The Singularities of Solutions of the Cauchy Problems for Systems Whose Characteristic Roots Are Non-Uniform Multiple

By

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§ 1. Introduction

R. Courant and P. D. Lax [1] and D. Ludwig [4] investigated the singularities of the solutions of the Cauchy problems for diagonalizable linear hyperbolic systems whose characteristic roots are real and uniform multiple. They constructed a uniform asymptotic solution and proved that the singularities of the solutions propagate only along the characteristic surfaces on which the singularities of the initial data lie. Secondly, D. Ludwig and B. Granoff [2] dropped the condition that the characteristic roots are uniform multiple. They defined their hyperbolicity for systems with constant coefficient in the principal part whose normal surface has self-intersection points and discussed the propagation of singularities by constructing a uniform asymptotic solution. An important feature of their results is that the singularities of the solutions propagate also along the characteristic surface which generally does not carry the singular support of the initial data. Geometrically, this is an enveloping surface generated by a family of surfaces which connect the two characteristic surfaces with intersection points. The complex versions corresponding to the results of [1] and [4] were done by Y. Hamada [3], C. Wagschal [6], especially for meromorphic Cauchy data.

The aim of this paper is to extend the results of [2] for a certain type of systems with variable coefficients in the complex domain. Our results include as a corollary the exactness of the asymptotic solution constructed by D. Ludwig and B. Granoff [2] in the real analytic case.

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§ 2. Assumptions and Result

Let C^{n+1} be the $n+1$-dimensional complex space and denote its point by $(t, x) = (t, x_1, \cdots, x_n)$. We also denote the covector at $x = (x_1, \cdots, x_n)$ by $\xi = (\xi_1, \cdots, \xi_n)$.

We consider a first order system:

$$(2.1) \qquad \mathcal{L}u = \partial u / \partial t + \sum_{\mu=1}^{n} A^{\mu}(t, x) \, \partial u / \partial x_{\mu} + B(t, x) u = 0,$$

where A^{μ}, B are $k \times k$-matrices whose components are holomorphic in a neighborhood of the origin and u is a vector function. For convenience we put $A^0 = I$ (the identity matrix), $x_0 = t$ and use the convention that repeated indices μ and ν are summed from 1 to n and 0 to n respectively.

Now we assume the following assumptions (I) $\sim$ (V).

(I)   For any $(t, x) \sim 0$ and $\xi \sim (1, 0, \cdots, 0)$, the matrix $-A^{\mu}(t, x) \xi_{\mu}$ has k (counting multiplicities) eigenvalues $\lambda^l(t, x; \xi)$ ($1 \leq l \leq k$) and the associated eigenvectors form a complete set.

(II)   Let $\lambda^{+}(t, x; \xi) = \lambda^{k-1}(t, x; \xi)$ and $\lambda^{-}(t, x; \xi) = \lambda^k(t, x; \xi)$ are the two eigenvalues which coincide at $(t, x) = 0$, $\xi = (1, 0, \cdots, 0)$. Then, for any $(t, x) \sim 0$ $\xi \sim (1, 0, \cdots, 0)$, each set of eigenvalues $\{\lambda^{+}(t, x; \xi), \lambda^l(t, x; \xi)\, ; 1 \leq l \leq k-2\}$, $\{\lambda^{-}(t, x; \xi), \lambda^l(t, x; \xi)\, ; 1 \leq l \leq k-2\}$ is mutually distinct.

(III)   The eigenvalues $\lambda^{\pm}(t, x; \xi)$ and the associated eigenvectors $R^{\pm}(t, x; \xi)$ are holomorphic in $(t, x) \sim 0$, $\xi \sim (1, 0, \cdots, 0)$.

(IV) λ^{+}, λ^{-} satisfy $\lambda_t^{+} - \lambda_t^{-} + \nabla_{\xi}\lambda^{+} \cdot \nabla_x\lambda^{-} - \nabla_{\xi}\lambda^{-} \cdot \nabla_x\lambda^{+} = 0$ for any $(t, x) \sim 0$ and $\xi \sim (1, 0, \cdots, 0)$.

From the assumption (III), there exist in a neighborhood of the origin regular and holomorphic phases $\varphi^{\pm}(t, x)$, $\varphi^l(t, x)$ ($l = 1, 2, \cdots, k-2$) defined by

$$\begin{cases} \varphi_t^{\pm} = \lambda^{\pm}(t, x; \nabla\varphi_x^{\pm}), \quad \varphi_t^l = \lambda^l(t, x; \nabla_x\varphi^l), \\ \varphi^{\pm}(0, x) = \varphi^l(0, x) = x_1. \end{cases}$$

We also define an auxiliary phase $\varPhi(t, x, \tau)$ which is regular and holomorphic in a neighborhood of the origin by

$$(2.2) \qquad \begin{cases} \varPhi_t \pm \varPhi_\tau = \lambda^{\pm}(t, x; \nabla_x\varPhi), \\ \varPhi(t, x, \mp t) = \varphi^{\pm}(t, x). \end{cases}$$

The existence of such $\varPhi(t, x, \tau)$ is assured by the assumption (IV) which corresponds to the integrability condition of the over determined system (2. 2).

The last assumption is as follows.

(V) $\varPhi(0, 0, \tau) \not\equiv 0$ $(\tau \sim 0)$.

Remark. It is possible to take another auxilary phase defined by an equation which slightly differs from (2. 2). In this case, we must assume the integrability condition corresponding to this equation instead of the assumption (IV). These conditions can be easily found if one examines the following proof.

The assumption (V) is satisfied, for example, if we impose the following assumption:

(2. 3) $\nabla_\xi(\lambda^+ - \lambda^-) \cdot \nabla_x(\lambda^+ - \lambda^-) = 0$ for any $(t, x) \sim 0$ and $\xi \sim (1, 0, \cdots, 0)$.

In fact $\varPhi_{\tau\tau} \neq 0$ $((t, x) \sim 0, \xi \sim (1, 0, \cdots, 0))$ follows from (2. 3).

We impose on the system (2. 1) an initial condition which has a pole on the hyperplane $x_1 = 0$. Since the assumptions (I) $\sim$ (V) and the fact that the Cauchy data has a pole do not depend upon a choice of a holomorphic basis for $\boldsymbol{C}^k$, we may assume that the Cauchy data $u(0, x)$ has the following form. Namely,

(2. 4) $u(0, x) = v^+(x) R^+(0, x; 1, 0, \cdots, 0) + v^-(x) R^-(0, x; 1, 0, \cdots, 0)$

$$+ \sum_{l=1}^{k-2} v^l(x) R^l(0, x; 1, 0, \cdots, 0),$$

where one of $v^\pm(x)$, $v^l(x)$ $(1 \leq l \leq k-2)$ has a pole on $x_1 = 0$ and $R^l(t, x; \xi)$ $(l = 1, 2, \cdots, k-2)$ denote the eigenvectors with eigenvalues $\lambda^l(t, x; \xi)$ $(l = 1, 2, \cdots, k-2)$. Furthermore, according to the principle of superposition, it is enough to consider the special case:

(2. 5) $\begin{cases} v^+(x) = (-1)^{q-1}(q-1)! w^+(x') / x_1^q, \\ v^-(x) = v^1(x) = \cdots = v^{k-2}(x) = 0, \end{cases}$

where $q \in \boldsymbol{N}$, $x' = (x_2, \cdots, x_n)$.

From the assumption (V) and Weierstrass' preparation theorem, there exist in a neighborhood of the origin a Weierstrass' polynomial

$P(t, x, \tau)$ of τ and a nonzero holomorphic function $\Psi(t, x, \tau)$ which satisfy

$$\Phi(t, x, \tau) = \Psi(t, x, \tau) P(t, x, \tau).$$

Since $\Phi(0, x, 0) = x_1$, $P(t, x, \tau)$ is an irreducible polynomial of τ. We denote the discriminant of P by ω.

In this situation we have the following theorem.

Theorem. *There exists in a neighborhood of the origin a unique solution $u(t, x)$ of the Cauchy problem for the system* (2.1) *with the initial condition* (2.5). *It is given in the following form*:

$$(2.6) \quad u(t, x) = F^+(t, x) / [\varphi^+(t, x)]^q + G^+(t, x) \log \varphi^+(t, x)$$

$$+ F^-(t, x) / [\varphi^-(t, x)]^q$$

$$+ G^-(t, x) \log \varphi^-(t, x) + \int_{-t}^{t} \{F(t, x, \tau) / [\Phi(t, x, \tau)]^q$$

$$+ G(t, x, \tau) \log \Phi(t, x, \tau)\} d\tau + \sum_{l=1}^{k-2} \{F^l(t, x) / [\varphi^l(t, x)]^q$$

$$+ G^l(t, x) \log \varphi^l(t, x)\} + H(t, x),$$

where $F^\pm(t, x)$, $G^\pm(t, x)$, $F^l(t, x)$, $H(t, x)$, $F(t, x, \tau)$, $G(t, x, \tau)$ are vector functions holomorphic at the origin. Moreover, if we prolong the righthand side of (2.6) *by an analytic continuation, we find that the solution $u(t, x)$ generally remifies around the analytic set $D = \{(t, x); \omega(t, x) = 0\}$ and also tends to infinity as (t, x) becomes close to D.*

Remark. The analytic set D is characteristic with respect to $\mathcal{L}$.

Next we give an example which satisfies all the assumptions of the theorem.

Example. Consider the Cauchy problem:

$$\begin{cases} \partial/\partial t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} t + x_2, & 0 \\ 0, & -t^2/2 \end{pmatrix} \partial/\partial x_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} t, & 0 \\ 0, & t+1 \end{pmatrix} \partial/\partial x_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ u_1(0, x) = 0, \quad u_2(0, x) = 1/x_1. \end{cases}$$

The solution is given by

$$u_1(t, x) = 1/x_1 \cdot \{\log\{(t+t^2/2+x_2-\sqrt{\psi})/(t+t^2/2+x_2+\sqrt{\psi})\}$$
$$-\log\{(x_2+t^2/2-\sqrt{\psi})/(x_2+t^2/2+\sqrt{\psi})\}\},$$

$$u_2(t, x) = (x_1-t^3/6)^{-1},$$

where $\psi = t^4/4 + 2/3 \cdot t^3 + 2x_2 t + 2x_1 + x_2^2$. $u_1(t, x)$ ramifies algebraically around the characteristic manifold $\psi=0$ and tends to infinity as (t, x) becomes close to $\psi=0$. However this characteristic manifold $\psi=0$ does not contain $\{t=x_1=0\}$.

The proof of the theorem consists of four parts. First we give the construction of the phases. Secondly we construct the formal solution (uniform asymptotic solution). Thirdly we prove the exactness of the formal solution. Finally we investigate the behavior of the solution $u(t, x)$ as a holomorphic function defined by (2. 6).

§ 3. Construction of the Phases

The phase $\varphi^\pm(t, x)$ and $\varphi^l(t, x)$ $(1 \leq l \leq k-2)$ can be constructed in the same way as in [3]. Thus we only show how to construct the auxiliary phase $\varPhi(t, x, \tau)$. To construct $\varPhi(t, x, \tau)$, we follow the way of D. Ludwig-B. Granoff [2]. Though their procedure were made in the constant coefficient case, it is still valid to our problem if we impose the integrability condition (IV) on (2. 2).

Let $F(t, x, s)$ be the solution of the Cauchy problem:

$$(3. 1) \qquad \begin{cases} F_t = \lambda^-(t, x; \nabla_x F), \\ F(s, x, s) = \varphi^+(s, x). \end{cases}$$

By Hamilton-Jacobi's theory, $F(t, x, s)$ is a holomorphic function of (t, x, s) in a neighborhood of the origin.

Differentiate $F(s, x, s) = \varphi^+(s, x)$ at $t=s$ by s. Then we obtain

$$(3. 2) \quad F_s(s, x, s) = \lambda^+(s, x; \nabla_x F(s, x, s)) - \lambda^-(s, x; \nabla_x F(s, x, s)).$$

Next consider the bicharacteristic curve (3. 3) $X=X(t, x)$, $X(s, x) = x$ associated with (3. 1). What we want to prove is that $F_s(t, x, s)$ and $\lambda^+(t, x; \nabla_x F(t, x, s)) - \lambda^-(t, x; \nabla_x F(t, x, s))$ are invariant along the curve (3. 3). In fact, we have $F_{st} - \nabla_\xi \lambda^- \cdot \nabla_x F_s = 0$ if we differentiate $F_t(t,$

$x, s) = \lambda^-(t, x; \nabla_x F(t, x, s))$ by s.　Thus $F_s(t, x, s)$ is invariant along this curve.　On the other hand, we have

$$d/dt \{\lambda^+(t, X(t, x); \nabla_x F(t, X(t, x), s)) - \lambda^-(t, X; \nabla_x F)\}$$

$$= \lambda_t^+ - \lambda_t^- + \nabla_\xi \lambda^+ \cdot \nabla_x \lambda^- - \nabla_\xi \lambda^- \cdot \nabla_x \lambda^+.$$

From the assumption (IV),

$$d/dt \{\lambda^+(t, X(t, x); \nabla_x F(t, X(t, x), s)) - \lambda^-(t, X; \nabla_x F)\} = 0.$$

Namely, $\lambda^+(t, x; \nabla_x F(t, x, s)) - \lambda^-(t, x; \nabla_x F(t, x, s))$ is invariant along the curve (3.3).　Since the mapping $(t, x) \mapsto (t, X(t, x))$ is locally analytic homeomorphism, (3.2) implies

$$(3.4) \quad F_s(t, x, s) = \lambda^+(t, x; \nabla_x F(t, x, s)) - \lambda^-(t, x; \nabla_x F(t, x, s))$$

for $(t, x, s) \sim 0$.

Now define $\varPhi(t, x, \tau)$ by $\varPhi(t, x, \tau) = F(t, x, (t+\tau)/2)$.　Then using (3.4) we can easily prove the following relations:

$$\begin{cases} \varPhi_t \pm \varPhi_\tau = \lambda^\pm(t, x; \nabla_x \varPhi), \\ \varPhi(t, x, t) = \varphi^+(t, x). \end{cases}$$

As for $\varPhi(t, x, -t) = \varphi^-(t, x)$, we observe

$$\begin{cases} \partial/\partial t \, F(t, x, 0) = \lambda^-(t, x; \nabla_x F(t, x, 0)), \quad F(0, x, 0) = \varphi^+(0, x), \\ \partial/\partial t \, \varphi^-(t, x) = \lambda^-(t, x; \nabla_x \varphi^-(t, x)), \quad \varphi^-(0, x) = \varphi^+(0, x). \end{cases}$$

Thus we have $F(t, x, 0) = \varphi^-(t, x)$ from the uniqueness theorem.　Namely, $\varPhi(t, x, -t) = \varphi^-(t, x)$.

§4.　Construction of the Formal Solution

In order to obtain a formal solution, let $\{f_j(\zeta)\}_{j=-1}^\infty$ be the wave forms defined by

$$(4.1) \quad \begin{cases} d/d\zeta \, f_j(\zeta) = f_{j-1}(\zeta), \\ f_0(\zeta) = (-1)^{q-1}(q-1)!/\zeta^q, \, f_1(\zeta) = (-1)^{q-2}(q-2)!/\zeta^{q-2}, \cdots, \\ f_{q-1}(\zeta) = 1/\zeta, \, f_q(\zeta) = \log \zeta, \, f_{q+m}(\zeta) = \zeta^m/m! \cdot \log \zeta - A_m/m! \cdot \zeta^m, \end{cases}$$

where $A_m = 1 + 1/2 + \cdots + 1/m$, $A_0 = 0$.　Then we seek an asymptotic solution of the form:

(4. 2) $u(t, x) = \sum_{j=0}^{\infty} [f_j(\varphi^+(t, x)) a_j^+(t, x) + f_j(\varphi^-(t, x)) a_j^-(t, x)$

$$+ \int_{-t}^{t} f_j(\varPhi(t, x, \tau)) b_j(t, x, \tau) d\tau + \sum_{j=0}^{\infty} \sum_{l=1}^{k-2} f_j(\varphi^l(t, x))$$

$$\times a_j^l(t, x),$$

where (4. 3) $b_j(t, x, \tau) = \beta_j^+(t, x, \tau) R^+(t, x; \nabla_x \varPhi(t, x, \tau)) + \beta_j^-(t, x, \tau)$
$\times R^-(t, x; \nabla_x \varPhi)$. We follow the argument of D. Ludwig-B. Granoff [2]
and make it complete by deriving the recurrence relations for $a_j^{\pm}$, a_j^l
and $\beta_j^{\pm}$.

Substitute (4. 2) into (2. 1) and calculate formally, we obtain

(4. 4) $\mathcal{L}u = f_{-1}(\varphi^+) A_+ a_0^+ + f_{-1}(\varphi^-) A_- a_0^- + \sum_{j=0}^{\infty} f_j(\varphi^+) [A_+ a_{j+1}^+$

$$+ \mathcal{L}(a_j^+) + b_j(t, x, t)] + \sum_{j=0}^{\infty} f_j(\varphi^-) [A_- a_{j+1}^- + \mathcal{L}(a_j^-)$$

$$+ b_j(t, x, -t)] + \sum_{j=0}^{\infty} \int_{-t}^{t} \{f_{j-1}(\varPhi) A^\nu \varPhi_\nu b_j + f_j(\varPhi) \mathcal{L}(b_j)\} d\tau$$

$$+ \sum_{l=1}^{k-2} \sum_{j=0}^{\infty} f_j(\varphi^l) [A_l a_{j+1}^l + \mathcal{L}(a_j^l)],$$

where $A_{\pm} = A^\nu \varphi_\nu^{\pm}$, $A_l = A^\nu \varphi_\nu^l$. Taking account of (2. 2) and (4. 1), an
integration by parts yields

$$\int_{-t}^{t} f_{j-1}(\varPhi) A^\nu \varPhi_\nu b_j d\tau = f_j(\varphi^+) [-\beta_j^+(t, x, t) R^+ + \beta_j^-(t, x, t) R^-]$$

$$+ f_j(\varphi^-) [\beta_j^+(t, x, -t) R^+ - \beta_j^-(t, x, -t) R^-]$$

$$+ \int_{-t}^{t} f_j(\varPhi) \partial/\partial\tau [\beta_j^+ R^+ - \beta_j^- R^-] d\tau .$$

Thus (4. 4) becomes

(4. 5) $\mathcal{L}u = f_{-1}(\varphi^+) A_+ a_0^+ + f_{-1}(\varphi^-) A_- a_0^- + \sum_{j=0}^{\infty} f_j(\varphi^+) [A_+ a_{j+1}^+$

$$+ \mathcal{L}(a_j^+) + 2\beta_j^-(t, x, t) R^-] + \sum_{j=0}^{\infty} f_j(\varphi^-) [A_- a_{j+1}^- + \mathcal{L}(a_j^-)$$

$$+ 2\beta_j^+(t, x, -t) R^+] + \sum_{j=0}^{\infty} \int_{-t}^{t} f_j(\varPhi) \{\mathcal{L}(b_j) + \partial/\partial\tau (\beta_j^+ R^+$$

$$- \beta_j^- R^-)\} d\tau + \sum_{l=1}^{k-2} f_{-1}(\varphi^l) A_l a_0^l + \sum_{l=1}^{k-2} \sum_{j=0}^{\infty} f_j(\varphi^l) [A_l a_{j+1}^l$$

$$+ \mathcal{L}(a_j^l)].$$

Next we absorb $\int_{-t}^{t} f_j(\mathbf{\Phi})\,\{*\}\,d\tau$ into $f_{j+1}(\varphi^{\pm})\,[*]$ by the following method. We require the coefficient of $f_0(\mathbf{\Phi})$ equals

$$2\mathbf{\Phi}_\tau\{(L^+\cdot\partial R^-/\partial\xi_\mu)\cdot\partial\beta_0^-/\partial x_\mu\cdot R^+ - (L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_0^+/\partial x_\mu\cdot R^-\}.$$

Integrate

$$\int_{-t}^{t} 2f_0(\mathbf{\Phi})\,\mathbf{\Phi}_\tau\{(L^+\cdot\partial R^-/\partial\xi_\mu)\cdot\partial\beta_0^-/\partial x_\mu\cdot R^+$$

$$- (L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_0^+/\partial x_\mu\cdot R^-\}\,d\tau$$

by parts and incorporate the results into $f_1(\varphi^\pm)\,[*]$ and $\int_{-t}^{t} f_1(\mathbf{\Phi})\,\{\,*\,\}\,d\tau$ of (4.5). Then we have

(4.6) $\mathcal{L}u = f_{-1}(\varphi^+)\,A_+a_0^+ + f_{-1}(\varphi^-)\,A_-a_0^- + f_0(\varphi^+)\,[A_+a_1^+ + \mathcal{L}(a_0^+)$

$\qquad + 2\beta_0^+(t,x,t)\,R^-] + f_1(\varphi^+)\,[A_+a_2^+ + \mathcal{L}(a_1^+) + 2\beta_1^-(t,x,t)\,R^-$

$\qquad + 2(L^+\cdot\partial R^-/\partial\xi_\mu)\cdot\partial\beta_0^-/\partial x_\mu(t,x,t)\,R^+ - 2(L^-\cdot\partial R^+/\partial\xi_\mu)$

$\qquad \times \partial\beta_0^+/\partial x_\mu(t,x,t)\,R^-] + \sum_{j=2}^{\infty} f_j(\varphi^+)\,[A_+a_{j+1}^+ + \mathcal{L}(a_j^+)$

$\qquad + 2\beta_j^-(t,x,t)\,R^-] + f_0(\varphi^-)[A_-a_1^- + \mathcal{L}(a_0^-) + 2\beta_0^+(t,x,-t)\,R^+]$

$\qquad + f_1(\varphi^-)[A_-a_2^- + \mathcal{L}(a_1^-) + 2\beta_1^+(t,x,-t)\,R^+ - 2(L^+\cdot\partial R^-/\partial\xi_\mu)$

$\qquad \times \partial\beta_0^-/\partial x_\mu(t,x,-t)R^+ + 2(L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_0^+/\partial x_\mu(t,x,-t)R^-]$

$\qquad + \sum_{j=2}^{\infty} f_j(\varphi^-)\,[A_-a_{j+1}^- + \mathcal{L}(a_j^-) + 2\beta_j^+(t,x,-t)\,R^+]$

$\qquad + \int_{-t}^{t} f_1(\mathbf{\Phi})\,[\mathcal{L}(b_1) + \partial/\partial\tau\,(\beta_1^+R^+ - \beta_1^-R^-) + 2\cdot\partial/\partial\tau\{-(L^+$

$\qquad \times \partial R^-/\partial\xi_\mu)\cdot\partial\beta_0^-/\partial x_\mu\cdot R^+ + (L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_0^+/\partial x_\mu\cdot R^-]\,d\tau$

$\qquad + \sum_{j=2}^{\infty} \int_{-t}^{t} f_j(\mathbf{\Phi})\,\{\mathcal{L}(b_j) + \partial/\partial\tau\,(\beta_j^+R^+ - \beta_j^-R^-)\}\,d\tau$

$\qquad + \sum_{l=1}^{k-2} f_{-1}(\varphi^l)\,A_l a_0^l + \sum_{l=1}^{k-2}\sum_{j=0}^{\infty} f_j(\varphi^l)\,[A_l a_{j+1}^l + \mathcal{L}(a_j^l)].$

Generally, we require the coefficient of $f_j(\mathbf{\Phi})$ equals

$$2\mathbf{\Phi}_\tau\{(L^+\cdot\partial R^-/\partial\xi_\mu)\,\partial\beta_j^-/\partial x_\mu\cdot R^+ - (L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu\cdot R^-\}.$$

Integrate

$$\int_{-t}^{t} 2f_j(\varPhi)\,\varPhi_\tau\{(L^+\cdot\partial R^-/\partial\xi_\mu)\cdot\partial\beta_j^-/\partial x_\mu\cdot R^+$$

$$-(L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu\cdot R^-\}d\tau$$

by parts and incorporate the results into $f_{j+1}(\varphi^\pm)\,[*]$ and $\int_{-t}^{t}f_{j+1}(\varPhi)\,[\;*\;]d\tau$ of (4. 6). Then the coefficients of $f_{j+1}(\varphi^\pm)$, $f_{j+1}(\varPhi)$ in (4. 6) become

$(4. 7)_+$ $A_+a_{j+2}^+ + \mathcal{L}(a_{j+1}^+) + 2\beta_{j+1}^-(t, x, t)R^- + 2(L^+\cdot\partial R^-/\partial\xi_\mu)$

$$\times\partial\beta_j^-/\partial x_\mu(t, x, t)R^+ - 2(L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu(t, x, t)R^- ,$$

$(4. 7)_-$ $A_-a_{j+2}^- + \mathcal{L}(a_{j+1}^-) + 2\beta_{j+1}^+(t, x, -t)R^+ - 2(L^+\cdot\partial R^-/\partial\xi_\mu)$

$$\times\partial\beta_j^-/\partial x_\mu(t, x, -t)R^+ + 2(L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu(t, x, -t)R^- ,$$

$\mathcal{L}(b_{j+1}) + \partial/\partial\tau\,(\beta_{j+1}^+R^+ - \beta_{j+1}^-R^-) + 2\cdot\partial/\partial\tau\{-(L^+\cdot\partial R^-/\partial\xi_\mu)$

$$\times\partial\beta_j^-/\partial x_\mu\cdot R^+ + (L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu\cdot R^-$$

respectively. Here we used the convention $\beta_j^\pm = 0$ $(j<0)$.

Now we require the conditions:

$$(4. 8)\quad\begin{cases} L^+\{\mathcal{L}(a_{j+1}^\pm) + 2\beta_{j+1}^\mp(t, x, \pm t)R^\mp \pm 2(L^+\cdot\partial R^-/\partial\xi_\mu) \\ \qquad \times\partial\beta_j^-/\partial x_\mu(t, x, \pm t)R^+ \\ \qquad \mp 2(L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu(t, x, \pm t)R^-\} = 0 , \\ L^-\{\mathcal{L}(a_{j+1}^\pm) + 2\beta_{j+1}^\mp(t, x, \pm t)R^\mp \pm 2(L^+\cdot\partial R^-/\partial\xi_\mu) \\ \qquad \times\partial\beta_j^-/\partial x_\mu(t, x, \pm t)R^+ \\ \qquad \mp 2(L^-\cdot\partial R^+/\partial\xi_\mu)\cdot\partial\beta_j^+/\partial x_\mu(t, x, \pm)R^-\} = 0 \end{cases}$$

to hold everywhere in order to guarantee the existence of $a_{j+2}^\pm$ which satisfy $(4. 7)_\pm = 0$ even if $A_\pm$ degenerate.

As for the coefficients of $f_j(\varphi^l)$ $(1\leq l\leq k-2)$, we argue in the same way as [4]. Then taking account of the initial condition (2. 5), we obtain the sufficient conditions on $a_j^\pm$, a_j^l, $\beta_j^\pm$ in order that (4. 2) will be an asymptotic solution. Namely, with the conventions $\beta_j^\pm(t, x, \tau) = 0$ $(j<0)$, $h_j^\pm(t, x) = h_j^l(t, x) = 0$ $(j<1)$, we obtain

$$(4. 9)\quad\begin{cases} a_j^\pm(t, x) = \alpha_j^\pm(t, x)R^\pm(t, x; \nabla_x\varphi^\pm(t, x)) + h_j^\pm(t, x) , \\ a_j^l(t, x) = \alpha_j^l(t, x)R^l(t, x; \nabla_x\varphi^l(t, x)) + h_j^l(t, x) , \end{cases}$$

$$(4.10)_+ \begin{cases} \partial/\partial t \ \alpha_j{}^+ - \nabla_\xi \lambda^+ \cdot \nabla_x \alpha_j{}^+ + L^+ \mathcal{L}(R^+) \alpha_j{}^+ = -L^+ \mathcal{L}(h_j{}^+) \\ \quad - 2(L^+ \cdot \partial R^- / \partial \xi_\mu) \cdot \partial \beta_{j-1}^- / \partial x_\mu(t, x, t), \\ 2\beta_j{}^-(t, x, t) = -L^- \mathcal{L}(\alpha_j{}^+ R^+ + h_j{}^+) + 2(L^- \cdot \partial R^+ / \partial \xi_\mu) \\ \quad \times \partial \beta_{j-1}^+ / \partial x_\mu(t, x, t), \end{cases}$$

$$(4.10)_- \begin{cases} \partial/\partial t \ \alpha_j{}^- - \nabla_\xi \lambda^- \cdot \nabla_x \alpha_j{}^- + L^- \mathcal{L}(R^-) \alpha_j{}^- = -L^- \mathcal{L}(h_j{}^-) \\ \quad - 2(L^- \cdot \partial R^+ / \partial \xi_\mu) \cdot \partial \beta_{j-1}^+ / \partial x_\mu(t, x, -t), \\ 2\beta_j{}^+(t, x, -t) = -L^+ \mathcal{L}(\alpha_j{}^- R^- + h_j{}^-) + 2(L^+ \cdot \partial R^- / \partial \xi_\mu) \\ \quad \times \partial \beta_{j-1}^- / \partial x_\mu(t, x, -t), \qquad \cdot \end{cases}$$

$$(4.11) \qquad \partial/\partial t \ \alpha_j{}^l - \nabla_\xi \lambda^l \cdot \nabla_x \alpha_j{}^l + L^l \mathcal{L}(R^l) \alpha_j{}^l = -L^l \mathcal{L}(h_j{}^l),$$

$$(4.12) \begin{cases} (\partial/\partial t + \partial/\partial \tau) \beta_j{}^+ - \nabla_\xi \lambda^+ \cdot \nabla_x \beta_j{}^+ + L^+ (\partial/\partial \tau \ R^+ + \mathcal{L}(R^+)) \beta_j{}^+ \\ \quad + L^+ (-\partial/\partial \tau \ R^- + \mathcal{L}(R^-)) \beta_j{}^- = 2L^+ \cdot \partial/\partial \tau \{(L^+ \cdot \partial R^- / \partial \xi_\mu) \\ \quad \times \partial \beta_{j-1}^- / \partial x_\mu \cdot R^+ - (L^- \cdot \partial R^+ / \partial \xi_\mu) \partial \beta_{j-1}^+ / \partial x_\mu \cdot R^- \}, \\ (\partial/\partial t - \partial/\partial \tau) \beta_j{}^- - \nabla_\xi \lambda^- \cdot \nabla_x \beta_j{}^+ + L^- (\partial/\partial \tau \ R^+ + \mathcal{L}(R^+)) \beta_j{}^+ \\ \quad + L^- (-\partial/\partial \tau \ R^- + \mathcal{L}(R^-)) \beta_j{}^- = 2L^- \cdot \partial/\partial \tau \{(L^+ \cdot \partial R^- / \partial \xi_\mu) \\ \quad \times \partial \beta_{j-1}^- / \partial x_\mu \cdot R^+ - (L^- \cdot \partial R^+ / \partial \xi_\mu) \cdot \partial \beta_{j-1}^+ / \partial x_\mu \cdot R^- \}, \end{cases}$$

$$(4.13) \qquad \alpha_j{}^+(0, x) R^+(0, x; 1, 0, \cdots, 0) + \alpha_j{}^-(0, x) R^-(0, x; 1, 0, \cdots, 0)$$

$$+ \sum_{l=1}^{k-2} \alpha_j{}^l(0, x) R^l(0, x; 1, 0, \cdots, 0)$$

$$= \begin{cases} w^+(x') R^+(0, x; 1, 0, \cdots, 0) & \text{for } j = 0, \\ -\{h_j{}^+(0, x) + h_j{}^-(0, x) + \sum_{l=1}^{k-2} h_j{}^l(0, x)\} & \text{for } j \geq 1, \end{cases}$$

where $\{L^1(t, x; \xi), \cdots, L^{k-2}(t, x; \xi), L^+(t, x; \xi), L^-(t, x; \xi)\}$ is the dual basis of $\{R^1(t, x; \xi), \cdots, R^{k-2}(t, x; \xi), R^+(t, x; \xi), R^-(t, x; \xi)\}$ and $h_j{}^\pm(t, x)$, $h_j{}^l(t, x)$ are particular solutions of

$$A_\pm a_j{}^\pm + \mathcal{L}(a_{j-1}^\pm) + 2\beta_{j-1}^\mp(t, x, \mp t) R^\pm$$

$$= \mp 2(L^+ \cdot \partial R^- / \partial \xi_\mu) \cdot \partial \beta_{j-2}^- / \partial x_\mu(t, x, \pm t) R^+$$

$$\pm 2(L^- \cdot \partial R^+ / \partial \xi_\mu) \cdot \partial \beta_{j-2}^+ / \partial x_\mu(t, x, \pm t) R^-$$

and

$$A_l a_j{}^l + \mathcal{L}(a_{j-1}^l) = 0$$

respectively.

Next we investigate more closely how $h_j{}^\pm$ and $h_j{}^l$ are determined by $\alpha_k{}^\pm$, $\alpha_k{}^l$, $\beta_k{}^\pm$ $(k \leq j-1)$. For this purpose we prove the following lemma.

Lemma 4. 1. *Set*

$$(4.14) \quad h_j{}^\pm(t, x) = \sum_{m=1}^{k-2} \sigma_{j,m}^\pm(t, x) R^m(t, x; \nabla_x \varphi^\pm(t, x)),$$

$$(4.15) \quad h_j{}^l(t, x) = \sum_{m=1, m \neq l}^{k-2} \sigma_{j,m}^l(t, x) R^m(t, x; \nabla_x \varphi^l(t, x)) + \sigma_{j,k-1}^l(t, x)$$

$$\times R^+(t, x; \nabla_x \varphi^l(t, x)) + \sigma_{j,k}^l(t, x) R^-(t, x; \nabla_x \varphi^l(t, x)).$$

Then the following recurrence relations hold. Namely, with the conventions $\alpha_j{}^\pm = \alpha_j{}^l = 0$ $(j < 0)$ *and* $\sigma_{j,m}^\pm = \sigma_{j,m}^l = 0$ $(j < 1)$,

$$(4.16) \quad \begin{cases} \sigma_{j,m}^\pm = M_{m,\pm}^1(t, x; \partial_x) \alpha_{j-1}^\pm + \sum_{h=1}^{k-2} N_{m,h,\pm}^1(t, x; \partial_t, \partial_x) \sigma_{j-1,h}^\pm, \\ \sigma_{j,m}^l = M_{m,l}^1(t, x; \partial_x) \alpha_{j-1}^\pm + \sum_{h=1, h \neq l}^{k} N_{m,h,l}^1(t, x; \partial_t, \partial_x) \sigma_{j-1,h}^l \end{cases}$$

hold for $j \geq 1$, *where* $M_{m,\pm}^p(t, x; \partial_x)$ *etc. denote holomorphic linear partial differential operators of order* $\leq p$ *with respect to the differentials indicated in the bracket and are independent of* j.

Proof. Taking account of (4.8), we have

$$(4.17) \quad A_\pm a_j{}^\pm = -\mathcal{L}(a_{j-1}^\pm) - 2\beta_{j-1}^\mp(t, x, \pm t) R^\mp \mp 2(L^+ \cdot \partial R^-/\partial \xi_\mu)$$

$$\times \partial \beta_{j-2}^-/\partial x_\mu(t, x, \pm t) R^+ \pm 2(L^- \cdot \partial R^+/\partial \xi_\mu)$$

$$\times \partial \beta_{j-2}^\pm/\partial x_\mu(t, x, \pm t) R^- = \sum_{m=1}^{k-2} \kappa_{j,m}^\pm(t, x) R^m(t, x; \nabla_x \varphi^\pm(t, x))$$

where $\kappa_{j,m}^\pm$ $(1 \leq m \leq k-2)$ are holomorphic functions defined in a neighborhood of the origin. On the other hand, apply $A_\pm$ to $h_j{}^\pm$. Then

$$(4.18) \quad A_\pm h_j{}^\pm(t, x) = \sum_{m=1}^{k-2} (\lambda^\pm(t, x; \nabla_x \varphi^\pm(t, x))$$

$$- \lambda^m(t, x; \nabla_x \varphi^\pm(t, x)) \kappa_{j,m}^\pm(t, x) R^m(t, x; \nabla_x \varphi^\pm(t, x))$$

because $R^m(t, x; \nabla_x \varphi^{\pm}(t, x))$ is the eigenvector of $-A^{\mu}(t, x) \varphi_{\mu}^{=}$ with the eigenvalue $\lambda^m(t, x; \nabla_x \varphi^{\pm}(t, x))$. Here $\lambda^{\pm}(t, x; \nabla_x \varphi^{\pm}(t, x)) - \lambda^m(t, x; \nabla_x \varphi^{\pm}(t, x)) \neq 0$ $(1 \leq m \leq k-2)$ for $(t, x) \sim 0$ by the assumption (II). Thus comparing the two equations (4. 17) and (4. 18), we have (4. 19) $\sigma_{j,m}^{\pm}(t, x) = \kappa_{j,m}^{\pm}(t, x) / (\lambda^{\pm} - \lambda^m)$.

Now it is almost clear from (4. 17) that $\kappa_{j,m}^{\pm}(t, x)$ is given by

$$\kappa_{j,m}^{\pm} = M_{m,\pm}^1(t, x; \partial_x) \alpha_{j-1}^{\pm} + \sum_{h=1}^{k-2} N_{m,h,\pm}^1(t, x; \partial_t, \partial_x) \sigma_{j-1,h}^{\pm},$$

where $M_{m,\pm}^1(t, x; \partial_x)$ etc. denote the same kind of operators mentioned above. Thus we obtain the first equation of (4. 16) if we combine this with (4. 19).

In a similar manner we can prove the second equation of (4. 16). We omit its proof.

Using (4. 14) and (4. 15), we can rewrite the recurrence relations (4. 9), (4. 11) as follows. Namely,

$$(4. 20)_+ \quad \partial/\partial t \; \alpha_j^+ = U_+^1(t, x; \partial_x) \alpha_j^+ + \sum_{m=1}^{k-2} V_{m,+}^1(t, x; \partial_x) \sigma_{j,m}^+$$
$$+ W_+^1(t, x; \partial_x) \beta_{j-1}^-(t, x, t),$$

$$(4. 20)_- \quad \partial/\partial t \; \alpha_j^- = U_-^1(t, x; \partial_x) \alpha_j^- + \sum_{m=1}^{k-2} V_{m,-}^1(t, x; \partial_x) \sigma_{j,m}^-$$
$$+ W_-^1(t, x; \partial_x) \beta_{j-1}^+(t, x, -t),$$

$$(4. 20)_l \quad \partial/\partial t \; \alpha_j^l = U_l^1(t, x; \partial_x) \alpha_j^l + \sum_{m=1, m \neq l}^{k} V_{m,l}^1(t, x; \partial_x) \sigma_{j,m}^l,$$

where $U_{\pm}^p(t, x; \partial_x)$ etc. are holomorphic linear partial differential operators of order $\leq p$ with respect to the differentials indicated in the bracket and are independent of j.

Further the initial conditions (4. 13) of $\alpha_j^{\pm}$ and α_j^l become

$$(4. 21)_+ \quad \alpha_j^+(0, x) = \begin{cases} w^+(x') & \text{for } j=0, \\ -\sum_{l=1}^{k-2} \sigma_{j,k-1}^l(0, x) & \text{for } j \geq 1, \end{cases}$$

$$(4. 21)_- \quad \alpha_j^-(0, x) = \begin{cases} 0 & \text{for } j=0, \\ -\sum_{l=1}^{k-2} \sigma_{j,k}^l(0, x) & \text{for } j \geq 1, \end{cases}$$

$$(4.21)_l \quad \alpha_j{}^l(0, x) = \begin{cases} 0 & \text{for} \quad j=0, \\ -\left(\sum_{m=1, m\neq l}^{k-2} \sigma_{j,l}^m(0, x) + \sigma_{j,l}^+(0, x) + \sigma_{j,l}^-(0, x) \right) & \text{for} \quad j\geq 1. \end{cases}$$

Here, we introduce a new independent variables $t_1 = 1/2 \cdot (t+\tau)$, $t_2 = 1/2 \cdot (t-\tau)$ and rewrite the equation (4.12) into a canonical form. Namely, using the same symbols $\beta^\perp$ for their (t_1, x, t_2)-space interpretation, we have

$$(4.22) \quad \begin{cases} \partial/\partial t_1 \ \beta_j{}^+ = K_+{}^1(t_1, x, t_2; \partial_x)\beta_j{}^+ + L_+{}^0(t_1, x, t_2)\beta_j{}^- \\ \qquad + P_+{}^2(t_1, x, t_2; \partial_{t_1}, \partial_x, \partial_{t_2})\beta_{j-1}^- + Q_+{}^1(t_1, x, t_2; \partial_x)\beta_{j-1}^+, \\ \partial/\partial t_2 \ \beta_j{}^- = K_-{}^1(t_1, x, t_2; \partial_x)\beta_j{}^- + L_-{}^0(t_1, x, t_2)\beta_j{}^+ \\ \qquad + P_-{}^2(t_1, x, t_2; \partial_{t_1}, \partial_x, \partial_{t_2})\beta_{j-1}^+ + Q_-{}^1(t_1, x, t_2; \partial_x)\beta_{j-1}^-, \\ \beta_j{}^+(0, x, t_2) = G_+{}^1(t_2, x; \partial_x)\alpha_j{}^-(t_2, x) \\ \qquad + \sum_{m=1}^{k-2} H_{m,+}^1(t_2, x; \partial_x)\sigma_{j,m}^-(t_2, x) + J_+{}^1(t_2, x; \partial_x)\beta_{j-1}^-(0, x, t_2), \\ \beta_j{}^-(t_1, x, 0) = G_-{}^1(t_1, x; \partial_x)\alpha_j{}^+(t_1, x) \\ \qquad + \sum_{m=1}^{k-2} H_{m,-}^1(t_1, x; \partial_x)\sigma_{j,m}^+(t_1, x) + J_-{}^1(t_1, x; \partial_x)\beta_{j-1}^+(t_1, x, 0), \end{cases}$$

where $K_\pm(t_1, x, t_2; \partial_x)$ etc. are differential operators of total order $\leq p$ whose order with respect to the variables t_1, t_2 are always not greater than one.

For convenience, let us introduce a definition.

Definition. We say that a linear partial differential operator majorizes another operator if each coefficient of the former operator majorizes the corresponding coefficient of the latter operator.

Now, the following lemma guarantees that $\alpha_j{}^\perp$, $\alpha_j{}^l$ and $\beta_j{}^\perp$ are determined successively by $(4.20)_{\pm,l}$, $(4.21)_\pm$ and (4.22).

Lemma 4.2.

$1°$ *There exists in a neighborhood of the origin a unique holomorphic solution $\alpha_j{}^\perp$ (resp. $\alpha_j{}^l$) of the Cauchy problem $(4.20)_\pm$ (resp. $(4.20)_l$) with the initial conditions $(4.21)_\pm$ (resp. $(4.21)_l$). Moreover, let the operators $\widetilde{U}_\pm{}^1$ (resp. $\widetilde{U}_l{}^1$) and the functions $F_j{}^\pm$ (resp. $F_j{}^l$)*

are the respective majorants of $U_\pm{}^1(\text{resp. } U_l{}^1)$ *and* $\sum_{m=1}^{k-2} V_{m,\pm}^1 \sigma_{j,m}^\pm + W_\pm{}^1 \beta_{j-1}^\pm$
(resp. $\sum_{m=1, m\neq l}^{k} V_{m,l}^1 \sigma_{j,m}^l$*).* *Then* $\tilde{\alpha}_j{}^\pm \gg \alpha_j{}^\pm$, $\tilde{\alpha}_j{}^l \gg \alpha_j{}^l$ *if* $\tilde{\alpha}_j{}^\pm$, $\tilde{\alpha}_j{}^l$ *satisfy the following conditions*:

$$(4.23)_\pm \quad \begin{cases} \partial/\partial t \ \tilde{\alpha}_j{}^\pm \gg \tilde{U}_\pm{}^1(t, x; \partial_x)\tilde{\alpha}_j{}^\pm + F_j{}^\pm(t, x), \\ \tilde{\alpha}_j{}^\pm(0, x) \gg \alpha_j{}^\pm(0, x), \end{cases}$$

$$(4.23)_l \quad \begin{cases} \partial/\partial t \ \tilde{\alpha}_j{}^l \gg \tilde{U}_l{}^1(t, x; \partial_x)\tilde{\alpha}_j{}^l + F_j{}^l(t, x), \\ \tilde{\alpha}_j{}^l(0, x) \gg \alpha_j{}^l(0, x). \end{cases}$$

2° *There exists in a neighborhood of the origin a unique holomorphic solution* $\beta_j{}^\pm$ *of the Cauchy problem* (4.22). *Moreover, let the operator* $\tilde{K}_\pm{}^1$, $\tilde{L}_\pm{}^0$ *and functions* $S_j{}^\pm$ *are majorants of* $K_\pm{}^1$, $L_\pm{}^0$ *and* $P_\pm{}^2\beta_{j-1}^\pm + Q_\pm{}^1\beta_{j-1}^\pm$. *Then* $\tilde{\beta}_j{}^\pm \gg \beta_j{}^\pm$ *if* $\tilde{\beta}_j{}^\pm$ *satisfy the following conditions*:

$$(4.24) \quad \begin{cases} \partial/\partial t_1 \ \tilde{\beta}_j{}^+ \gg \tilde{K}_+{}^1(t_1, x, t_2; \partial_x)\tilde{\beta}_j{}^+ + \tilde{L}_+{}^0(t_1, x, t_2)\tilde{\beta}_j{}^- + S_j{}^+, \\ \partial/\partial t_2 \ \tilde{\beta}_j{}^- \gg \tilde{K}_-{}^1(t_1, x, t_2; \partial_x)\tilde{\beta}_j{}^- + \tilde{L}_-{}^0(t_1, x, t_2)\tilde{\beta}_j{}^+ + S_j{}^-, \\ \tilde{\beta}_j{}^+(0, x, t_2) \gg \beta_j{}^+(0, x, t_2), \quad \tilde{\beta}_j{}^-(t_1, x, 0) \gg \beta_j{}^-(t_1, x, 0). \end{cases}$$

Proof. Since 1° is well known, we omit its proof. As for 2°, the first part can be proved by iteration process. In order to prove the second part expand $\beta_j{}^\pm$ into a power series: $\beta_j{}^\pm(t_1, x, t_2) = \sum_{l,m=0}^\infty \beta_{l,m}^\pm(x) \times t_1{}^l t_2{}^m$. It is enough to show that $\beta_{l,m}^\pm(x)$ can be determined successively from (4.22). However, this is easily proved by the double induction with respect to the index l and m. Since the argument is elementary, we omit further illustration.

Consequently, we have constructed an asymptotic solution (4.2).

§5. The Exactness of the Formal Solution

Since the formal solution (4.2) satisfies the equation (2.1) and the initial condition (2.5), it remains to prove its convergence and uniqueness. However the uniqueness follows from the Cauchy-Kowalevsky theorem. We only have to prove its convergence.

If we calculate formally reminding the definition of the wave forms, we can rewrite the formal (asymptotic) solution (4. 2) in the form:

$$(5.1) \quad u(t, x) = F^{+}(t, x)/(\varphi^{+})^{q} + (\log \varphi^{+}) \sum_{j=q}^{\infty} (\varphi^{+})^{j-q} a_{j}^{+}/(j-q)!$$

$$- \sum_{j=q+1}^{\infty} A_{j-q}(\varphi^{+})^{j-q} a_{j}^{+}/(j-q)! + F^{-}(t, x)/(\varphi^{-})^{q}$$

$$+ (\log \varphi^{-}) \sum_{j=q}^{\infty} (\varphi^{-})^{j-q} a_{j}^{-}/(j-q)! - \sum_{j=q+1}^{\infty} A_{j-q}(\varphi^{-})^{j-q} a_{j}^{-}/(j-q)!$$

$$+ \sum_{l=1}^{k-2} \{ F^{l}(t, x)/(\varphi^{l})^{q} + (\log \varphi^{l}) \sum_{j=q}^{\infty} (\varphi^{l})^{j-q} a_{j}^{l}/(j-q)!$$

$$- \sum_{j=q+1}^{\infty} A_{j-q}(\varphi^{l})^{j-q} a_{j}^{l}/(j-q)! \} + \int_{-t}^{t} F(t, x, \tau)/\Phi^{q} \, d\tau$$

$$+ \int_{-t}^{t} \{ (\log \Phi) \sum_{j=q}^{\infty} \Phi^{j-q} b_{j}/(j-q)! - \sum_{j=q+1}^{\infty} A_{j-q} \Phi^{j-q} b_{j}/(j-q)! \} d\tau$$

where $F^{\pm}(t, x)$, $F^{l}(t, x)$ and $F(t, x, \tau)$ are vector functions which are holomorphic in a neighborhood of the origin.

We have by a simple calculation $A_{j-q}/(j-q)! \leq 1/(j-q)! + 1/(j-q-1)!$ for $j \geq q+1$. Thus it is enough to show that

$$\sum_{j=q}^{\infty} (\varphi^{\pm})^{j-q} a_{j}^{\pm}/(j-q)!, \qquad \sum_{j=q+1}^{\infty} (\varphi^{\pm})^{j-q} a_{j}^{\pm}/(j-q-1)!,$$

$$\sum_{j=q}^{\infty} (\varphi^{l})^{j-q} a_{j}^{l}/(j-q)!, \qquad \sum_{j=q+1}^{\infty} (\varphi^{l})^{j-q} a_{j}^{l}/(j-q-1)!,$$

$$\sum_{j=q}^{\infty} \Phi^{j-q} b_{j}/(j-q)!, \qquad \sum_{j=q+1}^{\infty} \Phi^{j-q} b_{j}/(j-q-1)!$$

converge absolutely and uniformly in a neighborhood of the origin.

Next we prepare some lemmas which are necessary to estimate $\alpha_{j}^{\pm}$, α_{j}^{l}, $\sigma_{j, m}^{\pm}$, $\sigma_{j, m}^{l}$ and $\beta_{j}^{\pm}$.

First we introduce a family of functions $\{\varphi_{j}(\zeta)\}_{j=0}^{\infty}$ originated by C. Wagschal [6] to establish successive majorants of functions defined by recurrence relations.

Lemma 5. 1. (C. Wagschal [6]). Let $\varphi_{j}(\zeta) = (d/d\zeta)^{j} \, 1/(r-\zeta)$ $= j!/(r-\zeta)^{j+1}$ for a constant $r > 0$ and nonnegative integers j. Then $\varphi_{j}(\zeta)$ $(j \geq 0)$ have the following properties.

$1°$ $\varphi_j(\zeta) \ll r\varphi_{j+1}(\zeta)/(j+1)$.

$2°$ Let $\zeta = \rho t + \sum_\mu x_\mu$ (resp. $\zeta = \rho(t_1 + t_2) + \sum_\mu x_\mu$), then $\partial/\partial t \ \varphi_j(\zeta) = \rho\varphi_{j+1}(\zeta)$ (resp. $\partial/\partial t_1 \ \varphi_j(\zeta) = \rho\varphi_{j+1}(\zeta)$, $\partial/\partial t_2 \ \varphi_j(\zeta) = \rho\varphi_{j+1}(\zeta)$).

$3°$ Let $Q^p(t,x;\partial_t,\partial_x)$ (resp. $Q^p(t_1,x,t_2;\partial_{t_1},\partial_x,\partial_{t_2})$) be a linear partial differential operator whose respective orders with respect to the variables (t,x) (resp. (t_1,x,t_2)) and t (resp. (t_1,t_2)) are p and q. We assume that the coefficients of Q^p are holomorphic in the polydisk $\{(t,x); |t|, |x_\mu| \leq R\}$ (resp. $\{(t_1,x,t_2); |t_1|, |t_2|, |x_\mu| \leq R\}$) and are majorized by

$$M/\{R - (t + \sum_\mu x_\mu)\} \quad (resp. \ M/\{R - (t_1 + t_2 + \sum_\mu x_\mu)\}).$$

Then there exists a constant $C_0 > 0$ which depends only upon $M > 0$, $R > 0$, r and p such that

$$v \ll \varphi_j(\zeta), \quad \zeta = \rho t + \sum_\mu x_\mu \ (resp. \ \zeta = \rho(t_1 + t_2) + \sum_\mu x_\mu), \quad \rho \geq 1$$

imply

$$Q^p v \ll C_0 \rho^q \varphi_{j+p}(\zeta).$$

Remark.   Henceforth, for convenience, we simply put the sign "$\sim$" on the symbols to denote their majorants both for functions and operators. Here we assume that $\widetilde{M}^1_{m,\pm}$, $\widetilde{U}_\pm{}^1$, $\widetilde{K}$ etc. simultaneously satisfy the property $3°$ of Lemma 5.1.   We also make a remark on the property $1°$.   That is the factor $(j+1)^{-1}$ which was neglected in [6] is very important for our calculations.

The following lemma can be easily proved.

Lemma 5.2.   Let $\varphi(\zeta)$ $(\zeta \in C^1)$ and $f(t,x,\tau)$ are functions which are holomorphic in a neighborhood of the origin.   Put

$$\tilde{f}(t_1,x,t_2) = f(t_1 + t_2, x, t_1 - t_2).$$

Then we have the followings.

$1°$ $\tilde{f}(t_1,x,t_2) \ll \varphi(\rho(t_1 + t_2) + \sum_\mu x_\mu)$ implies $f(t,x,\pm t) \ll \varphi(\rho t + \sum_\mu x_\mu)$.

$2°$ $f(t, x, t) \ll \varphi(\rho t + \sum_\mu x_\mu)$ implies $\tilde{f}(t_1, x, 0) \ll \varphi(\rho t_1 + \sum_\mu x_\mu)$.

$3°$ $f(t, x, -t) \ll \varphi(\rho t + \sum_\mu x_\mu)$ implies $\tilde{f}(0, x, t_2) \ll \varphi(\rho t_2 + \sum_\mu x_\mu)$.

Now we are ready to prove the following key lemma.

Lemma 5. 3. *There exist constants $C_1 > 0$ and $\rho \geq 1$ such that*

$$(5.2) \quad \begin{cases} \sigma_{j+1,m}^{\pm} \ll C_0 C_1^{2j+2} \varphi_{2j+1}/j!, \quad \sigma_{j+1,m}^l \ll C_0 C_1^{2j+2} \varphi_{2j+1}/j!, \\ \alpha_j^{\pm} \ll C_1^{2j+1} \varphi_{2j}/j!, \quad \alpha_j^l \ll C_1^{2j+1} \varphi_{2j}/j!, \quad \beta_j^{\pm} \ll C_1^{2j+2} \varphi_{2j+1}/j! \end{cases}$$

for $0 < r < R$ and $j \geq 0$. Here φ_j stands both for $\varphi_j(\rho t + \sum_\mu x_\mu)$ and $\varphi_j(\rho(t_1 + t_2) + \sum_\mu x_\mu)$.

Remark. The estimates (5. 2) will hold, for example, if C_1 and ρ satisfy the following conditions:

$$C_1 \geq \max(M, C_0 R(k-1)/2), \quad C_0\{1 + C_0 R(k-1)/(2C_1)\} \leq \rho,$$

$$C_0\{1 + C_0 R(k-1)/(2C_1)\} \leq C_1, \quad C_0\{1 + R/2 + R(R+4\rho)/(8C_1^2)\} \leq \rho,$$

$$1/C_1 + C_0 \rho R(k-1)/(2C_1^2) \leq 1.$$

We assume these conditions and $C_0 \geq 1$ in the proof.

Proof. We follow the way of C. Wagschal [6; p388~390]. However our case is more complicated.

Set

$$\tilde{\sigma}_{j+1,m}^{\pm} = C_0 C_1^{2j+2} \varphi_{2j+1}/j!, \quad \tilde{\sigma}_{j+1,m}^l = C_0 C_1^{2j+2} \varphi_{2j+1}/j!, \quad \tilde{\alpha}_j^{\pm} = C_1^{2j+1} \varphi_{2j}/j!,$$

$$\tilde{\alpha}_j^l = C_1^{2j+1} \varphi_{2j}/j!, \quad \hat{\beta}_j^{\pm} = C_1^{2j+2} \varphi_{2j+1}/j!.$$

Then taking Lemma 4. 2 and Lemma 5. 2 into account, it is enough to prove that the following relations hold for some $C_1 > 0$ and $\rho \geq 1$. Namely,

$$(5.3)_\pm \quad \partial/\partial t \; \tilde{\alpha}_j^{\pm} \gg \tilde{U}_\pm^1 \tilde{\alpha}_j^{\pm} + \sum_{m=1}^{k-2} \tilde{V}_{m,\pm}^1 \tilde{\sigma}_{j,m}^{\pm} + \tilde{W}_\pm^1 \hat{\beta}_{j-1}^{\pm}(t, x, \pm t),$$

$$(5.3) \quad \partial/\partial t \; \tilde{\alpha}_j^l \gg \tilde{U}_l^1 \tilde{\alpha}_j^l + \sum_{m=1, m \neq l}^{k} \tilde{V}_{m,l}^1 \tilde{\sigma}_{j,m}^l,$$

$$(5.4) \quad \begin{cases} \partial/\partial t_1 \; \tilde{\beta}_j^+ \gg \tilde{K}_+^1 \tilde{\beta}_j^+ + \tilde{L}_+^0 \tilde{\beta}_j^- + \tilde{P}_+^2 \tilde{\beta}_{j-1}^- + \tilde{Q}_+^1 \tilde{\beta}_{j-1}^+, \\ \partial/\partial t_2 \; \tilde{\beta}_j^- \gg \tilde{K}_-^+ \tilde{\beta}_j^- + \tilde{L}_-^0 \tilde{\beta}_j^+ + \tilde{P}_-^2 \tilde{\beta}_{j-1}^+ + \tilde{Q}_-^1 \tilde{\beta}_{j-1}^-, \end{cases}$$

$$(5.5)_+ \quad \tilde{\alpha}_j{}^+(0,x) \gg \begin{cases} w^+(x') & \text{for } j=0, \\ \sum_{l=1}^{k-2} \tilde{\sigma}_{j,k-1}^l(0,x) & \text{for } j\geq 1, \end{cases}$$

$$(5.5)_- \quad \tilde{\alpha}_j{}^-(0,x) \gg \begin{cases} 0 & \text{for } j=0, \\ \sum_{l=1}^{k-2} \tilde{\sigma}_{j,k}^l(0,x) & \text{for } j\geq 1, \end{cases}$$

$$(5.5)_l \quad \tilde{\alpha}_j{}^l(0,x) \gg \begin{cases} 0 & \text{for } j=0, \\ \sum_{m=1, m\neq l}^{k-2} \tilde{\sigma}_{j,l}^m(0,x) + \tilde{\sigma}_{j,l}^+(0,x) + \tilde{\sigma}_{j,l}^-(0,x) & \text{for } j\geq 1, \end{cases}$$

$$(5.6) \quad \begin{cases} \tilde{\beta}_j{}^+(0,x,t_2) \gg \widetilde{G}_+{}^1 \tilde{\alpha}_j{}^-(t_2,x) + \sum_{m=1}^{k-2} \widetilde{H}_{m,+}^1 \tilde{\sigma}_{j,m}^-(t_2,x) + \tilde{J}_+{}^1 \tilde{\beta}_{j-1}^-(0,x,t_2), \\ \tilde{\beta}_j{}^-(t_1,x,0) \gg \widetilde{G}_-{}^1 \tilde{\alpha}_j{}^+(t_1,x) + \sum_{m=1}^{k-2} \widetilde{H}_{m,-}^1 \tilde{\sigma}_{j,m}^+(t_1,x) + \tilde{J}_-{}^1 \tilde{\beta}_{j-1}^+(t_1,x,0), \end{cases}$$

$$(5.7)_\pm \quad \tilde{\sigma}_{j,m}^\pm \gg \widetilde{M}_{m,\pm}^1 \tilde{\alpha}_{j-1}^\pm + \sum_{h=1}^{k-2} \widetilde{N}_{m,h,\pm}^1 \tilde{\sigma}_{j-1,h}^\pm,$$

$$(5.7)_l \quad \tilde{\sigma}_{j,m}^l \gg \widetilde{M}_{m,l}^1 \tilde{\alpha}_{j-1}^l + \sum_{h=1, h\neq l}^{k} \widetilde{N}_{m,h,l}^1 \tilde{\sigma}_{j-1,h}^l.$$

From Lemma 5.1 the right hand side of $(5.3)_\pm$ are majorized by

$$C_0\{C_1^{2j+1}\varphi_{2j+1}/j! + C_0 C_1^{2j}(k-2)\varphi_{2j}/(j-1)! + C_1^{2j}\varphi_{2j}/(j-1)!\}.$$

Since

$$\varphi_{2j}/(j-1)! \ll R/2 \cdot \varphi_{2j+1}/j! \quad \text{for} \quad j\geq 1$$

by Lemma 5.1, this is majorized by

$$C_0 C_1^{2j+1}\{1 + C_0 R(k-1)/(2C_1)\}\varphi_{2j+1}/j!.$$

On the other hand $\partial/\partial t\, \tilde{\alpha}_j{}^\pm = \rho C_1^{2j+1}\varphi_{2j+1}/j!$. Since $C_0\{1 + C_0 R(k-1)/(2C_1)\} \leq \rho$, $(5.3)_\pm$ hold. Similary we can prove $(5.3)_l$.

Using Lemma 5.1 we see that the right hand side of (5.4) are majorized by

$$C_0(C_1^{2j+2}\varphi_{2j+2}/j! + C_1^{2j+2}\varphi_{2j+1}/j! + \rho C_1^{2j}\varphi_{2j+1}/(j-1)! + C_1^{2j}\varphi_{2j}/(j-1)!).$$

Since

$$\varphi_{2j+1}/(j-1)!, \quad \varphi_{2j+1}/j! \ll R/2 \cdot \varphi_{2j+2}/j!,$$

$$\varphi_{2j}/(j-1)! \ll R^2/8 \cdot \varphi_{2j+2}/j!$$

for $j \geq 1$, $j \geq 0$ and $j \geq 1$ respectively by Lemma 5.1, this is majorized by

$$C_0 C_1^{2j+2} \{ 1 + R/2 + R(R + 4\rho) / (8C_1^2) \} \varphi_{2j+2}/j! .$$

On the other hand $\partial/\partial t_1 \, \tilde{\beta}_j^+$, $\partial/\partial t_2 \, \tilde{\beta}_j^- = \rho C_1^{2j+2} \varphi_{2j+2}/j!$. Since

$$C_0 \{ 1 + R/2 + R(R + 4\rho) / (8C_1^2) \} \leq \rho ,$$

(5. 4) holds.

Next, we assume $w^\tau(x') \ll M/(R - \sum_\mu x_\mu)$. Then

$$M/(R - \sum_\mu x_\mu) \ll Mr/R(r - \sum_\mu x_\mu) \ll \tilde{\alpha}_0^+(0, x),$$

because $C_1 \geq M$. Thus (5. 5) $_+$ holds for $j = 0$. For $j \geq 1$, observe that the right hand side of (5. 5) $_+$ is majorized by

$C_0 C_1^{2j}(k - 2) \varphi_{2j-1}/(j-1)!$. Since $\varphi_{2j-1}/(j-1)! \ll R/2 \cdot \varphi_{2j}/j!$, this is majorized by $C_0 C_1^{2j} R(k-2)/2 \cdot \varphi_{2j}/j!$. On the other hand

$$\tilde{\alpha}_j^+(0, x) = C_1^{2j+1} \varphi_{2j}/j! . \quad \text{Since } C_1 \geq C_0 R(k-1)/2, \quad (5. 5)_+$$

holds for $j \geq 1$.

Similary, we can prove (5. 5) $_-$ and (5. 5) $_l$.

As for (5. 6), the right hand side of (5. 6) are both majorized by

$$C_0 C_1^{2j+1} \varphi_{2j+1}/j! + (k-2) C_0 C_1^{2j} \varphi_{2j}/(j-1)! + C_1^{2j} \varphi_{2j}/(j-1)! .$$

Since

$$\varphi_{2j}/(j-1)! \ll R/2 \cdot \varphi_{2j+1}/j! \quad \text{for} \quad j \geq 1$$

by Lemma 5.1, this is majorized by

$$C_0 C_1^{2j+1} \{ 1 + C_0 R(k-1) / (2C_1) \} \varphi_{2j+1}/j! .$$

On the other hand $\tilde{\beta}_j^+(0, x, t_2)$, $\tilde{\beta}_j^-(t_1, x, 0) = C_1^{2j+2} \varphi_{2j+1}/j!$. Therefore reminding

$$C_0 \{ 1 + C_0 R(k-1) / (2C_1) \} \leq C_1 ,$$

(5. 6) holds.

Finally, let us investigate the validity of (5. 7) $_\pm$ and (5. 7) $_l$. By the same reasoning,

$$C_0 C_1^{2j} \{ 1/C_1 + C_0 \rho R(k-2) / (2C_1^2) \} \varphi_{2j-1}/(j-1)!$$

majorizes the right hand side of (5. 7) $_\pm$. Since

$$\tilde{\sigma}_{j,m}^{\pm} = C_0 C_1^{2j} \varphi_{2j-1}/(j-1)! \quad \text{and} \quad 1/C_1 + C_0 \rho R(k-1)/(2C_1^2) \leq 1$$

$(5.7)_{\pm}$ hold. Similary, we can prove $(5.7)_t$.

Thus we have completed the proof.

Let us return to the proof of convergence. Since the arguments are same, we only prove the convergence of $\sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}a_j^{\pm}/(j-q)!$. From (4.9),

$$\sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}a_j^{\pm}/(j-q)! = \sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}\left(\alpha_j^{\pm}R^{\pm} + \sum_{m=1}^{k-2}\sigma_{j,m}^{\pm}R^m\right)/(j-q)!\ .$$

Thus it is enough to prove the convergence of $\sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}\alpha_j^{\pm}/(j-q)!$ and $\sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}\sigma_{j,m}^{\pm}/(j-q)!$. From Lemma 5.3,

$$\sum_{j=q}^{\infty}|(\varphi^{\pm})^{j-q}\alpha_j^{\pm}|/(j-q)! \leq \sum_{j=q}^{\infty}(2j)!/j!(j-q)!\cdot(C_1/(r-\hat{\zeta}))^{2j+1}|\varphi^{\pm}|^{j-q}$$

where $\hat{\zeta} = \rho|t| + \sum_{\mu}|x_{\mu}|$. Therefore $\sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}\alpha_j^{\pm}/(j-q)!$ converges absolutely and uniformly in any compact subset of the domain $4C_1^2|\varphi^{\pm}(t,x)| < (r-\hat{\zeta})^2$. Similary, we can prove the convergence of $\sum_{j=q}^{\infty}(\varphi^{\pm})^{j-q}\sigma_{j,m}^{\pm}/(j-q)!$. Thus we have proved the exactness of the formal solution (4.2). Further, (4.2) can be rewritten in the form (2.6).

§ 6. Interpretation of the Results

In this section we investigate the behavior of the functions

$$\int_{-t}^{t}F(t,x,\tau)/(\varPhi(t,x,\tau))^q d\tau \quad \text{and} \quad \int_{-t}^{t}G(t,x,\tau)\log\varPhi(t,x,\tau)d\tau$$

given in (2.6).

Integration by parts yields

$$\int_{-t}^{t}G(t,x,\tau)\log\varPhi(t,x,\tau)d\tau = (\log\varphi^+(t,x))\int_{0}^{t}G(t,x,\sigma)d\sigma$$

$$+ (\log\varphi^-(t,x))\int_{-t}^{0}G(t,x,\sigma)d\sigma - \int_{-t}^{t}\left\{\int_{0}^{\tau}G(t,x,\sigma)d\sigma\right\}$$

$$\times \varPhi_{\tau}(t,x,\tau)/\varPhi(t,x,\tau)d\tau\ .$$

Therefore, it is enough to investigate the integral

$$I_q(t, x) = \int_{-t}^{t} F/\varPhi^q d\tau = \int_{-t}^{t} H(t, x, \tau)/(P(t, x, \tau))^q d\tau$$

where $H(t, x, \tau) = F(t, x, \tau)(\varPsi(t, x, \tau))^{-q}$.

For this purpose let us recall the well known fact "Let ω be the resultant of pseudo-polynomials D and E whose respective orders are d and e. Then there exist pseudo-polynomials A and B whose respective orders are not greater than $e-1$ and $d-1$ such that $AD+BE=\omega$ is valid." Then, using this fact we can prove the latter half of the theorem if we adapt the argument of integrating rational functions of one independent variable. Since the argument is elementary, we omit further illustrations.

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