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Global Existence and Asymptotics of the Solutions of the Second-Order Quasilinear Hyperbolic Equations with the First-Order Dissipation

By

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Introduction

In this paper, we first consider the following Cauchy problem for the quasilinear hyperbolic equations

(1) $$L(u)=u_{tt}-\sum_{i,j=1}^{n} a_{ij}(x,t,Du)u_{ij}+\alpha u_t+b(Du)=0$$

$$\begin{cases} u(x,0)=\phi(x) \\ u_t(x,0)=\psi(x) \end{cases}$$

where $x\in \boldsymbol{R}^n$, $t\geq 0$, $\alpha>0$, $u_i=\dfrac{\partial u}{\partial x_i}$, $u_t=\dfrac{\partial u}{\partial t}$ and

$$Du=(u,u_t,u_1,u_2,\dots,u_n).$$

Here the coefficients a_{ij} are smooth and satisfy

$$\sum_{i,j} a_{ij}(x,t,y)\xi_i\xi_j\geq a(y)\sum_i \xi_i^2, \qquad a(0)>0$$

for all $x\in\boldsymbol{R}^n$, $t\in\boldsymbol{R}^1$, $y\in\boldsymbol{R}^{n+2}$, $\xi\in\boldsymbol{R}^n$.

Recently, we investigated the global existence and decay of the solutions of the semilinear wave equations

(2) $$u_{tt}-\Delta u+\alpha u_t+b(Du)=0 \qquad x\in\boldsymbol{R}^n, \quad t\geq 0, \quad \alpha>0$$

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with the small data in [2]. For the space dimension $n=1$, Nishida [6] showed that the quasilinear equations

$$(3) \qquad u_{tt}-\frac{\partial}{\partial x}\sigma(u_x)+\alpha u_t=0 \qquad x\in \boldsymbol{R}^n, \quad t\geq 0, \quad \alpha>0$$

have the global smooth solutions for the small data. But his argument is not applicable to the cases $n\geq 2$. In §2 we establish the global existence and decay theorem of (1) for general cases $n\geq 1$ with small data and boundedness of some coefficients (Theorem 2).

Next we consider the following initial-boundary value problem;

$$(1)' \qquad L(u)=\varepsilon f(x, t) \qquad x\in\Omega, \quad t\geq 0$$

$$\begin{cases} u(x, 0)=\phi(x) \\ u_t(x, 0)=\psi(x) \\ u|_{\partial\Omega}=0 \end{cases}$$

where Ω is a bounded open set in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$ and ε is a sufficiently small constant. For the semilinear equations

$$u_{tt}-\sum_{i,j}a_{ij}(x)u_{ij}+\alpha u_t=b(u) \qquad x\in\Omega, \quad t\geq 0, \quad \alpha>0,$$

Sattinger [7] discussed the global existence and stability with small data. In §3 we establish the global existence and decay theorem (Theorem 3) even for general quasilinear equations (1)′ under the assumptions stated in Theorem 3.

Moreover, at the end of §3, we mention the results of the existence, uniqueness and stability of the time periodic solutions for

$$(1)'' \qquad L(u)=\varepsilon f(x, t) \qquad x\in\Omega, \quad t\in\boldsymbol{R}^1$$

$$u|_{\partial\Omega}=0$$

where Ω is a bounded open set in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$ and ε is a sufficiently small constant. For the semilinear equations

$$u_{tt}-\sum_{i,j}a_{ij}(x)u_{ij}+\alpha u_t=\varepsilon f(x, t, Du) \qquad x\in\Omega, \ t\in\boldsymbol{R}^1, \ \alpha>0,$$

Rabinowitz [8] showed existence and stability of the time periodic solutions. Moreover he [9] showed existence only for the nonlinear equations

$$u_{tt}-u_{xx}+\alpha u_t=\varepsilon f(x,\, t,\, Du,\, u_{tt},\, u_{tx},\, u_{xx})$$

$$x\in(a,\, b)\subset \boldsymbol{R}^1,\quad t\in \boldsymbol{R}^1,\quad \alpha>0.$$

Although our (1)″ are quasilinear, we can establish not only existence for more general space dimension $n\geq 1$ but also stability (Corollary of Theorem 3).

Notations and Preliminaries

In this paper, all functions are real valued. Let Ω be $\boldsymbol{R}^n$ or a bounded open set in $\boldsymbol{R}^n$ with the C^∞-boundary $\partial\Omega$. We denote by $L^p(\Omega)$ $(1\leq p\leq\infty)$ the space of measurable functions u on Ω whose p-th powers are integrable with the norm

$$\|u\|_p=\left(\int |u(x)|^p dx\right)^{\frac{1}{p}},\quad \|u\|_\infty=\operatorname*{ess.\,sup}_{x\in\Omega}|u(x)|\,.$$

If $p=2$, we write $\|\cdot\|$. Let $f(z)$ be a function of $z\in\boldsymbol{R}^r$ (r is some positive integer). Then $D_z^k f$ (resp. $\bar{D}_z^k f$) (k is some positive integer) represents the vector which has

$$\frac{(k+r)!}{k!r!}\qquad \left(\text{resp. } \frac{(k+r)!}{k!r!}-1\right)$$

components,

$$D_z^k f=\left\{\left(\frac{\partial}{\partial z}\right)^\alpha f\right\},\qquad 0\leq|\alpha|\leq k$$

$$\left(\text{resp. } \bar{D}_z^k f=\left\{\left(\frac{\partial}{\partial z}\right)^\alpha f\right\},\qquad 1\leq|\alpha|\leq k\right)$$

where $\alpha=(\alpha_1,\, \alpha_2,\ldots,\, \alpha_r)$ and $|\alpha|=\sum_{i=1}^r \alpha_i$.

Especially, $D^k f$ and $D^{k,m} f$ represent

$$D^k f=\left\{\left(\frac{\partial}{\partial x}\right)^\alpha\left(\frac{\partial}{\partial t}\right)^i f\right\},\qquad 0\leq i+|\alpha|\leq k,$$

$$D^{k,m}f=\left\{\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial t}\right)^{i}f\right\}, \qquad 0\leq|\alpha|\leq k,\ 0\leq i\leq m.$$

$\bar{D}^k f$ and $\bar{D}^{k,m}f$ are similarly defined as $\bar{D}^k_z f$. If $k=1$ for $D^k_z f$, we write simply $D_z f$. Moreover $D^k_z f\cdot D^k_z g$ (resp. $\bar{D}^k_z f\cdot \bar{D}^k_z g$) represents the usual vector inner product for $D^k_z f$ and $D^k_z g$ (resp. $\bar{D}^k_z f$ and $\bar{D}^k_z g$).

For some set G in R^r, $C^k(G)$ is the space of the real valued functions on G that are k-times continuously differentiable. $C^\infty_0(\Omega)$ denotes the space of $C^\infty(\Omega)$ functions with compact support in Ω. H^k denotes the space of functions all of whose derivatives of order $\leq k$ belong to $L^2(\Omega)$ and the norm of H^k is equal to $\|D^k_x\cdot\|$. The completion of the space of $C^\infty_0(\Omega)$ functions by the H^k norm is denoted by $\mathring{H}^k$.

Let X be a Banach space on Ω. Then $u(x, t)\in\mathscr{E}^k_t(X)$ (resp. $L^\infty_t(X)$) $(t_0\leq t\leq t_1)$ means that $u(\cdot, t)$ belongs to X for every fixed t and u is k-times continuously differentiable (resp. bounded) with respect to t in X-topology on $t_0\leq t\leq t_1$.

We use c_i as the constants, especially use c for the constants which we need not distinguish and write $c_i(X)$ when we emphasize its dependence on X. We denote by $h_i(\tau)$ the continuous nonnegative and nondecreasing functions on $\tau\geq 0$.

We note the next Sobolev's inequalities.

Lemma 1 (Mizohata [3] Chapter 7). *We suppose Ω is as in the above.*

i) *If $u\in H^{\left[\frac{n}{2}\right]+1+m}$ $(m\geq 0)$, we have*

$$\|D^m_x u\|_\infty\leq c_0(m, n, \Omega)\|D_x^{\left[\frac{n}{2}\right]+1+m}u\|.$$

ii) *If $u\in H^{\left[\frac{n}{2}\right]+1+m}$ $(m\geq 0)$, we have for $m+1\leq|\alpha|\leq m+1+\left[\frac{n}{2}\right]$*

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}u\in L^p$$

and

$$\left\|\left(\frac{\partial}{\partial x}\right)^{\alpha}u\right\|_p\leq c_0(p, m, n, \alpha, \Omega)\|D_x^{\left[\frac{n}{2}\right]+1+m}u\|$$

where

$$\begin{cases} \dfrac{1}{p} \in \left[\dfrac{|\alpha|}{n} - \dfrac{m+1}{n}, \dfrac{1}{2}\right] - 0 & (n = even) \\ \dfrac{1}{p} \in \left[\dfrac{|\alpha|}{n} - \dfrac{2m+1}{n}, \dfrac{1}{2}\right] & (n = odd). \end{cases}$$

§1. Basic Estimates

In this section we show the estimates of $a_{ij}(x, t, Du)$ and $b(Du)$. We list up the following assumptions $\left(s = \left[\frac{n}{2}\right] + 2\right)$.

Assumption 1.

i) $a_{ij}(x, t, y) \in C^{s+1}(\boldsymbol{R}^n \times \boldsymbol{R}^1 \times \boldsymbol{R}^{n+2})$ for $1 \leq i, j \leq n$.

ii) $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$.

iii) $\sum_{i,j} a_{ij}(x, t, y)\xi_i\xi_j \geq a(y) \sum_i \xi_i^2$, $a(0) = a_0 > 0$

for all $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^1$, $y \in \boldsymbol{R}^{n+2}$, $\xi \in \boldsymbol{R}^n$

where $a(y) \in C^0(\boldsymbol{R}^{n+2})$.

iv) $\sup_{\boldsymbol{R}^n \times \boldsymbol{R}^1} \sum_{i,j} |D_{x,t,y}^{s+1} a_{ij}(x, t, y)| \leq h_0(|y|)$.

Assumption 2.

$$\sup_{\boldsymbol{R}^n \times \boldsymbol{R}^1} \sum_{i,j} |\bar{D}_x a_{ij}(x, t, y)| \leq |\tilde{y}| h_0(|y|) \quad \text{for} \quad |\tilde{y}| \leq 1$$

where $\tilde{y} = (0, y_2, y_3, \ldots, y_{n+2})$.

Assumption 3.

i) $b(y) \in C^{s+1}(\boldsymbol{R}^{n+2})$, $|D_y^{s+1} b(y)| \leq h_1(|y|)$.

ii) $D_y b(0) = 0$.

Assumption 4.

i) $b(Du) = b_1(u) + b_2(Du)$.

ii) $b_1(u)u \geq 0$.

iii) $b_2(y) \leq |\tilde{y}|^2 h_1(|y|)$ for $|\tilde{y}| \leq 1$.

Remark. Throughout this paper s represents $\left[\frac{n}{2}\right]+2$ and $|\cdot|$ denotes the usual Euclidian norm.

By Assumption 1, we can choose the positive constants γ_0 and a_1 $(0<a_1<a_0)$ such that

$$a(y)>a_1>0 \quad \text{if} \quad |y| \leq \gamma_0. \tag{4}$$

Moreover, by Lemma 1, we can choose a positive constant e_0 such that

$$\sup_{0\leq t\leq T} \|Du(t)\|_\infty \leq \gamma_0 \quad \text{if} \quad \sup_{0\leq t\leq T} \|D^s u(t)\| \leq e_0 \tag{5}$$

where T is any positive constant. So we define the space of $u(x, t)$, $\mathscr{D}(0, T|e)$, for $0<e\leq e_0$ by

$$\mathscr{D}(0, T|e)=\{u(x, t)|D^{s+1}u(x, t)\in \mathscr{E}_t^0(L^2) \ (0\leq t\leq T) \quad \text{and} \quad \sup_{0\leq t\leq T} \|D^{s+1}u(t)\| \leq e \quad (0<e\leq e_0)\}. \tag{6}$$

Now we define $E_v\{u(t)\}$ by

$$E_v\{u(t)\}=\int \frac{\lambda}{2}|D^s u|^2+\lambda D^s u\cdot D^s u_t+\frac{1}{2}|D^s u_t|^2 + \frac{1}{2}\sum a_{ij}(x, t, Dv)D^s u_i\cdot D^s u_j dx \qquad (0<\lambda<1).$$

Then we note the following under Assumption 1.

Lemma 2. *If* $v\in\mathscr{D}(0, T|e)$, $E_v\{u(t)\}$ *is equivalent to* $\|D^{s+1}u(t)\|^2$ *for* $0\leq t\leq T$, *that is,*

$$c_1\|D^{s+1}u(t)\|^2 \leq E_v\{u(t)\} \leq c_2\|D^{s+1}u(t)\|^2 \qquad \text{for} \quad 0\leq t\leq T,$$

where c_1 *and* c_2 *depend only on* e_0, c_0, a_1, h_0 *and* λ.

This lemma is easily verified by Assumption 1 and (4)~(6).

In order to estimate $\sum a_{ij}u_{ij}$ and b, we note the following estimates of the composite functions.

Lemma 3.

i) *Suppose that*

$$\begin{cases} f(x,\ t,\ y)\in C^{s+1}(\boldsymbol{R}^n\times\boldsymbol{R}^1\times\boldsymbol{R}^{n+2}) \\ \sup_{\boldsymbol{R}^n\times\boldsymbol{R}^1}|D^{s+1}_{x,t,y}f(x,\ t,\ y)|\le h_2(|y|) \\ v(x,\ t)\in\mathscr{E}^i_t(H^{s+1-i})\ (0\le i\le s+1),\quad w(x,\ t)\in\mathscr{E}^i_t(H^{s-1-i}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0\le i\le s-1). \end{cases}$$

Then it follows for $1\le k\le s$ *that*

$$\|D^k\{f(x,\ t,\ Dv(x,\ t))w(x,\ t)\}-f(x,\ t,\ Dv(x,\ t))D^kw(x,\ t)\|$$

$$\le c\{f_0+(\|D^{s+1}v\|+\|D^{s+1}v\|^s)h_2(\|Dv\|_\infty)\}\|D^{k-1}w\| \tag{7}$$

$$\le c(1+\|D^{s+1}v\|^s)\|D^{s-1}w\|h_2(\|Dv\|_\infty) \tag{8}$$

where

$$f_0=\sup_{\boldsymbol{R}^n\times\boldsymbol{R}^1}|\bar{D}^sf(x,\ t,\ 0)|.$$

ii) *Suppose that*

$$\begin{cases} g(y)\in C^{s+1}(\boldsymbol{R}^{n+2}),\quad g(0)=0 \\ |D^{s+1}_y g(y)|\le h_3(|y|) \\ v(x,\ t)\in\mathscr{E}^i_t(H^{s+1-i})\quad (0\le i\le s+1). \end{cases}$$

Then it follows for $0\le k\le s$ *that*

$$\|D^kg(Dv(x,\ t))\|$$

$$\le c\{g_0+(\|D^{s+1}v\|+\|D^{s+1}v\|^{s-1})h_3(\|Dv\|_\infty)\}\|D^{k+1}v\| \tag{7$'$}$$

$$\le c(\|D^{s+1}v\|+\|D^{s+1}v\|^s)h_3(\|Dv\|_\infty) \tag{8$'$}$$

where

$$g_0=|D_yg(0)|.$$

Remark. We can get (7)$\sim$(8)$'$ by using Lemma 1 and especially Taylor's formula for (7), (7)$'$ (refer to Chapter I and II in Dionne [1]). In this paper, we use more precise forms (7), (7)$'$ rather than (8), (8)$'$.

Now, defining ε_1 by

$$\varepsilon_1 = \sup_{\boldsymbol{R}^n \times \boldsymbol{R}^1} \sum |\bar{D}^s a_{ij}(x, t, 0)|,$$

we have the following (we omit $\sum$ for simplicity)

Lemma 4. *Suppose Assumption* 1, *then for* $u(x, t)$ *and* $v(x, t) \in \mathscr{E}_t^i(H^{s+1-i})$ $(0 \leq i \leq s+1)$ *we have the following*:

i) $\|D^k(a_{ij}u_{ij}) - a_{ij}D^k u_{ij}\|$

$$\leq c(\|D^{s+1}v\| h_0 + \|D^{s+1}v\|^s h_0 + \varepsilon_1) \|\bar{D}^{k+1}u\|$$

where $a_{ij} = a_{ij}(x, t, Dv)$, $h_0 = h_0(\|Dv\|_\infty)$, $1 \leq k \leq s$.

ii) $\|(a_{ij}D^k u_i \cdot D^k u_t)_j - a_{ij}D^k u_{ij} \cdot D^k u_t - a_{ij}D^k u_i \cdot D^k u_{tj}\|_1$

$$+ \left\| \frac{1}{2}(a_{ij}D^k u_i \cdot D^k u_j)_t - a_{ij}D^k u_i \cdot D^k u_{jt} \right\|_1$$

$$\leq c(\|D^{s+1}v\| h_0 + \varepsilon_1) \|\bar{D}^{k+1}u\|^2 \qquad 1 \leq k \leq s.$$

iii) $\|(a_{ij}D^k u_i \cdot D^k u)_j - a_{ij}D^k u_{ij} \cdot D^k u - a_{ij}D^k u_i \cdot D^k u_j\|_1$

$$\leq c(\|D^{s+1}v\| h_0 + \varepsilon_1) \|D^{k+1}u\|^2 \qquad 1 \leq k \leq s.$$

iii)$'$ *Suppose in addition Assumption* 2. *Then, left hand side of* iii) $\leq c\|\bar{D}^{s+1}v\| \, \|\bar{D}^{s+1}u\| \, \|D^{s+1}u\| h_0$.

iv) $\|D^k\{a_{ij}(Du) - a_{ij}(Dv)\}u_{ij}\|$

$$\leq c\|D^{k+1}(u-v)\| \, \|D^{s+1}u\| (1 + \|D^{s+1}u\|^{s-2} + \|D^{s+1}v\|^{s-2})$$

$$\times h_0(\|Du\|_\infty + \|Dv\|_\infty) \qquad 0 \leq k \leq s-1.$$

v) *Suppose in addition* $v \in \mathscr{D}(0, T|e)$. *Then,*

$$\|D_x^2 u\| \leq c(e_0, h_0, a_1, \Omega) \{\|a_{ij}(Dv)u_{ij}\| + \|D_x^1 u\|\} \qquad \textit{for} \quad 0 \leq t \leq T.$$

Lemma 5. *Suppose Assumption* 3, *then for* $u(x, t)$ *and* $v(x, t)$

$\in \mathscr{E}_t^i(H^{s+1-i})$ $(0 \le i \le s+1)$ *we have the following*:

i) $\|D^k b(Du)\| \le c\|D^{k+1}u\| (\|D^{s+1}u\| + \|D^{s+1}u\|^{s-1})h_1(\|Du\|_\infty) \quad 0 \le k \le s.$

ii) $\|\bar{D}^s b(Du)\| \le c\|D^{s+1}u\| \, \|\bar{D}^{s+1}u\| (1 + \|D^{s+1}u\|^{s-2})h_1(\|Du\|_\infty).$

iii) $\|D^k\{b(Du) - b(Dv)\}\|$

$$\le c\|D^{k+1}(u-v)\| (\|D^{s+1}u\| + \|D^{s+1}v\|)$$

$$\times (1 + \|D^{s+1}u\|^{s-2} + \|D^{s+1}v\|^{s-2})h_1(\|Du\|_\infty + \|Dv\|_\infty) \quad 0 \le k \le s.$$

iv) *Suppose in addition Assumption* 4. *Then,*

$$\|b_2(Du)\| \le c\|\bar{D}u\|^2 h_1(\|Du\|_\infty).$$

Remarks. Lemmas 4 and 5 (except v) of Lemma 4) are given by using Lemmas 1 and 3 (refer to the Chapter I and II in Dionne [1]) and v) of Lemma 4 is shown by the strong (uniform) ellipticity of $\sum a_{ij}(Dv)u_{ij}$ with $v \in \mathscr{D}(0, T|e)$.

§2. Cauchy Problem

In this section, we consider the Cauchy problem

$$(1) \qquad L(u) = u_{tt} - \sum_{i,j} a_{ij}(x, t, Du)u_{ij} + \alpha u_t + b(Du) = 0$$

$$x \in \boldsymbol{R}^n, \quad t \ge 0, \quad \alpha > 0$$

$$\begin{cases} u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x). \end{cases}$$

We put $\alpha = 1$ without loss of generality. We suppose $\phi \in H^{s+1}$, $\psi \in H^s$ and put

$$\|D_x^{s+1}\phi\| + \|D_x^s\psi\| = \varepsilon.$$

By using the equation (1), we can determine

$$\left(\frac{\partial}{\partial t}\right)^k u(x, 0) \qquad (2 \le k \le s+1)$$

successively beginning with ϕ, ψ and it follows

$$\|D^{s+1}u(0)\| \leq \varepsilon h_4(\varepsilon) \tag{9}$$

where h_4 depends only on h_0 and h_1. Then we have the following

Theorem 1 (local existence). *We suppose Assumptions* 1 *and* 3. *Moreover we suppose* $\phi \in H^{s+1}$, $\psi \in H^s$ *and that* $D^{s+1}u(0)$ *satisfies*

$$\|D^{s+1}u(0)\| \leq c_3 e$$

where $c_3 = \left(\frac{c_1}{4c_2}\right)^{\frac{1}{2}}$ *and* $0 < e \leq e_0$. *Then there exists a positive constant* t_0 *such that Cauchy problem for* (1) *has a unique solution*

$$u(x, t) \in \mathscr{D}(0, t_0|e).$$

Remark. This Theorem is due to the Theorem in Chapter V of Dionne [1], although we modified the formulation. We only note the following: If $v \in \mathscr{D}(0, t_0|e)$, the linear equation

$$u_{tt} - \sum a_{ij}(x, t, Dv)u_{ij} + u_t = -b(Dv)$$

is strictly hyperbolic on $0 \leq t \leq t_0$ so that we get the energy inequality

$$\|D^{s+1}u(t)\|^2 \leq \exp(ct)\left\{\frac{c_2}{c_1}\|D^{s+1}u(0)\|^2 + \int_0^t \|D^s b(Dv)\|^2 d\tau\right\}$$

where c depends only on h_0 and e_0. On the other hand, it follows from $v \in \mathscr{D}(0, t_0|e)$ that

$$\|D^s b(Dv(\tau))\|^2 \leq ce^2 \qquad \text{for} \quad 0 \leq \tau \leq t_0.$$

Choosing t_0 sufficiently small in the above inequalities, we have

$$\sup_{0 \leq t \leq t_0} \|D^{s+1}u(t)\| \leq e \qquad \text{for} \quad D^{s+1}u(0) \quad \text{as in the Theorem.}$$

Therefore we can perform the iteration arguments. For more details, refer to the Appendix.

In order to show the global existence, we establish the following

Lemma 6 (*a priori* estimate). *We suppose Assumptions* $1 \sim 4$. *More-*

over we suppose that (1) *has the solution* $u \in \mathscr{D}(0, T|e)$ (*T is any positive constant*) *for*

$$\begin{cases} \|D_x^{s+1}\phi\| + \|D_x^s\psi\| = \varepsilon \\ \sup\limits_{R^n \times R^1} \sum |\bar{D}^{s+1} a_{ij}(x, t, 0)| = \varepsilon_1. \end{cases}$$

Then there exist the positive constants δ_0, $\delta_1(e)$ *such that*

$$u \in \mathscr{D}(0, T|c_3 e) \qquad \text{for} \quad 0 < e, \ \varepsilon_1 \leq \delta_0, \ 0 < \varepsilon \leq \delta_1(e).$$

Here δ_0, δ_1 *do not depend on* T.

Proof. For $u \in \mathscr{D}(0, T|e) \cap \mathscr{E}_t^i(H^{s+2-i})$ $(0 \leq i \leq s+2)$ and $v \in \mathscr{D}(0, T|e)$ we first estimate the following

$$\begin{aligned} I = &\int D^s\{L_v(u) + b(Du)\} \cdot D^s u_t dx = \int D^s u_{tt} \cdot D^s u_t dx \\ &- \int D^s \sum a_{ij}(Dv) u_{ij} \cdot D^s u_t dx + \int D^s u_t \cdot D^s u_t dx + \int D^s b(Du) \cdot D^s u_t dx \\ = &I_1 - I_2 + I_3 + I_4 \end{aligned}$$

where

$$L_v(u) = u_{tt} - \sum a_{ij}(x, t, Dv) u_{ij} + u_t.$$

By Lemmas 4 and 5, we have

$$I_1 = \frac{1}{2} \frac{d}{dt} \int |D^s u_t|^2 dx$$

$$\begin{aligned} -I_2 = &\frac{1}{2} \frac{d}{dt} \int \sum a_{ij}(Dv) D^s u_i \cdot D^s u_j dx + \int \sum \{D^s(a_{ij} u_{ij}) - a_{ij} D^s u_{ij}\} \cdot D^s u_t dx \\ &- \int \frac{1}{2} \sum (a_{ij}(Dv) D^s u_i \cdot D^s u_j)_t + \sum a_{ij}(Dv) D^s u_i \cdot D^s u_{jt} dx \\ &+ \int \sum (a_{ij}(Dv) D^s u_i \cdot D^s u_t)_j - \sum a_{ij}(Dv) D^s u_{ij} \cdot D^s u_t \\ &- \sum a_{ij}(Dv) D^s u_i \cdot D^s u_{jt} dx \\ \geq &\frac{1}{2} \frac{d}{dt} \int \sum a_{ij}(Dv) D^s u_i \cdot D^s u_j dx \end{aligned}$$

$$-c\{\|D^{s+1}v\|h_0(\|Dv\|_\infty)+\|D^{s+1}v\|^s h_0(\|Dv\|_\infty)+\varepsilon_1\}\ \|\bar{D}^{s+1}u\|^2$$

$$I_3=\int|D^s u_t|^2dx=\|D^s u_t\|^2$$

$$I_4=\int\bar{D}^s b(Du)\cdot\bar{D}^s u_t+b(Du)u_t dx$$

$$\geq\int b_1(u)u_t dx-\|\bar{D}^s b(Du)\|\ \|\bar{D}^s u_t\|-\|b_2(Du)\|\ \|u_t\|$$

$$\geq\frac{d}{dt}\int B(u)\,dx-c(1+\|D^{s+1}u\|^{s-1})\|D^{s+1}u\|\|\bar{D}^{s+1}u\|^2 h_1(\|Du\|_\infty)$$

where $B(u)=\int_0^u b_1(v)dv$. From the above estimates, we have

$$I\geq\frac{d}{dt}\left\{\int\frac{1}{2}|D^s u_t|^2+\frac{1}{2}\sum a_{ij}(Dv)\,D^s u_i D^s u_j+B(u)\,dx\right\} \tag{10}$$

$$+\|D^s u_t\|^2-c\{\varepsilon_1+(\|D^{s+1}v\|+\|D^{s+1}v\|^s)h_0(\|Dv\|_\infty)$$

$$+(\|D^{s+1}u\|+\|D^{s+1}u\|^s)h_1(\|Du\|_\infty)\}\ \|\bar{D}^{s+1}u\|^2.$$

Next we estimate

$$I'=\int D^s\{L_v(u)+b(Du)\}\cdot D^s u dx=\int D^s u_{tt}\cdot D^s u dx$$

$$-\int D^s(\sum a_{ij}u_{ij})\cdot D^s u dx+\int D^s u_t\cdot D^s u dx+\int D^s b(Du)\cdot D^s u dx$$

$$=I_1'-I_2'+I_3'+I_4'.$$

We have

$$I_1'=\frac{d}{dt}\int D^s u\cdot D^s u_t dx-\|D^s u_t\|^2$$

$$-I_2'=\int\sum a_{ij}D^s u_i\cdot D^s u_j dx$$

$$+\int\sum a_{ij}D^s u_{ij}\cdot D^s u-D^s(\sum a_{ij}u_{ij})\cdot D^s u dx$$

$$+\int\sum(a_{ij}D^s u_i\cdot D^s u)_j-\sum a_{ij}D^s u_i\cdot D^s u_j-\sum a_{ij}D^s u_{ij}\cdot D^s u dx$$

$$\geq a_1 \sum \|D^s u_i\|^2 - c\{\varepsilon_1 + \|D^{s+1}u\| h_0(\|Dv\|_\infty) + \|D^{s+1}v\| h_0(\|Dv\|_\infty)$$

$$+ \|D^{s+1}v\|^s h_0(\|Dv\|_\infty)\} (\|\bar{D}^{s+1}u\|^2 + \|\bar{D}^{s+1}v\|^2)$$

$$I_3' = \frac{1}{2} \frac{d}{dt} \int |D^s u|^2 dx$$

$$I_4' = \int \bar{D}^s b(Du) \cdot \bar{D}^s u + b_1(u)u + b_2(Du)u dx$$

$$\geq - \|\bar{D}^s b(Du)\| \, \|\bar{D}^s u\| - \|b_2(Du)\| \, \|u\|$$

$$\geq -c(\|D^{s+1}u\| + \|D^{s+1}u\|^s) \|\bar{D}^{s+1}u\|^2 h_1(\|Du\|_\infty).$$

Therefore we have

$$(11) \qquad I' \geq \frac{d}{dt}\left(\int \frac{1}{2} |D^s u|^2 + D^s u \cdot D^s u_t dx\right) + a_1 \sum \|D^s u_i\|^2 - \|D^s u_t\|^2$$

$$-c[\varepsilon_1 + \{h_1(\|Du\|_\infty) + h_0(\|Dv\|_\infty)\} (\|D^{s+1}u\| + \|D^{s+1}v\|$$

$$+ \|D^{s+1}u\|^s + \|D^{s+1}v\|^s)] (\|\bar{D}^{s+1}u\|^2 + \|\bar{D}^{s+1}v\|^2).$$

Choosing some positive number $0 < \lambda < \frac{1}{2}$, we get from (10) and (11)

$$(12) \qquad \int_0^t I + \lambda I' d\tau \geq \left[E_v\{u(\tau)\} + \int B(u(\tau)) dx \right] \Big|_0^t$$

$$+ \int_0^t \frac{1}{2} \|D^s u_t(\tau)\|^2 + \lambda a_1 \sum \|D^s u_i(\tau)\|^2$$

$$-c(1+\lambda)\{\varepsilon_1 + (e + e^s) h(e_0)\} \{\|\bar{D}^{s+1}u(\tau)\|^2 + \|\bar{D}^{s+1}v(\tau)\|^2\} d\tau$$

where $h(e_0) = h_0(ce_0) + h_1(ce_0)$.

Now denote by $u_\delta(x, t)$ the function $(\phi_\delta * u)(x, t)$ where $\phi_\delta *$ is Friedrichs' mollifier with respect to x. Then we note that for the solution $u \in \mathscr{D}(0, T|e)$ of (1), it follows

$$u_\delta(x, t) \in \mathscr{D}(0, T|e) \cap \mathscr{E}_t^i(H^{s+2-i}) \qquad (0 \leq i \leq s+2).$$

Applying $\phi_\delta *$ to (1),

$$L_u(u_\delta) + b(Du_\delta) = C_\delta u + C'_\delta u$$

where

$$C_\delta u = \sum [\phi_\delta * \{a_{ij}(Du)u_{ij}\} - a_{ij}(Du)u_{\delta ij}]$$

$$C'_\delta u = b(Du_\delta) - \phi_\delta * b(Du).$$

From (12), it follows

$$\left[E_u\{u_\delta(\tau)\} + \int B(u_\delta(\tau))dx\right]\Big|_0^t + \int_0^t \frac{1}{2}\|D^s u_{\delta t}\|^2 + \lambda a_1 \sum \|D^s u_{\delta i}\|^2$$

$$- c(1+\lambda)\{\varepsilon_1 + (e+e^s)h(e_0)\}(\|\bar{D}^{s+1}u\|^2 + \|\bar{D}^{s+1}u_\delta\|^2)d\tau$$

$$\leq \int_0^t (\|D^s C'_\delta u\| + \|D^s C'_\delta u\|)(\|D^s u_{\delta t}\| + \lambda \|D^s u_\delta\|)d\tau.$$

Then we have

$$(13) \qquad E_u\{u_\delta(\tau)\} \longrightarrow E_u\{u(\tau)\}, \quad \int B(u_\delta(\tau))dx \longrightarrow \int B(u(\tau))dx$$

$$(14) \qquad \|D^{s+1}u_\delta(\tau)\| \longrightarrow \|D^{s+1}u(\tau)\|$$

$$(15) \qquad \|D^s C'_\delta u(\tau)\| \longrightarrow 0$$

$$(16) \qquad \|D^s C_\delta u(\tau)\| \longrightarrow 0$$

for every $0 \leq \tau \leq t$ when $\delta \to 0$. In fact, (13)~(15) are easily verified and for (16) refer to Chapter 6 of [3]. Therefore, taking $\delta \to 0$, we get

$$(17) \qquad \left[E_u\{u(\tau)\} + \int B(u(\tau))dx\right]\Big|_0^t$$

$$+ \int_0^t \frac{1}{2}\|D^s u_t(\tau)\|^2 + \lambda a_1 \sum \|D^s u_i(\tau)\|^2$$

$$- c(1+\lambda)\{\varepsilon_1 + (e+e^s)h(e_0)\}\|\bar{D}^{s+1}u(\tau)\|^2 d\tau \leq 0.$$

Choosing a small constant δ_0 which satisfies

$$c(1+\lambda)\{\varepsilon_1 + (e+e^s)h(e_0)\} \leq \frac{1}{2}\min\left(\frac{1}{2}, \lambda a_1\right) \qquad \text{for} \quad 0 < \varepsilon_1,\ e \leq \delta_0,$$

we have

$$(18)\quad \left[E_u\{u(\tau)\}+\int B(u(\tau))\,dx\right]\Big|_0^t+\frac{1}{2}\min\left(\frac{1}{2},\ \lambda a_1\right)\int_0^t\|\bar{D}^{s+1}u(\tau)\|^2d\tau\leq 0.$$

Since

$$\int B(\phi)dx\leq\int|\phi|\sup_{|v|\leq|\phi|}|b_1(v)|dx\leq c\varepsilon^3h_1(\varepsilon),$$

it follows from (18) and (9) that

$$(19)\qquad \|D^{s+1}u(t)\|^2\leq c_1^{-1}E_u\{u(t)\}$$

$$\leq c_1^{-1}E_u\{u(0)\}+c_1^{-1}\int B(\phi)dx$$

$$\leq c_1^{-1}c_2h_4(\varepsilon)\varepsilon+ch_1(\varepsilon)\varepsilon^3.$$

Therefore, choosing a constant δ_1 so small that

right hand side of (19) $\leq c_3^2e^2$ *for* $0<\varepsilon\leq\delta_1$,

it follows consequently

$u\in\mathscr{D}(0,\ T|c_3e)$ for $0<e,\ \varepsilon_1\leq\delta_0,\ 0<\varepsilon\leq\delta_1(e)$.

Q. E. D.

By Theorem 1 and Lemma 6, we have the following

Theorem 2. *We suppose Assumptions* $1\sim 4$, *that is,*

i) $a_{ij}(x,\ t,\ y)\in C^{s+1}(\boldsymbol{R}^n\times\boldsymbol{R}^1\times\boldsymbol{R}^{n+2})$.

ii) $a_{ij}=a_{ji}$.

iii) $\sum a_{ij}(x,\ t,\ y)\xi_i\xi_j\geq a(y)\sum\xi_i^2,\quad a(0)>0$

for all $x\in\boldsymbol{R}^n,\ t\in\boldsymbol{R}^1,\ y\in\boldsymbol{R}^{n+2},\ \xi\in\boldsymbol{R}^n$

where $a(y)\in C^0(\boldsymbol{R}^{n+2})$.

iv) $$\begin{cases}\sup_{\boldsymbol{R}^n\times\boldsymbol{R}^1}\sum|D_{x,t,y}^{s+1}a_{ij}(x,\ t,\ y)|\leq h_0(|y|)\\ \sup_{\boldsymbol{R}^n\times\boldsymbol{R}^1}\sum|\bar{D}_xa_{ij}(x,\ t,\ y)|\leq|\tilde{y}|h_0(|y|)\quad \textit{for}\quad |\tilde{y}|\leq 1.\end{cases}$$

v) $b(y)\in C^{s+1}(R^{n+2}),\quad |D_y^{s+1}b(y)|\leq h_1(|y|)$.

vi) $D_y b(0)=0$.

vii) $b(Du)=b_1(u)+b_2(Du)$,

$$\begin{cases} b_1(u)u\geq 0 \\ |b_2(y)|\leq|\tilde{y}|^2 h_1(|y|) \qquad for \quad |\tilde{y}|\leq 1. \end{cases}$$

Here s represents $\left[\frac{n}{2}\right]+2$. *Moreover we suppose* $\phi\in H^{s+1}$, $\psi\in H^s$ *and put*

$$\begin{cases} \|D_x^{s+1}\phi\|+\|D_x^s\psi\|=\varepsilon \\ \sup_{R^n\times R^1}\sum|\bar{D}^s a_{ij}(x,t,0)|=\varepsilon_1. \end{cases}$$

Then there exists a positive constant ε_0 *such that the Cauchy problem for*

$$u_{tt}-\sum a_{ij}(x,t,Du)u_{ij}+u_t+b(Du)=0$$

$$\begin{cases} u(0)=\phi \\ u_t(0)=\psi \end{cases}$$

has a unique solution $u(x,t)\in\mathscr{D}(0,+\infty|e_0)$ *for* $0<{}^\forall\varepsilon$, ${}^\forall\varepsilon_1\leq\varepsilon_0$. *Furthermore u satisfies*

$$\|u(t)\|_\infty+\|\bar{D}^s u(t)\|\longrightarrow 0 \quad as \quad t\longrightarrow+\infty.$$

Corollary 1. *In addition to the assumptions of Theorem 2, we further suppose the following*:

i) $B(u)=\int_0^u b_1(v)dv\leq ub_1(u)h_5(|u|)$.

ii) $|b_2(y)|\leq|y_2||\tilde{y}|h_1(|y|)$.

iii) $\sum a_{ij}u_{ij}=\sum a'_{ij}(Du)u_{ij}+\sum\{f^i(u_i)\}_i$

$$\begin{cases} f^i(0)=0, \quad \frac{\partial}{\partial u_i}f^i(0)=0 \\ \sum_{k\neq 2}\left|\frac{\partial}{\partial y_k}a'_{ij}(y)\right|\leq|y_2|h_0(|y|). \end{cases}$$

Then solution u that is obtained by the Theorem 2 satisfies

$$\|\bar{D}u(t)\|^2+\int B(u(t))dx\leq ct^{-1}.$$

Corollary 2. *In addition to the assumptions of Theorem 2, we further suppose the following*:

i) $b_1(u)=0$.

ii) $$\begin{cases} |D^s_{y_1}b_2(y)|\leq|y_2|^2h_1(|y|) \\ \left|\frac{\partial}{\partial\bar{y}}b_2(y)\right|\leq|y_2|h_1(|y|). \end{cases}$$

iii) $\sum a_{ij}u_{ij}=\sum a'_{ij}(Du)u_{ij}+\sum\{f^i(u_i)\}_i$

$$\begin{cases} f^i(0)=0, \quad \frac{\partial}{\partial u_i}f^i(0)=0 \\ \sum_{k\neq 2}\left|\frac{\partial}{\partial y_k}a'_{ij}(y)\right|\leq|y_2|h_0(|y|). \end{cases}$$

Then the solution u that is obtained by Theorem 2 satisfies

$$\|\bar{D}^{s+1}u(t)\|^2\leq ct^{-1},$$

$$\|u(t)\|_\infty\leq ct^{-\gamma} \qquad \left(\gamma=\frac{n}{4(s-1)}\right).$$

Proof of the Theorem 2. From (9) and Theorem 1, we can choose a positive constant δ_2 as

$$u\in\mathscr{D}(0,\ t_0|e),\quad 0<e\leq\delta_0 \qquad \text{for}\quad 0<{}^\forall\varepsilon\leq\delta_2.$$

Now if we choose $\varepsilon_0=\min(\delta_2,\ \delta_1(e))$, it follows by Lemma 6

$$u\in\mathscr{D}(0,\ t_0|c_3e) \qquad \text{for}\quad 0<{}^\forall\varepsilon,\ {}^\forall\varepsilon_1\leq\varepsilon_0.$$

By using Theorem 1 again with $\phi=u(t_0)$, $\psi=u_t(t_0)$, the solution

(20) $$u\in\mathscr{D}(0,\ 2t_0|e)$$

exists for $0<{}^\forall\varepsilon,\ {}^\forall\varepsilon_1\leq\varepsilon_0$. By Lemma 6, (20) immediately implies

$$u\in\mathscr{D}(0,\ 2t_0|c_3e).$$

Thus repeating the same arguments, we have the solution

$$u\in\mathscr{D}(0,\ +\infty|c_3e)\subset\mathscr{D}(0,\ +\infty|e_0).$$

Next, by the same way as we got (18), we have

$$\|\bar{D}^{s+1}u(t)\|^2+\int_0^t\|\bar{D}^{s+1}u(\tau)\|^2+\int b_1(u(\tau))u(\tau)dxd\tau\leq c. \tag{21}$$

(21) and the following Nirenberg's inequality ([5])

$$\|u\|_\infty\leq c\|\bar{D}_x^{s-1}u\|^\alpha\|u\|^{1-\alpha},\quad \alpha=\frac{n}{2(s-1)} \tag{22}$$

give

$$\|\bar{D}^s u(t)\|+\|u(t)\|_\infty\longrightarrow 0\quad\text{as}\quad t\longrightarrow+\infty.$$

Q. E. D.

Proof of the Corollary 1. Define $E_1(t)$ by

$$E_1(t)=\int\frac{1}{2}u_t^2+\frac{1}{2}\sum a'_{ij}\ (0)u_iu_j+\sum F^i(u_i)+B(u)dx$$

where

$$F^i(u_i)=\int_0^{u_i}f^i(v)dv.$$

Estimating

$$\int L(u)u_tdx=0$$

by using the assumptions, it follows that

$$\frac{d}{dt}E_1(t)+\gamma\|u_t(t)\|^2\leq 0\quad(\gamma>0)\qquad\text{for}\quad 0<{}^\forall\varepsilon\leq\varepsilon_0$$

which implies

$$E_1(t)\leq E_1(\tau)\qquad\text{for}\quad t\geq\tau. \tag{23}$$

It follows by integrating (23) and using (21) that

$$tE_1(t)\leq\int_0^t E_1(\tau)d\tau$$

$$\leq\int_0^t c\|\bar{D}u(\tau)\|^2+\int cb_1(u(\tau))u(\tau)dxd\tau$$

$$\leq c$$

which gives

$$\|\bar{D}u(t)\|^2+\int B(u(t))dx\leq ct^{-1}.$$

Q.E.D.

Proof of the Corollary 2. For this case we can give a proof by estimating

$$\int D^s\{L(u)\}\cdot D^s u_t+\lambda\bar{D}^s\{L(u)\}\cdot\bar{D}^s u dx=0 \qquad (0<\lambda<1)$$

similarly as in the previous arguments and using Nirenberg's inequality (22). We omit the details. Q.E.D.

§3. Initial-Boundary Value Problem and Periodic Solutions

We consider the following initial-boundary value problem

$$(1)' \qquad L(u)=\varepsilon_2 f(x,t) \qquad x\in\Omega,\ t\geq 0,\ 0<\varepsilon_2\leq 1$$

$$\begin{cases} u(0)=\phi \\ u_t(0)=\psi \\ u|_{\partial\Omega}=0 \end{cases}$$

where Ω is a bounded open set in $\boldsymbol{R}^n$ with C^∞-boundary $\partial\Omega$. For the term $f(x, t)$, we assume

Assumption 5.

i) $f(x,t)\in C^s(\boldsymbol{R}^n\times\boldsymbol{R}^1)$.

ii) $\sup_{t\in\boldsymbol{R}^1}\|D^s f(t)\|\leq M<+\infty$.

Moreover we assume the following compatibility condition;

(24) $$u^{k-1}\in H^{s-k+2}\cap \mathring{H}^{s-k+1},\ u^k\in \mathring{H}^{s-k+1}\qquad \text{for}\quad 1\leq k\leq s$$

where

$$u^k=\left(\frac{d}{dt}\right)^k u(x, 0)$$

which are determined successively by (1)′ beginning with ϕ and ψ. We note that if $u\in\mathscr{D}(0, T|e_0)$ is the solution of (1)′, it follows

(25) $$\|D^{k+1}u\|\leq c(h_0, h_1, e_0, a_1, c_0, M, \Omega)(\|D_t^{k+1}u\|+\|D^{1,k}u\|+\varepsilon_2\|D^s f\|)$$

$$\text{for}\quad 1\leq k\leq s$$

by using the Lemmas 4 and 5 (especially v) of Lemma 4). We show the local arguments to the simple case ($b=0, f=0$ and $\alpha=0$) at the Appendix.

We have the following

Theorem 3. *We suppose Assumptions* 1, 3 *and* 5, *that is,*

i) $a_{ij}(x, t, y)\in C^{s+1}(\boldsymbol{R}^n\times\boldsymbol{R}^1\times\boldsymbol{R}^{n+2})$.

ii) $a_{ij}=a_{ji}$.

iii) $\sum a_{ij}(x, t, y)\xi_i\xi_j\geq a(y)\sum\xi_i^2,\qquad a(0)>0$

for all $x\in\boldsymbol{R}^n$, $t\in\boldsymbol{R}^1$, $y\in\boldsymbol{R}^{n+2}$, $\xi\in\boldsymbol{R}^n$
where $a(y)\in C^0(\boldsymbol{R}^{n+2})$.

iv) $\sup_{\boldsymbol{R}^n\times\boldsymbol{R}^1}\sum|D_{x,t,y}^{s+1}a_{ij}(x, t, y)|\leq h_0(|y|)$.

v) $b(y)\in C^{s+1}(\boldsymbol{R}^{n+2})$, $|D_y^{s+1}b(y)|\leq h_1(|y|)$.

vi) $D_y b(0)=0$.

vii) $$\begin{cases} f(x, t)\in C^s(\boldsymbol{R}^n\times\boldsymbol{R}^1) \\ \sup_{t\in\boldsymbol{R}^1}\|D^s f(t)\|\leq M<+\infty. \end{cases}$$

Here s *represents* $\left[\frac{n}{2}\right]+2$. *Moreover we suppose* $\phi\in H^{s+1}\cap \mathring{H}^s$ *and* $\psi\in\mathring{H}^s$ *satisfy the compatibility condition* (24), *and put*

$$\begin{cases} \|D_x^{s+1}\phi\|+\|D_x^s\psi\|=\varepsilon \\ \sup\limits_{R^n\times R^1}\sum|\bar{D}^s a_{ij}(x,\,t,\,0)|=\varepsilon_1. \end{cases}$$

Then there exists a positive constant ε_0 *such that the initial-boundary value problem for*

$$u_{tt}-\sum a_{ij}(x,\,t,\,Du)u_{ij}+u_t+b(Du)=\varepsilon_2 f(x,\,t)$$

$$\begin{cases} u(0)=\phi \\ u_t(0)=\psi \\ u|_{\partial\Omega}=0 \end{cases}$$

has a unique solution $u(x,\,t)$ *which satisfies*
$u(x,\,t)\in\mathscr{D}(0,\,+\infty|e_0)$ *and* $D_t^s u(x,\,t)\in\mathscr{E}_t^0(\mathring{H}^1)$ *for* $0<{}^\forall\varepsilon,\ {}^\forall\varepsilon_1,\ {}^\forall\varepsilon_2\le\varepsilon_0$. *Furthermore* u *satisfies*

$$\|D^{s+1}u(t)\|\le c\|D^{s+1}u(0)\|\exp(-\gamma t)+c\varepsilon_2\sup_{t\in R^1}\|D^s f(t)\|$$

where γ *is some positive constant.*

Corollary 3 (Periodic solutions). *We suppose Assumptions* 1, 3 *and* 5. *Moreover we suppose that* $a_{ij}(x,\,t,\,y)$ *and* $f(x,\,t)$ *are* ω*-time-periodic, that is,*

$$a_{ij}(x,\,t+\omega,\,y)=a_{ij}(x,\,t,\,y),\quad f(x,\,t+\omega)=f(x,\,t)$$

for all $x,\,t,\,y$. *Then there exists a positive constant* ε_0 *such that*

$$(1)'' \qquad L(u)=\varepsilon_2 f(x,\,t)\qquad x\in\Omega,\ t\in\mathbb{R}^1$$

$$u|_{\partial\Omega}=0$$

has a unique ω*-time-periodic solution* $u(x,\,t)$ *which satisfies*

$u(x,\,t)\in\mathscr{D}(-\infty,\,+\infty|e_0)$ *and* $D_t^s u(x,\,t)\in\mathscr{E}_t^0(\mathring{H}^1)$ *for*

$0<{}^\forall\varepsilon_1,\ {}^\forall\varepsilon_2\le\varepsilon_0$.

Furthermore, for the time periodic solution $u(x, t)$ as we got above, any solution $v(x, t)$ of (1)″ *which satisfies*

$$\begin{cases} v(x, t) \in \mathscr{D}(0, +\infty|e_0) \quad \text{and} \quad D_t^s v(x, t) \in \mathscr{E}_t^0(\mathring{H}^1) \\ \|D^{s+1}v(0)^s\| = \varepsilon \end{cases}$$

is asymptotic to $u(x, t)$ exponentially as $t \to +\infty$, that is,

$$\|D^s(u-v)(t)\| \leq c \exp(-\gamma t) \qquad (\gamma > 0)$$

for $0 < {}^\forall\varepsilon, {}^\forall\varepsilon_1, {}^\forall\varepsilon_2 \leq \varepsilon_0$.

Proof of Theorem 3. Recalling the arguments in §2, it is sufficient only to show the *apriori* estimate for

$$u \in \mathscr{D}(0, +\infty|e) \cap \mathscr{E}_t^\infty(H^{s+1}) \cap \mathscr{E}_t^\infty(\mathring{H}^s).$$

If we want to show the estimate for $u \in \mathscr{D}(0, +\infty|e)$ and $D_t^s u \in \mathscr{E}_t^0(\mathring{H}^1)$, we may use the mollifier with respect to t for this case. Then estimating

$$\int D_t^s\{L(u)\} \cdot D_t^s u_t dx + \lambda \int D_t^s\{L(u)\} \cdot D_t^s u \, dx$$

$$= \varepsilon_2 \int D_t^s f \cdot D_t^s u_t dx + \lambda\varepsilon_2 \int D_t^s f \cdot D_t^s u \, dx$$

by the same way as in Theorem 2, we get

$$\text{(26)} \qquad \frac{d}{dt}\{E(t)\} + \frac{1}{2}\|\bar{D}_t^{s+1}u(t)\|^2 + \lambda a_1\|\bar{D}^{1,s}u(t)\|^2$$

$$-c(1+\lambda)\{\varepsilon_1+\varepsilon_2+(e+e^s)h(e_0)\}\|D^{s+1}u(t)\|^2 \leq c\varepsilon_2(1+\lambda)\|D_t^s f(t)\|^2$$

where

$$E(t) = \int \frac{\lambda}{2}|D_t^s u|^2 + \lambda D_t^s u \cdot D_t^s u_t + \frac{1}{2}|D_t^s u_t|^2$$

$$+ \frac{1}{2}\sum a_{ij}(x, t, Du) D_t^s u_i \cdot D_t^s u_j dx \qquad \left(0 < \lambda < \frac{1}{2}\right).$$

By Poincaré's inequality

$$\|u\| \leq c(\Omega)\, \|\bar{D}_x u\| \tag{27}$$

and (25), it follows

$$\|D^{s+1}u\| \leq c(\|\bar{D}_t^{s+1}u\| + \|\bar{D}^{1,s}u\| + \varepsilon_2\|D^s f\|).$$

Therefore choosing δ small, (26) implies

$$\frac{d}{dt}\{E(t)\} + \frac{1}{4}\|\bar{D}_t^{s+1}u(t)\|^2 + \frac{\lambda}{2}a_1\|\bar{D}^{1,s}u(t)\|^2 \tag{28}$$

$$\leq c\varepsilon_2\|D^s f(t)\|^2 \qquad \text{for} \quad 0 < e,\ \varepsilon_1,\ \varepsilon_2 \leq \delta.$$

By Lemma 2 and (27), we get

$$E(t) \leq c\{\|\bar{D}_t^{s+1}u(t)\|^2 + \|\bar{D}^{1,s}u(t)\|^2\}$$

so that by (28)

$$\frac{d}{dt}\{E(t)\} + 2\gamma E(t) \leq c\varepsilon_2\|D^s f(t)\|^2 \quad (\gamma > 0)$$

which implies

$$E(t) \leq cE(0)\exp(-2\gamma t) + c\varepsilon_2 \sup_{t \in R^1} \|D^s f(t)\|^2.$$

Hence we get the estimate

$$\|D^{s+1}u(t)\| \leq c\|D^{s+1}u(0)\| \exp(-\gamma t) + c\varepsilon_2 \sup_{t \in R^1} \|D^s f(t)\|$$

which become *apriori* estimate. Q.E.D.

Proof of Corollary 3. We consider the following initial-boundary value problem;

$$L(u^m) = \varepsilon_2 f^m(x, t) \qquad (m = 0, 1, 2, \ldots) \tag{29}$$

$$\begin{cases} u^m(x, -m) = 0 \\ u_t^m(x, -m) = 0 \\ u^m|_{\partial\Omega} = 0 \end{cases}$$

where $f^m(x, t)$ satisfies the following conditions:

i) $f^m(x, t) \in C^s(\boldsymbol{R}^n \times \boldsymbol{R}^1)$.

ii) $\sup_{t \in \boldsymbol{R}^1} \|D^s f^m(t)\| \leq cM$ where c is independent of m.

iii) $\begin{cases} f^m(x, t) \equiv f(x, t) & \text{for} \quad t \geq -m+1, \\ D^s f^m(x, t) \equiv 0 & \text{for} \quad t \leq -m. \end{cases}$

Applying Theorem 3 to (29), we have the solution of (29) as

$$(30) \qquad \begin{cases} u^m(x, t) \in \mathscr{D}(-m, +\infty | e_0) \quad \text{and} \quad D_t^s u^m(x, t) \in \mathscr{E}_t^0(\mathring{H}^1) \\ \sup_{t \geq -m} \|D^{s+1} u^m(t)\| \leq c\varepsilon_2 \qquad \text{for} \quad 0 < \varepsilon_1, \ \varepsilon_2 \leq {}^{\exists}\varepsilon_0 \end{cases}$$

where we emphasize that c and ε_0 are independent of m. Putting $u^m(x, t) \equiv 0$ for $t \leq -m$ we can extend $u^m(x, t)$ on $-\infty < t < +\infty$ as

$$(31) \qquad \begin{cases} u^m(x, t) \in \mathscr{D}(-\infty, +\infty | e_0) \quad \text{and} \quad D_t^s u^m(x, t) \in \mathscr{E}_t^0(\mathring{H}^1) \\ \sup_{t \in \boldsymbol{R}^1} \|D^{s+1} u^m(t)\| \leq c\varepsilon_2 \qquad \text{for} \quad 0 < \varepsilon_1, \ \varepsilon_2 \leq \varepsilon_0. \end{cases}$$

Then estimating

$$\int D_t^{s-1}\{L(u^{m+1}) - L(u^m)\} \cdot \{D_t^{s-1}(u_t^{m+1} - u_t^m) + \lambda D_t^{s-1}(u^{m+1} - u^m)\} dx = 0 \qquad (t \geq -m+1)$$

by using Lemmas 4, 5 and (27) as before, we have

$$(32) \qquad \frac{d}{dt}\{E_{u^m}(u^{m+1} - u^m)\} + 2\gamma E_{u^m}(u^{m+1} - u^m) \leq 0$$

for $0 < \varepsilon_1, \varepsilon_2 \leq {}^{\exists}\varepsilon_0$ and $t \geq -m+1$ where γ is some positive constant independent of m and

$$E_v(u) = \int \frac{\lambda}{2} |D_t^{s-1} u|^2 + \lambda D_t^{s-1} u \cdot D_t^{s-1} u_t + \frac{1}{2} |D_t^{s-1} u_t|^2 + \frac{1}{2} \sum a_{ij}(x, t, Dv) D_t^{s-1} u_i \cdot D_t^{s-1} u_j \, dx.$$

From (32), we have

$$(33) \qquad \|D^s(u^{m+1} - u^m)(t)\| \leq c \|D^s(u^{m+1} - u^m)(\tau)\| \exp\{-\gamma(t - \tau)\}$$

for all $t \geq \tau \geq -m+1$. Let T be any fixed finite number. Then we can suppose $T \geq -m+1$ by taking m large. So it follows from (31) and (33)

$$\|D^s(u^{m+1}-u^m)(T)\| \leq c \exp\{-\gamma(T+m-1)\}.$$

This gives

$$\|D^s(u^{m+1}-u^m)(T)\| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty. \tag{34}$$

Moreover we have from (33)

$$\|D^s(u^{m+1}-u^m)(t)\| \leq c\|D^s(u^{m+1}-u^m)(T)\| \exp\{-\gamma(t-T)\} \qquad \text{for} \quad t \geq T. \tag{35}$$

Therefore it follows from (34) and (35) that

$$\sup_{t \geq T} \|D^s(u^{m+1}-u^m)(t)\| \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty \tag{36}$$

for any fixed finite number T. On the other hand, it is clear that

$$\sup_{t \geq T} \|D^s(f^m-f)(t)\| \longrightarrow 0 \qquad m \longrightarrow +\infty \tag{37}$$

for any finite number T. Hence (31), (36) and (37) give the existence of a solution of (1)″ (refer to the last of Appendix for the regularity).

Now we will show the uniqueness. We suppose two solutions u and v to (1)″ exist. Then putting $w=u-v$, we have

$$\|D^s w(t)\| \leq c\|D^s w(\tau)\| \exp\{-\gamma(t-\tau)\} \qquad \text{for} \quad t \geq \tau \tag{38}$$

by the same way as we got (33). Now if $w \not\equiv 0$, there exists some t_0 such that

$$\|D^s w(t_0)\| \neq 0. \tag{39}$$

From (38) we get

$$\|D^s w(t_0)\| \leq c \exp\{-\gamma(t_0-\tau)\} \qquad \text{for all} \quad \tau \leq t_0. \tag{40}$$

If we choose τ negatively large enough, (40) contradicts to (39) and this implies the uniqueness. Therefore from the existence and uniqueness, it is clear that if a_{ij} and f are periodic, the solution is periodic. Finally we can get the stability from (38). Q.E.D.

Appendix

We consider the local solution of the following initial-boundary value problem;

(41) $$u_{tt}-\sum_{i,j=1}^{n}a_{ij}(x,\,t,\,Du)u_{ij}=0 \qquad x\in\Omega \quad t\geq 0$$

$$\begin{cases} u(x,\,0)=\phi \\ u_t(x,\,0)=\psi \\ u|_{\partial\Omega}=0 \end{cases}$$

where Ω is $\boldsymbol{R}^n$ or a bounded open set in $\boldsymbol{R}^n$ with C^∞-boundary $\partial\Omega$. We assume ϕ and ψ satisfies the compatibility condition in the sense of (24).

First we consider the following linear problem;

(42) $$L_v(u)\equiv u_{tt}-\sum_{i,j}a_{ij}(x,\,t,\,Dv(x,\,t))u_{ij}=f(x,\,t) \qquad x\in\Omega \quad t\geq 0$$

$$\begin{cases} u(x,\,0)=\phi \\ u_t(x,\,0)=\psi \\ u|_{\partial\Omega}=0. \end{cases}$$

Then we have the following

Proposition 1. *We suppose Assumption* 1. *Moreover we suppose that* $\phi\in H^{s+1}\cap\mathring{H}^s$ *and* $\psi\in\mathring{H}^s$ *satisfy the compatibility condition and that*

(43) $$\begin{cases} v\in\mathscr{D}(0,\,+\infty\,|\,e) \\ f\in\mathscr{E}_t^i(H^{s-i}) \qquad (0\leq i\leq s). \end{cases}$$

Then (42) *has a unique solution* $u(x,\,t)$ *which satisfies*

$$u(x,\,t)\in\mathscr{E}_t^0(H^{s+1}\cap\mathring{H}^s)\cap\mathscr{E}_t^i(\mathring{H}^{s+1-i}) \qquad (1\leq i\leq s+1)$$

and the following inequality holds: *For* $1\leq l\leq s$

(44) $\|D^{l+1}u(t)\|^2 \leq c\exp(ct)\{\|D^{l+1}u(0)\|^2 + \|D^{l-1}f(0)\|^2 + \int_0^t \|D^l f(\tau)\|^2 d\tau\}$.

Corollary 4. *In Proposition* 1, *we further suppose* $f \equiv 0$.

Then there exist the positive constants t_0 *and* $\delta\,(<1)$ *such that for* $\|D^{s+1}u(0)\| \leq \delta e$ (42) *has a unique solution*

$$u(x, t) \in \mathscr{D}(0, t_0|e) \cap \mathscr{E}_t^0(H^{s+1} \cap \mathring{H}^s) \cap \mathscr{E}_t^i(\mathring{H}^{s+1-i}) \qquad (1 \leq i \leq s+1)$$

where t_0 *and* δ *depend on* e_0 *but not on* e.

Proof of Proposition 1. Extending v on $-1 < t \leq 0$ properly, we have from Assumption 1 and (43) that

$$\tilde{a}_{ij}(x, t) \equiv a_{ij}(x, t, Dv(x, t)) \in \mathscr{E}_t^i(H^{s-i}) \qquad (0 \leq i \leq s)$$

$$\sum \tilde{a}_{ij}(x, t)\xi_i\xi_j \geq a_1 \sum \xi_i^2 \qquad a_1 > 0$$

$$\tilde{a}_{ij}(x, t) = \tilde{a}_{ji}(x, t)$$

for all $x \in \Omega$, $\xi \in R^n$ and $t > -1$. Therefore, taking care of regularity of $\tilde{a}_{ij}$, we have from the arguments in [4] that if $\phi \in H^2 \cap \mathring{H}^1$, $\psi \in \mathring{H}^1$ and $f \in \mathscr{E}_t^1(L^2)$ (42) has a unique solution $u(x, t)$ which satisfies

(45) $u(x, t) \in \mathscr{E}_t^0(H^2 \cap \mathring{H}^1) \cap \mathscr{E}_t^1(\mathring{H}^1) \cap \mathscr{E}_t^2(L^2)$

$$\|D^2u(t)\|^2 \leq c\exp(ct)\{\|D^2u(0)\|^2 + \|f(0)\|^2 + \int_0^t \|D_t^1 f(\tau)\|^2 d\tau\}.$$

So let us show the regularity of the solution. Now we put

$$w^0 = u_t, \quad w^k = u_k \qquad (1 \leq k \leq n).$$

Then differentiating (42), we have

(46) $$L_v(w^k) = -\sum_{i,j=1}^{n} a_{ij}^k w_j^i + f_k \qquad \text{for} \quad 0 \leq k \leq n$$

where $a_{ij}^0 = \frac{\partial}{\partial t}a_{ij}$, $a_{ij}^k = \frac{\partial}{\partial x_k}a_{ij}$ $(1 \leq k \leq n)$ and $f_0 = f_t$. In order to solve the (46) we make the sequences $\{w^{k,m}\}$ $(m \geq 0)$ as follows; for $m = 0$

$$w^{0,0} \equiv \psi, \quad w^{k,0} \equiv \phi_k \qquad (1 \leq k \leq n),$$

for $m\geq 1$

$$L_v(w^{k,m})=-\sum_{i,j=1}^{n}a_{ij}^{k}w_j^{i,m-1}+f_k \qquad (0\leq k\leq n)$$

$$\begin{cases} w^{0,m}(0)=\psi \\ w_t^{0,m}(0)=-\sum a_{ij}^0(x,0)\phi_{ij}+f(0) \\ w^{0,m}|_{\partial\Omega}=0, \end{cases} \begin{cases} w^{k,m}(0)=\phi_k \\ w_t^{k,m}(0)=\psi_k \\ w^{k,m}|_{\partial\Omega}=0. \end{cases} \qquad (1\leq k\leq n)$$

Using the assumptions, we have from (45) that

$$w^{k,m}(x,t)\in\mathscr{E}_t^0(H^2\cap\mathring{H}^1)\cap\mathscr{E}_t^1(\mathring{H}^1)\cap\mathscr{E}_t^2(L^2) \qquad (m\geq 0,\ 0\leq k\leq n),$$

$$\sum_{k=0}^{n}\|D^2(w^{k,m+1}-w^{k,m})(t)\|^2\leq c\int_0^t\|D_t^1(\sum_{\substack{k=0\\ i,j=1}}^{n}a_{ij}^k(w^{k,m}-w^{k,m-1}))(\tau)\|^2d\tau$$

$$(47) \qquad \leq c\int_0^t\sum_{k=0}^{n}\|D^2(w^{k,m}-w^{k,m-1})(\tau)\|^2d\tau,$$

$$\sum_{k=0}^{n}\|D^2w^{k,m}(t)\|^2\leq c\exp(ct)\{\sum_{k=0}^{n}\|D^2w^{k,m}(0)\|^2+\|D^1f(0)\|^2$$
$$+\int_0^t\{\sum_{k=0}^{n}\|D^2w^{k,m-1}(\tau)\|^2+\|D^2f(\tau)\|^2\}d\tau\} \quad (m\geq 1).$$

From (47) we can get the solutions of (46). So (42) has a solution

$$u(x,t)\in\mathscr{E}_t^0(H^3\cap\mathring{H}^2)\cap\mathscr{E}_t^{i+1}(\mathring{H}^{2-i}) \qquad (0\leq i\leq 2).$$

Furthermore (45) and (47) give

$$\|D^3u(t)\|^2\leq c\exp(ct)\left\{\|D^3u(0)\|^2+\|D^1f(0)\|^2+\int_0^t\|D^2f(\tau)\|^2d\tau\right\}.$$

Similarly we can get the regularity up to the order $s+1$ and the energy inequality (44) step by step. Q.E.D.

Proof of Corollary 4. It follows from (44) that

$$(48) \qquad \|D^{s+1}u(t)\|^2\leq c\exp(ct)\|D^{s+1}u(0)\|^2$$

where c depends on e_0, h_0 and a_1 but not on e. Therefore choosing t_0

and δ such that

$$\exp(ct_0) \leq 2, \quad 2c\delta^2 \leq 1,$$

we have

$$\sup_{0 \leq t \leq t_0} \|D^{s+1}u(t)\| \leq e.$$

Q.E.D.

Under the above preparations, we have the following

Theorem 4. *We suppose Assumption* 1. *Moreover we suppose that* $\phi \in H^{s+1} \cap \mathring{H}^s$ *and* $\psi \in \mathring{H}^s$ *satisfy the compatibility condition. Then there exist the positive constants* t_0 *and* $\delta\,(<1)$ *such that if* $\|D^{s+1}u(0)\| \leq \delta e\ (0 < e \leq e_0)$, *then the initial-boundary problem for* (41) *has a unique solution* $u(x, t)$ *which satisfies*

$$u(x, t) \in \mathscr{D}(0, t_0|e) \quad \text{and} \quad D^{k,l}u(x, t) \in \mathscr{E}_t^0(\mathring{H}^1) \qquad (0 \leq k+l \leq s,\ k \neq s).$$

Proof. We first note that we can construct some $w(x, t) \in \mathscr{E}_t^i(H^{s+1-i})$ $(0 \leq i \leq s+1)$ satisfying

$$D^{s+1}w(0) = D^{s+1}u(0) \quad \text{and} \quad \delta\|D^{s+1}w(t)\| \leq \|D^{s+1}w(0)\| \qquad (0 \leq t \leq t_0).$$

Then we construct the approximate sequence as follows;

$$u^0 = w,$$

$$\text{(49)} \qquad L_{u^{m-1}}(u^m) = 0 \qquad (m \geq 1)$$

$$\begin{cases} u^m(0) = \phi \\ u_t^m(0) = \psi \\ u^m|_{\partial\Omega} = 0. \end{cases}$$

Since $D^{s+1}u^m(0) = D^{s+1}w(0)$ for all $m \geq 0$, it follows by Corollary 4 that

$$\text{(50)} \qquad u^m(x, t) \in \mathscr{D}(0, t_0|e) \cap \mathscr{E}_t^0(H^{s+1} \cap \mathring{H}^s) \cap \mathscr{E}_t^i(\mathring{H}^{s+1-i}) \qquad (1 \leq i \leq s+1)$$

for all $m \geq 0$. From (49) we have

$$L_{u^m}(u^{m+1}-u^m)=A^m(u^m-u^{m-1})$$

where $A^m(u^m-u^{m-1})=\sum\{a_{ij}(x, t, Du^m)-a_{ij}(x, t, Du^{m-1})\}u^m_{ij}$. So from (44) we have

$$\|D^s(u^{m+1}-u^m)(t)\|^2\leq c\int_0^t\|D^{s-1}A^m(u^m-u^{m-1})(\tau)\|^2d\tau. \tag{51}$$

On the other hand we have from iv) of Lemma 4

$$\|D^{s-1}A^m(u^m-u^{m-1})\|^2\leq c(e_0)\|D^s(u^m-u^{m-1})\|^2. \tag{52}$$

Substituting (52) to (51), we have

$$\|D^s(u^{m+1}-u^m)(t)\|^2\leq c\int_0^t\|D^s(u^m-u^{m-1})(\tau)\|^2d\tau \qquad \text{for all} \quad m\geq 1. \tag{53}$$

Therefore (50) and (53) give a solution of (41) satisfying

$$D^{s+1}u(x, t)\in L_t^\infty(L^2) \quad \text{and} \quad D^{k,l}u(x, t)\in L_t^\infty(\mathring{H}^1) \qquad (0\leq k+l\leq s,\ k\neq s).$$

Finally the uniqueness follows from the similar energy inequality as (53) and the regularity $D^{s+1}u\in\mathscr{E}_t^0(L^2)$ follows from

$$\sup_{0\leq t\leq t_0}\|D^{s+1}(\varphi_\delta*u-\varphi_{\delta'}*u)(t)\|\to 0, \quad \text{as} \quad \delta,\ \delta'\to 0$$

where $\varphi_\delta*$ is the mollifier with respect to t and u is extended properly on $-\varepsilon<t<t_0+\varepsilon$. Q.E.D.

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References

[1] Dionne, P., Sur les problème de Cauchy hyperboliques bien posés, *J. Analyse Math.*, **10** (1962), 1–90.
[2] Matsumura, A., On the asymptotic behavior of solutions of semilinear wave equations, *Publ. RIMS, Kyoto Univ.*, **12** (1976), 169–189.
[3] Mizohata, S., *Theory of partial differential equations*, Cambridge UP., 1973.
[4] ———, Quelque problèmes au bord, du type mixte, pour des équations

hyperboliques. *Séminaire sur les équations aux derivées partielles,* Collège de France, (1966–1967), 23–60.

[5] Nirenberg, L., On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa.,* **13** (1959), 115–162.

[6] Nishida, T., Global smooth solutions for the second-order quasilinear wave equations with the first-order dissipation (*to appear*).

[7] Sattinger, D., Stability of nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.,* **28** (1968), 226–244.

[8] Rabinowitz, P., Periodic solutions of nonlinear partial differential equations, *Comm. Pure Appl. Math.,* **20** (1967), 145–205.

[9] ———, Periodic solutions of nonlinear partial differential equations II, *Comm. Pure Appl. Math.,* **22** (1969), 15–39.