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# A Five-Square Theorem

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It is clear that for every even integer $2n>0$ there is a natural number $s$ such that $2n$ is representable in the form

(1) $$2n=\sum_{i=1}^{s} x_i^2 \quad \text{with the condition} \quad \sum_{i=1}^{s} x_i=0\,,$$

where the $x_i$ $(1\leqq i\leqq s)$ are rational integers. We denote by $s(2n)$ for a given $2n$ the smallest possible value of such $s$. Of course, no representations of that kind are possible for odd integers.

We have evidently $2\leqq s(2n)\leqq 8$ for all $2n>0$. The purpose of this note is to prove the following

**Theorem.** *We have*

$$s(2n)\leqq 5 \quad \textit{for all} \quad 2n>0$$

*with the equality exclusively for the integers* $2n$ *of the form*

(2) $$4^k(32l+28) \quad (k\geqq 0,\ l\geqq 0).$$

The problem of determining the value of

$$\max_{n\geqq 1} s(2n)$$

has been (orally) communicated to the writer by Professor S. Hitotumatu in Kyoto University, who was led to this problem in the course of his study of 'translatable complete $l$-th power configurations.' Our result gives a satisfactory solution for the problem proposed.

It should be noted, however, that a general problem on the solvability of the system of Diophantine equations

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(3) $$n=\sum_{i=1}^{s} x_i^2\,, \quad m=\sum_{i=1}^{s} x_i\,,$$

has been treated by G. Pall [5], who showed in particular that if $s=4$ equations (3) are solvable in integers $x_i$, if and only if

$$n\equiv m \,(\mathrm{mod}\ 2), \text{ and } 4n-m^2=\text{a sum of three squares},$$

whereas if $s=5$ the conditions

$$n\equiv m \quad (\mathrm{mod}\ 2), \ 4n-m^2\geqq 0$$

are necessary and sufficient for the solvability of (3) in integers $x_i$. (The case of $s=4$ is due substantially to A. L. Cauchy [2]. See also [4].) Our theorem is an immediate consequence of these results; but, this notwithstanding, we shall present here another simple and direct proof of the theorem.

An analogue to (1) for the representation of an odd integer $2n+1>0$ will be

(4) $$2n+1=\sum_{i=1}^{s} x_i^2 \quad \text{with} \quad \sum_{i=1}^{s} x_i=1\,,$$

where the $x_i$ are again rational integers. If we denote by $s(2n+1)$ for a given $2n+1$ the smallest possible value of $s$ in the representation (4), then it can be shown that we have

$$s(2n+1)\leqq 4 \quad \text{for all} \quad 2n+1>0.$$

This result also is a special case of Pall's [5].

**1.** In order to prove the theorem we require some auxiliary results which we formulate in the following lemma (cf. e.g. [1]).

**Lemma.** *Let $m$ be a positive integer. The integer $m$ can be represented in the form*

$$m=x^2+y^2+z^2$$

*with some integers $x$, $y$, $z$, if and only if $m$ is not of the form*

(5) $$4^k(8l+7) \quad (k\geqq 0,\ l\geqq 0);$$

*the integer m can be represented in the form*

$$m = x^2 + y^2 + 2z^2$$

*with some integers x, y, z, if and only if m is not of the form*

$$4^k(16l+14) \quad (k \geqq 0,\ l \geqq 0). \tag{6}$$

As a matter of fact, the first part of the lemma is a classical theorem proved by G. L. Dirichlet, and the second part is also a well-known result which, as has been noted by L. E. Dickson [3], can be derived easily from the first part.

Now, let there be given an even integer $2n > 0$. We shall first show that every number $2n$ of the form (2) admits a representation of the type (1) with $s=5$. In fact, it will obviously suffice to prove that an even integer $2n$ of the form $32l+28\,(l \geqq 0)$ is representable in that form. Since $16l+4$ is not of the form (6), we have in virtue of the lemma

$$16l+4 = x^2 + y^2 + 2z^2$$

for some integers $x$, $y$, $z$, and

$$\begin{aligned} 2n &= 32l+28 = 2(16l+4)+20 \\ &= (x+z+1)^2 + (-x+z+1)^2 + (y-z+1)^2 \\ &\quad + (-y-z+1)^2 + (-4)^2, \end{aligned}$$

as required.

Next, we shall show that if $2n$ has the form (2), then it cannot be represented in the form (1) with $s \leqq 4$. Indeed, if we had

$$2n = \sum_{i=1}^{4} x_i^2 \quad \text{and} \quad \sum_{i=1}^{4} x_i = 0,$$

then we would have

$$\begin{aligned} 2n &= x_1^2 + x_2^2 + x_3^2 + (-x_1 - x_2 - x_3)^2 \\ &= (x_1+x_2)^2 + (x_2+x_3)^2 + (x_3+x_1)^2; \end{aligned}$$

but, this is impossible in view of the lemma since $2n$ is an integer of the form (5).

Finally, we prove that if $2n$ is not of the form (2), then it is representable in the form (1) with $s \leqq 4$. We distinguish two cases according as $n$ is odd or even.

If $n$ is odd, then by the lemma there are integers $x, y, z$ such that

$$n = x^2 + y^2 + 2z^2$$

and so

$$2n = (x+z)^2 + (-x+z)^2 + (y-z)^2 + (-y-z)^2.$$

If $n$ is even, $(2n)/4$ is an integer which is not of the form (5) and, again by the lemma, we have

$$2n = 4(x^2+y^2+z^2) = (2x)^2 + (2y)^2 + (2z)^2$$

for some integers $x, y, z$, whence, putting $2x = x_1 + x_2$, $2y = x_2 + x_3$, $2z = x_3 + x_1$, we obtain

$$2n = (x_1+x_2)^2 + (x_2+x_3)^2 + (x_3+x_1)^2$$
$$= x_1^2 + x_2^2 + x_3^2 + (-x_1 - x_2 - x_3)^2.$$

This completes the proof of our theorem.

A simple consequence of the theorem is that the positive quaternary quadratic form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

represents all positive integers (and 0 trivially).

**2.** We have proved that $s(2n) = 5$ if and only if the integer $2n$ has the form (2). For the sake of completeness we should like to give a description of the properties that characterize those integers $2n$ for which we have $s(2n) = 2$, 3 or 4. To this end, it will be convenient to introduce the symbol $q(m)$ for an integer $m \geqq 1$ to denote $m/e^2$, where $e^2$ is the largest square divisor of the integer $m$. $q(m)$ is thus squarefree for all $m$.

We have:

$s(2n) = 2$ if and only if $q(2n) = 2$;

$s(2n) = 3$ if and only if $q(2n)$ is even, is greater than 2, and does not contain any prime factor $p \equiv 5 \pmod 6$.

$s(2n) = 4$ if and only if either $q(2n)$ is odd and $2n$ is not of the form (2), or $q(2n)$ is even and is divisible by some prime number $p \equiv 5$ (mod 6).

Note that if $q(2n)$ is an even integer then the integer $2n$ is not of the form (2).

## References

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