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On the First Initial-Boundary Value Problem of Compressible Viscous Fluid Motion

By

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Up to the present day many kinds of mathematical discussions on incompressible viscous fluid motion have fully developed (cf. [32, 36]). As for compressible viscous one, however, there have been only a few works on it. In 1959 Serrin [50] proved the uniqueness theorem in a bounded domain, making use of the classical energy method. In 1962 Nash [44] tried to show the existence theorem in R^3, but it seems to the author that he has failed. Independently of them Itaya succeeded to prove the existence and the uniqueness theorems on the Cauchy problem for it in [24–28], using Tikhonov's fixed point theorem.

Now in the present paper, we shall show that the first initial-boundary value problem for it can uniquely be solved under suitable assumptions for the initial-boundary data and for the boundary of the domain, from the classical point of view.

In § 1 an exact statement and the main theorem (Theorem 1) will be found. In §2 we perform the characteristic transformation and mention the theorem of the transformed problem (Theorem 2). Firstly we prove Theorem 2 and then show that Theorem 2 implies Theorem 1 in the last section § 8. In §§ 3–5 linear equations connected with the transformed equations are treated. In more detail, in § 3. 1 we briefly state some basic results for a fundamental solution in the whole space R^3 due to Eidel'man [9, 18] and Pogorzelski [46–48] (cf. [25]). In § 3. 2 we check the basic condition of uniform solvability due to Solomjak [52, 54], which is essential for the study of the boundary value problem in applied mathematics, corresponding to the Lopatinsky condition for the

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elliptic system in the sense of Petrowsky [37-39] and the complementing condition for the elliptic system in the Douglis-Nirenberg sense [1, 8, 53]. Once it is shown that this condition holds, we can construct the Poisson kernel and the Green matrix [1, 10-12, 14, 15, 22, 29-31, 54]. We estimate the Green matrix in § 4 and the solution in § 5. Making use of these results we give a proof of the existence in § 6 by the method of successive approximation and that of the uniqueness in § 7, therefore the proof of Theorem 2 is completed.

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§ 1. Introduction

1.1. Statement of the Problem. Compressible viscous isotropic Newtonian fluid motion is described by five differential equations corresponding to the law of mass, momentum and energy as follows (as for kinematics, see, for example, [34, 35, 40, 51, 55]):

$$\frac{D\rho}{Dt}=-\rho\frac{\partial v_k}{\partial x_k}, \tag{1.1}$$

$$\frac{Dv_i}{Dt}=\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\mu'\frac{\partial v_k}{\partial x_k}\right)+\frac{1}{\rho}\frac{\partial}{\partial x_k}\left[\mu\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)\right] -\frac{1}{\rho}\frac{\partial p}{\partial x_i}+f_i \quad (i=1, 2, 3), \tag{1.2}$$

$$\frac{DS}{Dt}=\frac{1}{\rho\theta}\frac{\partial}{\partial x_k}\left(\kappa\frac{\partial\theta}{\partial x_k}\right)+\frac{\mu}{2\rho\theta}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2+\frac{\mu'}{\rho\theta}\left(\frac{\partial v_k}{\partial x_k}\right)^2, \tag{1.3}$$

where ρ, density; $v=(v_1, v_2, v_3)$, velocity; μ, coefficient of viscosity; μ', second coefficient of viscosity; κ, coefficient of heat conduction; p, pressure; $f=(f_1, f_2, f_3)$, outer force; S, entropy; θ, absolute temperature; $D/Dt =\partial/\partial t+v_k\cdot\partial/\partial x_k$.

The summation convention will always be used unless the contrary is stated explicitly.

By the physical structure of fluid we can assume that μ, μ', κ, p and S are functions of ρ and θ such that $\mu'+\frac{2}{3}\mu\geqq 0$ and $\mu, \kappa, p, S_\theta>0$.

If S be smooth, then using the equation (1.1) we have

$$\text{(1.4)}\qquad \frac{DS}{Dt}=S_\theta\frac{D\theta}{Dt}+S_\rho\frac{D\rho}{Dt}$$
$$=S_\theta\frac{D\theta}{Dt}-\rho S_\rho\frac{\partial v_k}{\partial x_k}.$$

Thus (1.3) has the form:

$$\text{(1.3)}'\qquad \frac{D\theta}{Dt}=\frac{1}{\rho\theta S_\theta}\frac{\partial}{\partial x_k}\left(\kappa\frac{\partial\theta}{\partial x_k}\right)+\frac{\mu}{2\rho\theta S_\theta}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2$$
$$+\frac{\mu'}{\rho\theta S_\theta}\left(\frac{\partial v_k}{\partial x_k}\right)^2+\frac{\rho S_\rho}{S_\theta}\frac{\partial v_k}{\partial x_k}.$$

Hereafter, we shall consider (1.1), (1.2), (1.3)′ under the following initial-boundary condition:

$$\text{(1.5)}\qquad \begin{cases} v(x,0)=v_0(x),\ \theta(x,0)=\theta_0(x),\ \rho(x,0)=\rho_0(x) & (x\in\Omega),\\ v(x,t)=0,\ \theta(x,t)=\theta_1(x,t) & ((x,t)\in\Gamma_T).\end{cases}$$

From now on we always assume that the compatibility conditions between the system of equations (1.1), (1.2), (1.3)′ and the initial-boundary condition (1.5) are satisfied.

As a final result for the problem (1.1), (1.2), (1.3)′, (1.5), we have the following main theorem (as for notations, see §§ 1.2, 1.3):

Theorem 1. *If Ω be a domain in R^3, bounded or undounded, and its boundary Γ belong to $C^{2+\alpha}$ and satisfy the Lyapunov conditions, then there uniquely exists* $(v,\theta,\rho)\in H^{2+\alpha}(\bar{Q}_{T'})\times H^{2+\alpha}(\bar{Q}_{T'})\times B^{1+\alpha}(\bar{Q}_{T'})$ *for some* $T'\in(0,T]$ *such that* (v,θ,ρ) *satisfies* (1.1), (1.2), (1.3)′, (1.5), *where* $(v_0,\theta_0,\rho_0)\in H^{2+\alpha}(\bar{\Omega})\times H^{2+\alpha}(\bar{\Omega})\times H^{1+\alpha}(\bar{\Omega})$ $(0<\bar{\rho}_0\equiv\inf_{x\in\bar{\Omega}}\rho_0(x)\leqq\rho_0(x)\leqq\bar{\bar{\rho}}_0\equiv\sup_{x\in\bar{\Omega}}\rho_0(x)<+\infty,\ 0<\bar{\theta}_0\equiv\inf_{x\in\bar{\Omega}}\theta_0(x)\leqq\theta_0(x)\leqq\bar{\bar{\theta}}_0\equiv\sup_{x\in\bar{\Omega}}\theta_0(x)<+\infty)$, $\theta_1(x,t)\in H^{2+\alpha}(\Gamma_T)$, $f\in B^{1+L}(\bar{Q}_T)$ *and* $\mu,\mu',\kappa,p,S\in\mathcal{O}^{2+L}_{\text{loc}}(\mathcal{D}_{\rho,\theta})$ *with prescribed properties.*

1.2. Basic Notations. For $s=(s_1,s_2,s_3)$ (s_i's non-negative integers), r (non-negative integer), we define

$$|s|=\sum_{i=1}^{3}s_i,\quad D_t^rD_x^s=\partial^{r+|s|}/\partial t^r\partial x_1^{s_1}\partial x_2^{s_2}\partial x_3^{s_3}.$$

$$\text{(1.6)}\qquad Q_T=\Omega\times(0,T],\ \bar{Q}_T=\bar{\Omega}\times[0,T],$$

$$\Gamma_T = \Gamma \times [0, T] \quad (T \in (0, +\infty)).$$

For any non-negative integer n and $\alpha \in (0, 1)$

(1.7) $$H^{n+\alpha}(\bar{Q}_T) = \{v(x,t) \mid \|v\|_T^{(n+\alpha)} \equiv \sum_{2r+|s|=0}^{n} |D_t^r D_x^s v|_T^{(0)} + \sum_{2r+|s|=(n-1)\vee 0}^{n} |D_t^r D_x^s v|_{t,T}^{(\alpha/2)} + \sum_{2r+|s|=n} |D_t^r D_x^s v|_{x,T}^{(\alpha)} < +\infty\},$$

where

(1.8) $$\begin{cases} |v|_T^{(0)} = \sup_{(x,t)\in\bar{Q}_T} |v(x,t)|, \\ |v|_{t,T}^{(\alpha/2)} = \sup_{(x,t),(x,t')\in\bar{Q}_T, t\neq t'} \dfrac{|v(x,t)-v(x,t')|}{|t-t'|^{\alpha/2}}, \\ |v|_{x,T}^{(\alpha)} = \sup_{(x,t),(x',t)\in\bar{Q}_T, x\neq x'} \dfrac{|v(x,t)-v(x',t)|}{|x-x'|^{\alpha}}, \end{cases}$$

(1.9) $$|v|_T^{(\alpha)} = |v|_{t,T}^{(\alpha/2)} + |v|_{x,T}^{(\alpha)}.$$

(1.10) $$H^{n+\alpha}(\bar{\Omega}) = \{u(x) \mid \|u\|^{(n+\alpha)} \equiv \sum_{|s|=0}^{n} |D_x^s u|^{(0)} + \sum_{|s|=n} |D_x^s u|^{(\alpha)} < +\infty\},$$

where

(1.11) $$\begin{cases} |u|^{(0)} = \sup_{x\in\bar{\Omega}} |u(x)|, \\ |u|^{(\alpha)} = \sup_{x,x'\in\bar{\Omega}, x\neq x'} \dfrac{|u(x)-u(x')|}{|x-x'|^{\alpha}}. \end{cases}$$

(1.12) $$B^{n+\alpha}(\bar{Q}_T) = \{w(x,t) \mid \sum_{r+|s|=0}^{n} |D_t^r D_x^s w|_T^{(0)} + \sum_{r+|s|=n} |D_t^r D_x^s w|_T^{(\alpha)} < +\infty\}.$$

For a vector function $g(x,t) = (g_i)_{i=1}^k$, $g \in H^{n+\alpha}(\bar{Q}_T)$ implies $g_i \in H^{n+\alpha}(\bar{Q}_T)$ $(i=1,2,\cdots,k)$ and $|g|_T^{(0)}$ denotes $\sum_{i=1}^{k} |g_i|_T^{(0)}$, etc. For the Hölder exponent $\alpha=1$, notations such as $|g|_T^{(L)}$ are used.

(1.13) $$\mathscr{D}_{\rho,\theta} = \{(\rho,\theta) \in (0,+\infty) \times (0,+\infty)\}.$$

For $n=0,1,2,\cdots$,

(1.14) $\mathcal{O}_{\text{loc}}^{n+L}(\mathscr{D}_{\rho,\theta}) = \{q(\rho,\theta) \mid q$ is defined on $\mathscr{D}_{\rho,\theta}$, n-times partially differentiable and all its n-th order derivatives are locally Lipschitz-continuous there$\}$.

Other notations, not described above, will be explained where they appear.

1.3. On the Domain. We shall always assume that the domain in R^3, bounded or unbounded, to be considered here has a boundary which satisfies the following Lyapunov conditions (cf. [21]):
1. At each point of the boundary there exists a well-defined tangent plane and hence a well-defined normal.
2. For any point $x \in \Gamma$ there exists a single fixed number $d>0$ such that the portion of the boundary inside a sphere of radius d with the center of x intersects lines parallel to the normal at x in at most one point.
3. If θ be the angle between the normals at x_1 and x_2, then

$$\theta \leqq a|x_1 - x_2|^\alpha,$$

where a and α are positive constants independent of x_1, x_2 and $\alpha \leqq 1$.

The sphere mentioned above we shall call by the Lyapunov sphere.

Let $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be a local rectangular coordinate system with the center of $\xi \in \Gamma$, i.e., we take the inner normal at ξ as the $\hat{x}_3$ axis and place the $\hat{x}_1, \hat{x}_2$-axes in the tangent plane at ξ. Because of condition 2, the portion of the boundary lying inside the Lyapunov sphere around ξ may be represented in the form:

$$\hat{x}_3 = F(\hat{x}_1, \hat{x}_2; \xi), \tag{1.15}$$

where $F(\hat{x}_1, \hat{x}_2; \xi)$ is a single-valued function in a certain domain of the $(\hat{x}_1, \hat{x}_2)$-plane. Conditions 1 and 3 imply that $F \in C^{1+\alpha}$.

We shall call that $\Gamma \in C^m (m>1)$ if it satisfies the above mentioned conditions and if in the neighborhood of each $\xi \in \Gamma$ it may be represented by (1.15) with $F \in C^m(K_0)$, where $K_0 = \{\hat{x}' \equiv (\hat{x}_1, \hat{x}_2) \mid |\hat{x}'| \equiv (\hat{x}_1^2 + \hat{x}_2^2)^{1/2} \leqq d/2\}$. It is easily seen that in K_0 the function F satisfies the inequalities

$$\left\{ \begin{aligned} &|F| \leqq \beta_1 |\hat{x}'|^{1+\alpha}, \\ &|\mathrm{grad}\, F| \leqq \beta_2 |\hat{x}'|^\alpha, \end{aligned} \right. \tag{1.16}$$

where β_1 and β_2 are positive constants uniformly in $\xi \in \Gamma$.

Let λ be any small positive number. It is well known (cf. [54])

that in $\bar{\Omega}$ we can construct two systems $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$ such that

1. $\omega^{(k)} \subset \Omega^{(k)} \subset \bar{\Omega}$, $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \bar{\Omega}$,

2. for any x, there exists $\omega^{(k)}$ such that $x \in \omega^{(k)}$ and $\operatorname{dist}(x, \bar{\Omega} - \omega^{(k)}) \geqq \beta_3$ $(\beta_3 > 0)$,

3. for any λ, there exists a number N_0 independent of λ such that $\bigcap_{k=1}^{N_0+1} \Omega^{(k)} = \emptyset$,

4 (i). if $\Omega^{(k)} \cap \Gamma = \emptyset$, in this case we shall denote $k = k'$, then $\omega^{(k')}$ and $\Omega^{(k')}$ are the cubes with the same center and with the length of their edges equal to $\lambda/2$ and λ, respectively,

4 (ii). if $\omega^{(k)} \cap \Gamma \neq \emptyset$, in this case $k = k''$, then for a local rectangular coordinate system $\{\hat{x}\}$ with the center of some $\xi^{(k'')} \in \Gamma$,

$$\omega^{(k'')} = \left\{ |\hat{x}_i| \leqq \frac{1}{2} \beta_4 \lambda \ \ (i=1,2),\ 0 \leqq \hat{x}_3 - F(\hat{x}_1, \hat{x}_2; \xi^{(k'')}) \leqq \beta_4 \lambda \right\},$$

$$\Omega^{(k'')} = \{ |\hat{x}_i| \leqq \beta_4 \lambda \ \ (i=1,2),\ 0 \leqq \hat{x}_3 - F(\hat{x}_1, \hat{x}_2; \xi^{(k'')}) \leqq 2\beta_4 \lambda \},$$

where F is a function describing the boundary Γ in the neighborhood of $\xi^{(k'')}$ and β_4 is a positive constant independent of λ.

By changing the variables in such a way that

$$\bar{x}_i = \hat{x}_i \ \ (i=1,2),\ \bar{x}_3 = \hat{x}_3 - F(\hat{x}'), \tag{1.17}$$

$\Omega^{(k'')}$ is transformed into a standard cube $K = \{ |\bar{x}_i| \leqq \beta_4 \lambda \ (i=1,2),\ 0 \leqq \bar{x}_3 \leqq 2\beta_4 \lambda \}$, and the boundary in $\Omega^{(k'')}$ is $K' = \{ |\bar{x}_i| \leqq \beta_4 \lambda \ (i=1,2),\ \bar{x}_3 = 0 \}$.

Furthermore, it is well known that there exist the smooth functions $\zeta^{(k)}(x)$ and $\eta^{(k)}(x)$ such that

$$\begin{cases} \zeta^{(k)}(x) = \begin{cases} 1 & \text{if} \quad x \in \omega^{(k)} \\ 0 & \text{if} \quad x \in \bar{\Omega} - \Omega^{(k)} \end{cases},\quad 0 \leqq \zeta^{(k)}(x) \leqq 1, \\ \eta^{(k)}(x) = 0 \quad \text{if} \quad x \in \bar{\Omega} - \Omega^{(k)},\ \sum_k \zeta^{(k)}(x)\eta^{(k)}(x) = 1, \\ |D_x^s \zeta^{(k)}(x)| \leqq A_1^{(|s|)} \lambda^{-|s|},\ |D_x^s \eta^{(k)}(x)| \leqq A_2^{(|s|)} \lambda^{-|s|}. \end{cases} \tag{1.18}$$

Let $\{\Gamma^{(j)}\}$ be a covering of Γ and set $\Gamma_T^{(j)} = \Gamma^{(j)} \times [0, T]$. As we saw above, $\Gamma^{(j)}$ can be represented by the local coordinate system $\{\bar{x}\}$ in the same way as K'. Thus the function defined on $\Gamma_T^{(j)}$ may be considered as a function $u^{(j)}(\bar{x}', t)$ defined on $\tilde{\Gamma}_T^{(j)} = \{ |\bar{x}'| \leqq \beta_4 \lambda \} \times [0, T]$.

$$H^{n+\alpha}(\Gamma_T) = \{ u(x,t) \mid \|u\|_{\Gamma_T}^{(n+\alpha)} \equiv \sup_j \|u^{(j)}\|_{\tilde{\Gamma}_T^{(j)}}^{(n+\alpha)} < +\infty \}. \tag{1.19}$$

§ 2. Preliminaries

2. 1. Characteristic Transformation. For $v\in H^{2+\alpha}(\bar{Q}_T)$ with $v|_{\Gamma_T}=0$ we consider a system of ordinary differential equations:

$$(2.1)\quad \begin{cases} \dfrac{d}{d\tau}\bar{x}(\tau;x,t)=v(\bar{x}(\tau;x,t),\tau), \\ \bar{x}(t;x,t)=x \quad (\forall(x,t)\in\bar{Q}_T,\ 0\leqq\tau\leqq t). \end{cases}$$

By the assumption that $v\in H^{2+\alpha}(\bar{Q}_T)$, we have a unique solution curve passing (x,t) that satisfies (2. 1). If we put

$$(2.2)\quad \bar{x}(0;x,t)=x_0(x,t),$$

then it is obvious that the transformation $(x,t)\mapsto(x_0(x,t),t_0=t)$ is a one-to-one mapping from $\bar{Q}_T$ onto $\bar{Q}_T$ and especially the boundary Γ_T is transformed onto Γ_T. We denote the inverse transformation by $(x(x_0,t_0),t=t_0)$. For any function $g(x,t)$ we define

$$(2.3)\quad \hat{g}(x_0,t_0)=g(x(x_0,t_0),t=t_0).$$

It is to be noted that (2. 1) implies

$$(2.4)\quad \begin{cases} \dfrac{d}{d\tau}x(x_0,\tau)=\hat{v}(x_0,\tau), \\ x(x_0,0)=x_0. \end{cases}$$

Thus $x(x_0,t_0)$ is expressed in such a way that

$$(2.5)\quad x=x_0+\int_0^{t_0}\hat{v}(x_0,\tau)\,d\tau.$$

Hence

$$(2.6)\quad \frac{\partial x}{\partial x_0}=I+\int_0^{t_0}\hat{v}_{x_0}(x_0,\tau)\,d\tau,$$

$$(2.7)\quad \frac{\partial}{\partial x}=\frac{\partial x_0}{\partial x}\cdot\frac{\partial}{\partial x_0}=\left(\frac{\partial x}{\partial x_0}\right)^{-1}\cdot\frac{\partial}{\partial x_0},$$

where I is an identity matrix.

For simplicity we put

$$(2.8)\quad \left(\frac{\partial x}{\partial x_0}\right)^{-1}=(g_{ij}).$$

The system of equations (1.1), (1.2), (1.3)′ implies

$$\frac{\partial}{\partial t_0}\hat{\rho} = -\hat{\rho} g_{ij}\partial_j \hat{v}_i \quad (\partial_j = \partial/\partial x_{0j}), \tag{2.9}$$

$$\begin{aligned}\frac{\partial}{\partial t_0}\hat{v}_i = {} & \frac{1}{\hat{\rho}}[(\mu+\mu')g_{ik}g_{ms} + \mu g_{jk}g_{js}\delta_{mi}]\partial_k\partial_s\hat{v}_m \\ & + \frac{\partial_j\hat{\rho}}{\hat{\rho}}[\mu_{\hat{\rho}}' g_{ij}g_{ks}\partial_s\hat{v}_k + \mu_{\hat{\rho}}(g_{kj}g_{ks}\partial_s\hat{v}_i + g_{kj}g_{is}\partial_s\hat{v}_k) - p_{\hat{\rho}}g_{ij}] \\ & + \frac{1}{\hat{\rho}}[\mu' g_{ik}\partial_s\hat{v}_m + \mu(g_{mk}\partial_s\hat{v}_i + g_{jk}\partial_s\hat{v}_j\delta_{mi})]\partial_k g_{ms} \\ & + \frac{1}{\hat{\rho}}[\mu_{\hat{\theta}}' g_{ij}g_{ks}\partial_j\hat{\theta}\partial_s\hat{v}_k + \mu_{\hat{\theta}} g_{jk}(g_{js}\partial_s\hat{v}_i + g_{is}\partial_s\hat{v}_j)\partial_k\hat{\theta} \\ & - p_{\hat{\theta}}g_{ij}\partial_j\hat{\theta} + f_i] \quad (i=1,2,3),\end{aligned} \tag{2.10}$$

$$\begin{aligned}\frac{\partial\hat{\theta}}{\partial t_0} = {} & \frac{\kappa}{\hat{\rho}\hat{\theta}S_{\hat{\theta}}} g_{ij}g_{ik}\partial_j\partial_k\hat{\theta} + \frac{\kappa_{\hat{\rho}}}{\hat{\rho}\hat{\theta}S_{\hat{\theta}}} g_{ij}g_{ik}\partial_k\hat{\theta}\partial_j\hat{\rho} \\ & + \frac{\kappa}{\hat{\rho}\hat{\theta}S_{\hat{\theta}}} g_{ij}\partial_k\hat{\theta}\partial_j g_{ik} + \frac{1}{\hat{\rho}\hat{\theta}S_{\hat{\theta}}}\Big[\kappa_{\hat{\theta}} g_{ij}g_{ik}\partial_j\hat{\theta}\partial_k\hat{\theta} + \frac{\mu}{2}(g_{jk}\partial_j\hat{v}_i \\ & + g_{ik}\partial_k\hat{v}_j)^2 + \mu'(g_{ij}\partial_j\hat{v}_i)^2 + \hat{\rho}^2\hat{\theta}S_{\hat{\theta}}g_{ij}\partial_j\hat{v}_i\Big].\end{aligned} \tag{2.11}$$

Condition (1.5) implies

$$\begin{cases} \hat{v}(x_0,0) = v_0(x_0),\ \hat{\theta}(x_0,0) = \theta_0(x_0),\ \hat{\rho}(x_0,0) = \rho_0(x_0) \quad (x_0 \in \Omega), \\ \hat{v}(x_0,t_0) = 0,\ \hat{\theta}(x_0,t_0) = \theta_1(x_0,t_0) \quad ((x_0,t_0) \in \Gamma_T). \end{cases} \tag{2.12}$$

Next we state the following well known fact (cf. [42]):

Lemma 2.1. *Let $\Gamma \in C^{m+\alpha}$ $(m \geqq 0)$ and $g \in C^{m+\alpha}(\Gamma_T)$. Then there exists at least one function $g^* \in C^{m+\alpha}(\bar{Q}_T)$ such that $g^* = g$ on Γ_T.*

Using this lemma, we can extend $\theta_1 \in H^{2+\alpha}(\Gamma_T)$ to $\theta_1^* \in H^{2+\alpha}(\bar{Q}_T)$, which we also denote by θ_1 without misunderstanding.

2.2. Restatement of the Problem. We shall use the notation (x, t) instead of (x_0, t_0) except the last section. If $\hat{v} \in H^{2+\alpha}(\bar{Q}_T)$ is given in (2.9), then $\hat{\rho}$ is uniquely determined, being expressed by

$$(2.13)\qquad \hat{\rho}(x,t)=\rho_0(x)\cdot\exp\left[-\int_0^t g_{ij}\partial_j\hat{v}_i(x,\tau)\,d\tau\right].$$

If in (2.10) and (2.11) we put

$$(2.14)\qquad \begin{cases} w_i(x,t)=\hat{v}_i(x,t)-v_{0i}(x) \quad (i=1,2,3), \\ w_4(x,t)=\hat{\theta}(x,t)-\theta_1(x,t)+\theta_1(x,0)-\theta_0(x), \end{cases}$$

then (2.10) and (2.11) can be written in the form

$$(2.15)\qquad \frac{\partial}{\partial t}w_i=\mathfrak{A}_{ij}^{km}(x,t,w)\partial_k\partial_m w_j+\mathfrak{B}_i(x,t,w) \quad (i=1,2,3,4)$$

and the initial-boundary condition (2.12) is transformed into

$$(2.16)\qquad w(x,0)=0,\ w(x,t)|_{\Gamma_T}=0.$$

As a second main theorem we shall prove

Theorem 2. *Under the same assumptions of Theorem* 1, *for some* $T'\in(0,T]$ *there uniquely exists* $(w,\hat{\rho})\in H^{2+\alpha}(\bar{Q}_{T'})\times B^{1+\alpha}(\bar{Q}_{T'})$ *such that* $(w,\hat{\rho})$ *satisfies* (2.9) *with* w_i+v_{0i} *in place of* $\hat{v}_i$, (2.15) *and* (2.16).

First of all we proceed to prove Theorem 2 and next to prove Theorem 1 briefly.

At first we consider the following linear problem connected with (2.15):

$$(2.17)\qquad \begin{cases} \dfrac{\partial\tilde{w}_i}{\partial t}=\mathfrak{A}_{ij}^{km}(x,t,w)\partial_k\partial_m\tilde{w}_j+\phi_i(x,t,w) \quad (i=1,2,3,4), \\ \tilde{w}(x,0)=0,\ \tilde{w}|_{\Gamma_T}=0, \end{cases}$$

where ϕ_i $(i=1,2,3,4)$ are any given functions which belong to $H^{\alpha}(\bar{Q}_T)$ and satisfy the compatibility condition. Here we assume that

$$(2.18)\qquad w\in\mathfrak{S}_T=\{w\in H^{2+\alpha}(\bar{Q}_T)\mid\ w|_{\Gamma_T}=0,$$

$$\langle w\rangle_T^{(2,\alpha)}\equiv\sum_{2r+|s|=0}^{2}|D_t^rD_x^sw|_T^{(0)}+\sum_{|s|=1}|D_x^sw|_{t,T}^{(\alpha/2)}<M_1\},$$

where M_1 is an arbitrary positive number smaller than $\bar{\theta}_0$.

Let M_0 be the positive root of $1-3x-6x^2-6x^3=0$. Then there exists $T_1 \in (0, T]$ such that

$$(2.19) \qquad (M_1 + \|v_0\|^{(2+\alpha)})T_1 < M_0, \ \bar{\theta}_0 - M_1 - \|\theta_1\|_{T_1}^{(2+\alpha)} \cdot T_1 > 0.$$

On this account, matrix (g_{ij}) is well-defined and the following inequality holds:

$$(2.20) \qquad |g_{ij} - \delta_{ij}| \leqq B_1(\bar{T}, \langle w \rangle_{\bar{T}}^{(2,\alpha)}) \quad (\bar{T} \in (0, T_1]),$$

where B_1 is a positive constant increasing in each argument and $B_1 \downarrow 0$ as $\bar{T} \downarrow 0$.

Lemma 2.2. *Assume* (2.18). *Then the system* (2.17) *is uniformly parabolic in the sense of Petrowsky, i.e., there exists a number* $\delta > 0$ *such that*

$$(2.21) \qquad \max_j \sup_{|\xi|=1} \operatorname{Re} \lambda_j(\xi, x, t) \leqq -\delta \quad (\forall (x, t) \in \bar{Q}_T),$$

where λ_j*'s are the roots of* $\det\{\mathfrak{A}_{sj}^{km}(i\xi_k)(i\xi_m) - \lambda\delta_{sj}\} = 0$.

Proof. Since

$$\det\{\mathfrak{A}_{sj}^{km}(i\xi_k)(i\xi_m) - \lambda\delta_{sj}\} = \left(\lambda + \frac{\mu}{\hat{\rho}} g_{jk} g_{jm} \xi_k \xi_m\right)^2$$

$$\times \left(\lambda + \frac{2\mu + \mu'}{\hat{\rho}} g_{jk} g_{jm} \xi_k \xi_m\right)\left(\lambda + \frac{\kappa}{\hat{\rho}\hat{\theta} S_\theta} g_{jk} g_{jm} \xi_k \xi_m\right),$$

we have

$$(2.22) \qquad \begin{cases} \lambda_1 = \lambda_2 = -\dfrac{\mu}{\hat{\rho}} g_{jk} g_{jm} \xi_k \xi_m, \\[2ex] \lambda_3 = -\dfrac{2\mu + \mu'}{\hat{\rho}} g_{jk} g_{jm} \xi_k \xi_m, \\[2ex] \lambda_4 = -\dfrac{\kappa}{\hat{\rho}\hat{\theta} S_\theta} g_{jk} g_{jm} \xi_k \xi_m. \end{cases}$$

Using (2.20) we obtain

$$g_{jk} g_{jm} \xi_k \xi_m \geqq \sum_j \xi_j^2 \sum_i \{g_{ij}^2 - \sum_{k \neq j} |g_{ij}| |g_{ik}|\}$$

$$\geqq (1 - 6B_1 - 3B_1^2)|\xi|^2.$$

Because of the property of B_1, there exists $T_2 \in (0, T_1]$ such that

$$0<B_1(T_2,M_1)<1,\ 1-6B_1(T_2,M_1)-3B_1(T_2,M_1)^2>0.$$

From (2. 13), (2. 14), (2. 18), (2. 20) and the properties of ρ_0 and θ_0 it follows that the domain considered in (1. 13) becomes

(2. 23)
$$\begin{aligned}\mathscr{D}^*_{\hat\rho,\hat\theta}=(&\bar\rho_0\cdot\exp[-(M_1+\|v_0\|^{(2+\alpha)})9(1+B_1)T_2],\\ &\bar{\bar\rho}_0\cdot\exp[(M_1+\|v_0\|^{(2+\alpha)})9(1+B_1)T_2])\\ &\times(\bar\theta_0-M_1-\|\theta_1\|^{(2+\alpha)}_{T_2}\cdot T_2,\ \bar{\bar\theta}_0+M_1+2\|\theta_1\|^{(2+\alpha)}_{T_2}).\end{aligned}$$

Hence for $\sigma\in\{\kappa,\mu,\mu',S_{\hat\theta}\}$ we can define

$$\bar\sigma=\min_{(\hat\rho,\hat\theta)\in\mathscr{D}^*\hat\rho,\hat\theta}\sigma(\hat\rho,\hat\theta),\quad \bar{\bar\sigma}=\max_{(\hat\rho,\hat\theta)\in\mathscr{D}^*\hat\rho,\hat\theta}\sigma(\hat\rho,\hat\theta).$$

Thus for any $(x,t)\in\bar Q_{T_2}$

(2. 24)
$$\begin{aligned}\max_j\sup_{|\xi|=1}\operatorname{Re}\lambda_j(\xi,x,t)&\leqq-(1-6B_1-3B_1^2)\\ &\times\min\left\{\frac{\bar\mu}{\bar{\bar\rho}_0},\ \frac{\bar\kappa}{\bar{\bar\rho}_0(\bar{\bar\theta}_0+M_1+2\|\theta_1\|^{(2+\alpha)}_{T_2})\bar{\bar S}_{\hat\theta}}\right\}\\ &\times\exp[-(M_1+\|v_0\|^{(2+\alpha)})9(1+B_1)T_2]\equiv-\delta.\end{aligned}$$

Q.E.D.

Hereafter, for simplicity we choose T from the beginning in such a way that $T=T_2$.

§ 3. Green Matrix of a Linear Problem

3. 1. Some Results of a Fundamental Solution in the Whole Space R^3.

We begin this section with a general, well-known lemma concerning the extention of functions ([19, 54]):

Lemma 3. 1. *Suppose* $\Gamma\in C^{\tilde m}$, $\tilde m=\max(1+\alpha, m)$ *and* $g\in H^m(\bar Q_T)$. *Then there exists an extention* $^*g$ *of* g, *which belongs to* $H^m(\bar R_T{}^3)$ *and satisfies the inequality* $\|^*g\|^{(m)}_{T,R^3}\leqq C\|g\|_T{}^{(m)}$, *where* $R_T{}^3=R^3\times(0,T]$ *and* $\|^*g\|^{(m)}_{T,R^3}$ *is defined by the formulas* (1. 7)-(1. 9) *with* $\bar R_T{}^3$ *instead of* $\bar Q_T$.

Henceforth we shall always denote the extention of a function g by $^*g$.

Since the properties of the fundamental solution in the whole space

R^3 are well known, we state them briefly.

First of all, in connection with (2. 17), we consider the following system of ordinary differential equations in t:

$$(3.1)\qquad \begin{cases} \dfrac{dV}{dt} = {}^*\mathfrak{A}(y, \tau, w; i\zeta)\cdot V(\zeta, t-\tau; y, \tau; w), \\ V(\zeta, t-\tau; y, \tau; w)|_{t=\tau} = I \ \ (\zeta \in C^3), \end{cases}$$

where ${}^*\mathfrak{A}(y, \tau, w; i\zeta) = ({}^*\mathfrak{A}_{sj}^{km}(y, \tau, w(y, \tau))(i\zeta_k)(i\zeta_m))$. V can be solved directly from (3. 1), i.e.,

$$(3.2)\qquad V(\zeta, t-\tau; y, \tau; w) = \exp[(t-\tau)\cdot {}^*\mathfrak{A}(y, \tau, w; i\zeta)].$$

Then the parametrix Z_0 of the equation

$$(3.3)\qquad \frac{\partial u}{\partial t} = {}^*\mathfrak{A}(x, t, w; D_x)u$$

is defined by the formula

$$(3.4)\qquad Z_0(x-\xi, t-\tau; y, \tau; w) = (2\pi)^{-3}\int_{R^3}\exp[i\xi_0\cdot(x-\xi)] \times V(\xi_0, t-\tau; y, \tau; w)\,d\xi_0.$$

Lemma 3. 2. ([13, 18]) *If A be any matrix of N-th order with complex components whose eigenvalues are $\lambda_1, \cdots, \lambda_N$, then*

$$\|\exp[tA]\| \leqq \sum_{k=0}^{N-1}(2t\|A\|)^k\cdot\exp[t\cdot\max_j \operatorname{Re}\lambda_j].$$

Introducing the notation $|A| = \sum_{i,j}|A_{ij}|$ for the matrix $A = (A_{ij})$, we have the following inequalities:

$$\frac{1}{2N^2}|A|^2 \leqq \max_j \sum_k |A_{kj}|^2 \leqq \|A\|^2 \leqq \sum_{j,k}|A_{kj}|^2 \leqq |A|^2.$$

Since for $\zeta = \xi + i\eta\ (\xi, \eta \in R^3)$

$$(3.5)\qquad \begin{cases} \operatorname{Re}\lambda_1(\zeta) = \operatorname{Re}\lambda_2(\zeta) = -\dfrac{\mu}{\hat\rho}g_{ij}g_{ik}\xi_j\xi_k + \dfrac{\mu}{\hat\rho}g_{ij}g_{ik}\eta_j\eta_k, \\ \operatorname{Re}\lambda_3(\zeta) = -\dfrac{2\mu+\mu'}{\hat\rho}g_{ij}g_{ik}\xi_j\xi_k + \dfrac{2\mu+\mu'}{\hat\rho}g_{ij}g_{ik}\eta_j\eta_k, \\ \operatorname{Re}\lambda_4(\zeta) = -\dfrac{\kappa}{\hat\rho\hat\theta S_\theta}g_{ij}g_{ik}\xi_j\xi_k + \dfrac{\kappa}{\hat\rho\hat\theta S_\theta}g_{ij}g_{ik}\eta_j\eta_k, \end{cases}$$

we have

$$\max_j \operatorname{Re} \lambda_j(\zeta, x, t) \leqq -\delta|\xi|^2 + B_2|\eta|^2, \tag{3.6}$$

where δ is a constant in (2. 24) and

$$B_2 = (1+6B_1+3B_1^2)\left[(2\bar{\bar{\mu}}+\bar{\bar{\mu}}')(\bar{\rho}_0)^{-1}\bar{\bar{\kappa}}\{\bar{\rho}_0(\bar{\theta}_0 - M_1 - \|\theta_1\|_T^{(2+\alpha)}\cdot T)\right.$$
$$\left.\times \bar{S}_\theta\}^{-1}\right]\cdot \exp[(M_1 + \|v_0\|^{(2+\alpha)})9(1+B_1)T].$$

On the other hand the inequality

$$|{}^*\mathfrak{A}| \leqq 10B_2|\zeta|^2$$

holds. Hence

$$|\exp[(t-\tau)\cdot {}^*\mathfrak{A}(y, \tau, w; i\zeta)]| \leqq C_0 \tag{3.7}$$
$$\times \exp\left[(t-\tau)\left\{-\frac{\delta}{2}|\xi|^2 + \left(B_2 + \frac{\delta}{2}\right)|\eta|^2\right\}\right],$$

where $C_0 = 4\sqrt{2}[1+2K(1, 0)B_2\delta^{-1} + 4K(2, 0)B_2^2\delta^{-2} + 8K(3, 0)B_2^3\delta^{-3}]$, and for $a \geqq 0$, $b \in [0, 1)$, $K(a, b) = \{a/(1-b)e\}^a$. Thus we obtain

$$|V(\zeta, t-\tau; y, \tau; w)| \leqq C_0 \tag{3.8}$$
$$\times \exp\left[\left\{-\frac{\delta}{2}|\xi|^2 + \left(B_2 + \frac{\delta}{2}\right)|\eta|^2\right\}(t-\tau)\right].$$

From (3. 4) and (3. 8) it follows that

Lemma 3. 3.

$$|D_t^r D_x^s Z_0| \leqq C_1^{(r,|s|)}(t-\tau)^{-(3+2r+|s|)/2} \tag{3.9}$$
$$\times \exp\left[-\frac{|x-\xi|^2}{8\delta_1(t-\tau)}\right] \quad \left(\delta_1 \equiv B_2 + \frac{\delta}{2}\right),$$

$$|\varDelta_x^{x'} D_t^r D_x^s Z_0| \leqq C_1^{(r,|s|+1)}|x-x'|(t-\tau)^{-(4+2r+|s|)/2} \tag{3.10}$$
$$\times \exp\left[-\frac{|x''-\xi|^2}{8\delta_1(t-\tau)}\right],$$
$$(x''=x' \text{ if } |x'-\xi| \leqq |x-\xi|;\ x''=x, \text{ otherwise}),$$

$$|\varDelta_t^{t'} D_t^r D_x^s Z_0| \leqq C_1^{(r+1,|s|)}(t-t')(t'-\tau)^{-(5+2r+|s|)/2} \tag{3.11}$$
$$\times \exp\left[-\frac{|x-\xi|^2}{8\delta_1(t-\tau)}\right] \quad (t>t'>\tau),$$

where we use the notations $\Delta_{x,t}^{x',t'} g(x,t) = g(x,t) - g(x',t')$, $\Delta_x^{x'} = \Delta_{x,t}^{x',t}$ *and* $\Delta_t^{t'} = \Delta_{x,t}^{x,t'}$.

Lemma 3.4.

$$(3.12)\quad |D_x^s Z_0(x-\xi, t-\tau; y, \tau; w) - D_x^s Z_0(x-\xi, t-\tau; y+h, \tau; w)| \leqq C_2^{(|s|)} |h|^{\alpha} (t-\tau)^{-(3+|s|)/2} \cdot \exp\left[-\frac{|x-\xi|^2}{8\delta_2(t-\tau)}\right],$$

$$(3.13)\quad |D_x^s Z_0(x-\xi, t-\tau; y, \tau; w) - D_x^s Z_0(x-\xi, t-\tau; y, \tau+h; w)| \leqq \bar{C}_2^{(|s|)} |h|^{\alpha/2} (t-\tau)^{-(3+|s|)/2} \cdot \exp\left[-\frac{|x-\xi|^2}{8\delta_2(t-\tau)}\right], \quad \left(\delta_2 \equiv \delta_1 + \frac{1}{4}\delta\right).$$

Next the unique fundamental solution $Z(x-\xi, t-\tau; \xi, \tau; w)$ of (3.1) in $H^{2+\alpha}(\bar{R}_T^3)$ is defined by

$$(3.14)\quad Z(x-\xi, t-\tau; \xi, \tau; w) = Z_0(x-\xi, t-\tau; \xi, \tau; w) + \int_\tau^t d\tau_0 \int_{R^3} Z_0(x-y, t-\tau;\ y, \tau_0; w)\, \Phi(y, \tau_0; \xi, \tau; w)\, dy \equiv Z_0 + Z',$$

where Φ is a solution of a Volterra type integral equation:

$$(3.15)\quad \begin{cases} \Phi(x, t; \xi, \tau; w) = K(x, t; \xi, \tau; w) \\ \qquad + \displaystyle\int_\tau^t d\tau_0 \int_{R^3} K(x, t; y, \tau_0; w)\, \Phi(y, \tau_0; \xi, \tau; w)\, dy, \\ K(x, t; \xi, \tau; w) = ({}^*\mathfrak{A}(x, t, w; D_x) \\ \qquad - {}^*\mathfrak{A}(\xi, \tau, w; D_x))\, Z_0(x-\xi, t-\tau; \xi, \tau; w), \end{cases}$$

which is given by the Neumann series

$$(3.16)\quad \begin{cases} \Phi(x, t; \xi, \tau; w) = \displaystyle\sum_{m=0}^{\infty} K_m(x, t; \xi, \tau; w), \\ K_m(x, t; \xi, \tau; w) = \displaystyle\int_\tau^t d\tau_0 \int_{R^3} K_0(x, t; y, \tau_0: w) \\ \qquad\qquad \times K_{m-1}(y, \tau_0; \xi, \tau; w)\, dy, \\ K_0(x, t; \xi, \tau; w) = K(x, t; \xi, \tau; w). \end{cases}$$

Lemma 3.5.

$$(3.17)\qquad |D_t^r D_x^s Z'| \leqq C_3^{(r,|s|)}(t-\tau)^{-(3+2r+|s|-\alpha)/2} \times \exp\left[-\frac{d_1}{36}\frac{|x-\xi|^2}{t-\tau}\right],\quad (2r+|s|\leqq 2),$$

$$(3.18)\qquad |\Delta_x^{x'} D_t^r D_x^s Z'| \leqq C_4^{(r,|s|)}|x-x'|^\alpha (t-\tau)^{-5/2} \times \exp\left[-\frac{d_1}{72}\frac{|x''-\xi|^2}{t-\tau}\right],\quad (2r+|s|=2),$$

$$(3.19)\qquad |\Delta_t^{t'} D_t^r D_x^s Z'| \leqq C_5^{(r,|s|)}(t-t')^{(2-2r-|s|+\alpha)/2}(t'-\tau)^{-5/2} \times \exp\left[-\frac{d_1}{72}\frac{|x-\xi|^2}{t-\tau}\right],\quad (0<2r+|s|\leqq 2,\ t>t'>\tau),$$

where $d_1 = 1/16\delta_1$.

Lemma 3.6.

$$(3.20)\qquad |D_t^r D_x^s Z| \leqq C_6^{(r,|s|)}(t-\tau)^{-(3+2r+|s|)/2} \times \exp\left[-\frac{d_1}{36}\frac{|x-\xi|^2}{t-\tau}\right],\quad (2r+|s|\leqq 2),$$

$$(3.21)\qquad |\Delta_x^{x'} D_t^r D_x^s Z| \leqq C_7^{(r,|s|)}\{|x-x'|^\gamma (t-\tau)^{-(5+\gamma)/2} + |x-x'|^\beta (t-\tau)^{-(5-\alpha+\beta)/2}\}\cdot\exp\left[-\frac{d_1}{72}\frac{|x''-\xi|^2}{t-\tau}\right],$$
$$(2r+|s|=2,\ \gamma\in[0,1],\ \beta\in[0,\alpha]),$$

$$(3.22)\qquad |\Delta_t^{t'} D_t^r D_x^s Z| \leqq C_8^{(r,|s|)}\{(t-t')(t'-\tau)^{-(5+2r+|s|)/2} + (t-t')^{(2-2r-|s|+\alpha)/2}(t'-\tau)^{-5/2}\}\exp\left[-\frac{d_1}{72}\frac{|x-\xi|^2}{t-\tau}\right],$$
$$(0<2r+|s|\leqq 2,\ t>t'>\tau).$$

3.2. The Basic Condition of Uniform Solvability. Let $\mathfrak{A}^{(k'')}(\xi^{(k'')}, \tau, w; D_{\bar{x}})$ be a matrix obtained from $\mathfrak{A}(\xi^{(k'')}, \tau, w; D_x)$ by the transformation of the coordinate system into the local rectangular coordinate system $\{\hat{x}\}$ around $\xi^{(k'')}\in\Gamma$ and the replacement of $D_{\hat{x}}$ by $D_{\bar{x}}$. Let $\hat{\mathfrak{A}}^{(k'')}(\xi^{(k'')}, \tau, w; i\bar{x}, \upsilon)$ be the matrix whose elements are the cofactors of $\mathfrak{A}^{(k'')}(\xi^{(k'')}, \tau, w; i\bar{x}) - \upsilon I$. Then we have

Lemma 3.7. *For any* $\bar{x}' = (\bar{x}_1, \bar{x}_2)\in R^2$ *and any* υ *such that*

$$\text{Re}\,\upsilon \geqq -\delta_3 \bar{x}'^2,\ |\upsilon|^2+\bar{x}'^4>0\ \ (0<\delta_3<\tilde{\delta}), \tag{3.23}$$

the row vectors of the matrix $\hat{\mathfrak{A}}^{(k'')}(\xi^{(k'')},\tau,w;i\bar{x},\upsilon)$ *are independent modulo*

$$M^+ = \prod_{j=1}^{4}(\bar{x}_3-\bar{x}_3^{+(j)}(\bar{x}',\upsilon)), \tag{3.24}$$

where $\bar{x}_3^{+(j)}$*'s are the roots in* $\bar{x}_3$ *of* $\det\{\mathfrak{A}^{(k'')}(\xi^{(k'')},\tau,w;i\bar{x})-\upsilon I\}=0$ *with positive imaginary parts, and* $\tilde{\delta}$ *is a constant appearing in* (2.24) *with* B_1C_9 *in place of* B_1 *if* $|D_x\hat{x}|\leqq C_9$.

Proof. If we put $(g_{ij})(D_x\hat{x})=(\tilde{g}_{ij})$, then $\mathfrak{A}^{(k'')}(\xi^{(k'')},\tau,w;i\bar{x})$ is obtained from $\mathfrak{A}(\xi^{(k'')},\tau,w;i\bar{x})$ by the replacement of g_{ij} by $\tilde{g}_{ij}$. That is to say from (2.22) it follows that

$$-\frac{\mu}{\hat{\rho}}\tilde{g}_{ij}\tilde{g}_{ik}\bar{x}_j\bar{x}_k=\upsilon,\quad -\frac{2\mu+\mu'}{\hat{\rho}}\tilde{g}_{ij}\tilde{g}_{ik}\bar{x}_j\bar{x}_k=\upsilon, \tag{3.25}$$

$$-\frac{\kappa}{\hat{\rho}\hat{\theta}S_{\theta}}\tilde{g}_{ij}\tilde{g}_{ik}\bar{x}_j\bar{x}_k=\upsilon.$$

Setting

$$(\mu/\hat{\rho})^{-1}=a_1=a_2,\ \ \{(2\mu+\mu')/\hat{\rho})\}^{-1}=a_3,\ \ (\kappa/\hat{\rho}\hat{\theta}S_{\theta})^{-1}=a_4, \tag{3.26}$$

we have

$$\tilde{g}_{ij}\tilde{g}_{ik}\bar{x}_j\bar{x}_k=-a_m\upsilon,\ \ (m=1,2,3,4). \tag{3.27}$$

Solving this equation in $\bar{x}_3$, we have

$$\bar{x}_3^{\pm(m)}=(\tilde{g}_{j3})^{-2}\{-\tilde{g}_{k3}(\bar{x}_1+\tilde{g}_{k2}\bar{x}_2)\pm(A_+^{(m)}+iB_+^{(m)})\},\qquad (m=1,2,3,4), \tag{3.28}$$

where

$$\begin{cases} B_+^{(m)}=\left\{\dfrac{1}{2}[\{\text{Re}\,D_m)^2+(\text{Im}\,D_m)^2\}^{1/2}-\text{Re}\,D_m]\right\}^{1/2}(>0),\\ A_+^{(m)}=\text{Im}\,D_m/2B_+^{(m)},\\ D_m=-\dfrac{1}{2}\{(\tilde{g}_{j3}\tilde{g}_{k1}-\tilde{g}_{k3}\tilde{g}_{j1})\bar{x}_1+(\tilde{g}_{j3}\tilde{g}_{k2}-\tilde{g}_{k3}\tilde{g}_{j2})\bar{x}_2\}^2-a_m(\tilde{g}_{j3})^2\upsilon. \end{cases} \tag{3.29}$$

Therefore M^+ defined by (3.24) is expressed by

$$M^+=(\bar{x}_3-\bar{x}_3^{+(1)})^2(\bar{x}_3-\bar{x}_3^{+(3)})(\bar{x}_3-\bar{x}_3^{+(4)}). \tag{3.30}$$

The matrix $\hat{\mathfrak{A}}^{(k'')}(\xi^{(k'')}, \tau, w; i\bar{x}, \upsilon) = (\hat{\mathfrak{A}}_{ij}^{(k'')})$ may be written as follows:

$$(3.31)\quad \hat{\mathfrak{A}}_{ij}^{(k'')} = \begin{cases} -\dfrac{(\tilde{g}_{m3})^6}{a_1 a_3 a_4}(\bar{x}_3-\bar{x}_3^{+(1)})(\bar{x}_3-\bar{x}_3^{-(1)})(\bar{x}_3-\bar{x}_3^{+(3)})(\bar{x}_3-\bar{x}_3^{-(3)}) \\ \quad\times(\bar{x}_3-\bar{x}_3^{+(4)})(\bar{x}_3-\bar{x}_3^{-(4)})\delta_{ij}+\dfrac{(\tilde{g}_{m3})^4}{a_1 a_4 a_5}(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2 \\ \quad+\tilde{g}_{i3}\bar{x}_3)(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2+\tilde{g}_{j3}\bar{x}_3)(\bar{x}_3-\bar{x}_3^{+(1)})(\bar{x}_3-\bar{x}_3^{-(1)}) \\ \quad\times(\bar{x}_3-\bar{x}_3^{+(4)})(\bar{x}_3-\bar{x}_3^{-(4)}), \quad \text{if} \quad i, j=1, 2, 3; \\ -\dfrac{(\tilde{g}_{m3})^6}{a_1^2 a_3}(\bar{x}_3-\bar{x}_3^{+(1)})^2(\bar{x}_3-\bar{x}_3^{-(1)})^2(\bar{x}_3-\bar{x}_3^{+(3)}) \\ \quad\times(\bar{x}_3-\bar{x}_3^{-(3)}), \quad \text{if} \quad i=j=4; \\ 0, \text{ otherwise}; \end{cases}$$

where $a_5 \equiv \{(\mu+\mu')/\hat{\rho}\}^{-1}$. Let $\check{\mathfrak{A}}_{ij}^{(k'')}$ be the remainder when we divide $\hat{\mathfrak{A}}_{ij}^{(k'')}$ by M^+. Then

$$(3.32)\quad \check{\mathfrak{A}}_{ij}^{(k'')} = \begin{cases} -\dfrac{(\tilde{g}_{m3})^6}{a_1 a_3 a_4}(\bar{x}_3^{-(1)}-\bar{x}_3^{-(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(3)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(4)}) \\ \quad\times(\bar{x}_3-\bar{x}_3^{+(1)})(\bar{x}_3-\bar{x}_3^{+(3)})(\bar{x}_3-\bar{x}_3^{+(4)})\delta_{ij} \\ \quad+\dfrac{(\tilde{g}_{m3})^4}{a_1 a_4 a_5}\{(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3^{+(1)})(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2 \\ \quad+\tilde{g}_{j3}\bar{x}_3^{+(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(4)}) \\ \quad-(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3^{+(3)})(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2+\tilde{g}_{j3}\bar{x}_3^{+(3)}) \\ \quad\times(\bar{x}_3^{+(3)}-\bar{x}_3^{-(1)})(\bar{x}_3^{+(3)}-\bar{x}_3^{-(4)})\}(\bar{x}_3^{+(1)}-\bar{x}_3^{+(3)})^{-1} \\ \quad\times\bar{x}_3(\bar{x}_3-\bar{x}_3^{+(1)})(\bar{x}_3-\bar{x}_3^{+(4)})+\dfrac{(\tilde{g}_{m3})^4}{a_1 a_4 a_5}\{\bar{x}_3^{+(1)}(\tilde{g}_{i1}\bar{x}_1 \\ \quad+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3^{+(3)})(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2+\tilde{g}_{j3}\bar{x}_3^{+(3)})(\bar{x}_3^{+(3)} \\ \quad-\bar{x}_3^{-(1)})(\bar{x}_3^{+(3)}-\bar{x}_3^{-(4)})-\bar{x}_3^{+(3)}(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3^{+(1)}) \\ \quad\times(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2+\tilde{g}_{j3}\bar{x}_3^{+(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)}) \\ \quad\times(\bar{x}_3^{-(1)}-\bar{x}_3^{-(4)})\}(\bar{x}_3^{+(1)}-\bar{x}_3^{+(3)})^{-1}(\bar{x}_3-\bar{x}_3^{+(1)}) \\ \quad\times(\bar{x}_3-\bar{x}_3^{+(4)}), \quad \text{if} \quad \upsilon\neq 0, \; i, j=1, 2, 3; \\ -\dfrac{(\tilde{g}_{m3})^6}{a_1 a_3 a_4}(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)})^3(\bar{x}_3-\bar{x}_3^{+(1)})^3\delta_{ij}+\dfrac{(\tilde{g}_{m3})^4}{a_1 a_4 a_5} \\ \quad\times(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3^{+(1)})(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2+\tilde{g}_{j3}\bar{x}_3^{+(1)}) \end{cases}$$

$$\times(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})^2(\bar{x}_3-\bar{x}_3{}^{+(1)})^2, \quad \text{if} \quad \upsilon=0,\ i,j=1,2,3;$$
$$-\frac{(\tilde{g}_{m3})^6}{a_1^2a_3}(\bar{x}_3{}^{+(4)}-\bar{x}_3{}^{-(1)})^2(\bar{x}_3{}^{+(4)}-\bar{x}_3{}^{-(3)})(\bar{x}_3-\bar{x}_3{}^{+(1)})^2$$
$$\times(\bar{x}_3-\bar{x}_3{}^{+(3)}), \quad \text{if} \quad i=j=4;$$
$$0, \text{ otherwise.}$$

(N.B. If $\upsilon=0$, then $\bar{x}_3{}^{+(1)}=\bar{x}^{+(3)}=\bar{x}_3{}^{+(4)}$, $\bar{x}_3{}^{-(1)}=\bar{x}_3{}^{-(3)}=\bar{x}_3{}^{-(4)}$.) If we put $\check{\mathfrak{A}}_{ij}^{(k'')}=\sum_{s=1}^{4}\alpha_{ij}^{(s)}\bar{x}_3{}^{s-1}$, then in order to show that the row vectors of $\check{\mathfrak{A}}^{(k'')}$ are linearly independent as polynomials of $\bar{x}_3$, it is sufficient to show that the rank of the matrix $(\alpha_{ij}^{(s)})$ is 4 for any $\bar{x}'\in R^2$ and υ satisfying (3.23). From (3.32) it follows that

$$(3.33) \qquad \alpha_{ij}^{(4)}=\begin{cases} -\dfrac{(\tilde{g}_{m3})^6}{a_1a_3a_4}(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(3)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(4)})\delta_{ij} \\ +\dfrac{(\tilde{g}_{m3})^4}{a_1a_4a_5}\{(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3{}^{+(1)})(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2 \\ +\tilde{g}_{j3}\bar{x}_3{}^{+(1)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(4)})-(\tilde{g}_{i1}\bar{x}_1 \\ +\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3{}^{+(3)})(\tilde{g}_{j1}\bar{x}_1+\tilde{g}_{j2}\bar{x}_2+\tilde{g}_{j3}\bar{x}_3{}^{+(3)})(\bar{x}_3{}^{+(3)} \\ -\bar{x}_3{}^{-(1)})(\bar{x}_3{}^{+(3)}-\bar{x}_3{}^{-(4)})\}(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{+(3)})^{-1}, \\ \qquad \text{if} \quad \upsilon\neq0,\ i,j=1,2,3; \\ -\dfrac{(\tilde{g}_{m3})^6}{a_1a_3a_4}(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})^3\delta_{ij}, \quad \text{if} \quad \upsilon=0,\ i,j=1,2,3; \\ -\dfrac{(\tilde{g}_{m3})^6}{a_1^2a_3}(\bar{x}_3{}^{+(4)}-\bar{x}_3{}^{-(1)})^2(\bar{x}_3{}^{+(4)}-\bar{x}_3{}^{-(3)}), \quad \text{if} \quad i=j=4; \\ 0, \text{ otherwise.} \end{cases}$$

Now let us consider $\det(\alpha_{ij}^{(4)})\equiv\varDelta$. By (3.33)

$$(3.34) \qquad \varDelta=\begin{cases} \dfrac{(\tilde{g}_{m3})^{20}}{a_1^5a_3^2a_4^3}(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})^2(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(3)})^2(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(4)})^4 \\ \times(\bar{x}_3{}^{+(4)}-\bar{x}_3{}^{-(3)})\Big[\dfrac{(\tilde{g}_{m3})^4}{a_3^2}(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(3)}) \\ \times(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(4)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{+(3)})-\dfrac{(\tilde{g}_{m3})^2}{a_3a_5} \\ \times\{(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3{}^{+(1)})^2(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(4)}) \\ -(\tilde{g}_{i1}\bar{x}_1+\tilde{g}_{i2}\bar{x}_2+\tilde{g}_{i3}\bar{x}_3{}^{+(3)})^2(\bar{x}_3{}^{+(3)}-\bar{x}_3{}^{-(1)})(\bar{x}_3{}^{+(3)}-\bar{x}_3{}^{-(4)})\} \end{cases}$$

$$\left. \begin{array}{l} -\frac{1}{a_5^2}\{(\tilde{g}_{i1}\tilde{g}_{j3}-\tilde{g}_{j1}\tilde{g}_{i3})\bar{x}_1+(\tilde{g}_{i2}\tilde{g}_{j3}-\tilde{g}_{j2}\tilde{g}_{i3})\bar{x}_2\}^2 \\ \times(\bar{x}_3{}^{+(3)}-\bar{x}_3{}^{-(4)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{+(3)})\Big](\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{+(3)})^{-1}, \\ \qquad \text{if} \quad \upsilon\neq 0; \\ \frac{(\tilde{g}_{m3})^{24}}{a_1^5 a_3^2 a_4^3}(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(1)})^{12}, \quad \text{if} \quad \upsilon=0. \end{array} \right.$$

In the case of $\upsilon=0$, it is clearly by (3. 34) that $\varDelta\neq 0$. Next in the case of $\upsilon\neq 0$, since by (3. 28) and (3. 29)

$$(3.35) \qquad (\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{-(3)})(\bar{x}_3{}^{+(1)}-\bar{x}_3{}^{+(3)})=(\tilde{g}_{m3})^{-4}[D_1-D_3]$$
$$=(a_3-a_1)(\tilde{g}_{m3})^{-2}\upsilon,$$

and by (3. 26) and the definition of a_5 in (3. 31)

$$(3.36) \qquad \frac{a_3-a_1}{a_3^2}+\frac{a_1}{a_3 a_5}=0,$$

we have for the product of the last two factors in (3. 34), $\varDelta_1$,

$$(3.37) \qquad \varDelta_1=-\frac{1}{a_5}(x_3{}^{+(3)}-x_3{}^{-(4)})\left[(a_3-a_1)^{-1}\{A_+{}^{(1)}+A_+{}^{(3)}\right.$$
$$+i(B_+{}^{(1)}+B_+{}^{(3)})\}^2+a_5^{-1}\{(\tilde{g}_{i1}\tilde{g}_{j3}-\tilde{g}_{j1}\tilde{g}_{i3})\bar{x}_1$$
$$\left.+(\tilde{g}_{i2}\tilde{g}_{j3}-\tilde{g}_{j2}\tilde{g}_{i3})\bar{x}_2\}^2\right]\neq 0.$$

Hence $\varDelta\neq 0$, i.e., rank $(\alpha_{jk}^{(s)})=4$. Q.E.D.

3. 2. On the Poisson Kernel. Making use of the fundamental solution Z constructed in § 3. 1, we shall find a Green matrix G for a system (2. 17) of the form

$$(3.38) \qquad G=Z-G_0,$$

where G_0 is a solution of a system of equations:

$$(3.39) \qquad \left\{ \begin{array}{l} D_t W=\mathfrak{A}(x,t,w;D_x)W, \\ W|_{t=\tau}=0, \quad W|_{\Gamma_{\tau,T}}=Z|_{\Gamma_{\tau,T}} \quad (0\leqq\tau\leqq t\leqq T). \end{array} \right.$$

Here we use the notations

$$\mathfrak{A}(x,t,w;D_x)=(\mathfrak{A}_{ij}^{km}\partial_k\partial_m), \quad \Gamma_{\tau,T}=\Gamma\times[\tau,T].$$

In connection with (3. 39), we consider the following system of equations:

$$(3.40)\quad \begin{cases} D_t W = \mathfrak{A}(x, t, w; D_x) W \quad \text{in} \quad Q_{\tau,\tau+h} \equiv \Omega \times (\tau, \tau+h], \\ W|_{t=\tau} = 0, \ W|_{\Gamma_{\tau,\tau+h}} = Z(x-\xi, t-\tau; \xi, \tau; w)|_{\Gamma_{\tau,\tau+h}}, \end{cases}$$

where (ξ, τ) is an arbitrary fixed point in Q_T and $0 < h \leqq T - \tau$.

For simplicity the transformation from x to $\hat{x}$[resp. $\bar{x}$] is denoted by $\Pi_x^{\hat{x}}$[resp. $\Pi_x^{\bar{x}}$] and the inverse transformation, $\Pi_{\hat{x}}^x$[resp. $\Pi_{\bar{x}}^x$]. Moreover the following notations are used.

$$\bar{f}(\bar{x}) = \Pi_x^{\bar{x}} f(x), \ \hat{f}(\hat{x}) = \Pi_x^{\hat{x}} f(x), \ \bar{f}(\bar{x}, \hat{\xi}) = \Pi_{x,\xi}^{\bar{x},\hat{\xi}} f(x, \xi),$$

$$\hat{f}(\hat{x}, \hat{\xi}) = \Pi_{x,\xi}^{\hat{x},\hat{\xi}} f(x, \xi), \ \Pi_{x,\xi}^{\bar{x},\hat{\xi}} = \Pi_x^{\bar{x}} \Pi_\xi^{\hat{\xi}} \ \text{etc.}$$

The regularizer R of

$$(3.41)\quad \begin{cases} D_t W = \mathfrak{A}(x, t, w; D_x) W + \psi, \\ W|_{t=\tau} = 0, \ W|_{\Gamma_{\tau,\tau+h}} = g, \end{cases}$$

introduced in [5] can be defined by the formulas:

$$(3.42)\quad \begin{cases} W(x, t) = R(\psi, g) \equiv \sum_k \eta^{(k)}(x) W^{(k)}(x, t), \\ W^{(k')}(x, t) = R^{(k')}\psi \equiv \int_\tau^t d\tau_0 \int_{\Omega^{(k')}} Z_0(x-y, t-\tau_0; \xi^{(k')}, \tau_0; w) \\ \qquad\qquad \times \zeta^{(k')}(y) \psi(y, \tau_0) dy, \\ W^{(k'')}(x, t) = \Pi_{\bar{x}}^x \overline{W}^{(k'')}(\bar{x}, t), \\ \overline{W}^{(k'')}(\bar{x}, t) = \bar{R}^{(k'')}\psi + \bar{R}_1^{(k'')} g, \\ \bar{R}^{(k'')}\psi \equiv \int_\tau^t d\tau_0 \int_K H_0^{(k'')}(\bar{x}, t; \bar{y}, \tau_0) \bar{\zeta}^{(k'')}(\bar{y}) \bar{\psi}(\bar{y}, \tau_0) d\bar{y}, \\ \bar{R}_1^{(k'')} g \equiv \int_\tau^t d\tau_0 \int_{K'} H_1^{(k'')}(\bar{x}-\bar{y}', t-\tau_0) \bar{\zeta}^{(k'')}(\bar{y}') \bar{g}(\bar{y}', \tau_0) d\bar{y}', \end{cases}$$

where $H_0^{(k'')}(\bar{x}, t; \bar{y}, \tau_0)$ and $H_1^{(k'')}(\bar{x}-\bar{y}', t-\tau_0)$ are called the Green matrix and the Poisson kernel, respectively, of the system

$$(3.43)\quad \begin{cases} D_t \overline{W} = \mathfrak{A}^{(k'')}(\xi^{(k'')}, \tau, w; D_{\bar{x}}) \overline{W} + \bar{\zeta}^{(k'')}(\bar{x}) \bar{\psi}(\bar{x}, t) \\ \overline{W}|_{t=\tau} = 0, \quad \overline{W}|_{\bar{x}_3=0} = \bar{\zeta}^{(k'')}(\bar{x}') \bar{g}(\bar{x}', t). \end{cases}$$

First of all, let us construct the Poisson kernel. Using the notations in § 3. 2, we define $\alpha_s^{(j,k)}$ $(j, k, s=1, 2, 3, 4)$ such that

$$(3.44)\quad \begin{cases} \alpha_s^{(j,k)}=0 \quad (s=1,2,3;\, j,k=1,2,3,4), \\ \sum_{j=1}^{4} \alpha_4^{(j,k)} \alpha_{i,j}^{(4)} = \delta_{ik} \quad (i,k=1,2,3,4). \end{cases}$$

m_i's are defined by the formula

$$M^+ = \sum_{i=0}^{4} m_i \bar{x}_3^{4-i},$$

i.e., from (3. 30) it follows that

$$(3.45)\quad \begin{cases} m_0=1, \; m_1=-(2\bar{x}_3^{+(1)}+\bar{x}_3^{+(3)}+\bar{x}_3^{+(4)}), \\ m_2=\bar{x}_3^{+(1)2}+2\bar{x}_3^{+(1)}(\bar{x}_3^{+(3)}+\bar{x}_3^{+(4)})+\bar{x}_3^{+(3)}\bar{x}_3^{+(4)}, \\ m_3=-\bar{x}_3^{+(1)2}(\bar{x}_3^{+(3)}+\bar{x}_3^{+(4)})-2\bar{x}_3^{+(1)}\bar{x}_3^{+(3)}\bar{x}_3^{+(4)}, \\ m_4=\bar{x}_3^{+(1)2}\bar{x}_3^{+(3)}\bar{x}_3^{+(4)}. \end{cases}$$

We define

$$(3.46)\quad \begin{cases} M_k^+ \equiv \sum_{i=0}^{k} m_i \bar{x}_3^{k-1}, \\ N_{jk} \equiv \sum_{s=1}^{4} \alpha_s^{(j,k)} M_{4-s}^+, \\ \tilde{H}_1^{(k'')}(\bar{x}, \tau_0) = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_+} \hat{\mathfrak{A}}^{(k'')}(\xi^{(k'')}, \tau, w;\, i\bar{x}', i\xi, \tau_0) \\ \qquad\qquad \times N(\bar{x}', \xi, \tau_0) \dfrac{\exp[i\bar{x}_3 \cdot \xi]}{M^+(\bar{x}', \xi, \tau_0)} d\xi, \end{cases}$$

where $N=(N_{jk})$ and γ_+ is a contour enclosing all $\bar{x}_3^{+(m)}$ $(m=1, 2, 3, 4)$.

On account of (3. 31) - (3. 33), (3. 44) - (3. 46), $\tilde{H}_1^{(k'')}$ can be expressed explicitly as follows:

$$\begin{cases} -\dfrac{(\tilde{g}_{n3})^6}{a_1 a_3 a_4} (\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(3)}) \\ \times (\bar{x}_3^{+(1)}-\bar{x}_3^{-(4)}) \alpha_4^{(j,m)} \cdot \exp(i\bar{x}_3^{+(1)}\bar{x}_3) \\ +\dfrac{(\tilde{g}_{n3})^4}{a_1 a_4 a_5} \sum_{s=1}^{3} \alpha_4^{(j,s)} \{(\tilde{g}_{s1}\bar{x}_1+\tilde{g}_{s2}\bar{x}_2+\tilde{g}_{s3}\bar{x}_3^{+(1)}) \\ \times (\tilde{g}_{m1}\bar{x}_1+\tilde{g}_{m2}\bar{x}_2+\tilde{g}_{m3}\bar{x}_3^{+(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)}) \\ \times (\bar{x}_3^{+(1)}-\bar{x}_3^{-(4)})-(\tilde{g}_{s1}\bar{x}_1+\tilde{g}_{s2}\bar{x}_2+\tilde{g}_{s3}\bar{x}_3^{+(3)}) \end{cases}$$

$$
(3.47)\quad (\tilde{H}_1^{(k'')})_{jm}=\begin{cases}
\times(\tilde{g}_{m1}\bar{x}_1+\tilde{g}_{m2}\bar{x}_2+\tilde{g}_{m3}\bar{x}_3^{+(3)})(\bar{x}_3^{+(3)}-\bar{x}_3^{-(1)}) \\
\times(\bar{x}_3^{+(3)}-\bar{x}_3^{-(4)})\}(\bar{x}_3^{+(1)}-\bar{x}_3^{+(3)})^{-2}(\bar{x}_3^{+(1)} \\
\times\exp(i\bar{x}_3^{+(1)}\bar{x}_3)-\bar{x}_3^{+(3)}\cdot\exp(i\bar{x}_3^{+(3)}\bar{x}_3)) \\
+\dfrac{(\tilde{g}_{n3})^4}{a_1a_4a_5}\sum_{s=1}^{3}\alpha_4^{(j,s)}\{\bar{x}_3^{+(1)}(\tilde{g}_{s1}\bar{x}_1+\tilde{g}_{s2}\bar{x}_2+\tilde{g}_{s3}\bar{x}_3^{+(3)}) \\
\times(\tilde{g}_{m1}\bar{x}_1+\tilde{g}_{m2}\bar{x}_2+\tilde{g}_{m3}\bar{x}_3^{+(3)})(\bar{x}_3^{+(3)}-\bar{x}_3^{-(1)}) \\
\times(\bar{x}_3^{+(3)}-\bar{x}_3^{-(4)})-\bar{x}_3^{+(3)}(\tilde{g}_{s1}\bar{x}_1+\tilde{g}_{s2}\bar{x}_2+\tilde{g}_{s3}\bar{x}_3^{+(1)}) \\
\times(\tilde{g}_{m1}\bar{x}_1+\tilde{g}_{m2}\bar{x}_2+\tilde{g}_3\bar{x}_{m3}^{+(1)})(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)}) \\
\times(\bar{x}_3^{+(1)}-\bar{x}_3^{-(4)})\}(\bar{x}_3^{+(1)}-\bar{x}_3^{+(3)})^{-2} \\
\times(\exp(i\bar{x}_3^{+(1)}\bar{x}_3)-\exp(i\bar{x}_3^{+(3)}\bar{x}_3)), \\
\qquad \text{if}\quad \upsilon\neq 0,\ \ j,m=1,2,3; \\
-\dfrac{(\tilde{g}_{n3})^6}{a_1a_3a_4}(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)})^3\alpha_4^{(j,m)}\cdot\exp(i\bar{x}_3^{+(1)}\bar{x}_3) \\
+\dfrac{(\tilde{g}_{n3})^4 i}{a_1a_4a_5}\sum_{s=1}^{3}\alpha_4^{(j,s)}(\tilde{g}_{s1}\bar{x}_1+\tilde{g}_{s2}\bar{x}_2+\tilde{g}_{s3}\bar{x}_3^{+(1)}) \\
\times(\tilde{g}_{m1}\bar{x}_1+\tilde{g}_{m2}\bar{x}_2+\tilde{g}_{m3}\bar{x}_3^{+(1)})\bar{x}_3^{+(1)}(\bar{x}_3^{+(1)}-\bar{x}_3^{-(1)})^2, \\
\qquad \text{if}\quad \upsilon=0,\ \ j,m=1,2,3; \\
-\dfrac{(\tilde{g}_{n3})^2}{a_1^2a_3}\alpha_4^{(4,4)}(\bar{x}_3^{+(4)}-\bar{x}_3^{-(1)})(\bar{x}_3^{+(4)}-\bar{x}_3^{-(3)}) \\
\times\exp(i\bar{x}_3^{+(4)}\bar{x}_3)=\exp(i\bar{x}_3^{+(4)}\bar{x}_3), \\
\qquad \text{if}\quad j=m=4; \\
0,\ \text{otherwise}.
\end{cases}
$$

Then $H_1^{(k'')}$ is defined by

$$
(3.48)\qquad H_1^{(k'')}(\bar{x},t)=\frac{1}{(2\pi)^3 i}\int_{R^2}\exp(i(\bar{x}',\zeta))\,d\zeta
$$

$$
\times\int_{\sigma-i\infty}^{\sigma+i\infty}\exp(\tau_0 t)\tilde{H}_1^{(k'')}(\zeta,\bar{x}_3,\tau_0)\,d\tau_0\quad(\sigma>-\delta_3\zeta^2).
$$

Hence the same arguments as used above imply that the Green matrix of (3.43) is defined by

$$
(3.49)\qquad H_0^{(k'')}(\bar{x},t;\bar{y},\tau)=Z_0^{(k'')}(\bar{x}-\bar{y}',t-\tau;\xi^{(k'')},\tau;w)
$$

$$-\int_{\tau}^{t} d\tau_0 \int_{R^2} H_1^{(k'')}(\bar{x}-\bar{y}_0', t-\tau_0) Z_0^{(k'')}(\bar{y}_0'-\bar{y}, \tau_0-\tau; \xi^{(k'')}, \tau; w) d\bar{y}_0',$$

where $Z_0^{(k'')}$ is a fundamental solution of (3. 43).

Now let us put

$$(3.50)\qquad \begin{cases} q=\upsilon+\delta_3\bar{x}'^2 \quad (0<\delta_3<\tilde{\delta}; \delta_3, \text{ fixed}), \\ A^{(k'')}\equiv\det\{\mathfrak{A}^{(k'')}(\xi^{(k'')}, \tau, w; i\bar{x})-(q-\delta_3\bar{x}'^2)I\}. \end{cases}$$

Then the following lemma clearly holds.

Lemma 3. 8. *For any* $\bar{x}'\in R^2$ *and any* $q\in C^1$ *with* $\operatorname{Re} q>0$, $A^{(k'')}=0$ *in* $\bar{x}_3$ *has four roots* $\bar{x}_3^{+(j)}$ *with* $\operatorname{Im}\bar{x}_3^{+(j)}>0$ *and four* $\bar{x}_3^{-(j)}$ *with* $\operatorname{Im}\bar{x}_3^{-(j)}<0$.

Since q_j and $\bar{x}_3^{\pm(j)}$ have homogeneous properties, i.e.,

$$q_j(\gamma\bar{x})=\gamma^2 q_j(\bar{x}),\ \bar{x}_3^{\pm(j)}(\gamma\bar{x}', \gamma q)=\gamma\bar{x}_3^{\pm(j)}(\bar{x}', q),$$

where q_j is the root of $A^{(k'')}=0$ with respect to q,

$$(3.51)\qquad |q_j(\bar{x})|\leqq C_{10}|\bar{x}|^2,\ |\bar{x}_3^{\pm(j)}|\leqq C_{11}(|\bar{x}'|^4+|q|^2)^{1/4}.$$

$$(3.52)\qquad \operatorname{Im}\bar{x}_3^{\pm(j)}\geqq C_{12}(|\bar{x}'|^4+|q|^2)^{1/4},$$

hence $|\bar{x}_3^{+(j)}|\geqq C_{12}(|\bar{x}'|^4+|q|^2)^{1/4}$ for $\bar{x}'\in R^2$, $q\in C^1$ with $\operatorname{Re} q>0$.

Lemma 3. 9. *There exists a positive constant* β_5 *such that if* $q\in C^1$ *and* $\bar{x}'\in R^2$ *satisfy* $\operatorname{Re} q\geqq-\beta_5|\operatorname{Im} q|$ *and* $|q|^2+|\bar{x}'|^4>0$, *then* $A^{(k'')}=0$ *has four roots with positive imaginary parts and four with negative ones.*

Proof. We have only to take $\beta_5=\dfrac{1}{2}(\tilde{\delta}-\delta_3)C_{10}^{-1}$ because of the parabolicity condition and (3. 51). Q.E.D.

For the sake of simplicity we assume that (3. 51) and (3. 52) hold for $(\bar{x}', q)$ stated in lemma 3. 9 with the same constants $C_{10}-C_{12}$.

Lemma 3. 10. *There exists a positive constant* β_6 *such that*

$A^{(k'')}=0$ *in* $\overline{x}_3$ *with* $\overline{x}'=\xi'+i\eta'\,(\xi',\eta'\in R^2)$ *has no real roots if*

$$\text{(3.53)}\qquad \operatorname{Re} q\geqq -\beta_5|\operatorname{Im} q|,\ \ |\xi'|^4+|q|^2>0,\ \ |\eta'|\leqq\beta_6(|\xi'|^4+|q|^2)^{1/4}.$$

Proof. Referring to (3.28) and (3.29) it is sufficient to find $\beta_6(>0)$ such that

$$\text{(3.54)}\qquad \{(\operatorname{Re}\overline{D}_m)^2+(\operatorname{Im}\overline{D}_m)^2\}^{1/2}-\operatorname{Re}\overline{D}_m-2\{\tilde{g}_{j3}(\tilde{g}_{j1}\eta_1+\tilde{g}_{j2}\eta_2)\}^2>0,$$
$$(\eta'\equiv(\eta_1,\eta_2)),$$

where

$$\text{(3.55)}\qquad \overline{D}_m=-\frac{1}{2}\{(\tilde{g}_{j3}\tilde{g}_{k1}-\tilde{g}_{k3}\tilde{g}_{j1})(\xi_1+i\eta_1)+(\tilde{g}_{j2}\tilde{g}_{k3}-\tilde{g}_{k2}\tilde{g}_{j3})$$
$$\times(\xi_2+i\eta_2)\}^2-a_m(\tilde{g}_{j3})^2[q-\delta_3\{(\xi_1+i\eta_1)^2+(\xi_2+i\eta_2)^2\}]$$
$$(\xi'\equiv(\xi_1,\xi_2)).$$

Denoting the left-hand side of (3.54) by $G(\eta)$, we have by Lemma 3.9

$$\text{(3.56)}\qquad G(0)\geqq 2C_{12}^2(|\xi'|^4+|q|^2)^{1/2}.$$

Utilizing the mean value theorem and the estimate

$$|G'(\eta)|\leqq 36(1+B_1)^2\{3(1+B_1)^2+\bar{a}\delta_3\}(|\xi'|+|\eta'|)$$
$$(\bar{a}\equiv\max(a_1,a_4)),$$

we have only to take

$$\text{(3.57)}\qquad \beta_6=\min\left\{\frac{1}{4}\left[-1+\left\{1+\frac{2C_{12}^2}{9(1+B_1)^2\{3(1+B_1)^2+\bar{a}\delta_3\}}\right\}^{1/2}\right],1\right\}.$$

Q.E.D.

It is easily seen that Lemmas 3.8–3.10 imply

Lemma 3.11. $\overline{x}_3^{\pm(k)}$ *defined by* (3.28) *with prescribed replacements are analytic with respect to* $\overline{x}'$ *and* q *in the domain* (3.53), *and* (3.51) *and* (3.52) *hold with suitable constants* $\overline{C}_i$ $(i=10,11,12)$.

Lemma 3.12. *There exist positive constants* $\beta_7(\leqq\beta_5)$ *and* $\beta_8(\leqq\beta_6)$ *such that* $\Delta(\overline{x}',q)\neq 0$ $(\overline{x}'=\xi'+i\eta')$ *for* $\overline{x}'$ *and* q *satisfying*

(3.58) $\quad \mathrm{Re}\, q \geqq -\beta_7 |\mathrm{Im}\, q|, \ |\xi'|^4 + |q|^2 > 0, \ |\eta'| \leqq \beta_8 (|\xi'|^4 + |q|^2)^{1/4}.$

Proof. First of all, we consider the case $\eta'=0$. From Lemma 3.9 it follows that for $\beta_7 = \min\left(\beta_5, \frac{1}{\sqrt{6}}\right)$ and some positive constant C_{13}

$$
\begin{aligned}
(3.59)\quad |\Delta_1|^2 = {} & a_5^{-2} (\bar{x}_3^{+(3)} - \bar{x}_3^{-(4)})^2 \{[(a_1 - a_3)^{-1} \{(B_+^{(1)} + B_+^{(3)})^2 \\
& - (A_+^{(1)} + A_+^{(3)})^2\} + a_5^{-1} \{(\tilde{g}_{i1} \tilde{g}_{j3} - \tilde{g}_{j1} \tilde{g}_{i3}) \xi_1 \\
& + (\tilde{g}_{i2} \tilde{g}_{j3} - \tilde{g}_{j2} \tilde{g}_{i3}) \xi_2\}^2]^2 + 4 (a_1 - a_3)^{-2} (A_+^{(1)} + A_+^{(3)})^2 \\
& \times (B_+^{(1)} + B_+^{(3)})^2\} \geqq C_{13} (|\xi'|^4 + |q|^2).
\end{aligned}
$$

Next in the same method as those used in the proof of Lemma 3.10 we obtain $\beta_8 = C_{13}/2C_{14}$, where

$$
\left\{
\begin{aligned}
C_{14} = {} & 32 [(a_1 - a_3)^{-1} (\bar{C}_{11}^2 + C_{15}^2 \bar{C}_{12}^{-2}) + a_5^{-1} C_{15}^2] [(a_1 - a_3)^{-1} \\
& \times \{\bar{C}_{11} \bar{C}_{12}^{-1} C_{15} + 4 \bar{C}_{12}^{-2} C_{15}^2 (1 + \bar{C}_{12}^{-2} C_{15})\} + 2 a_5^{-1} C_{15}] \\
& + 16 [2 (a_1 - a_3)^{-1} \bar{C}_{11} \bar{C}_{12}^{-1} C_{15} + a_5^{-1} C_{15}] \cdot [2 (a_1 - a_3)^{-1} \\
& \times \{2 \bar{C}_{11} \bar{C}_{12} C_{15} (1 + \bar{C}_{12}^{-2} C_{15}) + \bar{C}_{12}^{-2} C_{15}^2\} + 2 a_5^{-1} C_{15}], \\
C_{15} = {} & 36 (1 + B_1)^4 + 9 \bar{a} (1 + B_1)^2 (1 + \delta_3).
\end{aligned}
\right.
$$

Q.E.D.

From Lemmas 3.8–3.12 and the formula (3.47) follows

Lemma 3.13. *$\tilde{H}_2^{(k'')}(\bar{x}, q) = \tilde{H}_1^{(k'')}(\bar{x}, q - \delta_3 \bar{x}'^2)$ is analytic with respect to $\bar{x}'$ and q in the domain defined by* (3.58) *and*

$$
(3.60)\quad D_{\bar{x}_3}^s \tilde{H}_2^{(k'')} (\alpha \bar{x}', \bar{x}_3, \alpha^2 q) = \alpha^s D_{\bar{x}_3}^s \tilde{H}_2^{(k'')} (\bar{x}', \alpha \bar{x}_3, q).
$$

As is well known in [54] (cf. [4]) $\tilde{H}_2^{(k'')}$ can be represented in the following form

$$
(3.61)\quad \tilde{H}_2^{(k'')} (\bar{x}', \bar{x}_3, q) = \hat{\mathfrak{A}}_1^{(k'')} \left(\xi^{(k'')}, \tau, w; i\bar{x}', \frac{d}{d\bar{x}_3}, q\right) V(\bar{x}', \bar{x}_3, q),
$$

where

$$
\hat{\mathfrak{A}}_1^{(k'')} (\xi^{(k'')}, \tau, w; i\bar{x}, q) = \hat{\mathfrak{A}}^{(k'')} (\xi^{(k'')}, \tau, w; i\bar{x}, q - \delta_3 \bar{x}'^2),
$$

$$
(3.62)\quad V(\bar{x}', \bar{x}_3, q) = \mathfrak{A}_1^{(k'')} \left(\xi^{(k'')}, \tau, w; i\bar{x}', \frac{d}{d\bar{x}_3}, q\right) W(\bar{x}', \bar{x}_3, q),
$$

$$\mathfrak{A}_1^{(k'')}(\xi^{(k'')}, \tau, w; i\bar{x}, q) = \mathfrak{A}^{(k'')}(\xi^{(k'')}, \tau, w; i\bar{x}) - (q - \delta_3 \bar{x}'^2) I.$$

$$(3.63) \quad W(\bar{x}, q) = \frac{1}{2\pi}\Big[\int_0^{\bar{x}_3} \tilde{H}_2^{(k'')}(\bar{x}', z, q)\, dz \int_{\gamma_+} \frac{\exp[iy(\bar{x}_3 - z)]}{A^{(k'')}(i\bar{x}', iy, q)} dy$$

$$- \int_{\bar{x}_3}^{\infty} \tilde{H}_2^{(k'')}(\bar{x}', z, q)\, dz \int_{\gamma_-} \frac{\exp[iy(\bar{x}_3 - z)]}{A^{(k'')}(i\bar{x}', iy, q)} dy\Big],$$

(γ_-, a contour enclosing the roots of $A^{(k'')}=0$ which belong to the lower half space).

Note that W is a solution of

$$(3.64) \quad \begin{cases} A^{(k'')}\Big(i\bar{x}', \dfrac{d}{d\bar{x}_3}, q\Big) W(\bar{x}', \bar{x}_3, q) = \tilde{H}_2^{(k'')}(\bar{x}', \bar{x}_3, q), \\ W \to 0 \quad \text{as} \quad \bar{x}_3 \to \infty, \end{cases}$$

and has the following properties:

$$(3.65) \quad \begin{cases} D_{\bar{x}_3}^s W(\alpha \bar{x}', \bar{x}_3, \alpha^2 q) = \alpha^{s-8} D_{\bar{x}_3}^s W(\bar{x}', \alpha \bar{x}_3, q), \\ |D_{\bar{x}_3}^s W(\bar{x}, q)| \leqq C_{16}(|\bar{x}'|^4 + |q|^2)^{(s-8)/2} \\ \qquad \times \exp\Big[-\dfrac{1}{4}\bar{C}_{12}\bar{x}_3(|\bar{x}'|^4 + |q|^2)^{1/4}\Big], \end{cases}$$

for $\forall s \geqq 0$ and $\forall (\bar{x}', q)$ satisfying (3.58).

As for V we have for $\forall (\bar{x}', q)$ satisfying (3.58) and $\forall s \geqq 0$

$$(3.66) \quad \begin{cases} D_{\bar{x}_3}^s V(\alpha \bar{x}', \bar{x}_3, \alpha^2 q) = \alpha^{s-6} D_{\bar{x}_3}^s V(\bar{x}', \alpha \bar{x}_3, q), \\ |D_{\bar{x}_3}^s V| \leqq C_{17}(|\bar{x}'|^4 + |q|^2)^{(s-6)/4} \\ \qquad \times \exp\Big[-\dfrac{1}{4}\bar{C}_{12}\bar{x}_3(|\bar{x}'|^4 + |q|^2)^{1/4}\Big]. \end{cases}$$

In (3.48) if we put $\sigma = -\delta_3 \zeta^2 + a\,(a > 0)$ and $q = \tau_0 + \delta_3 \zeta^2$, then

$$(3.48)' \quad H_1^{(k'')}(\bar{x}, t) = \frac{1}{(2\pi)^3 i} \int_{R^2} \exp(i(\bar{x}', \zeta) - \delta_3 \zeta^2 t)\, d\zeta$$

$$\times \int_{a - i\infty}^{a + i\infty} \exp(qt)\, \tilde{H}_2^{(k'')}(\zeta, \bar{x}_3, q)\, dq\,.$$

By (3.61) we obtain

$$(3.67) \quad H_1^{(k'')}(\bar{x}, t) = \hat{\mathfrak{A}}^{(k'')}(\xi^{(k'')}, \tau, w; D_{\bar{x}}, D_t) K(\bar{x}, t),$$

where

$$(3.68)\qquad K(\bar{x},t)=\frac{1}{(2\pi)^3 i}\int_{R^2}\exp(i(\bar{x}',\zeta)-\delta_3\zeta^2 t)\,d\zeta \times\int_{a-i\infty}^{a+i\infty}\exp(qt)\,V(\zeta,\bar{x}_3,q)\,dq$$

In connection with K, we define for non-negative integer ν

$$(3.69)\qquad K^{(\nu)}(\bar{x},t)=\frac{1}{(2\pi)^3 i}\int_{R^2}\exp(i(\bar{x}',\zeta)-\delta_3\zeta^2 t)\,d\zeta \times\int_{a-i\infty}^{a+i\infty}\exp(qt)\,(q+\zeta^2)^{-\nu}V(\zeta,\bar{x}_3,q)\,dq.$$

From these it follows at once that the following formula holds.

$$(3.70)\qquad K(\bar{x},t)=[D_t-(1+\delta_3)\Delta_{(2)}]K^{(\nu)}(\bar{x},t),$$

where $\Delta_{(2)}\equiv D^2_{\bar{x}_1}+D^2_{\bar{x}_2}$.

Lemma 3.14.

$$(3.71)\qquad |D_t^r D_{\bar{x}}^s K^{(\nu)}(\bar{x},t)|\leqq C_{18}^{(r,|s|)}t^{-(2r+|s|-2-2\nu)/2}\cdot\exp\left[-2d_2\frac{|\bar{x}|^2}{t}\right].$$

Proof.

$$(3.72)\qquad D_t^r D_{\bar{x}}^s K^{(\nu)}=\frac{1}{(2\pi)^3 i}\int_{R^2}\exp(i(\bar{x}',\zeta)-\delta_3\zeta^2 t)\,(i\zeta)^{s'}d\zeta \times\int_{a-i\infty}^{a+i\infty}\exp(qt)\cdot\frac{(q-\delta_3\zeta^2)^r}{(q+\zeta^2)^\nu}D^{s_3}_{\bar{x}_3}V(\zeta,\bar{x}_3,q)\,dq$$
$$=\frac{1}{(2\pi)^3 i}t^{-(2r+|s|-2-2\nu)/2}\int_{R^2}\exp(i(\bar{y}',\xi)-\delta_3\xi^2)\,(i\xi)^{s'}d\xi \times\int_{R^2}\exp(q)\frac{(q-\delta_3\xi^2)^r}{(q+\xi^2)^\nu}D^{s_3}_{\bar{x}_3}V(\xi,\bar{y}_3,q)\,dq$$
$$\equiv t^{-(2r+|s|-2-2\nu)/2}\cdot H_3(\bar{y}),\quad (s=(s_1,s_2,s_3)=(s',s_3)),$$

because $D^{s_3}_{\bar{x}_3}V$ is analytic and (3.66) holds.

At first let us estimate $H_3(\bar{y})$. Utilizing Cauchy's integral theorem and changing the variables from ξ and q to $\xi\bar{y}_3$ and $q\bar{y}_3^2$ respectively we have

$$(3.73)\qquad H_3(\bar{y})=\frac{1}{(2\pi)^3 i}\int_{R^2}\exp(i(\bar{y}',\xi)-\delta_3\xi^2)(i\xi)^{s'}d\xi$$
$$\times\int_{L(a)}e^q(q-\delta_3\zeta^2)^r(q+\xi^2)^{-v}D^{s_3}_{\bar{y}_3}V(\xi,\bar{y}_3,q)\,dq$$
$$=\frac{1}{(2\pi)^3 i}\bar{y}_3{}^{2r+|s|-2-2v}\int_{R^2}\exp(i(\bar{y}',\xi)-\delta_3\xi^2\bar{y}_3{}^2)(i\xi)^{s'}d\xi$$
$$\times\int_{L(a)}\exp(q\bar{y}_3{}^2)(q-\delta_3\xi^2)^r(q+\xi^2)^{-v}D^{s_3}_{\bar{y}_3}V(\xi,\bar{y}_3{}^2,q)\,dq,$$

where $L(a)$ is a contour $\mathrm{Re}\,q=-\beta_7|\mathrm{Im}\,q|+a$. On $L(a)$

$$|q+\xi^2|^2\geqq\frac{1}{2}(|q|^2+\xi^4).$$

If we define

$$M=\max\left\{0,\ \frac{2r-2v-6+s_3}{2}\right\},$$

then

$$(q-\delta_3\xi^2)^r(q+\xi^2)^{-v}(|q|^2+\xi^4)^{(s_3-6)/4}\leqq C_{19}(|q|^2+\xi^4)^M,$$

where

$$C_{19}=2^{(r/2)+v}(1+\delta_3)^{r/2}\{1+[a(1+\beta_7{}^2)^{-1/2}]^{\min\{(2r-2v-6+s_3)/2,\,0\}}\}.$$

Hence

$$(3.74)\qquad \left|\int_{L(a)}\exp(q\bar{y}_3{}^2)(q-\delta_3\xi^2)^r(q+\xi^2)^{-v}D^{s_3}_{\bar{y}_3}V(\xi,\bar{y}_3{}^2,q)\,dq\right|$$
$$\leqq 2^{M-1}C_{17}C_{19}(1+\beta_7{}^2)^{1/4}\{(a^2+\xi^4)^M\beta_7{}^{-1}\bar{y}_3{}^{-2}+\Gamma(2M+1)$$
$$\times\beta_7{}^{-2M-1}\bar{y}_3{}^{-4M-2}\}\cdot\exp\left[\left\{a-\frac{1}{4}\bar{C}_{12}a^{1/2}(1+\beta_7{}^2)^{-1/4}\right\}\bar{y}_3{}^2\right].$$

Here we can choose $a>0$ such that

$$(3.75)\qquad a-\frac{1}{8}\bar{C}_{12}a^{1/2}(1+\beta_7{}^2)^{-1/4}<0.$$

Thus we obtain

$$(3.76)\qquad |H_3(\bar{y})|\leqq C_{20}(1+\bar{y}_3{}^{2r+2v-6+s_3})\exp\left[-\frac{1}{8}\bar{C}_{12}a^{1/2}(1+\beta_7{}^2)^{-1/4}\bar{y}_3{}^2\right].$$

We can also estimate in (3.74) as follows

(3.77) $$\left|\int_{L(a)} \exp(q\bar{y}_3^2)(q-\delta_3\xi^2)^r(q+\xi^2)^{-\nu} D_{\bar{y}_3}^{s_3} V(\xi, \bar{y}_3^2, q)\,dq\right|$$
$$\leqq 2^{M-1}(1+\beta_7^2)^{1/4} C_{17} C_{19} \{(a^2+\xi^4)^M \beta_7^{-1} + \Gamma(2M+1)\beta_7^{-2M-1}\},$$

therefore

(3.78) $$|H_3(\bar{y})| \leqq C_{21}.$$

From (3.76) and (3.78) follows

(3.79) $$|H_3(\bar{y})| \leqq C_{22} \cdot \exp\left[-\frac{1}{16}\bar{C}_{12} a^{1/2}(1+\beta_7^2)^{-1/4}\bar{y}_3^2\right],$$

where

$$C_{22} = [\max\{2C_{20}, C_{21}\}] \cdot \exp\left[\frac{1}{8}\bar{C}_{11} a^{1/2}(1+\beta_7^2)^{-1/4}\right].$$

Next in (3.73) if we use $\xi|\bar{y}'|$ and $q|\bar{y}'|^2$ in place of ξ and q respectively and proceed the same arguments as above, we have

(3.80) $$|H_3(\bar{y})| \leqq C_{23} \cdot \exp\left[-\frac{1}{2}\gamma\beta_8 a^{1/2}(1+\beta_7^2)^{-1/4}|\bar{y}'|^2\right],$$

provided that

(3.81) $$a + \delta_3\gamma^2\beta_8^2 a(1+\beta_7^2)^{-1/2} - \frac{1}{2}\gamma\beta_8 a^{1/2}(1+\beta_7^2)^{-1/4} < 0,$$

where γ is a sufficiently small positive number.

We choose a such that both (3.75) and (3.81) hold and then fix it. Therefore (3.79) and (3.80) imply that

(3.82) $$|H_3(\bar{y})| \leqq C_{24} \cdot \exp[-2d_2|\bar{y}|^2].$$

Thus from (3.72) and (3.82) follows (3.71) with $C_{18}^{(r,|s|)} = C_{24}$.

Q.E.D.

By (3.67), (3.70) and (3.71) we can easily obtain

Lemma 3.15.

(3.83) $$|D_t^r D_{\bar{x}}^s H_1^{(k'')}(\bar{x}, t)| \leqq C_{25}^{(r,|s|)} t^{-(4+2r+|s|)/2} \cdot \exp\left[-2d_2\frac{|\bar{x}|^2}{t}\right],$$

(3.84) $$|\Delta_{\bar{x}}^{\bar{x}_0} D_t^r D_{\bar{x}}^s H_1^{(k'')}(\bar{x}, t)| \leqq C_{25}^{(r,|s|+1)} |\bar{x}-\bar{x}_0| t^{-(5+2r+|s|)/2}$$

$$\times \exp\left[-2d_2\frac{|\bar{x}''|^2}{t}\right],$$

$$(\bar{x}''=\bar{x} \quad \text{if} \quad |\bar{x}|\leqq|\bar{x}_0|\,;\, \bar{x}''=\bar{x}_0, \ \text{otherwise}),$$

(3. 85) $$|\varDelta_t^{t_0} D_t^r D_{\bar{x}}^s H_1^{(k'')}(\bar{x}, t)|\leqq C_{25}^{(r+1,|s|)}(t-t_0)t_0^{-(6+2r+|s|)/2}$$

$$\times \exp\left[-2d_2\frac{|\bar{x}|^2}{t}\right], \ (t>t_0>0).$$

3. 4. Construction of Green Matrix. We shall find the solution of (3. 40) in the form of $G_0=Ru$ where $u=(u_0, u_1)$ and R is a regularizer defined by (3. 42). u will be a solution of

(3. 86) $$u=Pu+g, \quad g=(0, Z|_{\Gamma_{\tau,\tau+h}}),$$

where P is a bounded operator on $H^{2+\alpha}(\underset{\sim}{Q}_{\tau,\tau+h})\times H^{2+\alpha}(\Gamma_{\tau,\tau+h})$ when h is small.

We go into more detail. u_0 and u_1 satisfy

(3. 86)′ $$u_0=P_0u, \quad u_1=Z|_{\Gamma_{\tau,\tau+h}},$$

where

(3. 87)
$$\begin{cases} P_0u=\sum_{k''}\eta^{(k'')}(x)\Pi_{\bar{x},\hat{\xi}}^{x,\xi}[\mathfrak{A}^{(k'')}(\xi^{(k'')},\tau,w;D_{\bar{x}}-\operatorname{grad}FD_{\bar{x}_3}) \\ \qquad -\mathfrak{A}^{(k'')}(\xi^{(k'')},\tau,w;D_{\bar{x}})]\bar{U}_{k''}(\bar{x},t;\hat{\xi},\tau)+\sum_k\eta^{(k)}(x) \\ \qquad \times[\mathfrak{A}(x,t,w;D_x)-\mathfrak{A}(\xi^{(k)},\tau,w;D_x)]U_k(x,t;\xi,\tau) \\ \qquad +\sum_k[\mathfrak{A}(x,t,w;D_x)\eta^{(k)}(x)-\eta^{(k)}(x)\mathfrak{A}(x,t,w;D_x)] \\ \qquad \times U_k(x,t;\xi,\tau), \\ U_{k'}=R^{(k')}u_0, \\ U_{k''}=\Pi_{\bar{x},\hat{\xi}}^{x,\xi}\bar{U}_{k''}(\bar{x},t;\hat{\xi},\tau), \\ \bar{U}_{k''}=\bar{R}^{(k'')}u_0+\bar{R}_1^{(k'')}u_1, \\ P=(P_0,0). \end{cases}$$

(3. 88) $$h=\chi\lambda^2 \quad (0<\chi\leqq 1;\ \lambda, \text{ sufficiently small}).$$

Then the solution of (3. 86)′ is given by

(3. 89) $$u=\sum_{v=0}^{\infty}u^{(v)}, \quad u^{(v)}=Pu^{(v-1)}.$$

The convergence and the estimates of (3. 89) will be found in the next section almost along the line of [14].

§ 4. Estimates for Green Matrix

4. 1. Lemmas of Integral Operators. Let us introduce the notations and some function classes.

$$Q_{\tau,\tau_1}=\Omega\times(\tau,\tau_1],\ Q^{(k)}_{\tau,\tau_1}=\Omega^{(k)}\times(\tau,\tau_1],\ \Gamma_{\tau,\tau_1}=\Gamma\times[\tau,\tau_1],$$

$$d(x,t;y,\tau)=|x-y|+|t-\tau|^{1/2},\ d(\xi,\Gamma)=\inf_{\eta\in\Gamma}d(\xi,\eta),$$

$$\Psi_d(x,t;\xi,\tau)=\exp\left[-d\frac{|x-\xi|^2}{t-\tau}\right],$$

$$\rho_d(t,\tau;\xi)=\exp\left[-d\frac{d^2(\xi,\Gamma)}{t-\tau}\right],$$

$$\Phi_d(x,t;\xi,\tau)=\Psi_d(x,t;\xi,\tau)\rho_d(t,\tau;\xi).$$

(i) $U^{k+\alpha,m,n}_{C,d}(Q_{\tau,\tau_1},\bar{Q}_T)=\{u(x,t;\xi,\tau)\,|\,(x,t)\in Q_{\tau,\tau_1},\ (\xi,\tau)\in\bar{Q}_T,$

(4. 1) $$|D_t^rD_x^su|\leqq Cd^{-3-2r-|s|-m-n}(x,t;\xi,\tau)(t-\tau)^{n/2}\times\Psi_d(x,t;\xi,\tau),\ (2r+|s|\leqq k),$$

(4. 2) $$|\Delta_x^{x_0}D_t^rD_x^su|\leqq C|x-x_0|^\alpha d^{-3-2r-|s|-\alpha}(x'',t;\xi,\tau)(t-\tau)^{n/2}\times\Psi_d(x'',t;\xi,\tau),$$
$$(2r+|s|\leqq k,\ |x''-\xi|=\min(|x-\xi|,|x_0-\xi|)),$$

(4. 3) $$|\Delta_t^{t_0}D_t^rD_x^su|\leqq C(t-t_0)^{(k-2r-|s|+\alpha)/2}d^{-3-2r-|s|-m-n-\alpha}\times(x,t_0;\xi,\tau)(t-\tau)^{n/2}\Psi_d(x,t;\xi,\tau),$$
$$(k-2<2r+|s|\leqq k,\ t>t_0>\tau)\}.$$

(ii) $H_d^{k+\alpha,m}(\bar{Q}_{\tau,\tau_1},\bar{Q}_T)$ [resp. $H_d^{k+\alpha,m}(\Gamma_{\tau,\tau_1},\bar{Q}_T)$] $=\{h(x,t;\xi,\tau)\,|$

$(x,t)\in\bar{Q}_{\tau,\tau_1}$ [resp. $(x,t)\in\Gamma_{\tau,\tau_1}$], $(\xi,\tau)\in\bar{Q}_T$,

(4. 4) $$|D_t^rD_x^sh|\leqq C(t-\tau)^{(k-2r-|s|+\alpha)/2}d^{-3-m-k-\alpha}(x,t;\xi,\tau)\times\Phi_d(x,t;\xi,\tau),\ (2r+|s|\leqq k),$$

(4. 5) $$|\Delta_x^{x_0}D_t^rD_x^sh|\leqq C|x-x_0|^\alpha(t-\tau)^{(k-2r-|s|/)2}\times d^{-3-m-k-\alpha}(x'',t;\xi,\tau)\Phi_d(x'',t;\xi,\tau)\ \ (2r+|s|\leqq k),$$

$$(4.6)\qquad |\Delta_t^{t_0} D_t^r D_x^s h| \leqq C(t-t_0)^{(k-2r-|s|+\alpha)/2} d^{-3-m-k-\alpha}(x, t_0; \xi, \tau)$$
$$\times \Phi_d(x, t; \xi, \tau) \quad (k-2<2r+|s|\leqq k)\}.$$

When $x \in \Gamma$, D_x^s means $D_{\bar{x}'}^s$. In $H_d^{k+\alpha,2}(\bar{Q}_{\tau,\tau_1}, \bar{Q}_T)$ and $H_d^{k+\alpha,1}(\Gamma_{\tau,\tau_1}, \bar{Q}_T)$ we add to the following conditions, respectively,

$$(4.7)\qquad |I^{(k)}(h)| = \left| \iint_{Q_1^{(k)}} \zeta^{(k)}(y) h(y, \tau_0; \xi, \tau) dy d\tau_0 \right| \leqq C\rho_d(t, \tau; \xi)$$

for any k,

$$(4.8)\qquad |J^{(k'')}(h)| = \left| \iint_{K_1'} \bar{\zeta}^{(k'')}(\bar{y}') \bar{h}(\bar{y}', \tau_0; \hat{\xi}, \tau) d\bar{y}' d\tau_0 \right| \leqq C\rho_d(t, \tau; \xi)$$

for any k'', where $Q_1^{(k)} = \bar{Q}_{\tau,t}^{(k)} \cap \left\{ d(y, \tau_0; \xi, \tau) \leqq \frac{1}{2}(t-\tau)^{1/2} \right\}$, $K_1' = K' \times [\tau, t] \cap \left\{ (\bar{y}', \tau_0) \,|\, d(\bar{y}', F(\bar{y}'), \tau_0; \hat{\xi}, \tau) \leqq \frac{1}{2}(t-\tau)^{1/2} \right\}$.

(iii) $\mathring{H}_d^{k+\alpha}(\bar{Q}_{\tau_1,\tau_2}, \bar{Q}_T)$ [resp. $\mathring{H}_d^{k+\alpha}(\Gamma_{\tau_1,\tau_2}, \bar{Q}_T)$] $= \{h(x, t; \xi, \tau) \,|$

$(x, t) \in \bar{Q}_{\tau_1,\tau_2}$[resp. $(x, t) \in \Gamma_{\tau_1,\tau_2}$], $(\xi, \tau) \in \bar{Q}_T$ $(\tau_1 > \tau)$,

$$(4.9)\qquad |D_t^r D_x^s h| \leqq C(t-\tau_1)^{(k-2r-|s|+\alpha)/2} \Phi_d(x, t; \xi, \tau) \quad (2r+|s|\leqq k),$$

$$(4.10)\qquad |\Delta_x^{x_0} D_t^r D_x^s h| \leqq C|x-x_0|^\alpha (t-\tau_1)^{(k-2r-|s|)/2} \Phi_d(x'', t; \xi, \tau)$$
$$(2r+|s|\leqq k),$$

$$(4.11)\qquad |\Delta_t^{t_0} D_t^r D_x^s h| \leqq C(t-t_0)^{(k-2r-|s|+\alpha)/2} \Phi_d(x, t; \xi, \tau)$$
$$(k-2<2r+|s|\leqq k, \quad \tau_2 > t \geqq t_0 > \tau_1)\}.$$

$\hat{H}_d^{k+\alpha}(\bar{Q}_{\tau_1,\tau_2}, \bar{Q}_T)$ [resp. $\hat{H}_d^{k+\alpha}(\Gamma_{\tau_1,\tau_2}, \bar{Q}_T)$]

$= \{h(x, t; \xi, \tau) \,|\, (x, t) \in \bar{Q}_{\tau_1,\tau_2}$[resp. $(x, t) \in \Gamma_{\tau_1,\tau_2}$],

$(\xi, \tau) \in \bar{Q}_T$, $(\tau_1 > \tau)$,

$$(4.12)\qquad |D_t^r D_x^s h| \leqq C\Phi_d(x, t; \xi, \tau) \quad (2r+|s|\leqq k),$$

$$(4.13)\qquad |\Delta_x^{x_0} D_t^r D_x^s h| \leqq C|x-x_0|^\alpha \Phi_d(x'', t; \xi, \tau) \quad (2r+|s|\leqq k),$$

$$(4.14)\qquad |\Delta_t^{t_0} D_t^r D_x^s h| \leqq C(t-t_0)^{(k-2r-|s|+\alpha)/2} \Phi_d(x, t; \xi, \tau)$$
$$(k-2<2r+|s|\leqq k)\}.$$

We define the norm $\|h\|$ of a function h in (ii) and (iii) by the minimum constant C in (4.4)-(4.14).

(iv) $\overset{\circ}{C}^{k+\alpha}(\bar{Q}_{\tau,\tau_1})$ [resp. $\overset{\circ}{C}^{k+\alpha}(\Gamma_{\tau,\tau_1})$] $= \Big\{ f(x,t) \in C_{x,t}^{k+\alpha,(k+\alpha)/2}(\bar{Q}_{\tau,\tau_1})$ [resp. $f(x,t) \in C_{x,t}^{k+\alpha,(k+\alpha)/2}(\Gamma_{\tau,\tau_1})$] $\mid D_t^r f(x,t)|_{t=\tau} = 0 \ \Big(r=0,1,\cdots,\Big[\frac{k}{2}\Big]\Big)\Big\}$,

$$C_0^{k+\alpha}(\bar{Q}_{\tau,\tau_1}) = \Big\{ f \in C_{x,t}^{k+\alpha,(k+\alpha)/2}(\bar{Q}_{\tau,\tau_1}) \mid D_t^r f|_{t=\tau, x\in\Gamma} = 0 \ \Big(r=0,1,\cdots,\Big[\frac{k}{2}\Big]\Big)\Big\},$$

where

$$C_{x,t}^{k+\alpha,(k+\alpha)/2}(\bar{Q}_T) = \{f \mid \|\|f\|\|_T^{(k+\alpha)} \equiv \sum_{2r+|s|=0}^{k} |D_t^r D_x^s f|_T^{(0)} + \sum_{2r+|s|=k} |D_t^r D_x^s f|_{x,T}^{(\alpha)} + \sum_{2r+|s|=\max(k-1,0)}^{k} |D_t^r D_x^s f|_{t,T}^{((k-2r-|s|+\alpha)/2)} < +\infty\},$$

$$C_{x,t}^{k+\alpha,(k+\alpha)/2}(\Gamma_T) = \{f \mid \|\|f\|\|_{\Gamma_T}^{(k+\alpha)} \equiv \sup_j \|\|f^{(j)}\|\|_{\tilde{\Gamma}_T^{(j)}}^{(k+\alpha)} < +\infty \ \ (\text{cf. } (1.19))\}.$$

At first we give some fromulas.

Lemma 4.1. *If λ be small, then for $x, y \in \bar{\Omega}$*

$$|\bar{x}-\bar{y}|^2 \geqq \frac{1}{2}|\hat{x}-\hat{y}|^2 = \frac{1}{2}|x-y|^2. \tag{4.15}$$

Corollary of Lemma 4.1.

$$\Psi_d(x,t;y,\tau_0)\Psi_d(y,\tau_0;\xi,\tau) \leqq \Psi_d(x,t;\xi,\tau) \quad (\tau<\tau_0<t;\ x,y,\xi\in\bar{\Omega}), \tag{4.16}$$

$$\begin{cases} d(\bar{x},t;\bar{y},\tau) \geqq \dfrac{1}{\sqrt{2}} d(\hat{x},t;\hat{\xi},\tau) = \dfrac{1}{\sqrt{2}} d(x,t;\xi,\tau), \\ \Phi_d(\bar{x},t;\bar{y},\tau_0) \leqq \Phi_{(1/2)d}(\hat{x},t;\hat{y},\tau_0) = \Phi_{(1/2)d}(x,t;y,\tau_0), \end{cases} \tag{4.17}$$

for $\bar{x}, \bar{y} \in K$ with sufficient small λ.

Lemma 4.2. *For any $k>0$*

$$(t-\tau)^{-k/2}\Psi_d(x,t;\xi,\tau) \leqq e^d K\Big(\frac{k}{2},\frac{1}{2}\Big) d^{-k}(x,t;\xi,\tau)\Psi_{(1/2)d}(x,t;\xi,\tau).$$

It is easily seen that Lemmas 3.3, 3.15, 4.1 and 4.2 imply that

$$Z_0 \in U_{C_{26},d_1}^{k_0+\alpha_0,0,n}(R^3\times(\tau,T],\ R^3\times[0,T)), \tag{4.18}$$

$$H_1^{(k'')} \in U_{C_{27},d_2}^{k_0+\alpha_0,1,n}(R_+^3\times(\tau,T],\ R^3\times[0,T)), \tag{4.19}$$

with any integer $k_0 \geqq 0$, $\forall \alpha_0 \in (0,1]$, $\forall n \geqq 0$, $\forall T>0$ $(R_+^3 \equiv \{\bar{x}_3 \geqq 0\})$.

According to the notations in (3.42) we consider the following operators:

$$(4.20) \qquad R^{(k)}h = \int_\tau^t d\tau_0 \int_{\Omega^{(k)}} R^{(k)}(x,t;y,\tau_0)\zeta^{(k)}(y)h(y,\tau_0;\xi,\tau)dy,$$

$$(4.21) \qquad R_1 h = \Pi_{\bar{x},\hat{\xi}}^{x,\xi}\bar{R}_1 h = \Pi_{\bar{x},\hat{\xi}}^{x,\xi}\int_\tau^t d\tau_0 \int_{K'} H_1^{(k'')}(\bar{x}-\bar{y}',t-\tau_0)\bar{\zeta}^{(k'')}(\bar{y}') \times \bar{h}(\bar{y}',\tau_0;\hat{\xi},\tau)d\bar{y}',$$

where

$$(4.22) \qquad R^{(k)}(x,t;y,\tau) = \begin{cases} Z_0(x-y,t-\tau;\xi^{(k')},\tau;w) & \text{if } k=k', \\ \Pi_{\bar{x},\bar{y}}^{x,y}H_0^{(k'')}(\bar{x},t;\bar{y},\tau) & \text{if } k=k''. \end{cases}$$

In order to estimate $H_0^{(k'')}$ we begin with the following lemma.

Lemma 4.3. *Suppose*

$$(4.23) \qquad \Gamma \in C^{2+k+\alpha}, \quad \tau_1-\tau=\chi\lambda^2,$$

where $\chi \leqq 1$ *and* λ *is sufficiently small so as to satisfy* (4.15). *Then* $v_1 = R_1 h \in H_{d'}^{k+\alpha,0}(\bar{Q}_{\tau,\tau_1}^{(k'')},\bar{Q}_T)$ *and* $\|v_1\| \leqq C_{28}\|h\|$ $\left(d'=\min\left(d,\frac{1}{2}d_2\right)\right)$ *provided* $h \in H_d^{k+\alpha,0}(\Gamma_{\tau,\tau_1},\bar{Q}_T)$.

Proof. It is sufficient to estimate $D_{\bar{x}}^s \bar{v}_1 = D_{\bar{x}}^s \bar{R}_1 h$ $(\bar{x}\in K)$. Let us make use of the following notations:

1) $\qquad (x,t)=P, \ (y,\tau_0)=Q, \ (\xi,\tau)=M,$

2) $\qquad (\bar{x},t)=\bar{P}, \ (\bar{y},\tau_0)=\bar{Q}, \ (\hat{\xi},\tau)=\widehat{M},$

3) $\qquad (\bar{x}',t)=\bar{P}', \ (\bar{y}',\tau_0)=\bar{Q}', \ (\bar{\xi}',\tau)=\overline{M}',$

4) $\qquad dyd\tau_0=dQ, \ d\bar{y}'d\tau_0=d\bar{Q}',$

5) $\qquad d_0=d(P,M)=|x-\xi|+(t-\tau)^{1/2}.$

Let the integral domain $K'\times[\tau,t]$ be divided into three parts K_i' $(i=1,2,3)$ such that $K'\times[\tau,t]=K_1'\cup K_2'\cup K_3'$, where

$$K_1' = K'\times[\tau,t]\cap\Big\{(\bar{y}',\tau_0)\,|\,d(\bar{y}',F(\bar{y}'),\tau_0;\widehat{M})$$

$$=d(Q, M)\leqq\frac{1}{2}(t-\tau)^{1/2}\Big\},$$

$$K_2'=K'\times[\tau, t]\cap\Big\{(\bar{y}', \tau_0)\,|\,d(\bar{x}', \bar{x}_3+F(\bar{x}'), t; \bar{y}', F(\bar{y}'), \tau_0)$$

$$=d(P, Q)\leqq\frac{1}{2}(t-\tau)^{1/2}\Big\},$$

$$K_3'=K'\times[\tau, t]-K_1'-K_2'.$$

Let $\bar{x}\in K$ and $\bar{x}_3>0$.

$$(4.24)\qquad D_{\bar{x}}^s\bar{v}_1(\bar{P}, \hat{M})=\iint_{K_1'}[D_{\bar{x}}^sH_1^{(k'')}(\bar{P}-\bar{Q}')-D_{\bar{x}}^sH_1^{(k'')}(\bar{P}''-\bar{M}')]$$
$$\times\bar{\zeta}^{(k'')}(\bar{y}')\bar{h}(\bar{Q}', \hat{M})\,d\bar{Q}'$$
$$+D_{\bar{x}}^sH_1^{(k'')}(\bar{P}''-\bar{M}')\iint_{K_1'}\bar{\zeta}^{(k'')}(\bar{y}')\bar{h}(\bar{Q}', \hat{M})\,d\bar{Q}'$$
$$+\iint_{K_2'}D_{\bar{x}}^sH_1^{(k'')}(\bar{P}-\bar{Q}')\bar{\zeta}^{(k'')}(\bar{y}')\bar{h}(\bar{Q}', \hat{M})\,d\bar{Q}'$$
$$+\iint_{K_3'}D_{\bar{x}}^sH_1^{(k'')}(\bar{P}-\bar{Q}')\bar{\zeta}^{(k'')}(\bar{y}')\bar{h}(\bar{Q}', \hat{M})\,d\bar{Q}'$$
$$\equiv\sum_{i=1}^{4}L_1^{(i)},$$

where $\bar{P}''=(\bar{x}', |\bar{x}_3-\bar{\xi}_3|, t)$ and $|s|\leqq k$. Moreover it is sufficient to consider the case $\xi\in\tilde{\Omega}^{(k'')}$ ($\tilde{\Omega}^{(k'')}$ is a neighborhood of $\Omega^{(k'')}$ contained in a Lyapunov sphere with $\xi^{(k'')}\in\Gamma$ its center). Indeed if $\xi\notin\tilde{\Omega}^{(k'')}$, then $K_1'=\emptyset$ for sufficiently small χ because $t-\tau\leqq\tau_1-\tau\leqq\chi\lambda^2$.

$L_1^{(i)}$ $(i=1, 2)$ can easily be estimated, i.e.,

$$(4.25)\qquad |L_1^{(i)}|\leqq C_{28}^{(i)}\|h\|(t-\tau)^{(k-|s|+\alpha)/2}d_0^{-3-k-\alpha}\Phi_{d'}(P, M)\quad(i=1, 2).$$

$$|L_1^{(4)}|\leqq C_{28}^{(4)}\|h\|(t-\tau)^{(2k-|s|+2\alpha)/2}\Phi_{d'}(P, M)$$
$$\times\iint_{K_3'}d^{-4-k-\alpha}(P, Q)\,d^{-3-k-\alpha}(Q, M)\,d\bar{Q}'.$$

$$\iint_{K_3'}d^{-4-k-\alpha}(P, Q)\,d^{-3-k-\alpha}(Q, M)\,d\bar{Q}'$$
$$=\iint_{K_3'\cap\{d(P,Q)\leqq(1/2)d_0\}}[\cdots]\,d\bar{Q}'+\iint_{K_3'\cap\{d(P,Q)>(1/2)d_0\}}[\cdots]\,d\bar{Q}'$$

$$\equiv L_1^{(4,1)}+L_1^{(4,2)}.$$

$$L_1^{(4,1)}\leqq C_{28}^{(4,1)}d_0^{-3-k-\alpha}(t-\tau)^{-(k+\alpha)/2},$$

$$L_1^{(4,2)}\leqq\begin{cases} C_{28}^{(4,2,1)}d_0^{-4-k-\alpha}\{(t-\tau)^{(1-k-2\alpha)/2} \\ \quad\times\displaystyle\iint_{d(\bar{Q}',\bar{M}')\leqq(1/2)(t-\tau)^{1/2}} d^{-4+\alpha}(\bar{Q}',\bar{M}')\,d\bar{Q}' \\ \quad+\displaystyle\iint_{d(\bar{Q}',\bar{M}')>(1/2)(t-\tau)^{1/2}} d^{-3-k-\alpha}(\bar{Q}',\bar{M}')\,d\bar{Q}'\}, \\ \qquad\text{if } 1-k-\alpha\leqq 0; \\ C_{28}^{(4,2,2)}d_0^{-4-k-\alpha}\displaystyle\iint_{d(\bar{Q}',\bar{M}')\leqq 2d_0} d^{-3-k-\alpha}(\bar{Q}',\bar{M}')\,d\bar{Q}' \\ \quad+\displaystyle\iint_{d(\bar{Q}',\bar{M}')>2d_0} d^{-4-k-\alpha}(\bar{P}',\bar{Q}')\,d^{-3-k-\alpha}(\bar{Q}',\bar{M}')\,d\bar{Q}', \\ \qquad\text{if } 1-k-\alpha>0, \end{cases}$$

$$\leqq C_{28}^{(4,2)}d_0^{-3-k-\alpha}(t-\tau)^{-(k+\alpha)/2}.$$

Therefore we have

$$(4.26)\qquad |L_1^{(4,2)}|\leqq C_{28}^{(4)}(C_{28}^{(4,1)}+C_{28}^{(4,2)})\|h\|(t-\tau)^{(k-|s|+\alpha)/2}d_0^{-3-k-\alpha}\Phi_{d'}(P,M).$$

In the next place as concerns $L_1^{(3)}$ we make use of the formulas (3.67) and (3.70). If we put

$$(4.27)\qquad \hat{\mathfrak{A}}^{(k'')}(\xi^{(k'')},\tau,w;D_{\bar{x}},D_t)K^{(\nu)}(\bar{x},t)=\tilde{K}^{(\nu)}(\bar{x},t),$$

then from Lemma 3.14 it easily follows that

$$(4.28)\qquad H_1^{(k'')}(\bar{x},t)=[D_t-(1+\delta_3)\Delta_{(2)}]^{\nu}\tilde{K}^{(\nu)}(\bar{x},t)$$
$$=\sum_{2\mu_0+|\mu'|=2\nu}a_{\mu_0,\mu'}D_t^{\mu_0}D_{\bar{x}'}^{\mu'}\tilde{K}^{(\nu)}(\bar{x},t),$$

$$(4.29)\qquad \tilde{K}^{(\nu)}(\bar{P}-\bar{Q}')\in U_{C_{29},d_2}^{k_0+\alpha_0,2(1-\nu)-1,n}(R^3\times(\tau_0,T],R^3\times[0,T))$$

$(\forall k_0=0,1,2,\cdots;\ \forall\alpha_0\in(0,1];\ \forall n\geqq 0;\ \forall T>0)$.

Now we assume that in (4.28) $\nu=\left[\frac{|s|}{2}\right]+1$. Let $m_i\leqq\mu_i$ $(i=0,1,2)$ hold and let $2m_0+|m'|$ take a maximum value not exceeding $|s|$, i.e., $2m_0+|m'|=2\left[\frac{|s|}{2}\right]$ or $|s|$. Integrations by parts imply that

$$(4.30)\quad L_1^{(3)}=\sum_{2\mu_0+|\mu'|=2\nu} a_{\mu_0,\mu'}\left\{\iint_{K_2'} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'} D_{\bar{x}}^{s}\tilde{K}^{(\nu)}(\bar{P}-\bar{Q}')\right.$$

$$\times D_{\tau_0}^{m_0} D_{\bar{y}'}^{m'}[\bar{\zeta}^{(k'')}(\bar{y}')\bar{h}(\bar{Q}', M)]d\bar{Q}'$$

$$-\sum_{\eta_0\leqq m_0-1}\int_{\Gamma_2'} D_t^{\mu_0-(\eta_0+1)} D_{\bar{x}'}^{\mu'} D_{\bar{x}}^{s}\circ\tilde{K}^{(\nu)} D_{\tau_0}^{\eta_0}[\bar{\zeta}^{(k'')}\bar{h}]\cos(\boldsymbol{n},\tau_0)\,d\Gamma_2'$$

$$\left.-\sum_{|\eta'|\leqq|m'|-1}\int_{\Gamma_2'} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-(\eta'+1)} D_{\bar{x}}^{s}\tilde{K}^{(\nu)} D_{\tau_0}^{m_0} D_{\bar{y}'}^{\eta'}[\bar{\zeta}^{(k'')}\bar{h}]\cos(\boldsymbol{n},\bar{y}')\,d\Gamma_2'\right\}$$

$$=\sum_{2\mu_0+|\mu'|=2\nu} a_{\mu_0,\mu'}(L_1^{(3,1)}+L_1^{(3,2)}+L_1^{(3,3)}),$$

where Γ_2' is a boundary of K_2' and $\boldsymbol{n}$ is an inner normal vector at each point of Γ_2'. $L_1^{(3,i)}$ $(i=2,3)$ can easily be estimated as follows:

$$(4.31)\quad |L_1^{(3,i)}|\leqq C_{28}^{(3,i)}\|h\|(t-\tau)^{(k-|s|+\alpha)/2} d_0^{-3-k-\alpha}\Phi_{d'}(P, M)\quad (i=2,3).$$

In order to estimate $L_1^{(3,1)}$, let $t_0=\min\{t\,|\,(\bar{x}', t)\in K_2'\}$, $t_i=t-2^{-i}(t-t_0)$, $K_{2i}'=K_2'\cap(t_i, t_{i+1})$. Then

$$(4.32)\quad L_1^{(3,1)}=\sum_{i=0}^{\infty}\iint_{K'_{2i}} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'} D_{\bar{x}}^{s}\tilde{K}^{(\nu)} D_{\tau_0}^{m_0} D_{\bar{y}'}^{m'}[\bar{\zeta}^{(k'')}\bar{h}]d\bar{Q}'$$

$$=\sum_{i=0}^{\infty}\sum_{|\eta'|\leqq|m'|} c_{m',\eta'}\iint_{K'_{2i}} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'} D_{\bar{x}}^{s}\tilde{K}^{(\nu)} D_{\bar{y}'}^{\eta'}\bar{\zeta}^{(k'')}$$

$$\times\Delta_{\tau_0}^{t_i} D_{\tau_0}^{m_0} D_{\bar{y}'}^{m'-\eta'}\bar{h}\,d\bar{Q}'+\sum_{i=0}^{\infty}\sum_{|\eta'|\leqq|m'|} c_{m',\eta'}\iint_{K'_{2i}} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'}$$

$$\cdot D_{\bar{x}}^{s}\tilde{K}^{(\nu)} D_{\bar{y}'}^{\eta'}\bar{\zeta}^{(k'')} D_{t_i}^{m_0} D_{\bar{y}'}^{m'-\eta'}\bar{h}\,d\bar{Q}'\equiv L_1^{(3,1,1)}+L_1^{(3,1,2)}.$$

$$(4.33)\quad |L_1^{(3,1,1)}|\leqq C_{28}^{(3,1,1)}\|h\|(t-\tau)^{(k-|s|+\alpha)/2} d_0^{-3-k-\alpha}\Phi_{d'}(P, M).$$

Next we transform $L_1^{(3,1,2)}$ into the form

$$(4.34)\quad L_1^{(3,1,2)}=\sum_{|\eta'|\leqq|m'|} c_{m',\eta'}\sum_{i=0}^{\infty}\left[\iint_{K'_{2i}} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'} D_{\bar{x}}^{s}\tilde{K}^{(\nu)} D_{\bar{y}'}^{\eta'}\bar{\zeta}^{(k'')}\right.$$

$$\times\Delta_{\bar{y}'}^{\bar{z}'} D_{t_i}^{m_0} D_{\bar{y}'}^{m'-\eta'}\bar{h}\,d\bar{Q}'+D_{t_i}^{m_0} D_{\bar{z}'}^{m'-\eta'}\bar{h}\left\{\iint_{K'_{2i}} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'-1}\right.$$

$$\cdot D_{\bar{x}}^{s}\tilde{K}^{(\nu)} D_{\bar{y}'}^{\eta'+1}\bar{\zeta}^{(k'')}d\bar{Q}'-\int_{\Gamma'_{2i}} D_t^{\mu_0-m_0} D_{\bar{x}'}^{\mu'-m'-1} D_{\bar{x}}^{s}\tilde{K}^{(\nu)}$$

$$\left.\left.\times D_{\bar{y}'}^{\eta'}\bar{\zeta}^{(k'')}\cos(\boldsymbol{n},\bar{y}')\,d\Gamma'_{2i}\right\}\right],$$

where $(\bar{z}',0)$ is a k''-local coordinate of $z\in\Gamma\cap\Omega^{(k'')}$ which is nearest of x and Γ'_{2i} is a boundary of K'_{2i}, if $2m_0+|m'|=|s|$. (N.B. in this case $2\mu_0+|\mu'|-2m_0-|m'|=2\left(1-\left\{\frac{|s|}{2}\right\}\right)$, $\left\{\frac{|s|}{2}\right\}=\frac{|s|}{2}-\left[\frac{|s|}{2}\right]$.) If $2m_0+|m'|=2\left[\frac{|s|}{2}\right]$, then $2\mu_0+|\mu'|-2m_0-|m'|=2$, hence $D_t^{\mu_0-m_0}D_{\bar{x}'}^{\mu'-m'}=D_t$ or $D_{\bar{x}'}^2$. In the first case we have

$$(4.35)\qquad L_1^{(3,1,2)}=\sum_{|\eta'|\leq|m'|}c_{m',\eta'}\sum_{i=0}^{\infty}\Big[\int_{K_{t_i}}D_{\bar{x}}{}^s\widetilde{K}^{(\nu)}D_{\bar{y}'}^{\eta'}\bar{\zeta}^{(k'')}\Delta_{t_i}^{t_{i-1}}D_{t_i}^{m_0}D_{\bar{y}'}^{m'-\eta'}\bar{h}\,d\bar{y}'$$

$$-\int_{\Gamma'_{2i}}D_{\bar{x}}{}^s\widetilde{K}^{(\nu)}D_{\bar{y}'}^{\eta'}\bar{\zeta}^{(k'')}D_{t_i}^{m_0}D_{\bar{y}'}^{m'-\eta'}\bar{h}\cos(\boldsymbol{n},\tau_0)\,d\Gamma'_{2i}\Big],$$

where $K_{t_i}=K'_{2i}\cap\{t=t_i\}$. In the second case, if we transfer $2\{q\}$ differentiation from $K^{(\nu)}$ to another term in (4.30) by integration by parts, then this case is reduced to the case $2m_0+|m'|=\frac{|s|}{2}$. In the sequel after some calculations we obtain

$$(4.36)\qquad |L_1^{(3,1,2)}|\leqq C_{28}^{(3,1,2)}\|h\|\,(t-\tau)^{(k-|s|+\alpha)/2}d_0^{-3-k-\alpha}\Phi_{d'}(P,M).$$

Hence

$$(4.37)\qquad |D_{\bar{x}}{}^s\bar{v}_1|\leqq C_{28,1}\|h\|\,(t-\tau)^{(k-|s|+\alpha)/2}d_0^{-3-k-\alpha}\Phi_{d'}(P,M),$$

where $C_{28,1}=C_{28}^{(1)}+C_{28}^{(2)}+C_{28}^{(3,1,1)}+C_{28}^{(3,1,2)}+C_{28}^{(4)}(C_{28}^{(4,1)}+C_{28}^{(4,2)})$.

As for $\Delta_t{}^{t_0}D_{\bar{x}}{}^s\bar{v}_1$ and $\Delta_x{}^{x_0}D_{\bar{x}}{}^s\bar{v}_1$, we can obtain the desired estimates by an analogous way as above. Q.E.D.

(4.8) implies that

$$(4.38)\qquad Z_0^{(k'')}|_{\bar{x}_3=0}\in H_{d_1}^{k_0+\alpha_0,0}(R^2\times(\tau,T],R_+{}^3\times[0,T))$$

$$(\forall k_0=0,1,2,\cdots;\ \forall\alpha_0\in(0,1]).$$

Since $\{\bar{x}_3=0\}\in C^\infty$ and $\zeta\equiv1$, by the same argument as in the proof of Lemma 4.3 (in this case, however, we do not need the condition $\tau_1-\tau\leqq\chi\lambda^2$) we obtain

$$(4.39)\qquad \int_\tau^t d\tau_0\int_{R^2}H_1^{(k'')}(\bar{x}-\bar{y}_0',t-\tau_0)\,Z_0^{(k'')}(\bar{y}_0'-\bar{y},\tau_0;\xi^{(k'')},\tau;w)\,d\bar{y}_0'$$

$$\in H_{d_3}^{k_0+\alpha_0,0}(R_+{}^3\times(\tau,T],R_+{}^3\times[0,T))$$

$$(d_3=\min(d_1,d_2/2)),$$

hence

(4. 40) $\quad H_0^{(k'')}(\bar{x}, t; \bar{y}, \tau_0) \in U_{C_{30}, d_3}^{k_0+\alpha_0, 0, n}(R_+{}^3 \times (\tau_0, T], R_+{}^3 \times [0, T))$

for $\forall k_0 = 0, 1, 2, \cdots; \forall \alpha_0 \in (0, 1]; \forall n \geqq 0; \forall T > 0$.

The following three lemmas can be proved in the almost same way as those used in Lemma 4. 3.

Lemma 4. 4. *Under the condition* (4. 23), $u_k \equiv R^{(k)}h \in H_{d''}^{2+m+\alpha, 0}(\bar{Q}_{\tau,\tau_1}^{(k'')}, \bar{Q}_T)$ *and* $\|R^{(k)}h\| \leqq C_{31}\|h\|$, *where* $d'' = \min(d, d_2/2)$ *provided that* $h \in H_d{}^{m+\alpha, 2}(\bar{Q}_{\tau,\tau_1}, \bar{Q}_T)$.

Lemma 4. 5. *Under the condition* (4. 23) *except that* $\tau_2 - \tau_1 = \chi\lambda^2$, $R^{(k)}h$ [resp. $R_1 h$] *defined by* (4. 20) [resp. (4. 21)], *where the integral interval with respect to* τ_0 *is* (τ_1, t) *in place of* (τ, t), *is a bounded operator on* $\mathring{H}_d{}^{m+\alpha}(\bar{Q}_{\tau_1,\tau_2}, \bar{Q}_T)$ [resp. $\mathring{H}_d{}^{m+\alpha}(\Gamma_{\tau_1,\tau_2}, \bar{Q}_T)$] *and* $R^{(k)}h \in \mathring{H}_{d'}^{2+m+\alpha}(\bar{Q}_{\tau_1,\tau_2}^{(k)}, \bar{Q}_T)$ [resp. $R_1 h \in \mathring{H}_{d''}^{m+\alpha}(\bar{Q}_{\tau_1,\tau_2}^{(k'')}, \bar{Q}_T)$].

Lemma 4. 6. *Under the condition* (4. 23), $R^{(k)}$ [*resp.* R_1] *is a bounded mapping from* $\mathring{C}^{m+\alpha}(\bar{Q}_{\tau,\tau_1})$ [*resp.* $\mathring{C}^{m+\alpha}(\Gamma_{\tau,\tau_1})$] *into* $\mathring{C}^{2+m+\alpha}(\bar{Q}_{\tau,\tau_1}^{(k)})$ [*resp.* $\mathring{C}^{m+\alpha}(\bar{Q}_{\tau,\tau_1}^{(k'')})$] *and* $|||R^{(k)}h|||_{\bar{Q}_{\tau,\tau_1}^{(k)}}^{(2+m+\alpha)} \leqq C_{32}|||h|||_{\bar{Q}_{\tau,\tau_1}}^{(m+\alpha)}$ [*resp.* $|||R_1 h|||_{\bar{Q}_{\tau,\tau_1}^{(k'')}}^{(m+\alpha)} \leqq C_{33} \times |||h|||_{\Gamma_{\tau,\tau_1}}^{(m+\alpha)}$], *where* $|||\cdot|||_{\bar{Q}_{\tau,\tau_1}^{(k)}}^{(m+\alpha)}$ [*resp.* $|||\cdot|||_{\Gamma_{\tau,\tau_1}}^{(m+\alpha)}$] *is defined in* (iv) *with* $\bar{Q}_{\tau,\tau_1}^{(k)}$ [*resp.* Γ_{τ,τ_1}] *in place of* $\bar{Q}_T$ [*resp.* Γ_T].

4. 2. Estimates for Green Matrix. As we have already seen in §§ 3. 3 and 3. 4, Green matrix in $Q_{\tau,\tau+h}$ coincides with the unique solution of (3. 86). For sufficiently small h, this solution is given by the series (3. 89). Thus it is sufficient to estimate each term of (3. 89). From Lemma 3. 6 follows that

(4. 41) $$Z(x-\xi, t-\tau; \xi, \tau; w) \in U_{C_{34}, d_4}^{2+\alpha, 0, n}(Q_{\tau,T}, \bar{Q}_T).$$

Let $u^{(0)} = (u_0{}^{(0)}, u_1{}^{(0)}) \equiv (0, Z|_{\Gamma_{\tau,\tau+h}})$ and let $u^{(\nu)}$ ($\nu \geqq 1$) define by (3. 89) and h satisfy (3. 88). Then we have

Lemma 4. 7. $u = (u_0, u_1) \in H_{d_5}^{\alpha, 2}(Q_{\tau,\tau+h}, \bar{Q}_T) \times H_{d_5}^{2+\alpha, 0}(\Gamma_{\tau,\tau+h}, \bar{Q}_T)$, *where* $d_5 = d_4/2$.

Proof. By (4. 41) it is clear that

(4. 42) $$u_1^{(0)} \in H_{d_5}^{2+\alpha,0}, \ \|u_1^{(0)}\| \leqq C_{35}.$$

Since $P_1=0$, we obtain

(4. 43) $$u_1^{(\nu)}=0 \quad \text{for} \quad \nu=1,2,3,\cdots.$$

From (3. 87), (3. 89), (4. 42) and Lemma 4. 3 follows that

(4. 44) $$|u_0^{(1)}| \equiv |P_0 u^{(0)}| \leqq C_{35} C_{36}^{(1)} \Lambda (t-\tau)^{\alpha/2} d^{-5-\alpha}(x,t;\xi,\tau) \Phi_{d_5}(x,t;\xi,\tau),$$

where $\Lambda = \chi^{\alpha/2}+\lambda^{\alpha}$. Similarly after some more calculations we have

(4. 45) $$\begin{cases} |\varDelta_x^{x_0} u_0^{(1)}| \leqq C_{35} C_{36}^{(2)} \Lambda |x-x_0|^{\alpha} d^{-5-\alpha}(x'',t;\xi,\tau) \Phi_{d_5}(x'',t;\xi,\tau), \\ |\varDelta_t^{t_0} u_0^{(1)}| \leqq C_{35} C_{36}^{(3)} \Lambda (t-t_0)^{\alpha} d^{-5-\alpha}(x,t_0;\xi,\tau) \Phi_{d_5}(x,t;\xi,\tau) \\ \qquad\qquad (t>t_0>\tau). \end{cases}$$

Next in order to estimate $I^{(k)}(u_0^{(1)})$ we transform $u_0^{(1)}$ in the form

(4. 46) $$\begin{aligned} u_0^{(1)}(x,t;\xi,\tau) &= \tilde{u}_0^{(1)}(x,t;\xi,\tau) \\ &+ \sum_{j''} \eta^{(j'')}(x) \sum_{|s|=2} \{ \Pi_{\overline{\overline{x}},\hat{\xi}}^{x,\xi} [\tilde{a}_s(\xi^{(j'')},\tau;\overline{\overline{x}}') D_{\overline{\overline{x}}'}^{s} U_{j''}^{(0)}(\overline{\overline{x}},t;\hat{\xi},\tau)] \\ &+ [a_s(\xi,\tau)-a_s(\xi^{(j'')},\tau)] D_x^s U_{j''}^{(0)}(x,t;\xi,\tau) \}, \end{aligned}$$

where

(4. 47) $$\begin{cases} |\tilde{u}_0^{(1)}| \leqq C_{35} C_{36}^{(4)} (1+\chi^{(1-\alpha)/2}\lambda^{-\alpha}) d^{-5+\alpha}(x,t;\xi,\tau) \Phi_{d_5}(x,t;\xi,\tau), \\ \tilde{a}_s(\xi^{(j'')},\tau;\overline{\overline{x}}') = \sum_{0<|\nu'|\leqq 2} \tilde{a}_{s\nu'}(\xi^{(j'')},\tau) (D_{\overline{\overline{x}}_1} F(\overline{\overline{x}}'))^{\nu_1} (D_{\overline{\overline{x}}_2} F(\overline{\overline{x}}'))^{\nu_2}, \\ \qquad\qquad (\nu'=(\nu_1,\nu_2)), \\ \tilde{a}_{s\nu'}(\xi^{(j'')},\tau) \text{ and } a_s \text{ are determined by the coefficients of (2. 18)}, \\ U_{j''}^{(0)} = \Pi_{\overline{\overline{x}},\hat{\xi}}^{x,\xi} \overline{\overline{U}}_{j''}^{(0)} = \overline{\overline{R}}_1 u_1^{(0)}, \end{cases}$$

and $\overline{\overline{x}}$ is an expression of x by the transformation (1. 17) of j''-local rectangular coordinate system $\hat{x}$.

From (4. 46) and (4. 47) we derive that

(4. 48) $$|I^{(k)}(u_0^{(1)})| \leqq C_{35} C_{36}^{(5)} \Lambda \rho_5(t,\tau;\xi) + I,$$

where

$$I = \sum_{j''} \sum_{|s|=2} \left| \iint_{Q_1^{(k)}} \zeta^{(k)}(y) \eta^{(j'')}(y) \{ \Pi_{\overline{\overline{y}},\hat{\xi}}^{y,\xi} [\tilde{a}_s(\xi^{(j'')},\tau;\overline{\overline{y}}') D_{\overline{\overline{y}}}^{s} \overline{\overline{U}}_{j''}^{(0)}(\overline{\overline{y}},\tau_0;\hat{\xi},\tau)] \right.$$

$$+[a_s(\xi,\tau)-a_s(\xi^{(j'')},\tau)]D_y^sU_{j''}^{(0)}(y,\tau_0;\xi,\tau)\}dyd\tau_0\Big|.$$

As for I, it is sufficient to estimate

$$(4.49)\qquad I''=\iint_{K_1}\bar{\zeta}^{(k'')}(\bar{y})\bar{\eta}^{(j'')}(\bar{y})\{\Pi_{\bar{\bar{y}},\hat{\xi}}^{\bar{y},\xi}[\tilde{a}_s(\xi^{(j'')},\tau:\bar{\bar{y}}')D_{\bar{\bar{y}}}^s\bar{\bar{U}}_{j''}^{(0)}(\bar{\bar{y}},\tau_0;\hat{\xi},\tau)]$$

$$+[a_s(\xi,\tau)-a_s(\xi^{(j'')},\tau)]\Pi_y^{\bar{y}}D_y^sU_{j''}^{(0)}(y,\tau_0;\xi,\tau)\}d\bar{y}d\tau_0,$$

where $K_1=\left\{(\bar{y},\tau_0)\in K\times[\tau,t]\,|\,|y-\xi|+|\tau_0-\tau|^{1/2}\leqq\frac{1}{2}(t-\tau)^{1/2}\right\}$. It is to be noted that the coordinates $\bar{y}$ and $\bar{\bar{y}}$ of $y\in\Omega^{(k'')}\cap\Omega^{(j'')}$ are connected with $\bar{\bar{y}}=h_1(\bar{y})$ and $\bar{y}=h_2(\bar{\bar{y}})$ having the same smoothness and F is independent of k'' and j''. The integrand of (4.49) can be expressed as follows:

$$(4.50)\qquad \Pi_{\bar{\bar{y}},\hat{\xi}}^{\bar{y},\xi}D_{\bar{\bar{y}}}^s\bar{\bar{U}}_{j''}^{(0)}(\bar{\bar{y}},\tau_0;\hat{\xi},\tau)$$

$$=\Pi_{\hat{\xi}}^{\xi}\sum_{|\nu|\leqq|s|}\tilde{b}_{s\nu}(\bar{y})D_{\bar{y}}^{\nu}\bar{\bar{U}}_{j''}^{(0)}(h_1(\bar{y}),\tau_0;\hat{h}_1(\hat{\xi}),\tau),$$

where $\tilde{b}_{s\nu}(\bar{y})$ depend on the function h_1 and the derivatives of h_2 and $\hat{\xi}=\hat{h}_1(\hat{\xi})$, and

$$(4.51)\qquad \Pi_y^{\bar{y}}D_y^sU_{j''}^{(0)}(y,\tau_0;\xi,\tau)$$

$$=\Pi_{\hat{\xi}}^{\xi}\sum_{|\nu|\leqq|s|}\tilde{c}_{s\nu}(\bar{y})D_{\bar{y}}^{\nu}\bar{\bar{U}}_{j''}^{(0)}(h_1(\bar{y}),\tau_0;\hat{h}_1(\hat{\xi}),\tau),$$

where $\tilde{c}_{s\nu}(\bar{y})$ depend on $F^{(k'')}$, h_1 and the derivatives of $F^{(j'')}$ and h_2. $F^{(k'')}$ is a representation of F by k''-local coordinate system. From (4.49)-(4.51) it follows that

$$(4.52)\qquad I''=\sum_{|\nu|\leqq|s|}\Pi_{\hat{\xi}}^{\xi}\iint_{K_1}\bar{\zeta}^{(k'')}(\bar{y})\bar{\eta}^{(j'')}(\bar{y})\{\tilde{a}_s(\xi^{(j'')},\tau;h_1'(\bar{y}))\tilde{b}_{s\nu}(\bar{y})$$

$$+[\tilde{a}_s(\xi,\tau)-\tilde{a}_s(\xi^{(j'')},\tau)]\tilde{c}_{s\nu}(\bar{y})\}$$

$$\times D_{\bar{y}}^{\nu}\bar{\bar{U}}_{j''}^{(0)}(h_1(\bar{y}),\tau_0;\hat{h}_1(\hat{\xi}),\tau)d\bar{y}d\tau_0.$$

If $|\nu|<2$, then

$$(4.53)\qquad |I''|\leqq C_{35}C_{36}^{(6)}\Lambda\rho_{d_5}(t,\tau;\xi).$$

If $|\nu|=2$, $\nu_3<2$ then

$$(4.54)\qquad |I''|\leqq\Big|\iint_{K_1}D_{\bar{y}'}[\bar{\zeta}^{(k'')}(\bar{y})\bar{\eta}^{(j'')}(\bar{y})\{\tilde{a}_s(\xi^{(j'')},\tau;h_1'(\bar{y}))\tilde{b}_{s\nu}(\bar{y})$$

$$+[a_s(\xi,\tau)-a_s(\xi^{(j'')},\tau)]\tilde{c}_{s\nu}(\bar{y})\}]D_{\bar{y}}\overline{\overline{U}}{}_{j''}^{(0)}d\bar{y}d\tau_0\Big|$$

$$+\Big|\int_{\Gamma_1}\bar{\zeta}^{(k'')}(\bar{y})\bar{\eta}^{(j'')}(\bar{y})\{\tilde{a}_s\tilde{b}_{s\nu}+[a_s(\xi,\tau)-a_s(\xi^{(j'')},\tau)]\tilde{c}_{s\nu}\}$$

$$\times D_{\bar{y}}\overline{\overline{U}}{}_{j''}^{(0)}\cos(\boldsymbol{n},\bar{y}')d\Gamma_1\Big|$$

$$\leqq C_{35}C_{36}^{(7)}\Lambda\rho_{d_5}(t,\tau;\xi).$$

If $|\nu|=\nu_3=2$, then we can reduce to the two previous cases. Thus we have

$$u_0^{(1)}\in H_{d_5}^{\alpha,2}(Q_{\tau,\tau+h},\bar{Q}_T),\ \|u_0^{(1)}\|\leqq C_{35}C_{36}\Lambda. \tag{4.55}$$

By induction we have

$$u_0^{(\nu)}\in H_{d_5}^{\alpha,2}(Q_{\tau,\tau+h},\bar{Q}_T),\ \|u_0^{(\nu)}\|\leqq C_{35}C_{36}^{\nu}\Lambda^{\nu}\quad(\nu=0,1,2,\cdots). \tag{4.56}$$

Choosing λ, χ and hence h, sufficiently small, we obtain that $u_0=\Sigma u_0^{(\nu)}$ converges uniformly in $Q_{\tau,\tau+h}$. (4.42), (4.43) and (4.56) imply that the assertion of the lemma holds. Q.E.D.

Lemma 4.8. $G(x,t;\xi,\tau;w)\in U_{C_{37},d_5}^{2+\alpha,0,0}(Q_{\tau,\tau+h},\bar{Q}_T)$.

Proof. It is obvious from (4.41) and Lemmas 4.3, 4.4, 4.7. Q.E.D.

Lemma 4.9. $G\in U_{C_{37},d_6}^{2+\alpha,0,0}(Q_{\tau,T},\bar{Q}_T)$ $(d_6=d_5/2)$.

Proof. The existence of the unique solution G_0 for (3.39) belonging to $C_{x,t}^{2+\alpha,(1+\alpha)/2}(\bar{Q}_{\tau,T})$ can easily be proved by the usual procedures. Thus we only have to estimate G_0 for $t>\tau+h$.

Let $\tau_1=\tau+\dfrac{h}{2}$ and let $\chi_h(t)$ be a smooth function such that

$$\chi_h(t)=\begin{cases}0, & \text{if } t\leqq h/6,\\ 1, & \text{if } t\geqq h/3.\end{cases}$$

Then $\omega(x,t;\xi,\tau;\tau_1)\equiv\chi_h(t-\tau_1)G_0(x,t;\xi,\tau;w)$ satisfies the system of equations:

$$(4.57)\quad \begin{cases} (D_t-\mathfrak{A})\,\omega=\chi_h'(t-\tau_1)G_0(x,t;\xi,\tau;w)\equiv g_0(x,t;\xi,\tau;\tau_1), \\ \omega|_{t=\tau_1}=0, \quad \omega|_{\Gamma_T}=\chi_h(t-\tau_1)Z|_{\Gamma_T}\equiv g_1(x,t;\xi,\tau;\tau_1). \end{cases}$$

ω is obtained as a solution of (4.57) by the uniqueness of the solution. The property of χ_h and (4.41) imply that $g_0=0$ for $t<\tau_1+\dfrac{h}{6}$ and $t>\tau_1+\dfrac{h}{3}$. Since

$$(4.58)\qquad G_0\in H^{2+\alpha,0}_{d_5}(Q_{\tau,\tau+h},\bar{Q}_T),$$

we have $g_0\in \mathring{H}^{\alpha}_{d_5}(\bar{Q}_{\tau_1,T},\bar{Q}_T)$. For (4.57), utilizing Lemmas 4.3–4.5 and setting $u_j^{(0)}=g_j\,(j=0,1)$, we can prove in the same way as in the proof of Lemma 4.7 that the solution ω of (4.57) defined in $\bar{Q}_{\tau_1,\tau_1+h_0}$ for some h_0 uniquely exists and belongs to $\mathring{H}^{2+\alpha}_{d_5}(\bar{Q}_{\tau_1,\tau_1+h_0},\bar{Q}_T)$. Since $\omega=G_0$ for $t>\tau_1+h/3$, $G_0\in \hat{H}^{2+\alpha}_{d_5}(\bar{Q}_{\tau_1,\tau_1+h_0},\bar{Q}_T)$. It is to be noted that h_0 is independent of τ_1.

In the next place, if we put $\omega(x,t;\xi,\tau;\tau_2)=\chi_{h_0}(t-\tau_2)G_0(x,t;\xi,\tau;w)$ $(\tau_2=\tau_1+h_0/2)$, then ω is a solution of a system of equations analogous to (4.57). Hence $G_0\in \hat{H}^{2+\alpha}_{d_5}(\bar{Q}_{\tau_1,\tau_2+h_0},\bar{Q}_T)$.

After a finite number of repetitions of this procedure, we obtain $G_0\in \hat{H}^{2+\alpha}_{d_5}(\bar{Q}_{\tau_1,T},\bar{Q}_T)$. From this and (4.58) we deduce $G_0\in H^{2+\alpha,0}_{d_6}(Q_{\tau,T},\bar{Q}_T)$. Q.E.D.

The following lemma is an immediate consequence of Lemma 4.9.

Lemma 4.10.

$$(4.59)\quad \begin{cases} |D_t^rD_x^sG|\leqq C_{38}^{(r,|s|)}(t-\tau)^{-(3+2r+|s|)/2} \\ \qquad\qquad \times\exp\left[-d_7\dfrac{|x-\xi|^2}{t-\tau}\right]\ (2r+|s|\leqq 2), \\ |\Delta_x^{x_0}D_t^rD_x^sG|\leqq C_{39}|x-x_0|^{\alpha}(t-\tau)^{-(5+\alpha)/2} \\ \qquad\qquad \times\exp\left[-d_7\dfrac{|x''-\xi|^2}{t-\tau}\right]\ (2r+|s|=2), \\ |\Delta_t^{t_0}D_t^rD_x^sG|\leqq C_{40}^{(r,|s|)}(t-t_0)^{(2-2r-|s|+\alpha)/2}(t_0-\tau)^{-(5+\alpha)/2} \\ \quad \times\exp\left[-d_7\dfrac{|x-\xi|^2}{t-\tau}\right]\ (t>t_0>\tau,\ 0<2r+|s|\leqq 2;\ d_7=d_6/2). \end{cases}$$

Remark. As for G_0, the same estimates hold with $|x-\xi|+d(\xi,\Gamma)$ and $|x''-\xi|+d(\xi,\Gamma)$ in place of $|x-\xi|$ and $|x''-\xi|$, respectively.

§ 5. Estimates for the Solution in $H^{2+\alpha}(\bar{Q}_T)$ of a Linear Problem (2.17)

The unique solution of a linear system (2.17) in $H^{2+\alpha}(\bar{Q}_T)$ is given by

$$\tilde{w}(x,t)=\int_0^t d\tau\int_\Omega G(x,t;\xi,\tau;w)\phi(\xi,\tau,w)d\xi, \tag{5.1}$$

for (5.1) certainly satisfies (2.17) and belongs to $H^{2+\alpha}(\bar{Q}_T)$ as we shall see below.

Let ε and h be sufficiently small positive numbers. We introduce the notations

$$U_\varepsilon(x^0)=\bar{\Omega}\cap\{|x-x^0|\leqq\varepsilon\},\ Q_{h\varepsilon}(x^0,t^0)=U_\varepsilon(x^0)\times[t^0+h,t^0+2h].$$

Then in order to prove $\tilde{w}\in H^{2+\alpha}(\bar{Q}_T)$, it is sufficient to show that $\tilde{w}\in H^{2+\alpha}(Q_{h\varepsilon}(x^0,t^0))$ for $\forall(x^0,t^0)\in\bar{\Omega}\times[-h,T-2h]$ and $\|\tilde{w}\|^{(2+\alpha)}_{Q_{h\varepsilon}(x^0,t^0)}\leqq C_{41}\|\phi\|_T^{(\alpha)}$, where C_{41} is a constant independent of (x^0,t^0).

Lemma 5.1.

$$|D_x^s\tilde{w}|\leqq C_{42}^{(|s|)}t^{(2-|s|+\alpha)/2}\|\phi\|_T^{(\alpha)}\quad\text{if}\quad Q_{h\varepsilon}(x^0,t^0)\cap\Gamma_T=\varnothing\quad(|s|\leqq2), \tag{5.2}$$

$$|D_x^s\tilde{w}|\leqq\bar{C}_{42}^{(|s|)}t^{(2-|s|)/2}(t+|x-x^*|^2)^{\alpha/2}\|\phi\|_T^{(\alpha)}\quad\text{if}\quad Q_{h\varepsilon}(x^0,t^0)\cap\Gamma_T\neq\varnothing\quad(|s|\leqq2), \tag{5.3}$$

where x^* is a nearest boundary point of x,

$$|\Delta_x^{x_0}D_x^s\tilde{w}|\leqq C_{43}|x-x_0|^\alpha\|\phi\|_T^{(\alpha)}\quad(|s|=2), \tag{5.4}$$

$$|\Delta_t^{t_0}D_x^s\tilde{w}|\leqq C_{44}^{(|s|)}(t-t_0)^{(2-|s|+\alpha)/2}\|\phi\|_T^{(\alpha)}\quad(0<|s|\leqq2). \tag{5.5}$$

Proof. Let $\chi_0(\tau)$ and $\chi_1(\xi)$ be smooth functions on R^1 and R^3, respectively, such that $0\leqq\chi_0,\chi_1\leqq1$ and

$$\chi_0(\tau)=\begin{cases}1 & \text{if}\quad \tau\geqq\frac{2}{3}h\\ 0 & \text{if}\quad \tau\leqq\frac{1}{2}h\end{cases},\quad \chi_1(\xi)=\begin{cases}1 & \text{if}\quad |\xi|\leqq\frac{3}{2}\varepsilon\\ 0 & \text{if}\quad |\xi|\geqq2\varepsilon\end{cases}. \tag{5.6}$$

$\tilde{w}$ is transformed into the form:

(5.7) $\tilde{w}(x,t) = \int_0^t d\tau \int_\Omega G\phi\chi(\xi,\tau)d\xi + \int_0^t d\tau \int_\Omega G\phi(1-\chi)d\xi = \tilde{w}_1 + \tilde{w}_2,$

where $\chi(\xi,\tau) = \chi_0(\tau - t^0)\chi_1(\xi - x^0)$. It is easily seen that

(5.8) $$\begin{cases} |D_x^s \tilde{w}_2| \leqq C_{42,2}^{(|s|)} t^{(2-|s|+\alpha)/2} \|\phi\|_T^{(\alpha)} \quad (|s| \leqq 2), \\ |\varDelta_x^{x_0} D_x^s \tilde{w}_2| \leqq C_{43,2} |x-x_0|^\alpha \|\phi\|_T^{(\alpha)} \quad (|s|=2), \\ |\varDelta_t^{t_0} D_x^s \tilde{w}_2| \leqq C_{44,2}^{(|s|)} (t-t_0)^{(2-|s|+\alpha)/2} \|\phi\|_T^{(\alpha)} \quad (0<|s|\leqq 2). \end{cases}$$

As for $\tilde{w}_1$, we shall estimate only when $x^0 \in \Gamma$, $t^0 = -h$. Other cases can be estimated similarly to this case more easily because of less singularities. For simplicity $U_{2\varepsilon}(x^0)$ and $Q_{h\varepsilon}(x^0, -h)$ are denoted by U^+ and Q_0, respectively. By (3.38) $\tilde{w}_1$ can be expressed in the following form:

(5.9) $$\tilde{w}_1 = \int_0^t d\tau \int_{U^+} Z\phi\chi_1(\xi - x^0)d\xi - \int_0^t d\tau \int_{U^+} G_0\phi\chi_1(\xi - x^0)d\xi = \tilde{w}_{1,1} - \tilde{w}_{1,2}.$$

(5.10) $$\tilde{w}_{1,1} = \int_0^t d\tau \int_{R^3} Z^*\phi\chi_1 d\xi - \int_0^t d\tau \int_{U^-} Z^*\phi\chi_1 d\xi = \tilde{w}_{1,1,1} - \tilde{w}_{1,1,2},$$

where $U^- = (R^3 - \Omega) \cap \{|\xi - x^0| \leqq 2\varepsilon\}$ (as for ${}^*\phi$, see Lemma 3.1). It is well known that (cf. [13, 25])

(5.11) $$\begin{cases} |D_x^s \tilde{w}_{1,1,1}| \leqq C_{42,1,1,1}^{(|s|)} t^{(2-|s|+\alpha)/2} \|{}^*\phi\|_{T,R^3}^{(\alpha)} \quad (|s| \leqq 2), \\ |\varDelta_x^{x_0} D_x^s \tilde{w}_{1,1,1}| \leqq C_{43,1,1,1} |x-x_0|^\alpha \|{}^*\phi\|_{T,R^3}^{(\alpha)} \quad (|s|=2), \\ |\varDelta_t^{t_0} D_x^s \tilde{w}_{1,1,1}| \leqq C_{44,1,1,1}^{(|s|)} (t-t_0)^{(2-|s|+\alpha)/2} \|{}^*\phi\|_{T,R^3}^{(\alpha)} \quad (0<|s|\leqq 2). \end{cases}$$

Since $\phi(x^*, 0) = 0$, we have for $|s| \leqq 2$

(5.12) $|D_x^s \tilde{w}_{1,1,2}| \leqq C_{42,1,1,2}^{(|s|)} t^{(2-|s|)/2} (t^{\alpha/2} + |x - x^*|^\alpha) \|{}^*\phi\|_{T,R^3}^{(\alpha)} (1 + t^{-(2-|s|)/2} |I_s|),$

where

$$I_s = \int_0^t d\tau \int_{U^-} D_x^s Z\chi_1 d\xi.$$

After some lengthy calculations, we obtain

(5.13) $$|I_s| \leqq C_{45}^{(|s|)} t^{(2-|s|)/2},$$

therefore

$$(5.14)\qquad |D_x^s\tilde{w}_{1,1,2}|\leqq\overline{C}_{42,1,1,2}^{(|s|)}(t+|x-x^*|^2)^{(2-|s|+\alpha)/2}\|{}^*\phi\|_{T,R^3}^{(\alpha)}.$$

For $t-t_0\geqq t_0+|x-x^*|^2$ we immediately derive from (5.14) that

$$(5.15)\qquad |\Delta_t^{t_0}D_x^s\tilde{w}_{1,1,2}|\leqq C_{44,1,1,2}^{(|s|)}(t-t_0)^{(2-|s|+\alpha)/2}\|{}^*\phi\|_{T,R^3}^{(\alpha)}.$$

For $t-t_0<t_0+|x-x^*|^2$ we put $t''=\max\{2t_0-t,0\}$ and then obtain

$$(5.16)\qquad \Delta_t^{t_0}D_x^s\tilde{w}_{1,1,2}=\int_{t''}^t d\tau\int_{U^-}D_x^sZ\chi_1\Delta_{\xi,\tau}^{x,t}{}^*\phi d\xi-\int_{t''}^t d\tau\int_{U^-}D_x^sZ\chi_1\Delta_{\xi,\tau}^{x,t_0}{}^*\phi d\xi$$
$$+\int_0^{t''}d\tau\int_{U^-}\Delta_t^{t_0}D_x^sZ\chi_1\Delta_{\xi,\tau}^{x,t_0}{}^*\phi d\xi+\Delta_t^{t_0}{}^*\phi\int_{t''}^t d\tau\int_{U^-}D_x^sZ\chi_1 d\xi$$
$$+\Delta_{x,t_0}^{x^*,0}{}^*\phi\,\Delta_t^{t_0}\int_0^t d\tau\int_{U^-}D_x^sZ\chi_1 d\xi\equiv\sum_{i=1}^5 J_i^{(1)}.$$

$J_i^{(1)}$ $(i=1,2,3)$ can directly be estimated by Lemma 3.6. As for $J_4^{(1)}$ we can derive the desired estimate from the following estimate obtained similarly to I_s

$$(5.17)\qquad \left|\int_{t''}^t d\tau\int_{U^-}D_x^sZ\chi_1 d\xi\right|\leqq C_{46}^{(|s|)}(t-t'')^{(2-|s|)/2}.$$

In order to estimate $J_5^{(1)}$ we transform this into the form

$$(5.18)\qquad \Delta_t^{t_0}\int_0^t d\tau\int_{U^-}D_x^sZ\chi_1 d\xi=\Delta_t^{t_0}\int_0^t d\tau\int_{U^-}D_x^sZ_0\chi_1 d\xi$$
$$+\Delta_t^{t_0}\int_0^t d\tau\int_{U^-}D_x^sZ'\chi_1 d\xi\equiv J_{5,1}^{(1)}+J_{5,2}^{(1)}.$$

$$J_{5,1}^{(1)}=\int_{t''}^t d\tau\int_{U^-}[D_x^sZ_0(x-\xi,t-\tau;\xi,\tau;w)$$
$$-D_x^sZ_0(x-\xi,t-\tau;z,t;w)|_{z=x}]\chi_1 d\xi$$
$$-\int_{t''}^{t_0}d\tau\int_{U^-}[D_x^sZ_0(x-\xi,t_0-\tau;\xi,\tau;w)$$
$$-D_x^sZ_0(x-\xi,t-\tau;z,t_0;w)_{|z=x}]\chi_1 d\xi$$
$$+\int_0^{t''}d\tau\int_{U^-}\Delta_t^{t_0}[D_x^sZ_0(x-\xi,t-\tau;\xi,\tau;w)$$
$$-D_x^sZ_0(x-\xi,t-\tau;z,t_0;w)_{|z=x}]\chi_1 d\xi$$
$$+\int_{t''}^t d\tau\int_{U^-}[D_x^sZ_0(x-\xi,t-\tau;z,t;w)$$
$$-D_x^sZ_0(x-\xi,t-\tau;z,t_0;w)]_{|z=x}\chi_1 d\xi$$

$$+\Delta_t^{t_0}\int_0^t d\tau\int_{U^-} D_x^s Z_0(x-\xi, t-\tau; z, t_0; w)_{|z=x}\chi_1 d\xi \equiv \sum_{i=1}^5 J_{5,1,i}^{(1)}.$$

We easily obtain by Lemma 3. 4

$$|J_{5,1,i}^{(1)}| \leqq C_{47,1,i}^{(|s|)}(t-t_0)^{(2-|s|+\alpha)/2} \quad (i=1, 2, 3, 4). \tag{5.19}$$

$$J_{5,1,5}^{(1)} = -\Delta_t^{t_0}\int_0^t d\tau\int_{U^-} D_\xi^s Z_0(x-\xi, t-\tau; x, t_0; w)\chi_1 d\xi. \tag{5.20}$$

In an analogous way as I_s we have

$$|J_{5,1,5}^{(1)}| \leqq C_{47,1,5}^{(|s|)}(t-t_0)^{(2-|s|+\alpha)/2}. \tag{5.21}$$

As for $J_{5,2}^{(1)}$, we rewrite $J_{5,2}^{(1)}$ according to the definition of Z'.

$$J_{5,2}^{(1)} = \Delta_t^{t_0} D_x^s\int_0^t d\tau_0\int_{R^3} Z_0(x-y, t-\tau_0; y, \tau_0; w)\Psi(y, \tau_0)dy, \tag{5.22}$$

where

$$\Psi(y, \tau_0) = \int_0^{\tau_0} d\tau\int_{U^-}\Phi(y, \tau_0; \xi, \tau; w)\chi_1(\xi - x^0)d\xi.$$

The fact that $\Psi \in H^\alpha(\bar{R}_T^3)$, which is proved by the same arguments as above, implies that

$$|J_{5,2}^{(1)}| \leqq C_{47,2}^{(|s|)}(t-t_0)^{(2-|s|+\alpha)/2}. \tag{5.23}$$

Hence we have

$$|\Delta_t^{t_0} D_x^s \tilde{w}_{1,1,2}| \leqq \bar{C}_{44,1,1,2}^{(|s|)}(t-t_0)^{(2-|s|+\alpha)/2}\|{}^*\phi\|_{T,R^3}^{(\alpha)} \tag{5.24}$$

Next, for clearness' sake, we assume that $x'' = x_0$.

$$\begin{aligned}\Delta_x^{x_0} D_x^s \tilde{w}_{1,1,2} = &\int_0^t d\tau\int_\sigma D_x^s Z(x-\xi, t-\tau; \xi, \tau; w)\chi_1\Delta_{\xi,\tau}^{x_0,t}{}^*\phi d\xi\\ &-\int_0^t d\tau\int_\sigma D_{x_0}^s Z(x_0-\xi, t-\tau; \xi, \tau; w)\chi_1\Delta_{\xi,\tau}^{x_0,t}{}^*\phi d\xi\\ &+\int_0^t d\tau\int_{U^- - \sigma}\Delta_x^{x_0} D_x^s Z\chi_1\Delta_{\xi,\tau}^{x_0,t}{}^*\phi d\xi\\ &+\int_0^t d\tau\int_{U^-}\Delta_x^{x_0} D_x^s Z\chi_1 d\xi\cdot\Delta_{x_0,t}^{x_0^*,t}{}^*\phi,\end{aligned}$$

where $\sigma = U^- \cap \{|\xi - x| \leqq 2|x-x_0|\}$ and x_0^* is a nearest boundary point of x_0. Applying Lemma 3. 6 to the first three integrals and using the same procedures as (5. 18) - (5. 23) we have

(5. 25) $$|\Delta_x^{x_0} D_x^s \tilde{w}_{1,1,2}| \leq C_{43,1,1,2} |x-x_0|^\alpha \| {}^*\phi \|_{T,R^3}^{(\alpha)} \quad (|s|=2).$$

Last of all, let us estimate $\tilde{w}_{1,2}$. Let ε be sufficiently small such that $U^+ \cap \Omega^{(k')} = \emptyset$ for any k'. For arbitrary $(x, t) \in Q_0$,

(5. 26) $$G_0(x, t; \xi, \tau; w) = \sum_{k''} \eta^{(k'')}(x) \times \Pi_{\bar{x}}^x \Big[\int_\tau^t d\tau_0 \int_K H_0^{(k'')}(\bar{x}, t; \bar{y}, \tau_0) \bar{\zeta}^{(k'')}(\bar{y}) \bar{u}_0(\bar{y}, \tau_0; \xi, \tau) d\bar{y} + \int_\tau^t d\tau_0 \int_{K'} H_1^{(k'')}(\bar{x}-\bar{y}', t-\tau_0) \bar{\zeta}^{(k'')}(\bar{y}') \bar{u}_1(\bar{y}', \tau_0; \xi, \tau) d\bar{y}' \Big]$$

(cf. § 3. 4),

hence $\tilde{w}_{1,2}$ can be expressed in the form

(5. 27) $$\tilde{w}_{1,2}(x, t) = \sum_{k''} \eta^{(k'')}(x) \times \Pi_{\bar{x}}^x \Big[\int_0^t d\tau_0 \int_K H_0^{(k'')}(\bar{x}, t; \bar{y}, \tau_0) \bar{\zeta}^{(k'')}(\bar{y}) (\bar{U}_0\phi)(\bar{y}, \tau_0) d\bar{y} + \int_0^t d\tau_0 \int_{K'} H_1^{(k'')}(\bar{x}-\bar{y}', t-\tau_0) \bar{\zeta}^{(k'')}(\bar{y}') (\bar{U}_1\phi)(\bar{y}', \tau_0) d\bar{y}' \Big],$$

where

(5. 28) $$\begin{cases} \bar{U}_j\phi = \Pi_y^{\bar{y}} U_j\phi \quad (j=0, 1), \\ U_j\phi = \displaystyle\int_0^{\tau_0} d\tau \int_{U^+} u_j(y, \tau_0; \xi, \tau) \phi(\xi, \tau) \chi_1(\xi - x^0) d\xi. \end{cases}$$

Assuming that

Lemma 5. 2. *If* $\phi \in C_0^\alpha(Q_T)$, *then* $U_0\phi \in \mathring{C}^\alpha(\bar{Q}_{0,h})$, $U_1\phi \in \mathring{C}^{2+\alpha}(\Gamma_{0,h})$ *and*

(5. 29) $$|||U_0\phi|||_{\bar{Q}_{0,h}}^{(\alpha)} \leq C_{48} |||\phi|||_T^{(\alpha)}, \quad |||U_1\phi|||_{\Gamma_{0,h}}^{(2+\alpha)} \leq C_{49} |||\phi|||_T^{(\alpha)}$$

holds and connecting this with Lemma 4. 6, we obtain $\tilde{w}_{1,2} \in \mathring{C}^{2+\alpha}(Q_0)$. Thus Lemma 5. 1 is proved. Q.E.D.

Now we proceed the *proof of Lemma* 5. 2. In the same way as in the proof of Lemma 4. 7 (cf. § 3. 4)

$$u_0(y, \tau_0; \xi, \tau) = \sum_{\nu=0}^\infty u_0^{(\nu)}(y, \tau_0; \xi, \tau), \quad u_1(y, \tau_0; \xi, \tau) = u_1^{(0)}(y, \tau_0; \xi, \tau).$$

Hence we have

(5. 30) $$U_0\phi=\sum_{\nu=0}^{\infty}U_0^{(\nu)}\phi,\quad U_1\phi=U_1^{(0)}\phi,$$

where

$$U_j^{(\nu)}\phi=\int_0^{\tau_0}d\tau\int_{U^+}u_j^{(\nu)}(y,\tau_0;\xi,\tau)\phi(\xi,\tau)\chi_1(\xi-x^0)d\xi.$$

From the estimates of $\tilde{w}_{1,1}$ in Lemma 5. 1 follows that

(5. 31) $$U_1^{(0)}\phi\in\mathring{C}^{2+\alpha}(\Gamma_{0,h}),\quad |||U_1^{(0)}\phi|||_{\Gamma_{0,h}}^{(2+\alpha)}\leqq C_{49}|||\phi|||_T{}^{(\alpha)}.$$

Utilizing the expression of $u_0^{(1)}$, Lemma 4. 6 and (5. 31), we obtain

(5. 32) $$U_0^{(1)}\phi\in\mathring{C}^{\alpha}(\bar{Q}_{0,h}),\quad |||U_0^{(1)}\phi|||_{\bar{Q}_{0,h}}^{(\alpha)}\leqq C_{33}C_{36}C_{49}\Lambda|||\phi|||_T{}^{(\alpha)}.$$

By induction we have

(5. 33) $$U_0^{(\nu)}\phi\in\mathring{C}^{\alpha}(\bar{Q}_{0,h}),\quad |||U_0^{(\nu)}|||_{\bar{Q}_{0,h}}^{(\alpha)}\leqq C_{49}C_{33}^{\nu}C_{36}^{\nu}\Lambda^{\nu}|||\phi|||_T{}^{(\alpha)}.$$

Hence the lemma holds with $C_{48}=C_{49}/(1-C_{33}C_{36}\Lambda)$ for sufficiently small h. Q.E.D.

Corollary of Lemma 5. 1. *For* $(x,t)\in Q_{h\varepsilon}(x^0,t^0)$

(5. 34) $$|D_t\tilde{w}|\leqq C_{50}\begin{cases} t^{\alpha/2}\|\phi\|_T{}^{(\alpha)}, & \textit{if}\quad Q_{h\varepsilon}(x^0,t^0)\cap\Gamma_T=\emptyset,\\ (t+|x-x^*|^2)^{\alpha/2}\|\phi\|_T{}^{(\alpha)}, & \textit{otherwise.}\end{cases}$$

Lemma 5. 3. $C_{42}^{(|s|)}$, $\bar{C}_{42}^{(|s|)}$, C_{43}, $C_{44}^{(|s|)}$ and C_{50} are positive functions in $\langle w\rangle_T{}^{(2,\alpha)}$, $\|v_0\|^{(2+\alpha)}$, $\|\theta_0\|^{(2+\alpha)}$, $\|\theta_1\|_T{}^{(2+\alpha)}$, $\|\rho_0\|^{(1+\alpha)}$, $(\bar{\rho}_0)^{-1}$, $\bar{\bar{\rho}}_0$, $(\bar{\theta}_0)^{-1}$, $(\bar{\sigma})^{-1}$, $\bar{\bar{\sigma}}$ and T and monotonically increasing in each argument.

Proof. We obtain this by tracing the lengthy calculations made in §§ 2–5. Q.E.D.

The condition (2. 18) implies that

(5. 35) $$\mathfrak{A}_{ij}^{km}\in H^{\alpha}(\bar{Q}_T),\quad \|\mathfrak{A}_{ij}^{km}\|_T{}^{(\alpha)}\leqq b_1(T,\langle w\rangle_T{}^{(2,\alpha)}).$$

From the definition of g_{ij} it follows that

(5. 36) $$\begin{cases} |D_xg_{ij}|,\ |D_tg_{ij}|,\ |D_xg_{ij}|_{t,T}^{(\alpha/2)}\leqq B_3(T,\langle w\rangle_T{}^{(2,\alpha)},\ \|v_0\|^{(2+\alpha)}),\\ |D_xg_{ij}|_{x,T}^{(\alpha)}\leqq B_4(T,\langle w\rangle_T{}^{(2,\alpha)},\ \|v_0\|^{(2+\alpha)})T|D_x^2w|_T{}^{(\alpha)},\end{cases}$$

where B_3 has the same property as B_1.
Hence we obtain

$$(5.37)\quad \begin{cases} \left|\dfrac{D_x\hat{\rho}}{\hat{\rho}}\right| \leqq (\bar{\rho}_0)^{-1}|\rho_0'|^{(0)} + 3(1+3B_1+3B_3)T(\langle w\rangle_T{}^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}), \\ \left|\dfrac{D_x\hat{\rho}}{\hat{\rho}}\right|_{x,T}^{(\alpha)} \leqq (\bar{\rho}_0)^{-2}|\rho_0'|^{(0)}\{3\bar{\bar{\rho}}_0 + |\rho_0'|^{(0)}\} + 9B_4T|D_x{}^2w|_T{}^{(\alpha)} \\ \qquad \times(\langle w\rangle_T{}^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}) + 3(1+3B_2)T(|D_x{}^2w|_T{}^{(\alpha)} \\ \qquad + \|v_0\|^{(2+\alpha)}), \\ \left|\dfrac{D_x\hat{\rho}}{\hat{\rho}}\right|_{t,T}^{(\alpha/2)} \leqq 3(1+B_1+B_3)(\langle w\rangle_T{}^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}), \end{cases}$$

$$(5.38)\quad \|\sigma\|_T{}^{(1+\alpha)} \leqq \|\sigma\|_T{}^{(2)}\|\hat{\rho}\|_T{}^{(\alpha)} + \|\sigma\|_T{}^{(2)}\|\hat{\theta}\|_T{}^{(\alpha)}$$

$$(5.39)\quad \begin{cases} \left|f\left(x+\displaystyle\int_0^t \hat{v}d\tau, t\right) - f\left(x_0+\displaystyle\int_0^t \hat{v}d\tau, t\right)\right| \leqq |f_x|_T{}^{(0)}|x-x_0| \\ \qquad \times\{1+(\langle w\rangle_T{}^{(2,\alpha)} + \|v_0\|^{(2+\alpha)})T\}, \\ \left|f\left(x+\displaystyle\int_0^t \hat{v}d\tau, t\right) - f\left(x+\displaystyle\int_0^{t_0} \hat{v}d\tau, t_0\right)\right| \\ \qquad \leqq \{|f_x|_T{}^{(0)}(\langle w\rangle_T{}^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}) + |f_t|_T{}^{(0)}\}(t-t_0). \end{cases}$$

From these follows that

$$(5.40)\quad \begin{aligned} &\mathfrak{B} \in H^\alpha(\bar{Q}_T), \\ &\|\mathfrak{B}\|_T{}^{(\alpha)} \leqq b_2(T, \langle w\rangle_T{}^{(2,\alpha)}) + T\cdot b_3(T, \langle w\rangle_T{}^{(2,\alpha)})|D_x{}^2w|_T{}^{(\alpha)}. \end{aligned}$$

From Lemma 5. 1 with $\phi = \mathfrak{B}$ we derive

$$(5.41)\quad \begin{cases} \langle \tilde{w}\rangle_T{}^{(2,\alpha)} \leqq C_{51}(T, \langle w\rangle_T{}^{(2,\alpha)})(b_2 + T\cdot b_3|D_x{}^2w|_T{}^{(\alpha)}), \\ |D_x{}^2\tilde{w}|_T{}^{(\alpha)} \leqq C_{52}(T, \langle w\rangle_T{}^{(2,\alpha)})(b_2 + T\cdot b_3|D_x{}^2w|_T{}^{(\alpha)}), \end{cases}$$

where $C_{51}\downarrow 0$ as $T\downarrow 0$. Therefore there exists $T_1 \in (0, T]$ such that

$$(5.42)\quad \begin{cases} C_{51}(T_1, M_1)\,b_2(T_1, M_1) < M_1 \\ C_{52}(T_1, M_1)\,b_3(T_1, M_1)\,T_1 < 1. \end{cases}$$

Moreover there exists $T_2 \in (0, T_1]$ such that

$$(5.43)\quad C_{52}(T_2, M_1)\,b_2(T_2, M_1) \leqq \frac{1 - C_{52}(T_2, M_1)\,b_3(T_2, M_1)\,T_2}{C_{51}(T_2, M_1)\,b_3(T_2, M_1)\,T_2}$$

$$\times \{M_1 - C_{51}(T_2, M_1) b_2(T_2, M_1)\}.$$

Hence there exists $M_2 > 0$ such that

$$\frac{C_{52}(T_2, M_1) b_2(T_2, M_1)}{1 - C_{52}(T_2, M_1) b_3(T_2, M_1) T_2} \leqq M_2 \leqq \frac{M_1 - C_{51}(T_2, M_1) b_2(T_2, M_1)}{C_{51}(T_2, M_1) b_3(T_2, M_1) T_2}. \tag{5.44}$$

From (5.41) and (5.44) follows that

$$\tilde{w} \in \mathfrak{S}_{T_2}^0 \equiv \{w \in \mathfrak{S}_{T_2} \mid |D_x^2 w|_{T_2}^{(\alpha)} \leqq M_2\}. \tag{5.45}$$

For simplicity we again choose $T = T_2$ from the beginning.

§ 6. The Existence of a Bounded Solution of (2.15)-(2.16)

We construct the sequence $\{w_n(x, t)\}$ of successive approximate solutions as follows:

$$\begin{cases} w_0(x, t) = 0 \in \mathfrak{S}_T^0 \\ w_n(x, t) \text{ is a solution of (2.17) with } \phi = \mathfrak{B}(x, t, w_{n-1}) \\ \text{assuming that } w = w_{n-1} \in \mathfrak{S}_T^0. \end{cases}$$

Then by the results in §§ 2-5 we have

$$w_n(x, t) = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau; w_{n-1}) \mathfrak{B}(\xi, \tau, w_{n-1}) d\xi, \tag{6.1}$$

which belongs to $\mathfrak{S}_T^0$.

By induction we obtain

Lemma 6.1. $w_n \in \mathfrak{S}_T^0$ *for* $n = 0, 1, 2, \cdots$.

Next, let us consider the difference $w_n - w_{n-1}$, which satisfies the equality

$$\begin{cases} D_t(w_n - w_{n-1}) = \mathfrak{A}(x, t, w_{n-1}; D_x)(w_n - w_{n-1}) \\ \qquad + \{\mathfrak{A}(x, t, w_{n-1}; D_x) - \mathfrak{A}(x, t, w_{n-2}; D_x)\} w_{n-1} \\ \qquad + \mathfrak{B}(x, t, w_{n-1}) - \mathfrak{B}(x, t, w_{n-2}), \\ (w_n - w_{n-1})|_{t=0} = 0, \quad (w_n - w_{n-1})|_{\Gamma_T} = 0. \end{cases} \tag{6.2}$$

From the definition of g_{ij} it follows that

$$(6.3)\quad \begin{cases} \|g_{ij}(w_{n-1})-g_{ij}(w_{n-2})\|_T{}^{(1)} \\ \qquad \leqq B_5(T, \langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\langle w_{n-2}\rangle_T{}^{(2,\alpha)})\langle w_{n-1}-w_{n-2}\rangle_T{}^{(2,\alpha)}, \\ |D_x\{g_{ij}(w_{n-1})-g_{ij}(w_{n-2})\}|_{x,T}^{(\alpha)} \\ \qquad \leqq T\cdot B_6(T, \langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\langle w_{n-2}\rangle_T{}^{(2,\alpha)})|D_x{}^2(w_{n-1}-w_{n-2})|_T{}^{(\alpha)}, \\ |D_x\{g_{ij}(w_{n-1})-g_{ij}(w_{n-2})\}|_{t,T}^{(\alpha/2)} \\ \qquad \leqq B_7(T, \langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\langle w_{n-2}\rangle_T{}^{(2,\alpha)})\langle w_{n-1}-w_{n-2}\rangle_T{}^{(2,\alpha)}, \end{cases}$$

where B_5, B_6 and B_7 are monotorically increasing in each argument and $B_5, B_7\downarrow 0$ as $T\downarrow 0$. Hence we have

$$(6.4)\quad \begin{cases} |\hat\rho_{n-1}-\hat\rho_{n-2}|\leqq\bar{\bar\rho}_0\cdot\exp[9(1+B_1)(\langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\langle w_{n-2}\rangle_T{}^{(2,\alpha)} \\ \qquad +\|v_0\|^{(2+\alpha)})T]\cdot T\{B_5+9(1+B_1)\}\langle w_{n-1}-w_{n-2}\rangle_T{}^{(2,\alpha)}, \\ |D_x(\hat\rho_{n-1}-\hat\rho_{n-2})|\leqq 3[|\rho_0'|^{(0)}+\bar{\bar\rho}_0\{1+9(1+B_1+B_3)(\langle w_{n-1}\rangle_T{}^{(2,\alpha)} \\ \qquad +\|v_0\|^{(2+\alpha)})\}]T\{B_5+3(1+B_1)\}\cdot\exp[9(1+B_1)(\langle w_{n-1}\rangle_T{}^{(2,\alpha)} \\ \qquad +\langle w_{n-2}\rangle_T{}^{(2,\alpha)}+\|v_0\|^{(2+\alpha)})T]\langle w_{n-1}-w_{n-2}\rangle_T{}^{(2,\alpha)}, \\ |D_t(\hat\rho_{n-1}-\hat\rho_{n-2})|\leqq\bar{\bar\rho}_0[270(1+B_1+B_3)(\langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\|v_0\|^{(2+\alpha)}) \\ \qquad \times\{B_5(\langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\|v_0\|^{(2+\alpha)})+1+B_1+B_3\}]\cdot\exp[9(1+B_1) \\ \qquad \times(\langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\langle w_{n-2}\rangle_T{}^{(2,\alpha)}+\|v_0\|^{(2+\alpha)})T] \\ \qquad \times\langle w_{n-1}-w_{n-2}\rangle_T{}^{(2,\alpha)}, \end{cases}$$

where

$$\hat\rho_{n-1}(x,t)=\rho_0(x)\cdot\exp\left[-\int_0^t g_{ij}(w_{n-1})\partial_j(w_{n-1,i}+v_{0,i})d\tau\right].$$

$$(6.5)\quad \begin{cases} \left|\dfrac{1}{\hat\rho_{n-1}}-\dfrac{1}{\hat\rho_{n-2}}\right|\leqq(\bar\rho_0)^{-2}\cdot\exp[9(1+B_1)(\langle w_{n-1}\rangle_T{}^{(2,\alpha)}+\langle w_{n-2}\rangle_T{}^{(2,\alpha)} \\ \qquad +\|v_0\|^{(2+\alpha)})T]\cdot|\hat\rho_{n-1}-\hat\rho_{n-2}|, \\ |\varDelta_x{}^{x_0}(\hat\rho_{n-1}^{-1}-\hat\rho_{n-2}^{-1})|\leqq 2|x-x_0|^\alpha\{|\hat\rho_{n-1}^{-2}\hat\rho_{n-2}^{-2}[D_x(\hat\rho_{n-1}-\hat\rho_{n-2})\cdot\hat\rho_{n-1}^2 \\ \qquad +D_x\hat\rho_{n-2}(\hat\rho_{n-1}^2-\hat\rho_{n-2}^2)]|_T{}^{(0)}+|\hat\rho_{n-1}^{-1}-\hat\rho_{n-2}^{-1}|_T{}^{(0)}\}, \\ |\varDelta_t{}^{t_0}(\hat\rho_{n-1}^{-1}-\hat\rho_{n-2}^{-1})|\leqq 2(t-t_0)^{\alpha/2}\{|D_t(\hat\rho_{n-1}^{-1}-\hat\rho_{n-2}^{-1})|_T{}^{(0)} \\ \qquad +|\hat\rho_{n-1}^{-1}-\hat\rho_{n-2}^{-1}|_T{}^{(0)}\}. \end{cases}$$

$$
(6.6)\quad \left\{
\begin{aligned}
&|\sigma(\hat\rho_{n-1},\hat\theta_{n-1})-\sigma(\hat\rho_{n-2},\hat\theta_{n-2})|\leqq|\sigma_{\hat\rho}|^{(0)}_{\mathscr{D}^*}|\hat\rho_{n-1}-\hat\rho_{n-2}|_T{}^{(0)}\\
&\qquad+|\sigma_{\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T{}^{(0)},\\
&|\varDelta_x{}^{x_0}\{\sigma(\hat\rho_{n-1},\hat\theta_{n-1})-\sigma(\hat\rho_{n-2},\hat\theta_{n-2})\}|\leqq|x-x_0|\,[|\sigma_{\hat\rho}|^{(0)}_{\mathscr{D}^*}|D_x(\hat\rho_{n-1}\\
&\qquad-\hat\rho_{n-2})|_T{}^{(0)}+|D_x\hat\rho_{n-2}|_T{}^{(0)}\{|\sigma_{\hat\rho\hat\rho}|^{(0)}_{\mathscr{D}^*}|\hat\rho_{n-1}-\hat\rho_{n-2}|_T{}^{(0)}\\
&\qquad+|\sigma_{\hat\rho\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T{}^{(0)}\}+|\sigma_{\hat\theta}|^{(0)}_{\mathscr{D}^*}|D_x(\hat\theta_{n-1}-\hat\theta_{n-2})|_T{}^{(0)}\\
&\qquad+|D_x\hat\theta_{n-2}|_T{}^{(0)}\{|\sigma_{\hat\rho\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\rho_{n-1}-\hat\rho_{n-2}|_T{}^{(0)}\\
&\qquad+|\sigma_{\hat\theta\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T{}^{(0)}\}],\\
&|\varDelta_t{}^{t_0}\{\sigma(\hat\rho_{n-1},\hat\theta_{n-1})-\sigma(\hat\rho_{n-2},\hat\theta_{n-2})\}|\leqq(t-t_0)\,[|\sigma_{\hat\rho}|^{(0)}_{\mathscr{D}^*}|D_t(\hat\rho_{n-1}\\
&\qquad-\hat\rho_{n-2})|_T{}^{(0)}+|D_t\hat\rho_{n-2}|_T{}^{(0)}\{|\sigma_{\hat\rho\hat\rho}|^{(0)}_{\mathscr{D}^*}|\hat\rho_{n-1}-\hat\rho_{n-2}|_T{}^{(0)}\\
&\qquad+|\sigma_{\hat\rho\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T{}^{(0)}\}+|\sigma_{\hat\theta}|^{(0)}_{\mathscr{D}^*}|D_t(\hat\theta_{n-1}-\hat\theta_{n-2})|_T{}^{(0)}\\
&\qquad+|D_t\hat\theta_{n-2}|_T{}^{(0)}\{|\sigma_{\hat\rho\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\rho_{n-1}-\hat\rho_{n-2}|_T{}^{(0)}\\
&\qquad+|\sigma_{\hat\theta\hat\theta}|^{(0)}_{\mathscr{D}^*}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T{}^{(0)}\}],
\end{aligned}\right.
$$

where

$$\mathscr{D}^*=\mathscr{D}^*_{\rho,\theta},\qquad |\sigma|^{(0)}_{\mathscr{D}^*}=\max_{\mathscr{D}^*}|\sigma|.$$

Since $(\hat\theta_{n-1}-\hat\theta_{n-2})(x,0)=0$, we have

$$(6.7)\qquad |(\hat\theta_{n-1}-\hat\theta_{n-2})(x,t)|\leqq t|D_t(\hat\theta_{n-1}-\hat\theta_{n-2})|_T{}^{(0)}.$$

Lemma 6.2. *If $g(x,t)$ be defined in $\bar Q_T$, g_x exist,*

$$|\varDelta_t{}^{t_0}g|\leqq A_3(t-t_0)^\alpha,\qquad |\varDelta_x{}^{x_0}D_xg|\leqq A_4|x-x_0|^\beta$$

and Ω satisfy the cone condition, then

$$|\varDelta_t{}^{t_0}D_xg|\leqq A_5(t-t_0)^{\alpha\beta/(1+\beta)}.$$

Proof. See, e.g., [33].

From (6.3)-(6.7) and Lemma 6.2 follows that

$$
(6.8)\quad \left\{
\begin{aligned}
&\mathfrak{A}^{km}_{ij}(x,t,w_{n-1})-\mathfrak{A}^{km}_{ij}(x,t,w_{n-2})\in H^\alpha(\bar Q_T),\\
&\|\mathfrak{A}^{km}_{ij}(x,t,w_{n-1})-\mathfrak{A}^{km}_{ij}(x,t,w_{n-2})\|_T{}^{(\alpha)}\leqq b_4(T,\langle w_{n-1}\rangle_T{}^{(2,\alpha)}\\
&\qquad+\langle w_{n-2}\rangle_T{}^{(2,\alpha)})\langle w_{n-1}-w_{n-2}\rangle_T{}^{(2,\alpha)},
\end{aligned}\right.
$$

where b_4 is monotonically increasing in each argument and $b_4 \downarrow 0$ as $T \downarrow 0$. Thus we have

$$\begin{aligned}(6.9)\qquad &\|\{\mathfrak{A}(x, t, w_{n-1}; D_x) - \mathfrak{A}(x, t, w_{n-2}; D_x)\} w_{n-1}\|_T^{(\alpha)} \\ &\leqq b_5(T, \langle w_{n-1}\rangle_T^{(2,\alpha)} + \langle w_{n-2}\rangle_T^{(2,\alpha)}, |D_x^2 w_{n-1}|_T^{(\alpha)}) \\ &\quad \times \langle w_{n-1} - w_{n-2}\rangle_T^{(2,\alpha)},\end{aligned}$$

where b_5 has the same property as b_4.

In the next place we have

$$(6.10)\quad \begin{cases} \left|\dfrac{D_x\hat{\rho}_{n-1}}{\hat{\rho}_{n-1}} - \dfrac{D_x\hat{\rho}_{n-2}}{\hat{\rho}_{n-2}}\right| \leqq T[|D_x(g_{ij}(w_{n-1}) - g_{ij}(w_{n-2}))|_T^{(0)} \\ \quad \times \{\langle w_{n-1}\rangle_T^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}\} + |D_x g_{ij}(w_{n-1})|_T^{(0)} \\ \quad \times \langle w_{n-1} - w_{n-2}\rangle_T^{(2,\alpha)} + |g_{ij}(w_{n-1}) - g_{ij}(w_{n-2})|_T^{(0)} \\ \quad \times \{\langle w_{n-1}\rangle_T^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}\} + |g_{ij}(w_{n-2})|_T^{(0)} \\ \quad \times \langle w_{n-1} - w_{n-2}\rangle_T^{(2,\alpha)}], \\ \left|\varDelta_x^{x_0}\left(\dfrac{D_x\hat{\rho}_{n-1}}{\hat{\rho}_{n-1}} - \dfrac{D_x\hat{\rho}_{n-2}}{\hat{\rho}_{n-2}}\right)\right| \leqq |x - x_0|^\alpha T[|D_x(g_{ij}(w_{n-1}) \\ \quad - g_{ij}(w_{n-2}))|_{x,T}^{(\alpha)} \times \langle w_{n-1}\rangle_T^{(2,\alpha)} + 4|D_x(g_{ij}(w_{n-1}) \\ \quad - g_{ij}(w_{n-2}))|_T^{(0)}\{\langle w_{n-1}\rangle_T^{(2,\alpha)} + \langle w_{n-2}\rangle_T^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}\} \\ \quad + \{|D_x g_{ij}(w_{n-2})|_{x,T}^{(\alpha)} + 4|D_x g_{ij}(w_{n-2})|_T^{(0)}\}\langle w_{n-1} - w_{n-2}\rangle_T^{(2,\alpha)} \\ \quad + 2\{|D_x(g_{ij}(w_{n-1}) - g_{ij}(w_{n-2}))|_T^{(0)} + |g_{ij}(w_{n-1}) \\ \quad - g_{ij}(w_{n-2}))|_T^{(0)}\}(\langle w_{n-1}\rangle_T^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}) + |g_{ij}(w_{n-1}) \\ \quad - g_{ij}(w_{n-2})|_T^{(0)}(|D_x^2 w_{n-1}|_{x,T}^{(\alpha)} + \|v_0\|^{(2+\alpha)}) \\ \quad + 2\{|D_x g_{ij}(w_{n-2})|_T^{(0)} + |g_{ij}(w_{n-2})|_T^{(0)}\}\langle w_{n-1} - w_{n-2}\rangle_T^{(2,\alpha)} \\ \quad + |g_{ij}(w_{n-2})|_T^{(0)}|D_x^2(w_{n-1} - w_{n-2})|_T^{(\alpha)}], \\ \left|\varDelta_t^{t_0}\left(\dfrac{D_x\hat{\rho}_{n-1}}{\hat{\rho}_{n-1}} - \dfrac{D_x\hat{\rho}_{n-2}}{\hat{\rho}_{n-2}}\right)\right| \leqq (t - t_0)[\{|D_x(g_{ij}(w_{n-1}) - g_{ij}(w_{n-2}))|_T^{(0)} \\ \quad + |g_{ij}(w_{n-1}) - g_{ij}(w_{n-2})|_T^{(0)}\}(\langle w_{n-1}\rangle_T^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}) \\ \quad + \{|D_x g_{ij}(w_{n-2})|_T^{(0)} + |g_{ij}(w_{n-2})|_T^{(0)}\}\langle w_{n-1} - w_{n-2}\rangle_T^{(2,\alpha)}]. \end{cases}$$

$$\begin{cases} |\sigma_{\hat{\rho}}(\hat{\rho}_{n-1}, \hat{\theta}_{n-1}) - \sigma_{\hat{\rho}}(\hat{\rho}_{n-2}, \hat{\theta}_{n-2})| \leqq |\sigma_{\hat{\rho}\hat{\rho}}|_{\mathscr{D}^*}^{(0)}|\hat{\rho}_{n-1} - \hat{\rho}_{n-2}|_T^{(0)} \\ \quad + |\sigma_{\hat{\rho}\hat{\theta}}|_{\mathscr{D}^*}^{(0)}|\hat{\theta}_{n-1} - \hat{\theta}_{n-2}|_T^{(0)}, \end{cases}$$

$$(6.11)\quad \begin{cases} |\Delta_x^{x_0}\{\sigma_{\hat\rho}(\hat\rho_{n-1},\hat\theta_{n-1})-\sigma_{\hat\rho}(\hat\rho_{n-2},\hat\theta_{n-2})\}|\leqq|x-x_0|[|\sigma_{\hat\rho\hat\rho}|_{\mathscr{D}^*}^{(0)}|D_x(\hat\rho_{n-1} \\ \quad -\hat\rho_{n-2})|_T^{(0)}+|D_x\hat\rho_{n-2}|_T^{(0)}|\sigma_{\hat\rho\hat\rho}|_{\hat\theta,\mathscr{D}^*}^{(L)}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T^{(0)} \\ \quad +|\sigma_{\hat\rho\hat\theta}|_{\mathscr{D}^*}^{(0)}|D_x(\hat\theta_{n-1}-\hat\theta_{n-2})|_T^{(0)}+|D_x\hat\theta_{n-2}|_T^{(0)}|\sigma_{\hat\rho\hat\theta}|_{\hat\rho,\mathscr{D}^*}^{(L)} \\ \quad \times|\hat\rho_{n-1}-\hat\rho_{n-2}|_T^{(0)}], \\ |\Delta_t^{t_0}\{\sigma_{\hat\rho}(\hat\rho_{n-1},\hat\theta_{n-1})-\sigma_{\hat\rho}(\hat\rho_{n-2},\hat\theta_{n-2})\}|\leqq(t-t_0)[|\sigma_{\hat\rho\hat\rho}|_{\mathscr{D}^*}^{(0)} \\ \quad \times|D_t(\hat\rho_{n-1}-\hat\rho_{n-2})|_T^{(0)}+|D_t\hat\rho_{n-2}|_T^{(0)}|\sigma_{\hat\rho\hat\rho}|_{\hat\theta,\mathscr{D}^*}^{(L)}|\hat\theta_{n-1}-\hat\theta_{n-2}|_T^{(0)} \\ \quad +|\sigma_{\hat\rho\hat\theta}|_{\mathscr{D}^*}^{(0)}|D_t(\hat\theta_{n-1}-\hat\theta_{n-2})|_T^{(0)}+|D_t\hat\theta_{n-2}|_T^{(0)}|\sigma_{\hat\rho\hat\theta}|_{\hat\rho,\mathscr{D}^*}^{(L)} \\ \quad \times|\hat\rho_{n-1}-\hat\rho_{n-2}|_T^{(0)}], \end{cases}$$

where

$$\begin{cases} |\sigma|_{\hat\rho,\mathscr{D}^*}^{(L)}=\max\limits_{(\hat\rho,\hat\theta),(\hat\rho',\hat\theta)\in\mathscr{D}^*,\hat\rho\neq\hat\rho'}\dfrac{|\sigma(\hat\rho,\hat\theta)-\sigma(\hat\rho',\hat\theta)|}{|\hat\rho-\hat\rho'|}, \\ |\sigma|_{\hat\theta,\mathscr{D}^*}^{(L)}=\max\limits_{(\hat\rho,\hat\theta),(\hat\rho,\hat\theta')\in\mathscr{D}^*,\hat\theta\neq\hat\theta'}\dfrac{|\sigma(\hat\rho,\hat\theta)-\sigma(\hat\rho,\hat\theta')|}{|\hat\theta-\hat\theta'|}. \end{cases}$$

As for $\sigma_{\hat\theta}$, we can obtain the similar estimates for it to those of $\sigma_{\hat\rho}$.

$$(6.12)\quad \begin{cases} \left|f\left(x+\int_0^t\hat v_{n-1}d\tau,t\right)-f\left(x+\int_0^t\hat v_{n-2}d\tau,t\right)\right| \\ \quad \leqq|f_x|_T^{(0)}T\langle w_{n-1}-w_{n-2}\rangle_T^{(2,\alpha)}, \\ \left|\Delta_x^{x_0}\left\{f\left(x+\int_0^t\hat v_{n-1}d\tau,t\right)-f\left(x+\int_0^t\hat v_{n-2}d\tau,t\right)\right\}\right| \\ \quad \leqq|x-x_0|T\{|f_x|_T^{(0)}+|f_x|_{x,T}^{(L)}[1+(\langle w_{n-2}\rangle_T^{(2,\alpha)}+\|v_0\|^{(2+\alpha)})T]\} \\ \quad\quad \times\langle w_{n-1}-w_{n-2}\rangle_T^{(2,\alpha)}, \\ \left|\Delta_t^{t_0}\left\{f\left(x+\int_0^t\hat v_{n-1}d\tau,t\right)-f\left(x+\int_0^t\hat v_{n-2}d\tau,t\right)\right\}\right| \\ \quad \leqq(t-t_0)^{\alpha/2}T^{1-\alpha/2}[|f_x|_T^{(0)}+|f_x|_{x,T}^{(L)}\{\langle w_{n-1}\rangle_T^{(2,\alpha)} \\ \quad\quad +\|v_0\|^{(2+\alpha)}\}T+|f_t|_{x,T}^{(L)}T]\langle w_{n-1}-w_{n-2}\rangle_T^{(2,\alpha)}. \end{cases}$$

From (2.20), (5.36), (5.37) with w_{n-1} and w_{n-2}, (6.3), (6.7), (6.10)-(6.12) and Lemma 6.2 it follows that

$$(6.13)\quad \|\mathfrak{B}(x,t,w_{n-1})-\mathfrak{B}(x,t,w_{n-2})\|_T^{(\alpha)}$$

$$\leqq b_6(T,\langle w_{n-1}\rangle_T^{(2,\alpha)}+\langle w_{n-2}\rangle_T^{(2,\alpha)},|D_x^2w_{n-1}|_T^{(\alpha)}$$

$$+|D_x^2 w_{n-2}|_T^{(\alpha)})\langle w_{n-1}-w_{n-2}\rangle_T^{(2,\alpha)}+b_7(T,\langle w_{n-1}\rangle_T^{(2,\alpha)}$$

$$+\langle w_{n-2}\rangle_T^{(2,\alpha)})|D_x^2(w_{n-1}-w_{n-2})|_T^{(\alpha)},$$

where b_6 and b_7 have the same property as b_4.

Hence a solution of (6. 2) is given by

$$(6.14)\quad (w_n-w_{n-1})(x,t)=\int_0^t d\tau\int_\Omega G(x,t;\xi,\tau;w_{n-1})$$
$$\times[\{\mathfrak{A}(\xi,\tau,w_{n-1};D_x)-\mathfrak{A}(\xi,\tau,w_{n-2};D_x)\}w_{n-1}$$
$$+\mathfrak{B}(\xi,\tau,w_{n-1})-\mathfrak{B}(\xi,\tau,w_{n-2})]d\xi\,.$$

From Lemma 5. 1 follows that (cf. (5. 41))

$$(6.15)\quad \begin{cases} \langle w_n-w_{n-1}\rangle_T^{(2,\alpha)} \\ \quad\leqq C_{51}(T,\langle w_{n-1}\rangle_T^{(2,\alpha)})\,(b_5+b_6)\langle w_{n-1}-w_{n-2}\rangle_T^{(2,\alpha)} \\ \qquad +C_{51}b_7|D_x^2(w_{n-1}-w_{n-2})|_T^{(\alpha)}, \\ |D_x^2(w_n-w_{n-1})|_T^{(\alpha)}\leqq C_{52}(b_5+b_6)\langle w_{n-1}-w_{n-2}\rangle_T^{(2,\alpha)} \\ \qquad +C_{52}b_7|D_x^2(w_{n-1}-w_{n-2})|_T^{(\alpha)}. \end{cases}$$

Now we denote $\langle w\rangle_T^{(2,\alpha)}+|D_x^2 w|_T^{(\alpha)}$ by $\langle\!\langle w\rangle\!\rangle_T^{(2,\alpha)}$. Then from (6. 15) we derive

$$(6.16)\quad \langle\!\langle w_n-w_{n-1}\rangle\!\rangle_T^{(2,\alpha)}\leqq C_{53}(T,\langle\!\langle w_{n-1}\rangle\!\rangle_T^{(2,\alpha)}+\langle\!\langle w_{n-2}\rangle\!\rangle_T^{(2,\alpha)})$$
$$\times\langle\!\langle w_{n-1}-w_{n-2}\rangle\!\rangle_T^{(2,\alpha)},$$

where C_{53} is monotonically increasing in each argument and $C_{53}\downarrow 0$ as $T\downarrow 0$. By induction we have

$$(6.17)\quad \langle\!\langle w_n-w_{n-1}\rangle\!\rangle_T^{(2,\alpha)}\leqq C_{53}^{n-1}\langle\!\langle w_1-w_0\rangle\!\rangle_T^{(2,\alpha)}.$$

The property of C_{53} implies that for some $T'\in(0,T]$

$$(6.18)\quad C_{53}(T',2M_1+2M_2)<1\,.$$

Moreover since

$$\langle\!\langle w_1-w_0\rangle\!\rangle_{T'}^{(2,\alpha)}=\langle\!\langle w_1\rangle\!\rangle_{T'}^{(2,\alpha)}\leqq 2(C_{51}(T')+C_{52}(T'))\,b_2(T')<+\infty\,,$$

we have

$$\sum_{n=1}^{\infty}C_{53}^{n-1}\langle\!\langle w_1-w_0\rangle\!\rangle_{T'}^{(2,\alpha)}<+\infty\,.$$

That is to say, $\{w_n\}$ converge to an element w of $H^{2+\alpha}(\bar{Q}_{T'})$ as $n\to\infty$. As is known the expression of ρ_n, $\{\rho_n\}$ converge to an element ρ of $B^{1+\alpha}(\bar{Q}_{T'})$ as $n\to\infty$. $\mathfrak{A}(x, t, w_n; D_x)$, $\mathfrak{B}(x, t, w_n)$ and $G(x, t; \xi, \tau; w_n)$ also converge to $\mathfrak{A}(x, t, w; D_x)$, $\mathfrak{B}(x, t, w)$ and $G(x, t; \xi, \tau; w)$, respectively. Consequently a solution of (2.15)-(2.16) is given by

$$(6.19)\qquad w(x, t) = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau; w)\mathfrak{B}(\xi, \tau, w)\, d\xi\,.$$

Remark. For w, Lemma 5.1 also holds with $\phi=\mathfrak{B}$.

§ 7. The Proof of Uniqueness

Now let us direct ourselves towards the problem of uniqueness concerning (2.15). We assume that there exist two solutions $(w, \hat{\rho})$ and $(w^*, \hat{\rho}^*)$ of (2.15) in $H^{2+\alpha}(\bar{Q}_T)\times B^{1+\alpha}(\bar{Q}_T)$ satisfying one and the same initial-boundary condition (2.16). The difference $w-w^*$ satisfies (6.2) as w_n and w_{n-1} are replaced by w and w^* respectively. Then $w-w^*$ can be uniquely expressed in the form (6.14) as w_n and w_{n-1} are replaced by w and w^* respectively, i.e.,

$$(7.1)\qquad (w-w^*)(x, t) = \int_0^t d\tau \int_\Omega G(x, t; \xi, \tau; w) \times \{[\mathfrak{A}(\xi, \tau, w; D_x) - \mathfrak{A}(\xi, \tau, w^*; D_x)]w^* + \mathfrak{B}(\xi, \tau, w) - \mathfrak{B}(\xi, \tau, w^*)\}\, d\xi\,.$$

As for $w-w^*$, in a way analogous to that used in the preceding section for $w_n - w_{n-1}$, we have for some constant C_{54} having the same property as C_{53}

$$(7.2)\qquad \langle\!\langle w-w^*\rangle\!\rangle_T^{(2,\alpha)} \leqq C_{54}(T, \langle\!\langle w\rangle\!\rangle_T^{(2,\alpha)} + \langle\!\langle w^*\rangle\!\rangle_T^{(2,\alpha)})\langle\!\langle w-w^*\rangle\!\rangle_T^{(2,\alpha)}.$$

Hence there exists $T_0\in(0, T]$ such that

$$(7.3)\qquad C_{54}(T_0) < 1\,.$$

From these we derive $w(x, t) = w^*(x, t)$ for $0\leqq t\leqq T_0\leqq T$. Hereafter, it remains only to make a finite number of repetitions of the same procedure. The uniqueness of ρ follows from that of w. Thus Theorem 2 is proved.

§8. The Proof of Theorem 1

By Theorem 2 and (2.15) there exists a unique solution $(\hat{v}, \hat{\theta}, \hat{\rho}) \in H^{2+\alpha}(\bar{Q}_{T'}) \times H^{2+\alpha}(\bar{Q}_{T'}) \times B^{1+\alpha}(\bar{Q}_{T'})$. Now according to the notations in § 2, we respectively define x and (v, θ, ρ) by (2.5) and

$$(8.1) \qquad \begin{cases} v(x,t) = \hat{v}(x_0(x,t), t_0 = t), \\ \theta(x,t) = \hat{\theta}(x_0(x,t), t_0 = t), \\ \rho(x,t) = \hat{\rho}(x_0(x,t), t_0 = t). \end{cases}$$

Since (v, θ, ρ) certainly satisfies (1.1), (1.2), (1.3)′ and (1.5), it is sufficient to prove $(v, \theta, \rho) \in H^{2+\alpha}(\bar{Q}_{T'}) \times H^{2+\alpha}(\bar{Q}_{T'}) \times B^{1+\alpha}(\bar{Q}_{T'})$ and the uniqueness. Here we do those of v only.

Lemma 8.1. *If* $\hat{v} \in H^{2+\alpha}(\bar{Q}_T)$, *then* $v \in H^{2+\alpha}(\bar{Q}_T)$.

Proof. We prove that $|D_x^2 v|_{t,T}^{(\alpha/2)} < +\infty$ only. Other estimates can be derived analogously.

$$\begin{aligned} |\Delta_t^{t'} D_x^2 v| &\leqq |\Delta_t^{t'} D_x g_{ij}|_T^{(0)} |D_{x_0}\hat{v}|_T^{(0)} + |D_x g_{ij}|_T^{(0)} |\Delta_t^{t'} D_{x_0}\hat{v}(x_0(x,t), t_0 = t)|_T^{(0)} \\ &\quad + |\Delta_t^{t'} (D_x x_0)^2|_T^{(0)} + |(D_x x_0)^2|_T^{(0)} |\Delta_t^{t'} D_{x_0}^2 \hat{v}|_T^{(0)} \\ &\leqq |t-t'|^{\alpha/2} [|D_{x_0}\hat{v}|_T^{(0)} B_3 T^{1-\alpha/2} \{\langle \hat{v} \rangle_T^{(2,\alpha)} + \|v_0\|^{(2+\alpha)}\} \\ &\quad + 9B_3 \{9(1+B_1) |\hat{v}|_T^{(0)} |D_{x_0}^2 \hat{v}|_T^{(0)} + 2|D_{x_0}\hat{v}|_T^{(0)} + |D_{x_0}\hat{v}|_{t,T}^{(\alpha/2)}\} \\ &\quad + 162 B_3 (1+B_1) T^{1-\alpha/2} |D_{x_0}^2 \hat{v}|_T^{(0)} + 81(1+B_1)^2 \{T^{\alpha/2} 9^\alpha (1+B_1)^\alpha \\ &\quad \times (|\hat{v}|_T^{(0)})^\alpha |D_{x_0}^2 \hat{v}|_{x,T}^{(\alpha)} + |D_{x_0}^2 \hat{v}|_{t,T}^{(\alpha/2)}\}], \end{aligned}$$

hence $|D_x^2 v|_{t,T}^{(\alpha/2)} < +\infty$. Q.E.D.

Lemma 8.2. *The mapping* $v = F\hat{v}$ *from* $H^{2+\alpha}(\bar{Q}_T)$ *onto itself is ond to one.*

Proof. Suppose that $v = v^*$, i.e., $F\hat{v} = F\hat{v}^*$. Then from (2.4) we derive

$$\begin{cases} \dfrac{d}{d\tau} x(x_0, \tau; \hat{v}) = \hat{v}(x_0, \tau) \equiv v(x(x_0, \tau; \hat{v}), \tau), \end{cases}$$

$$(8.2)\quad \begin{cases} & x(x_0, 0; \hat{v}) = x_0; \\ \dfrac{d}{d\tau} x(x_0, \tau; \hat{v}^*) = \hat{v}^*(x_0, \tau) \equiv v^*(x(x_0, \tau; \hat{v}^*), \tau), \\ & x(x_0, 0; \hat{v}^*) = x_0. \end{cases}$$

Therefore we have

$$(8.3)\quad \begin{cases} \dfrac{d}{d\tau}(x(x_0, \tau; \hat{v}) - x(x_0, \tau; \hat{v}^*)) \\ \qquad = v(x(x_0, \tau; \hat{v}), \tau) - v(x(x_0, \tau; \hat{v}^*), \tau), \\ x(x_0, 0; \hat{v}) - x(x_0, 0; \hat{v}^*) = 0, \end{cases}$$

hence

$$(8.4)\quad \begin{cases} \dfrac{1}{2}\dfrac{d}{d\tau}(x(x_0, \tau; \hat{v}) - x(x_0, \tau; \hat{v}^*))^2 \\ \qquad \leqq |v_x|_T^{(0)}(x(x_0, \tau; \hat{v}) - x(x_0, \tau; \hat{v}^*))^2, \\ x(x_0, 0; \hat{v}) - x(x_0, 0; \hat{v}^*) = 0. \end{cases}$$

From (8.4) it follows that $x(x_0, \tau; \hat{v}) = x(x_0, \tau; \hat{v}^*)$, hence $\hat{v}(x_0, t) = \hat{v}^*(x_0, t)$. Q.E.D.

By Lemmas 8.1 and 8.2 the proof of Theorem 1 is now completed.

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