Publ. RIMS, Kyoto Univ.
13 (1977), 497–521

A New Class of Domains of Holomorphy (I)

(The concepts of boundary resolutions and L-manifolds)

By

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§1. Introduction

The present paper is the first part of our study of a class of domains of holomorphy which includes certain complex manifolds with non-Stein algebras, i.e., the algebra of holomorphic functions on the complex manifold is not a Stein algebra.

Following H. Kerner [7], we can define the concept of domains of holomorphy for holomorphically separable manifolds. In what follows $\underline{X}$ is assumed to be a holomorphically separable manifold which is a domain of holomorphy (see Definition (2.5)). If $\underline{X}$ admits a fibre discrete holomorphic mapping with empty branched locus $\Phi: \underline{X} \to \mathbb{C}^n$, then the classical fundamental Oka Theorem states that $\underline{X}$ is a Stein manifold. Unfortunately $\underline{X}$ does not always have such a fibre discrete mapping. In this situation we encounter with tremendous difficulties. For example, there exists a non-pseudoconvex domain with a non-Stein algebra (for the definition of pseudoconvex domains, see Definition (2.8)).

Then we have the following problems: What are the necessary conditions of domains of holomorphy? and what are the good sufficient conditions of domains of holomorphy?

The purpose of the present paper is to give a class of domains of holomorphy including (1) non-holomorphically convex manifolds, (2) non-pseudoconvex domains and (3) manifolds with non-Stein algebras.

Communicated by S. Nakano, October 1, 1976.
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In order to discuss these problems, we consider general complex manifolds at first. Let X be a complex manifold and let $\mathcal{O}(X)$ denote the algebra of the holomorphic functions on X. Following H. Grauert [3], we introduce an equivalence relation on X as follows: For any pair of two points p and q in X, $p\sim q \Leftrightarrow f(p)=f(q)$ for any $f\in\mathcal{O}(X)$. Then the quotient space W thus obtained has a structure of a ringed space, which is denoted by $\mathrm{Spec}\,\mathcal{O}(X)$ (see Definition (2.2)). The natural projection is denoted by $\varpi\colon X\to\mathrm{Spec}\,\mathcal{O}(X)$. Note that $\mathrm{Spec}\,\mathcal{O}(X)$ does not always admit the structure of a complex space. By Γ we denote the smallest closed set such that $\mathrm{Spec}\,\mathcal{O}(X)-\Gamma$ admits the structure of a complex space. A complex manifold X is called a resolution manifold if $X-\varpi^{-1}(\Gamma)\cong\mathrm{Spec}\,\mathcal{O}(X)-\Gamma$ holds. Now we introduce the notion of B-resolution (or boundary-resolution) of a holomorphically separable manifold $\underline{X}$ as follows: A resolution manifold X is called a B-resolution of $\underline{X}$ if $\mathrm{Spec}\,\mathcal{O}(X)-\Gamma\cong\underline{X}$. When X is a Stein manifold, we see that $\mathrm{Spec}\,\mathcal{O}(X)=X$, i.e., $\Gamma=\emptyset$. So every Stein manifold $\underline{X}$ has a trivial B-resolution $X=\underline{X}$. As will be shown, it may be interesting to consider a complex manifold $\underline{X}$ which has a B-resolution X with non-empty Γ.

In what follows we assume

(A-1) A B-resolution X of $\underline{X}$ is given,

(A-2) Each fibre of ϖ is connected.

By using the notion of B-resolutions, we can give a class of holomorphically separable complex manifolds, which are called L-manifolds (see Definition (3.11)) and prove the following theorems:

Theorem I. *Every* L-*manifold is a domain of holomorphy.*

Theorem II. *Every Stein manifold is an* L-*manifold and there exist* L-*manifolds which are neither holomorphically convex nor pseudoconvex with respect to any representation* (*see Definition* (2.4)). *Moreover, there exist* L-*manifolds with non-Stein algebras.*

Finally we are concerned with several examples due to H. Grauert

[3] and M. Otuki [10] respectively and some discussions will be given.

In the second part of our study, we will construct B-resolutions for certain domains of holomorphy on a certain 3-dimensional Stein space with an isolated singular point and will prove that they are L-manifolds. By using this construction, we can systematically make examples of L-manifolds.

The author would like to express his hearty thanks to Professors S. Nakano, S. Iitaka and H. Fujimoto and Mr. T. Sasaki, Mr. H. Omoto and Mr. M. Otuki during the preparation of the present paper.

§2. Basic Properties of Spec $\mathcal{O}(X)$ and Generalities on K-Complete Manifolds

Let X be a complex manifold. As in §1 we consider the equivalence relation and denote the quotient space by W. Following H. Grauert [3], we define a structure of a ringed space as follows: Let $\underline{U}$ be an open set in W and consider a continuous function $\underline{f}$ on $\underline{U}$ which has the following expression:

$$\varpi^*\underline{f} = \sum_{i_1, i_2, \dots, i_m}^{\infty} a_{i_1, i_2, \dots, i_m} \cdot f_1^{i_1} \cdot f_2^{i_2} \cdots f_m^{i_m}, \tag{2.1}$$

where $f_1, f_2, \dots, f_m$ are holomorphic functions on X. Convergence means compact convergence on $U = \varpi^{-1}(\underline{U})$. Then $\{\underline{f}\}$ make a certain subalgebra of continuous functions on $\underline{U}$ and from this we get a sheaf $\mathscr{A}$ as usual. Thus we obtain a ringed space $(W, \mathscr{A})$.

Definition (2.2). *The ringed space* $(W, \mathscr{A})$ *is written* Spec $\mathcal{O}(X)$ *for simplicity.* $\mathscr{A}(W)$ *denotes the algebra of global sections of* $\mathscr{A}$.

Here we state several basic properties of Spec $\mathcal{O}(X)$ (see H. Grauert [3]):

Theorem 1. *Let* $A_x = \varpi^{-1}(\varpi(x))$ *for* $x \in X$. *Then* (1) $A_r = \{x \in X : \dim_x A_x \geqq r\}$ *is an analytic set in* X *for each* r. *Therefore* $A = \{x \in X : \operatorname{rk} \varpi > \operatorname{rk}_x \varpi\}$ *is also an analytic set, where* $\operatorname{rk}_x \varpi = \operatorname{codim} A_x$ *and* $\operatorname{rk} \varpi = \sup_{x \in X} \operatorname{rk}_x \varpi$. (2) *Suppose that* $\operatorname{rk}_{x_0} \varpi$ *is locally constant on some small* *neighborhood of x_0. Then there exist neighborhoods U of x_0 and $\underline{U}$ of $\varpi(x_0)$ such that (i) $(\underline{U}, \mathscr{A}_{|\underline{U}})$ is isomorphic to some complex space and (ii) $\varpi: U \to \underline{U}$ is a holomorphic mapping with respect to the induced structure in (i).* (3) *$\mathcal{O}(X) \cong \mathscr{A}(W)$.* (4) *$W$ is $\mathscr{A}$-separable, i.e., for any pair of two points x and y in W with $x \neq y$, there exists an $f \in \mathscr{A}(W)$ such that $f(x) \neq f(y)$.*

Remark. By (2) in Theorem 1, $(W-A, \mathscr{A}_{|W-A})$ is isomorphic to some complex space. Hence $\varpi^{-1}(\Gamma)$ is contained in A (for the definition of Γ, see Introduction). This implies that $\varpi^{-1}(\Gamma)$ is a thin set in X. In what follows we write $\underline{X} = W - \Gamma$, $\mathcal{O}_{\underline{X}} = \mathscr{A}_{|\underline{X}}$.

As for the complex structure we have the following

Proposition (2.3). *If $\varpi^{-1}(\varpi(x_0))$ is compact for x_0, then there exists a neighborhood of $\varpi(x_0)$ in W which admits a complex structure.*

Proof. By definition, we have

$$\varpi^{-1}(\varpi(x_0)) = \bigcap_{f \in \mathcal{O}(X)} \{x : f(x) = f(x_0)\}.$$

Then for any point $p \in A_{x_0}$, there exist a neighborhood U and a finite number of holomorphic functions $f_1, f_2, \dots, f_r$ satisfying

$$A_{x_0} \cap U = \bigcap_{j=1}^{r} \{x : f_j(x) = f_j(x_0)\}.$$

By the compactness of A_{x_0}, there exist a finite covering $\{U_j\}$ of a small neighborhood of A_{x_0} and finite number of basis of the defining equations $f_j^{(1)}, f_j^{(2)}, \dots, f_j^{(r_j)}$ $(j = 1, 2, \dots, k)$ of $A_{x_0} \cap U_j$ satisfying

$$A_{x_0} \cap U_j = \bigcap_{s=1}^{r_j} \{x : f_j^{(s)}(x) = f_j^{(s)}(x_0)\}.$$

Hence we have

$$A_{x_0} = \bigcap_{j=1}^{k} \bigcap_{s=1}^{r_j} \{x : f_j^{(s)}(x) = f_j^{(s)}(x_0)\}.$$

Thus we have a holomorphic mapping $\Phi: \underline{X} \to \mathbf{C}^N$, $\Phi = (f_1^{(1)}, \dots, f_1^{(r_1)},$

$f_2^{(1)},\ldots,f_k^{(r_k)})$ satisfying $\Phi^{-1}(0)=A_{x_0}$. Therefore Φ is a proper mapping on V_ε, where $V_\varepsilon=\Phi^{-1}(D_\varepsilon)$ for a small polydisk D_ε with center $0\in\mathbf{C}^N$. We find that because of the compactness of $\Phi^{-1}(\Phi(p))$, the connected component of $\Phi^{-1}(\Phi(p))$ containing $\varpi^{-1}(\varpi(p))$ is nothing but $\varpi^{-1}(\varpi(p))$. By using (A–2) and adding more functions, we may assume

$$\Phi^{-1}(\Phi(p))=\varpi^{-1}(\varpi(p)) \qquad \text{for} \quad p\in V_\varepsilon.$$

By the Stein factorization theorem, we have a complex structure $(U_\varepsilon, \mathcal{O})$ on U_ε, where $U_\varepsilon=\Phi(V_\varepsilon)$. Referring to (2.1) and taking account that every element of Φ is a holomorphic function on X, we see that $(\underline{V}_\varepsilon, \mathscr{A}_{|\underline{V}_\varepsilon})\cong(U_\varepsilon, \mathcal{O})$, where $\underline{V}_\varepsilon=\varpi(V_\varepsilon)$.

In the rest of this section, we assemble notations on K-complete manifolds. The definition of K-complete manifolds is given as follows:

Definition (2.4). *$\underline{X}$ is called a* K-*complete manifold if there exists a system of holomorphic functions $f_1, f_2,\ldots, f_n$, where $n=\dim\underline{X}$ such that $\Phi=(f_1, f_2,\ldots, f_n)\colon \underline{X}\to\mathbf{C}^n$ is a fibre discrete mapping. Φ is called a representation.*

It is well known that every holomorphically separable manifold is K-complete (see R. Iwahashi [5]).

Following H. Kerner [7], we can define the concept of domains of holomorphy for K-complete manifolds as follows:

Theorem 2. *Let $\underline{X}$ be a* K-*complete complex space. Then there exists one and only one complex space $\underline{X}^*$ such that* (1) *there exists a fibre discrete mapping $\gamma\colon \underline{X}\to\underline{X}^*$ such that for any holomorphic function $f\in\mathcal{O}(\underline{X})$ there exists a holomorphic function $f^*\in\mathcal{O}(\underline{X}^*)$ satisfying $f=f^*\circ\gamma$,* (2) *for any $\underline{Y}$ satisfying* (1) *with respect to $\gamma'\colon \underline{X}\to\underline{Y}$ there exists a fibre discrete mapping $\tau\colon \underline{Y}\to\underline{X}^*$ such that (i) $\gamma=\tau\circ\gamma'$ and (ii) for any $f\in\mathcal{O}(\underline{Y})$ there exists an $\tilde{f}\in\mathcal{O}(\underline{X}^*)$ satisfying $f=\tilde{f}\circ\gamma$ and* (3) *$\underline{X}^*$ is a* K-*complete space.*

Definition (2.5). *$\underline{X}^*$ is called the* K-*convex hull of $\underline{X}$. If $\underline{X}\cong\underline{X}^*$, $\underline{X}$ is called a domain of holomorphy.*

We fix a representation $\Psi: \underline{X} \to \mathbf{C}^n$ and describe the definitions of boundary points of $(\underline{X}, \Psi)$ and pseudoconvex domains. Following H. Grauert and R. Remmert [4], we make

Definition (2.6). Let $(\underline{X}, \Psi)$ be a representation of $\underline{X}$. A filter r of open sets in $\underline{X}$ is called a boundary point if the following conditions are satisfied: (1) r has no cluster sets in $\underline{X}$, (2) $\{\Psi(U), U \in r\}$ determines one and only one point $\underline{r}$ in $\mathbf{C}^n$ and (3) for any neighborhood $\underline{U}$ of $\underline{r}$, one of connected components of $U = \Psi^{-1}(\underline{U})$ is contained in r and moreover, r is equivalent to a filter generated by such open sets. The set of all boundary points $\{r\}$ is called the boundary of $(\underline{X}, \Psi)$, which is denoted by $\partial \underline{X}$.

We write $\underline{\hat{X}} = \underline{X} \cup \partial \underline{X}$ and introduce a topology on $\underline{\hat{X}}$ as follows: A neighborhood W_r of $r \in \partial \underline{X}$ is a union of $W_0 \in r$ and all the boundary points determined by filters containing at least one open set $W \subset W_0$ as an element. With respect to this topology, $\underline{\hat{X}}$ is a Hausdorff space and has countable basis at infinity. Set $\hat{\Psi}(r) = \underline{r}$ for $r \in \partial \underline{X}$ and $\hat{\Psi}(x) = \Psi(x)$ for $x \in \underline{X}$, we have a continuous extension of Ψ. We infer that every boundary point is accessible.

Definition (2.7). *Let* $\mathrm{D} = \{w \in \mathbf{C}: |w| \leqq 1\}$ *and* $\mathrm{I} = \{t \in \mathbf{R}: 0 \leqq t \leqq 1\}$. *Then* (1) $\sigma: \mathrm{D} \times \mathrm{I} \to \underline{\hat{X}}$ *is called a continuous family of disks if* σ *is a non-constant continuous mapping and* $\Psi \circ \sigma(w, t_0)$ *is a holomorphic mapping of* w *for fixed* t_0 $(0 \leqq t_0 \leqq 1)$, (2) *a continuous family of disks is called an Oka family if* $\{\sigma(w, t): |w| \leqq 1\} \subset \underline{X}$ *for* $0 \leqq t < 1$ *and* $\{\sigma(w, t): |w| = 1\} \subset \underline{X}$ *for* $0 \leqq t \leqq 1$.

Definition (2.8). $r \in \partial \underline{X}$ *is called a pseudoconvex boundary point if there exists a neighborhood* $U(r) \subset \underline{\hat{X}}$ *such that for any Oka family* $\sigma \subset U(r)$, $\{\sigma(w, t): |w| \leqq 1$ *and* $0 \leqq t \leqq 1\} \subset \underline{X}$ *holds.* $(\underline{X}, \Psi)$ *is called a pseudoconvex domain if every boundary point is pseudoconvex.*

Remark. The definition of (pseudoconvex) boundary points depends on the choice of the representation.

Here we prepare two propositions which will be used later:

Proposition (2.9). *Let $\underline{X}$ be a holomorphically separable manifold. $\underline{X}$ is a domain of holomorphy if for any representation Ψ: $\underline{X}\to\mathbf{C}^n$ and for any boundary point r of $(\underline{X}, \Psi)$ there exists a sequence $\{q_k\}$ with $q_k\to r$ $(k\to\infty)$ and a holomorphic function $f\in\mathcal{O}(\underline{X})$ such that $|f(q_k)|\to\infty$ $(k\to\infty)$.*

The proof is easy.

Proposition (2.10). *Let X be a* B*-resolution of $\underline{X}$. Assume that Γ is non-empty and that $\mathcal{O}(X-\varpi^{-1}(\Gamma))\cong\mathcal{O}(X)$. Then for any representation $\Psi=(f_1, f_2,\dots, f_n)$, we have* (1) *every point of Γ can be regarded as a boundary point of $(\underline{X}, \Psi)$,* (2) *if* $\operatorname{codim} A\geqq 2$, *then every boundary point in Γ is not pseudoconvex and* (3) *$\underline{X}$ is not holomorphically convex.*

Proof. (1) First we note that in view of $\mathcal{O}(X-\varpi^{-1}(\Gamma))\cong\mathcal{O}(X)$, f_j $(j=1, 2,\dots, n)$ can be considered as an element of $\mathcal{A}(W)$. Choosing an arbitrary point $\underline{p}_0\in\Gamma$, we consider an open neighborhood $\underline{U}$ of $\underline{p}_0$ in $\operatorname{Spec}\mathcal{O}(X)$ in the following form:

$$\underline{U}_\varepsilon=\{\underline{p}\in\operatorname{Spec}\mathcal{O}(X)\colon |f_j(\underline{p})-f_j(\underline{p}_0)|<\varepsilon, j=1, 2,\dots, n\}.$$

The connected component of $\underline{U}_\varepsilon$ containing $\underline{p}_0$ is denoted by $\underline{\hat{U}}_\varepsilon$. Set $\underline{V}_\varepsilon=\underline{\hat{U}}_\varepsilon\cap\underline{X}$. Then $\{\underline{V}_\varepsilon\}$ generates a filter which satisfies the conditions (1)$\sim$(3) in Definition (2.6). So $\{\underline{V}_\varepsilon\}$ determines one and only one point $r\in\partial\underline{X}$. Proof of (2). Put $\varpi^{-1}(\underline{V}_\varepsilon)=V_\varepsilon$. Then V_ε is a neighborhood of $A_{\underline{p}_0}=\varpi^{-1}(\underline{p}_0)$. First we construct an Oka family in V_ε. Take a point p_0 in $A_{\underline{p}_0}$ and a small neighborhood $U(U\subset V_\varepsilon)$ of p_0. We fix a certain system of local coordinates on U. Then there exists a linear space L through p_0 in U satisfying (1) $A\cap L=\{p_0\}$ in U and (2) $\operatorname{codim} A=\dim L$. Note that $L-\{p_0\}\subset X-\varpi^{-1}(\Gamma)$, where $A\supseteqq A_{\underline{p}_0}$. By assumption $\dim L\geqq 2$. So there exists an Oka family such that $\{\sigma(w, t)\colon |w|\leqq 1$ and $0\leqq t<1\}\subset X-\varpi^{-1}(\Gamma)$, $\{\sigma(w, t)\colon 0<|w|\leqq 1$ and $t=1\}\subset X-\varpi^{-1}(\Gamma)$ and $\sigma(0, 1)\not\subset X-\varpi^{-1}(\Gamma)$. Pulling $\sigma(w, t)$ down on $\operatorname{Spec}\mathcal{O}(X)$ and identifying p_0 with r, we obtain an Oka family $\underline{\sigma}$ in $\underline{X}$ satisfying $\underline{\sigma}(0, 1)=r$, which implies that r is not pseudoconvex. Proof of (3). Fix a point $\underline{r}\in\Gamma$. Choose a point $p\in A_{\underline{r}}=\varpi^{-1}(\underline{r})$ and a local coordinate neighborhood U

of p. Then there exists a linear subspace L through p as in (2). For a small ε we choose a polydisk $D_\varepsilon \Subset L$ with center p and a compact set K which contains the Silov boundary of D_ε and which does not intersect with $A_{\underline{r}} \cap D_\varepsilon$. By assumption every holomorphic function $f \in \mathcal{O}(\underline{X})$ can be extended to $\hat{f} \in \mathcal{O}(X)$. So the holomorphically convex hull of K, $\hat{K}$ must intersect with A, which implies that $\hat{K} \cap \partial \underline{X} \neq \emptyset$.

§3. Two Classes of Homorphically Separable Manifolds Which Are Domains of Holomorphy

Let $\underline{X}$ be a holomorphically separable manifold and let X be a B-resolution of $\underline{X}$ satisfying (A–1) and (A–2). We shall give two kinds of holomorphically separable manifolds which are domains of holomorphy. Manifolds in the former class are called H-manifolds, which are extensions of holomorphically convex manifolds (see Definition (3.3)). Manifolds in the latter class are called L-manifolds, which are certain weakly 1-complete manifolds (see Definition (3.11)) with special kinds of positive line bundles.

First we fix notations. Let S be an analytic set of X of codim 1. Let $S = \bigcup_{j \in I} S_j$ be the irreducible decomposition of S. With some open covering $\{U_\lambda\}$, $S_j \cap U_\lambda$ are defined by the minimal defining equation $f_{j,\lambda} \in \mathcal{O}(U_\lambda)$ as follows:

$$S_j \cap U_\lambda = \{f_{j,\lambda} = 0\}.$$

Take a set m of positive integers m_j for $j \in I$ and define a complex line bundle $[S]^m$ as follows: On U_λ where $S \cap U_\lambda \neq \emptyset$, there exists a finite number of irreducible components $S_{j_1}, S_{j_2}, \dots, S_{j_\lambda}$ on U_λ. Put

$$\phi_\lambda^m = \begin{cases} \prod_{k=1}^{j_\lambda} f_{k,\lambda}^{m_k} & \text{on } U_\lambda \text{ where } S \cap U_\lambda \neq \emptyset, \\ 1 & \text{on } U_\lambda \text{ where } S \cap U_\lambda = \emptyset, \end{cases}$$

and

$$f_{\lambda\mu} = \phi_\lambda^m / \phi_\mu^m \qquad \text{on } U_\lambda \cap U_\mu.$$

Then $\{f_{\lambda\mu}\}$ defines a complex line bundle, which is denoted by $[S]^m$.

The dual line bundle of $[S]^m$ is defined by $\{f_{\lambda\mu}^{-1}\}$ and is denoted by $[S]^{-m}$. Also for a positive integer r, a complex line bundle defined by $\{f_{\lambda\mu}^{-r}\}$ is denoted by $[S]^{-rm}$.

Consider a general complex line bundle E which is represented as $\{e_{\lambda\mu}\}$ with respect to some open covering $\{U_\lambda\}$. A system of positive C^∞-functions $\{a_\lambda\}$ on U_λ is called a metric of E if

$$a_\mu = |e_{\lambda\mu}|^2 a_\lambda \qquad \text{on} \quad U_\lambda \cap U_\mu$$

are satisfied. Particularly when $E=[S]^{-m}$, a metric $\{a_\lambda\}$ of $[S]^{-m}$ induces a C^∞-function on X:

$$h = a_\lambda^{-1} |\phi_\lambda^m|^2. \tag{3.1}$$

A line bundle E is called positive if there exists a metric $\{a_\lambda\}$ such that $|\gamma_{\lambda,\alpha\bar\beta}|$ defined by

$$-\partial\bar\partial \log a_\lambda = \sum_{\alpha,\beta=1}^{n} \gamma_{\lambda,\alpha\bar\beta} dz_\lambda^\alpha \wedge d\bar z_\lambda^\beta$$

is positive definite on each U_λ. $\mathcal{O}(E)$ denotes the sheaf of germs of holomorphic sections of E. For a section $\varphi \in H^0(X, \mathcal{O}(E))$, the following is a global C^∞-function on X:

$$|||\varphi||| = a_\lambda |\varphi_\lambda|^2,$$

where $\varphi = \{\varphi_\lambda\}$. We write also

$$|||\varphi|||_K = \sup_{p\in K} |||\varphi(p)|||\,.$$

Definition (3.2). *X is called $[S]^{-m}$-convex except Σ if*

(1) Σ is an analytic set in X,

(2) There exists a divisor S and a complex line bundle $[S]^{-m}$ such that for any compact set K there exists a closed set $\hat K$ satisfying (i) $\hat K - \Sigma$ is relatively compact in X and (ii) for any point $p \in X-(\hat K \cup \Sigma)$ and for any pair of two positive numbers ε and l there exists a section $\varphi \in H^0(X, \mathcal{O}([S]^{-m}))$ satisfying

$$|||\varphi|||_K < \varepsilon \quad \textit{and} \quad |||\varphi(p)||| > l.$$

Remark. The definition does not depend on the choices of metrics.

Suppose that X is $[S]^{-m}$-convex except Σ and that Σ' is an analytic set of X with $\Sigma\subset\Sigma'$. Then X is also $[S]^{-m}$-convex except Σ'. In what follows, we assume that S is contained in Σ.

Definition (3.3). *$\underline{X}$ is called an* H-*manifold if there exists a divisor S on the* B-*resolution manifold X such that X is $[S]^{-m}$-convex except Σ.*

Theorem 3. *If $\underline{X}$ is an* H-*manifold, then $\underline{X}$ is a domain of holomorphy.*

Proof. It is sufficient to verify the condition in Proposition (2.9). Fix a representation $\Psi: \underline{X}\to \mathbf{C}^n$. Choose a boundary point r and a sequence $\{\underline{q}_n\}$, $\underline{q}_n\in\underline{X}$ with $\underline{q}_n\to r$. Then we have a sequence $\{q_n\}$ in $X-\varpi^{-1}(\Gamma)$ where $q_n=\varpi^{-1}(\underline{q}_n)$. Now we replace the sequence $\{q_n\}$ by $\{q_n^*\}$ satisfying the following three conditions: (1) $\{q_n^*\}$ is a divergent sequence in X, (2) $\varpi(q_n^*)\to r$ $(n\to\infty)$ and (3) $\{q_n^*\}$ is not contained in $\varpi^{-1}(\Gamma)\cup\Sigma$. By Proposition (2.3) we see that $\varpi^{-1}(r)$ is non-compact. So the replacement can be always done. Then in order to prove the theorem, it suffices to show the following

Lemma (3.4). *Let $\{q_n\}$ be a divergent sequence in X with $\{q_n\}\subset X-\Sigma$. Then there exists a subsequence $\{q_{n_j}\}$ and a holomorphic function $f\in\mathcal{O}(X)$ satisfying*

$$|f(q_{n_j})|\longrightarrow\infty \qquad \textit{for}\ \ j\longrightarrow\infty.$$

Proof. Fix a compact exhausion $\{K_j\}$ in the following manner: Take a compact set $\mathring{K}_1$ and an element $q_{n_1}\in K_1$ of the sequence where $\mathring{K}$ denotes the open kernel of K. Choose K_2 satisfying $\overline{\mathring{K}_1-\Sigma}\subset K_2$ and an element $q_{n_2}\in\mathring{K}_2-\mathring{K}_1$. Repeating this process we have $\{K_j\}$ and $\{q_{n_j}\}$, where $q_{n_j}\in\mathring{K}_{j+1}-\mathring{K}_j$ and $K_j\subset K_{j+1}$. We denote the $\{q_{n_j}\}$ by $\{q_j\}$ simply. By assumption we have $S\subset\Sigma$. So we get the following two sequences:

$$\begin{aligned}\delta_j&=h(p_j)^{1/2}\\ M_j&=\max_{p\in K_j}h(p)^{1/2}\end{aligned}\qquad (j=1, 2,\dots).$$

For an arbitrary positive sequence $\{\beta_j\}$ there exists a sequence of sections $\{\varphi_j\}$, $\varphi_j \in H^0(X, \mathcal{O}([S]^{-m}))$ satisfying

$$\begin{aligned} &|||\varphi_j|||_{K_j}^{1/2} < 2^{-j}M^{-1}, \\ (3.5)\qquad &|||\varphi_j(q_{j+1})|||^{1/2} > \beta_j \cdot \delta_{j+1}^{-1}. \end{aligned}$$

Set

$$(3.6)\qquad f = \sum_{j=1}^{\infty} \varphi_{j,\lambda}\phi_\lambda^n, \quad \text{where} \quad \varphi_j = \{\varphi_{j,\lambda}\}.$$

Then $f \in \mathcal{O}(X)$. In fact, on arbitrary fixed K_μ, f can be expressed as follows:

$$(3.7)\qquad f = \sum_{j<\mu+1} \varphi_{j,\lambda}\phi_\lambda^n + \sum_{j \geqq \mu+1} \varphi_{j,\lambda}\phi_\lambda^n.$$

Referring to

$$(3.8)\qquad |\varphi_{j,\lambda}\phi_\lambda^n|^2 = |\varphi_{j,\lambda}|^2 a_\lambda a_\lambda^{-1} |\phi_\lambda^n|^2 = |||\varphi_j|||^2 h,$$

the second term of (3.7) converges uniformly on K_μ by (3.5), which proves the assertion. Now choosing $\{\beta_j\}$ inductively, we may assume

$$(3.9)\qquad |f(q_{j+1})| \geqq j \qquad (j = 1, 2, 3, \ldots).$$

In fact, we note that

$$|f| \geqq |\varphi_{\mu,\lambda}\phi_\lambda^n| - \sum_{j<\mu} |\varphi_{j,\lambda}\phi_\lambda^n| - \sum_{j>\mu} |\varphi_{j,\lambda}\phi_\lambda^n|.$$

By using (3.5) and (3.8), we have

$$|f| \geqq |||\varphi_\mu|||^{1/2} \cdot h^{1/2} - \sum_{j<\mu} |\varphi_{j,\lambda} \cdot \phi_\lambda^n| - 2.$$

When $\mu = 1$, we have $|f(q_2)| \geqq \beta_2 - 2$. So we prove (3.9) in this case. Assume that (3.9) holds for $k = 1, 2, \ldots, \mu - 1$. We note that $\sum_{j<\mu} |\varphi_{j,\lambda}\phi_\lambda^n|$ depends only on $\beta_1, \beta_2, \ldots, \beta_{\mu-1}$, which is denoted by $\Phi_\mu(\beta_1, \beta_2, \ldots, \beta_{\mu-1})$. Then we find that $|f(q_{\mu+1})| \geqq \beta_\mu - \Phi_\mu - 2$. Choosing β_μ sufficiently large, we get (3.9) for $k = \mu$. This completes the proof of (3.9).

Here we define L-manifolds. The following is due to S. Nakano [8]:

Definition (3.10). *X is called a weakly 1-complete manifold if there exists a complete pseudoconvex function ψ on X of C^∞-class, where ψ is called a complete function if $X_c=\{\psi<c\}$ is relatively compact in X for each c.*

Definition (3.11). *$\underline{X}$ is called an L-manifold if the B-resolution X of $\underline{X}$ satisfies the following conditions: (1) X is a weakly 1-complete manifold and (2) there exists a complex line bundle $[S]^{-m}$ such that (i) $[S]^{-m}$ is positive and (ii) $[S]^{-rm}\otimes K_X^{-1}$ is also positive with some r, where K_X denotes the canonical line bundle of X.*

In this section the following theorems are essential, which are due to S. Nakano [8] and H. Kazama [6] respectively:

Theorem 4. *Let X be a weakly 1-complete manifold. For a positive line bundle B, we have*

$$H^q(X, \mathcal{O}(B\otimes K))=0 \qquad \text{for} \quad q\geqq 1, \tag{1}$$

(2) *Fix X_c for a constant c. Then for any compact set E in X_c and for any positive constant ε, we have the following: For any section $\varphi\in H^0(\bar{X}_c, \mathcal{O}(B\otimes K))$, there exists a section $\tilde{\varphi}\in H^0(X, \mathcal{O}(B\otimes K))$ satisfying $|||\varphi-\tilde{\varphi}|||_E<\varepsilon$.*

For simple proofs, see O. Suzuki [12]*.

Now we prove the following

Theorem 5. *If $\underline{X}$ is an L-manifold, then $\underline{X}$ is an H-manifold.*

Proof. We prove that X is $[S]^{-m}$-convex except S. First note that in view of the positivity of $[S]^{-m}$, ψ may be assumed to be s-pseudoconvex on $X-S$ by replacing ψ by $\psi+h$. For the proof of Theorem 4, it suffices to show that for any compact set K, the holomorphically convex hull $\hat{K}$ of K satisfies

* Theorem 2 in O. Suzuki [12] must be replaced by the statement (2). This correction is due to Professor S. Nakano. The author thanks him for his correction.

$$\hat{K} - S \subset \overline{X}_c, \quad \text{where} \quad c = \max_{p \in K} \psi(p). \tag{3.12}$$

Choose a point $p \in X - (\overline{X}_c \cup S)$. Let $\psi(p) = c'$. Then we see that $c' > c$. Take c''' with $c' > c''' > c$ and set $E = \overline{X}_{c'''}$. Since $X_{c'}$ is s-pseudoconvex at p, there exist a neighborhood U and a continuous family of divisors $\{L_t\}$ $(c' \leqq t \leqq c'')$ in U with the following properties: (1) For any t, $L_t \cap X_{c'} = \emptyset$, (2) $L_t \cap \partial X_{c'} = \emptyset$ for $t \neq c'$ and $L_{c'} \cap \partial X_{c'} = \{p\}$ (3) $L_t = \{f_t = 0\}$, where f_t is a continuous function of t. For the proof, see R. Narasimhan [9, Lemma, p. 357]. Making c'' near c', we may assume that L_t is a divisor on $X_{c''}$ for each t. Let $\mathfrak{U} = \{U_\lambda\}$ be a covering of $X_{c''}$ which contains U as an element. We may assume that every element U_λ satisfying $U_\lambda \cap E \neq \emptyset$ has no common points with U. In the following we denote U by U_0. Consider a 1-cocycle $\{\varphi^{(t)}_{\lambda\mu}\}$ which is defined by

$$\varphi^{(t)}_{\lambda\mu} = \varphi^{(t)}_\lambda - f^{-m}_{\lambda\mu} \cdot \varphi^{(t)}_\mu,$$

where
$$\varphi^{(t)}_\lambda = \begin{cases} 1/f_t & \text{on} \quad U_0 \\ 0 & \text{on} \quad U_\lambda \ (\lambda \neq 0). \end{cases}$$

Then there exists a C^∞-cochain $\{\eta^{(t)}_\lambda\}$ which is a continuous function of t on U_λ satisfying

$$\varphi^{(t)}_{\lambda\mu} = \eta^{(t)}_\mu - f^{-m}_{\lambda\mu} \cdot \eta^{(t)}_\lambda \qquad \text{on} \quad U_\lambda \cap U_\mu.$$

Then

$$g^{(t)} = \bar{\partial} \eta^{(t)}_\lambda$$

gives a Dolbault form corresponding to $\{\varphi^{(t)}_{\lambda\mu}\}$, which is continuous with respect to t.

Here we prepare a lemma. For a $[S]^{-m}$-valued form f and a convex increasing function χ, we set

$$\|f\|^2_\chi = \int_{X_{c''}} e^{-\chi(\Psi_{c''})} f \wedge \bar{*} f,$$

where $*$ denotes the usual star operation and $\Psi_{c''} = 1/(1 - \psi/c'')$. By $\mathscr{L}^2_{p,q}(X_{c''}, [S]^{-m}, \chi)$ we denote the Hilbert space with respect to the above norm. Then we have the following lemma:

Lemma (3.13). *For any convex increasing function* χ, *we have the following*: *For any* t, *there exists a* C^∞-*section* $u^{(t)}$ *of* $[S]^{-m}$ *such that*

(1) $\bar{\partial}u^{(t)}=g^{(t)}$ *and* (2) $\|u^{(t)}-u^{(c')}\|_\chi \longrightarrow 0 \quad (t \longrightarrow c')$.

Proof. Let C be the minimum of the eigenvalues of the curvature form of the positive metric of $[S]^{-m}\otimes K_X^{-1}$ on $\bar{X}_{c''}$. Then C is a positive constant. Consider

$$\mathscr{L}^2_{p,q}(X_{c''}, [S]^{-m}, \chi) \underset{\mathfrak{d}_\chi}{\overset{\bar{\partial}}{\rightleftarrows}} \mathscr{L}^2_{p,q+1}(X_{c''}, [S]^{-m}, \chi) \underset{\mathfrak{d}_\chi}{\overset{\bar{\partial}}{\rightleftarrows}} \mathscr{L}^2_{p,q+2}(X_{c''}, [S]^{-m}, \chi),$$

where $\bar{\partial}$ denotes the extension of the usual $\bar{\partial}$-operation in the sense distribution and $\mathfrak{d}_\chi$ denotes the adjoint operator of $\bar{\partial}$ in the theory of Hilbert spaces. Then by O. Suzuki [12], we have

$$\|f\|^2_\chi \leqq C(\|\bar{\partial}f\|^2_\chi + \|\mathfrak{d}_\chi f\|^2_\chi) \qquad \text{for } f\in D_{p,q+1}(\bar{\partial}) \cap D_{p,q+1}(\mathfrak{d}_\chi),$$

where

$$D_{p,q+1}(\bar{\partial}) = \{f\in \mathscr{L}^2_{p,q+1}(X_{c''}, [S]^{-m}, \chi): \ \bar{\partial}f\in \mathscr{L}^2_{p,q+2}(X_{c''}, [S]^{-m}, \chi)\},$$

$$D_{p,q+1}(\mathfrak{d}_\chi) = \{f\in \mathscr{L}^2_{p,q+1}(X_{c''}, [S]^{-m}, \chi): \ \mathfrak{d}_\lambda f\in \mathscr{L}^2_{p,q}(X_{c''}, [S]^{-m}, \chi)\}.$$

Let $\square_\chi$ be the Laplace-Beltrami operator. Then for any $g\in \mathscr{L}^2_{p,q+1}(X_{c''}, [S]^{-m}, \chi)$ there exists a unique $h\in D(\bar{\partial})\cap D(\mathfrak{d}_\chi)$ satisfying

$$\square_\chi h = g.$$

So we write $h=G_\chi(g)$. By Andreotti-Vesentini [2, p. 96], we find

$$\|\bar{\partial}h\|^2_\chi + \|\mathfrak{d}_\chi h\|^2_\chi \leqq 4C\|g\|^2_\chi. \tag{3.14}$$

Now we apply the above formula to our $g^{(t)}$. Then owing to the closedness of $g^{(t)}$, we obtain

$$g^{(t)} = \bar{\partial}\mathfrak{d}_\chi h^{(t)}, \quad \text{where} \quad h^{(t)} = G_\chi(g^{(t)}).$$

So setting

$$u^{(t)} = \mathfrak{d}_\chi h^{(t)},$$

we see that $\bar{\partial}u^{(t)}=g^{(t)}$. Moreover by the construction of $g^{(t)}$, we see that $\|g^{(t)}-g^{(c')}\|_{\chi}^{2}\to 0\ (t\to c')$. Here by (3.14) we have $\|u^{(t)}-u^{(c')}\|_{\chi}^{2}\to 0$ $(t\to 0)$. So we prove our lemma.

By this lemma we shall prove the assertion of Theorem 5. We set

$$\tilde{\varphi}_{\lambda}^{(t)}=u_{\lambda}^{(t)}-\eta_{\lambda}^{(t)}.$$

Then by the constructions of $\{\eta_{\lambda}^{(t)}\}$ and $u^{(t)}$ and by using the Cauchy inequality, we find a positive constant C_0 and t_0 such that

$$|||\tilde{\varphi}^{(t)}|||\leqq C_0 \quad \text{on } E \quad \text{and} \quad |\tilde{\varphi}_{0}^{(t)}(p)|\leqq C_0 \qquad \text{for} \quad t\leqq t_0.$$

Now we set

$$\phi_{\lambda}^{(t)}=\varphi_{\lambda}^{(t)}+\tilde{\varphi}_{\lambda}^{(t)}.$$

Then $\{\phi_{\lambda}^{(j)}\}$ gives a meromorphic section of $[S]^{-m}$ on $X_{c''}$. We infer that

$$\phi_{\lambda}^{(t)}=\begin{cases} 1/f_t+\tilde{\varphi}_{0}^{(t)} & \text{on} \quad U_0, \\ \tilde{\varphi}_{\lambda}^{(t)} & \text{on} \quad U_{\lambda}. \end{cases}$$

Hence we can find a positive constant C_1 which does not depend on t satisfying

$$\sup_{K}|||\phi^{(t)}|||\leqq C_1 \qquad \text{for} \quad t\leqq t_0.$$

Now take a pair of positive constants ε and l. Multiplying $\phi^{(t)}$ by a suitable constant, we may assume

$$\sup_{K}|||\phi^{(t)}|||<\varepsilon-\delta \qquad \text{for} \quad t\leqq t_0,$$

where δ is a sufficiently small positive constant. Choosing a $t^*(t^*\leqq t_0)$ sufficiently near c', we have

$$|||\phi_{0}^{(t^*)}(p)|||>l+\delta.$$

Taking $c^*, c^{*\prime}(c^{*\prime}>c^*>c')$ such that $\bar{X}_{c^{*\prime}}\cap\{f_{t^*}=0\}=\emptyset$. We apply Kazama's Theorem to $\phi^{(t^*)}$ on $\bar{X}_{c^{*\prime}}$. Then we can find a section ϕ of $[S]^{-m}$ on X such that

$$|||\phi-\phi^{(t^*)}|||_{\overline{X}_{c^*}}<\delta,$$

which proves the assertion.

Here we consider $\mathcal{O}(X)$ for the resolution X of an L-manifold $\underline{X}$.

Definition (3.15). *An algebra $\mathscr{A}$ is called a Stein algebra if there exists a Stein space Y satisfying $\mathscr{A}\cong\mathcal{O}(Y)$.*

Theorem 6. *Assume that $\underline{X}$ is an* L-*manifold. Then the following hold*: (1) *In the case where Γ is empty, $\mathcal{O}(\underline{X})\cong\mathcal{O}(X)$. So $\mathcal{O}(X)$ is a Stein algebra.* (2) *If there exist an irreducible component A of $\varpi^{-1}(p)$ for a point $p\in \operatorname{Spec}\mathcal{O}(X)$ and a divisor D in X satisfying the following three conditions*: (*i*) $A\subset S$, (*ii*) $A\cap D\neq\emptyset$, $A\not\subset D$ *and* (*iii*) $[D]^{-1}\geqq 0$, *then $\mathcal{O}(X)$ is a non-Stein algebra.*

Proof of (1). By the definition of the resolution manifold, we see that $X\cong\underline{X}$. So X is a K-complete and weakly 1-complete manifold. Then by the theorem of A. Andreotti and R. Narasimhan [1], we see that X is a Stein manifold.

Proof of (2). In the remainder of this section, we write $\Gamma(X,\mathcal{O})=\mathcal{O}(X)$. For the proof of (2), we prepare a Lemma:

Lemma (3.16). *Let $\mathscr{I}$ denote the ideal sheaf of A. By $\Gamma(X,\mathscr{I}^m)$ we denote the sections of the sheaf $\mathscr{I}^m$. If there exists an integer m ($m\geqq 1$) satisfying*

$$\dim_{\mathbf{C}}\Gamma(X,\mathcal{O})/\Gamma(X,\mathscr{I}^{m+1})=\infty, \tag{3.17}$$

then $\mathcal{O}(X)$ is not a Stein algebra.

Proof of Lemma (3.16). Assume that $\Gamma(X,\mathcal{O})$ is a Stein algebra. Then by the theorem of H. Grauert [3], the character ideal $I(p)$ for $p\in X$ must be finitely generated over $\Gamma(X,\mathcal{O})$, where $I(p)=\{f\in\Gamma(X,\mathcal{O}): f(p)=0\}$. In the following we consider $I(p)$ for a certain $p\in A$. The generators of $I(p)$ are denoted by $f_1, f_2,\dots, f_r$. By the choice of A, we see that

(3.18) $$f_j \in \Gamma(X, \mathscr{I}) \qquad (j=1, 2,\dots, r)$$

and that

(3.19) $$\dim_{\mathbf{C}} \Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I})=1.$$

So (3.17) implies the existence of k satisfying

(3.20) $$\dim_{\mathbf{C}} \Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^k)<+\infty$$

and

(3.21) $$\dim_{\mathbf{C}} \Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^{k+1})=\infty.$$

Therefore we see that

(3.22) $$\dim_{\mathbf{C}} \Gamma(X, \mathscr{I}^k)/\Gamma(X, \mathscr{I}^{k+1})=\infty.$$

Let π_k: $\Gamma(X, \mathcal{O})\to\Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^{k+1})$ and let $\pi_k(I(p))=I_k(p)$. Then $I_k(p)$ is also finitely generated over $\Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^{k+1})$ and every element of $\underline{f}\in I_k(p)$ can be expressed as follows:

(3.23) $$\underline{f}=\sum_{j=1}^{r} \gamma_j \underline{f}_j,$$

where $\underline{f}_j=\pi_k(f_j)$ and $\gamma_j\in\Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^{k+1})$.

In view of $I(p)=\Gamma(X, \mathscr{I})$, we see that

$$\Gamma(X, \mathscr{I}^k)/\Gamma(X, \mathscr{I}^{k+1})\subset I_k(p).$$

So (3.22) implies that

(3.24) $$\dim_{\mathbf{C}} I_k(p)=\infty.$$

Now consider the following exact sequence of $\mathbf{C}$-vector spaces:

$$0 \longrightarrow \Gamma(X, \mathscr{I}^k)/\Gamma(X, \mathscr{I}^{k+1}) \longrightarrow \Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^{k+1}) \longrightarrow \Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^k) \longrightarrow 0.$$

We denote the $\mathbf{C}$-basis of $\Gamma(X, \mathscr{I}^k)/\Gamma(X, \mathscr{I}^{k+1})$ and $\Gamma(X, \mathcal{O})/\Gamma(X, \mathscr{I}^k)$ by $\{h_i\}$ and $\{g_n\}$ respectively. By (3.20), $\{g_n\}$ is finite. Then $\underline{\gamma}_j$ in (3.23) is expressed as

$$\underline{\gamma}_j = \sum \alpha_j^{(i)} h_i + \sum \beta_j^{(n)} g_n,$$

where $\alpha_j^{(i)}$ and $\beta_j^{(n)}$ are constants.
Referring to (3.18), we see that

$$\underline{f} = \sum \beta_j^{(n)} g_n \underline{f}_j,$$

which implies that $\dim_{\mathbf{C}} I_k(p) < \infty$. This contradicts (3.24).

Now we prove (2) in Theorem 6. For this we shall prove the existence of m which satisfies the assumption of Lemma (3.16). Fix an arbitrary point $p_0 \in D \cap A$ and consider the monoidal transform at p_0. The manifold obtained is denoted by X^* and the projection is denoted by $Q: X^* \to X$. The following are well known: (M–1) Let K_{X^*} denote the canonical line bundle of X^*. Then $K_{X^*} = Q^*(K_X) \otimes [N]^{n-1}$, (M–2) $N = Q^{-1}(p_0)$ is isomorphic to $\mathbf{P}^{n-1}$ and $[N]^{-1}_{|N} > 0$, (M–3) if X is a weakly 1-complete manifold, then X^* is also a weakly 1-complete manifold and (M–4) for any complex line bundle E on X, $H^0(X, \mathcal{O}(E)) \cong H^0(X^*, \mathcal{O}(Q^*(E))$.

First we consider metrics of $[N]$.

Proposition (3.25). (1) *There exists relatively compact open neighbor hoods V and U of N with $V \Subset U$ and a metric $\{a_\lambda\}$ of $[N]$ such that $\partial\bar{\partial} \log a_\lambda$ is positive definite on V and $\partial\bar{\partial} \log a_\lambda = 0$ on $X - U$.* (2) *Let E be a complex line bundle with $E > 0$ and $E \otimes K_X^{-1} > 0$. Then there exists a positive integer r_0 such that*

$$E_r^* = Q^*(E^r) \otimes K_{X^*}^{-1} \otimes [N]^{-1}$$

are positive on X^* for $r \geqq r_0$.

Proof. We fix a covering of X as follows: For any point $a \in N$ there is a neighborhood U_p of p satisfying $N \cap U_p = \{\phi = 0\}$ with some holomorphic function on U_p. We cover N by a finite open sets $\{U_j\}$ with this property on each U_j. We may assume that $U_0 = \cup U_j$ is relatively compact in X. Choose open neighborhoods U' and U of N with

$\overline{U}' \subset \overline{U} \subset \overline{U}_0$. Then we have a covering $\{U_j\} \cup (X-\overline{U})$ of X, which is denoted by $\{U_\lambda\}$. By (M–2), we have a metric $\{a_\lambda\}$ of $[N]$ which is negative on some small neighborhood V of N with $V \Subset U$. Define a C^∞-function $\sigma(p)$ on X as follows:

$$\sigma(p) = \begin{cases} 0 & \text{on } U' \\ \log a_\mu & \text{on } U_\mu = X - \overline{U}. \end{cases}$$

Then the metric defined by $a_\lambda e^{-\sigma(p)}$ on U_λ gives a desired one. Owing to (M–1) and the construction of the metric $\{a_\lambda\}$, we can find r_0 in (2).

We set

$$E^*_{r,k} = Q^*([S]^{-rm} \otimes D^{-k}) \qquad \text{for } r \geqq 1 \text{ and } k \geqq 1.$$

Consider

$$0 \longrightarrow \mathcal{O}(E^*_{r,k} \otimes [N]^{-1}) \longrightarrow \mathcal{O}(E^*_{r,k}) \longrightarrow \mathcal{O}(E^*_{r,k}|_N) \longrightarrow 0.$$

By Proposition (3.25) and Definition (3.11), $E^*_{r,k} \otimes K^{-1}_{X^*} \otimes N^{-1}$ is positive for $r \geqq r_0$ and $k \geqq 1$. Thus by using (M–3) and Theorem 4, we have

$$H^1(X^*, \mathcal{O}(E^*_{r,k} \otimes [N]^{-1}) = 0 \qquad \text{for } r \geqq r_0 \text{ and } k \geqq 1.$$

Referring to $H^0(N, \mathcal{O}(E^*_{r_0,k}|_N)) \cong H^0(N, \mathcal{O})$, we see that for any non-zero section $\varphi^{(k)} \in H^0(N, \mathcal{O}(E^*_{r_0,k}|_N))$, there exists an extension $\tilde{\varphi}^{(k)} \in H^0(X^*, \mathcal{O}(E^*_{r_0,k}))$ for any $k \geqq 1$. By (M–4) we have a section $\varphi^{(k)} \in H^0(X, \mathcal{O}([S]^{-r_0m} \otimes [D]^{-k}))$ with $\varphi^{(k)}(p) \neq 0$ for any k $(k \geqq 1)$. Multiplying the defining equations of S and D, we have

$$f^{(k)} = \varphi^{(k)}_\lambda \cdot \phi_\lambda^{r_0m} \cdot \eta_\lambda^k \qquad (k \geqq 1),$$

where $U_\lambda \cap S = \{\phi_\lambda = 0\}$ and $U_\lambda \cap D = \{\eta_\lambda = 0\}$. Referring to $\varphi^{(k)}(p_0) \neq 0$ and by the assumptions (i) and (ii) in Theorem 6, we see that there exists an integer m_0 such that

$$f^{(k)} \in \Gamma(X, \mathcal{J}^{m_0}) / \Gamma(X, \mathcal{J}^{m_0+1}).$$

Thus we obtain an infinite dimensional vector space $\{f^{(k)}\}$ in $\Gamma(X, \mathcal{O})/\Gamma(X, \mathcal{J}^{m_0+1})$, which implies (3.17) for $m = m_0$. So we complete the proof of Theorem 6.

§4. Examples

In this section we are concerned with several examples due to H. Grauert [3] and M. Otuki [10] respectively. First we fix notations. Let R be a compact Riemann surface of genus g $(g \geqq 1)$ and let F be a topological trivial line bundle on R, which is expressed as $\{f_{\lambda\mu}\}$ with respect to some open covering $\{V_\lambda\}$. $\pi': F \rightarrow R$ denotes the natural projection and ζ_λ denotes the fibre coordinate on V_λ. By a well known lemma, we may assume that $|f_{\lambda\mu}|=1$ on $V_\lambda \cap V_\mu$. So $f=|\zeta_\lambda|^2$ is a global function on F and $V_\varepsilon=\{f<\varepsilon\}$ gives a fundamental neighborhood system of the zero section. F is called of finite order if there exists a positive integer k such that F^k is analytically trivial. Otherwise, it is called of infinite order. Also we consider a negative line bundle G on R. With respect to the same covering $\{V_\lambda\}$, G is expressed as $\{g_{\lambda\mu}\}$ whose fibre coordinate on V_λ is denoted by η_λ. By the negativity of G, there exists a metric $\{a_\lambda\}$ such that $\partial\bar{\partial}\log a_\lambda > 0$. By this we get a pseudoconvex function $g=a_\lambda|\eta_\lambda|^2$ on G which is s-pseudoconvex except the zero section. The following Lemma due to H. Grauert [3] is essential in this section:

Lemma (4.1). *In the case of finite order, there exists a proper and fibre connected holomorphic mapping $\Psi: V_\varepsilon \rightarrow \mathbf{D}$ where $\mathbf{D}$ is the unit disk such that $\mathcal{O}(V_\varepsilon) \cong \mathcal{O}(\mathbf{D})$. In the case of infinite order, $\mathcal{O}(V_\varepsilon) \cong \mathbf{C}$.*

For the proof of Lemma (4.1), see H. Grauert [3] (or O. Suzuki [11]).

Example 1 (M. Otuki). Let $V=F \oplus G$. π denotes the natural projection $\pi: V \rightarrow R$. In a natural manner the fibre coordinates of F and G are regarded as fibre coordinates of V. Also f and g are considered as functions on V. $\{\eta_\lambda=0\}$ or $\{\zeta_\lambda=0\}$ determines a divisor on V which is denoted by D or S respectively. We set $\varphi=f+g$ and $V_c=\{\varphi<c\}$. Then we have the following

Proposition (4.2). *For any c we have* (1) $[S]^{-1}$ *is positive on V_c,* (2) $[S]^{-m} \otimes K_{V_c}^{-1}$ *is also positive for $m \geqq m_0$ with some m_0,* (3) $[D]^{-1}$

is positive semi-definite on V_c *and* (4) V_c *is a weakly* 1-*complete manifold.*

Proof. With natural identification, $\{a_\lambda\}$ can be considered as a metric of $[S]$. So define a metric $\{a_\lambda^*\}$ of $[S]$ by $a_\lambda^* = a_\lambda e^\varphi$. Then we have a negative metric. Choosing a metric of the canonical line bundle K_V and restricting to V_c, we get a metric of K_{V_c}. So we can find m_0 as in (2). $[D]$ is expressed $\{f_{\lambda\mu}\}$. So as a metric of $[D]$, $\{1\}$ can be chosen, which proves (3). Set $\psi = 1/(1-\varphi/c)$. Then ψ is a complete pseudoconvex function on V_c.

Corollary. *In the case of infinite case,* $\mathcal{O}(V_c)$ *is not a Stein algebra.*

This follows from Theorem 6 and the following

Proposition (4.3). (i) *In the case of infinite order,* (1) $\mathcal{O}(V_c - S) \cong \mathcal{O}(V_c)$, (2) *Every holomorphic function is constant on* S *and* (3) $V_c - S$ *is holomorphically separable.* (*ii*) *In the case of finite order,* $\varpi^{-1}(\varpi(p))$ *is always a compact set, where* $\varpi\colon V_c \to \operatorname{Spec}\mathcal{O}(V_c)$.

Proof. $\pi^{-1}(V_\lambda) \cap (V_c - S)$ is a circular domain. So $f \in \mathcal{O}(V_c - S)$ is expressed as follows:

$$f = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} a_\lambda^{(j,k)} \zeta_\lambda^j \eta_\lambda^k,$$

$$\text{where} \quad \{a_\lambda^{(j,k)}\} \in H^0(R, \mathcal{O}(F^{-j} \otimes G^{-k})).$$

When k is negative, $F^{-j} \otimes G^{-k}$ is negative. Thus $a_\lambda^{(j,k)} = 0$ for k with $k < 0$, which implies (1) in (i). (2) in (i) is a direct consequence of Lemma (4.1). By Proposition (4.2) and Theorem 4, we prove (3) in (i) and (ii). Now consider $\operatorname{Spec}\mathcal{O}(V_c)$. By Theorem 4 local coordinate parameters on a small neighborhood of $p \in V_c - S$ can be chosen as global holomorphic functions on V_c which vanish on S of high order. So by the definition of $\mathscr{A}$ (see H. Grauert [3, p. 390]) we see that $V_c - S \cong \operatorname{Spec}\mathcal{O}(V_c) - \varpi(S)$. Now we prove the following

Proposition (4.4). *In the case of infinite order,* $\varpi(S)$ *does not* *admit any complex structure.*

Proof. Assume that $\varpi(S)$ would admit a complex structure, i.e., there exists an analytic set H in an ε-ball $D_\varepsilon=\{(z_1, z_2,\dots, z_N): \sum_{k=1}^{N}|z_k|^2<\varepsilon\}$ and a neighborhood $\underline{U}$ of $\varpi(S)$ such that $(\underline{U}, \mathscr{A}|_{\underline{U}})\cong(H, \mathcal{O}_H)$, where $\mathcal{O}_H=\mathcal{O}(D_\varepsilon)/\mathscr{I}(H)$ and $\mathscr{I}(H)$ is the ideal sheaf of H. Note that $\mathcal{O}(H)$ is a Stein algebra. With natural identification, we may assume that $\varpi: U\to H$ is a holomorphic mapping, where $U=\varpi^{-1}(\underline{U})$. Put $\varpi^*(z_j)=f_j$. Then $f_j\in\mathcal{O}(U)$ for each j. We define

$$\eta_\varepsilon=1/(1-\eta/\varepsilon), \quad \text{where} \quad \eta=\sum_{j=1}^{N}|f_j|^2$$

and

$$\varphi_c=1/(1-\varphi/c).$$

Then U is a weakly 1-complete manifold with respect to $\tilde{\psi}=\varphi_c+\eta_\varepsilon$. So by Theorem 6, $\mathcal{O}(U)$ is a non-Stein algebra. Moreover, $U-S\cong H-\{0\}$ and $\mathcal{O}(U-S)\cong\mathcal{O}(U)$. This implies that $\mathcal{O}(U)\cong\mathcal{O}(H)$, which is a contradiction.

Therefore we see that V_c is a B-resolution of $\underline{V}_c=\operatorname{Spec}\mathcal{O}(V_c)-\Gamma$, where $\Gamma=\varpi(S)$. By Proposition (4.2) and its Corollary, $\underline{V}_c$ is an L-manifold and $\mathcal{O}(\underline{V}_c)$ is a non-Stein algebra. Also by Proposition (2.10) $\underline{V}_c$ is not holomorphically convex. In the case of finite order, $\varpi: V_c\to\operatorname{Spec}\mathcal{O}(V_c)$ is a proper mapping. So by Proposition (2.3), $\operatorname{Spec}\mathcal{O}(V_c)$ is a complex analytic space and by the theorem of A. Andreotti and R. Narasimhan [1], it is a Stein space.

Example 2. Let $G^{(i)}$ $(i=1, 2)$ be negative line bundles on R whose fibre coordinates are denoted by $\eta_\lambda^{(i)}$ on V_λ. Negative metrics are denoted by $\{a_\lambda^{(i)}\}$ respectively. We set $g^{(i)}=a_\lambda^{(i)}|\eta_\lambda^{(i)}|^2$. Consider

$$V=F\oplus G^{(1)}\oplus G^{(2)}.$$

$\tau: V\to R$ denotes the natural projection. $(\zeta_\lambda, \eta_\lambda^{(1)}, \eta_\lambda^{(2)})$ gives a system of fibre coordinates on V_λ. Set $\varphi=h+g^{(1)}+g^{(2)}$ and define $V_c=\{\varphi<c\}$. Then the following propositions which are analogous to Propositions (4.2) and (4.3) can be proved:

Proposition (4.5). (i) V_c *is a weakly* 1-*complete manifold for each* c, (ii) $[S^{(i)}]^{-1}$ *is positive, where* $S^{(i)}=\{\eta_\lambda^{(i)}=0\}$ *for* $i=1, 2$, (iii) $[S^{(i)}]^{-m}\otimes K_{V_c}^{-1}$ *is also positive for* $m \geqq m_0$ *with some* m_0 *for* $i=1, 2$ *and* (iv) *let* $D=\{\zeta_\lambda=0\}$. *Then* $[D]^{-1}$ *is positive semi definite.*

Proofs are almost the same as ones in Propositions (4.2) and (4.3).

In the case where F is of infinite order, V_c is a B-resolution of $V_c=\operatorname{Spec}\mathcal{O}(V_c)-\Gamma$, where $\Gamma=\varpi(E)$ and $E=S^{(1)}\cap S^{(2)}$. So we see that $\underline{V}_c$ is an L-manifold. In this case, $\operatorname{codim} E=2$. So by Proposition (2.10), $\underline{V}_c$ can never be a pseudoconvex domain for any representation. By Theorem 6 we see that $\mathcal{O}(\underline{V}_c)$ is not a Stein algebra. These examples show the following

Theorem 7. *There exist* L-*manifolds which are neither holomorphically convex nor pseudoconvex for any representation. Moreover, there exist* L-*manifolds which have non-Stein algebras.*

Example 3 (H. Grauert [3, p. 383]). We use the notations of H. Grauert.

Proposition (4.6). *As for* $T=\{p\in F: h<1\}$, *where* $h=\|p\|$, *the following hold*: (1) $\mathcal{O}(T-M\cup\mathfrak{O})\cong\mathcal{O}(T)$, (2) *Every holomorphic function is constant on* $M\cup\mathfrak{O}$, (3) $T-M\cup\mathfrak{O}$ *is holomorphically separable*, (4) $[M]^{-1}\otimes[\mathfrak{O}]^{-m}$ *is positive for* $m\geqq m_0$ *and* $[M]^{-n}\otimes[\mathfrak{O}]^{-nm}\otimes K_T^{-1}$ *is positive for* $n\geqq n_0$ *and* $m\geqq m_0$, *where* m_0 *and* n_0 *are positive integers*, (5) $[\mathfrak{O}]^{-1}\geqq 0$ *and* (6) T *is a weakly* 1-*complete manifold.*

Corollary. $\mathcal{O}(T)$ *is not a Stein algebra.*

Proof. Except (4), (5) and (6), proofs are given in [3] and (6) is obvious. First we prove (5). By $\pi^*(F^{-1})=[\mathfrak{O}]^{-1}$, we see that (5) holds. We express the metric of F as $\{\underline{a}_\lambda\}$ with respect to a certain covering of X. Next we prove (4). Put $\tilde{a}_\lambda=a_\lambda\cdot e^h$, where $a_\lambda=\pi^*(\underline{a}_\lambda)$. Then $\{\tilde{a}_\lambda\}$ is a positive semi definite metric of $[\mathfrak{O}]^{-1}$ which is positive on $F-M$. Owing to $[\tilde{\mathfrak{O}}]^{-1}|_{\tilde{\mathfrak{O}}}>0$, we have a metric of $[\tilde{\mathfrak{O}}]^{-1}$ which is positive near $\tilde{\mathfrak{O}}$. Pulling up this metric by π and multiplying e^h, we obtain a metric of $[M]^{-1}=\pi^*([\tilde{\mathfrak{O}}]^{-1})$ which is positive near M. So choosing m_0 sufficiently large we get a positive metric of $[\mathfrak{D}]^{-m}\otimes M^{-1}$ on T for $m\geqq m_0$. The latter part of (4) is obvious and is omitted.

By this Proposition, we see that T is a B-resolution of $\underline{T}=\mathrm{Spec}\,\mathcal{O}(T)-\varpi(M\cup\mathfrak{D})$ and that $\underline{T}$ is an L-manifold.

Example 4. Finally we remark that for a holomorphically separable manifold $\underline{X}$, the B-resolution can not always be determined uniquely. Let F and G be line bundles given in Example 1. Set $H=F\otimes G^{-1}$ and $V'=H\oplus G$, where F is of infinite order. Then we have

Proposition (4.7). *Let* $S'=\{\eta'_\lambda=0\}$, *where* η'_λ *denotes the fibre coordinate of* G. *Then* (1) $[S']^{-1}>0$ *and* (2) $\mathcal{O}(V'-S')\cong\mathcal{O}(V')$.

The proof is similar to the one in Proposition (4.3).

We define Φ: $V'\to V$ by

$$\begin{cases} \zeta_\lambda=\zeta'_\lambda\eta'_\lambda \\ \eta_\lambda=\eta'_\lambda \end{cases}$$

where $(\zeta'_\lambda, \eta'_\lambda)$ denotes the fibre coordinates of V'. By Φ, we see that $V'-S'\cong V-S$. Referring to $\mathcal{O}(V'-S')\cong\mathcal{O}(V')$ and $\mathcal{O}(V-S)\cong\mathcal{O}(V)$, we see that $\mathrm{Spec}\,\mathcal{O}(V')-\varpi'(S')=\mathrm{Spec}\,\mathcal{O}(V)-\varpi(S)$, where ϖ': $V'\to\mathrm{Spec}\,\mathcal{O}(V')$.

Remark. Let $V'_c=\Phi^{-1}(V_c)$, then V'_c can never be an L-manifold for any c. In fact, $\Phi^{-1}(R)=S'$ is an s-pseudoconcave manifold. So complete pseudoconvex functions can never be admitted.

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