{q_1}, \dots, X^{q_{\nu-1}}$ be a q_1 -jet on $I(p^{q_1-1}\mathcal{O}), \dots$, a $q_{\nu-1}$ -jet on $I(p^{q_{\nu-1}-1}\mathcal{O})$ respectively such that $\rho_i^{q_i} X^{q_i} = X_0$ and $\rho_{q_j}^{q_i} X^{q_i} = X^{q_j}$ for any pair $\{q_i, q_j\}$ with $q_i \geq q_j$ ($q_1, \dots, q_{\nu-1} \geq l$). We denote by \tilde{X}_0 the q -jet on $I(p^{q-1}\mathcal{O})$ determined by $\rho_{q_i}^q \tilde{X}_0 = X^{q_i}$ ($1 \leq i \leq \nu-1$), where $q = \max\{q_i; 1 \leq i \leq \nu-1\}$. Let $\mathcal{A}(\mathfrak{F}_1), \dots, \mathcal{A}(\mathfrak{F}_{\nu-1})$ be $\nu-1$ distinct Monge characteristic systems of \mathcal{O} over a neighbourhood of X_0 in $I\mathcal{O}$.

Our method of integration is stated as follows:

“If each $\mathcal{A}(\mathfrak{P}_\beta)$ possesses $\mu_\beta + 1$ functionally independent integrals of order $\leq q_\beta$ defined on a neighbourhood of X^{q_β} such that μ_β functions of them form a system independent of \mathcal{O} over X_0 ($1 \leq \beta \leq \nu - 1$), then for any non-characteristic curve \mathfrak{F} such that \mathfrak{F}^a passes sufficiently near \tilde{X}_0 , Cauchy's problem for \mathcal{O} with initial curve \mathfrak{F} can be solved by integrating a completely integrable Pfaffian system.”

Applying Theorem VI in § 9, we can construct a new involutive system admitting \mathfrak{F}^a as an integral curve whose solutions are those of \mathcal{O} and to which the method of integration stated in 1° can be applied. The solution is obtained by applying to the new involutive system the method explained in 1°. The proof is carried out quite easily in just the same way as in the previous case (see Kakié [9], § 7), and hence we omit the detail.

Some remarks concerning our method of integration are found in the last section of our previous work [9].

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