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A New Class of Domains of Holomorphy (II)

(Domains of holomorphy on a three dimensional Stein space with an isolated singularity)

By

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Introduction

The present paper is the continuation of O. Suzuki [10]. There we defined the concept of L-manifolds (see Definition (3.11) in O. Suzuki [10]) and showed that every L-manifold is a domain of holomorphy in the sense of H. Kerner [7] (see Definition (2.5) in O. Suzuki [10]). Moreover, we showed that there exist L-manifolds which are neither holomorphically convex nor pseudoconvex manifolds and there exist L-manifolds which admit non-Stein algebras (see Definition (3.15) in O. Suzuki [10]). These results are summarized in Theorems I and II in Introduction in O. Suzuki [10]. Unfortunately, only two examples are given there.

In this paper we shall prove that under the condition (A) certain domains of holomorphy (which will be called simple domains) on a certain three dimensional Stein space with an isolated singularity are in fact L-manifolds. By this we can systematically construct many examples of domains of holomorphy which are not Stein manifolds.

Let $\underline{M}$ be a Stein space with an isolated singularity p_0. As will be shown in § 3, every domain of holomorphy $\underline{\Delta}$ which does not contain p_0 as a boundary point is a Stein space. But in the case where $p_0 \in \partial\underline{\Delta}$, the situation is not simple. There we can find many domains of

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holomorphy which have the properties far from Stein spaces. In order to make the points of difficulties clear, we have to restrict our considerations to the $\underline{M}$ with the following condition (A) and special kinds of domains of holomorphy on $\underline{M}$. The condition (A) is stated as follows:

(A) There exists a resolution of the singularity M of $\underline{M}$, $\tau: M\rightarrow\underline{M}$ with the following properties: There exist a non-singular compact algebraic curve A and two complex line bundles F and G such that M is isomorphic to $F\oplus G$ and $\tau^{-1}(p_0)$ is nothing but the zero section.

We write the natural projection $\pi: F\oplus G\rightarrow A$. In the following we fix a fine covering $\{V_\lambda\}$ of A and z_λ denotes the local coordinate parameter on V_λ and ζ_λ (resp. η_λ) denotes the fibre coordinate of F (resp. G). Also by H and $\underline{H}$, we denote the divisor which is defined by $\eta_\lambda=0$ and the analytic set $\tau(H)$ in $\underline{M}$ respectively. For the description of desired domains, we prepare the following definition:

Definition (0.1). (1) *A function Φ defined on $U-A$, where U is a neighborhood of the zero section, is called a conoid function of type (k, l) along A if it is expressed as*

$$\Phi=a_\lambda\frac{|\zeta_\lambda^2|^k}{|\eta_\lambda^2|^l}+b_\lambda|\eta_\lambda^2|^s+c_\lambda|\zeta_\lambda^2|^t \quad \text{on} \quad \pi^{-1}(V_\lambda)\cap U,$$

where $\{a_\lambda\}$ is a C^∞-metric on U of $\pi^*(F^k\otimes G^{-l})$ and $\{b_\lambda\}$ (resp. $\{c_\lambda\}$) is a non-negative C^∞-section on U of $\pi^*(G^s\otimes\bar{G}^s)$ (resp. $\pi^*(F^t\otimes\bar{F}^t)$) with the following condition, where $\bar{F}$ denotes the conjugate bundle of F:

$$a_\lambda=a_\lambda(z_\lambda, |\zeta_\lambda^2|, |\eta_\lambda|^2), \quad b_\lambda=b_\lambda(z_\lambda, |\zeta_\lambda^2|, |\eta_\lambda^2|)$$
$$\text{and} \quad c_\lambda=c_\lambda(z_\lambda, |\zeta_\lambda^2|, |\eta_\lambda^2|)$$

and k, l are positive integers and s, t are non-negative integers with $(s, t)\neq(0, 0)$. (2) *A function $\underline{\Phi}$ defined on $\underline{U}-p_0$, where $\underline{U}$ is a neighborhood of p_0, is called a conoid function at p_0, if $\Phi=\tau^*\underline{\Phi}$ is a conoid function along A.*

By this we make the following definition:

Definition (0.2). *$\underline{\Delta}$ is called a simple domain if there exists a conoid function $\underline{\Phi}$ at p_0 such that for every small neighborhood $\underline{U}$ of p_0, (i) $\underline{\Delta} \cap \underline{U} = \{\underline{\Phi} < \varepsilon\}$ for some positive constant ε and (ii) there exists one and only one connected component $\underline{\Delta}'$ of $\underline{\Delta} \cap \underline{U}$ with $p_0 \in \partial \underline{\Delta}'$.*

In §2, we will give several examples of simple domains which are explicitly written on $\underline{M}$. Although the definition of simple domains looks artificial, it seems to the author that other domains of holomorphy which are easily constructed may be Stein in a small neighborhood of p_0.

Now we state our Main Theorem:

Main Theorem. *Under the condition (A), simple domains are domains of holomorphy if and only if they are* L*-manifolds.*

Here we describe the outline of the proof of Main Theorem. First we show that simple domains of holomorphy are normal conoids (see Theorem I in §1). As for the definition of normal conoids, see Definition (1.6). We note that it is defined only by using the property of M. From this, by using the resolution of the singularity of indeterminancy of the characteristic function ϕ^* (see Definition (1.6)), we can construct the B-resolution Δ_* of $\underline{\Delta}$ in the canonical manner. Secondly we shall make a weakly 1-complete function on Δ_* by using Lemma (3.5) (see Theorem II, IV). Finally we discuss the algebra of holomorphic functions on Δ_* by using the results obtained in the previous paper (see Theorem III in §1).

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§1. Statements of Main Results

Let $\iota:\underline{M}\to\mathbf{C}^N$ be the imbedding of $\underline{M}$ in $\mathbf{C}^N$ and let

$$\omega = \iota^*(\omega'), \quad \text{where} \quad \omega' = |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2. \tag{1.1}$$

Then we have an s-pseudoconvex function ω on $\underline{M}$. Let

$$(1.2) \qquad \underline{V}_{\delta}(p_0)=\{p\in\underline{M}\colon \omega(p)<\delta\}.$$

Then we have a neighborhood system of p_0 on $\underline{M}$. Let

$$(1.3) \qquad V_{\delta}(A)=\tau^{-1}(\underline{V}_{\delta}(p_0)).$$

In the following, a domain on a complex manifold is assumed to be a relatively compact domain without mentioning it.

Definition (1.4). *Let Ω be a domain on M.* (1) *Ω is called a simple domain along A of type (k, l) if there exist a conoid function $\phi_{k,l}$ of type (k, l) and a positive constant ε such that for every small positive δ, we have* (*i*)

$$\Omega\cap V_{\delta}(A)=\{\phi_{k,l}<\varepsilon\}\cap V_{\delta}(A)$$

and (*ii*) *there exists one and only one connected component Ω'_{δ} of $\Omega\cap V_{\delta}(A)$ satisfying $A\subset\partial\Omega'_{\delta}$.*
(2) *Ω is called a simple conoid along A if there exist a conoid function such that* (*i*) *there exist ε and δ satisfying*

$$\{\phi_{k,l}<\varepsilon\}\cap V_{\delta}(A)\subset\Omega\cap V_{\delta}(A)$$

and (*ii*) *for any point $p\in\Omega\cap V_{\delta}(A)$, there exists δ_0 (which may depend on the choice of p) satisfying*

$$\Gamma_p^{\delta_0,\delta}\subset\Omega\cap V_{\delta}(A),$$

where $\Gamma_p^{\delta}=\{q\in V_{\delta}(A)\colon \phi_{k,l}(q)=\phi_{k,l}(p)\}$ and $\Gamma_p^{\delta_0,\delta}=\Gamma_p^{\delta}\cap V_{\delta_0}(A)$ and (*iii*) *for every small δ, there exists one and only one connected component Ω'_{δ} of $\Omega\cap V_{\delta}$ with $A\subset\partial\Omega'_{\delta}$.*

We note that $A\subset\partial\Omega$. In the following, a simple domain Ω on M is assumed to have a boundary of a real submanifold of C^{∞}-class of codimension one except A.

Here we define special kinds of conoid functions and conoid domains which are determined only by the resolution manifold M: Let

$$c_1(F^{-1})=f \quad \text{and} \quad c_1(G^{-1})=g,$$

where $c_1(E)$ is the first chern class of a complex line bundle E on A. Choosing a pair of natural numbers k_0 and l_0 by the condition that $fk_0=gl_0$ and k_0 and l_0 have no common multiple other than 1, we consider a complex line bundle

$$E_M=F^{-k_0}\otimes G^{l_0}.$$

Then we see that E_M is a topologically trivial line bundle on A. By a well known lemma, we can choose suitable fibre coordinates ζ_λ and η_λ of F and G respectively so that E_M is expressed as $\{e_{\lambda\mu}\}$ with $|e_{\lambda\mu}|=1$. In the following we fix such coordinates. Then

$$\phi^*=|\zeta_\lambda^2|^{k_0}/|\eta_\lambda^2|^{l_0} \tag{1.5}$$

is a pseudoconvex function on $M-H$.

Definition (1.6). (1) *ϕ^* is called the characteristic function of M.* (2) *$\Delta\subset M-H$ is called a normal conoid if Δ is a simple conoid with respect to the conoid function ϕ^* and $\underline{\Delta}\subset\underline{M}-\underline{H}$ is called a normal conoid if $\Delta=\tau^{-1}(\underline{\Delta})$ is a normal conoid.*

In what follows we assume that $\underline{\Delta}$ is a simple domain which is a domain of holomorphy.

Definition (1.7). (1) *A domain Ω on a complex manifold is called a pseudoconvex domain if the following holds for any boundary point $p\in\partial\Omega$: There exist a neighborhood $U(p)$ of p and a pseudoconvex function φ on $U(p)$ such that $\Omega\cap U=\{\varphi<0\}$.* (2) *$\Omega$ is called a domain of holomorphy on M if there exists a holomorphic function f which cannot be continued analytically across the boundary of Ω.*

Remark. (1) If Ω is a domain of holomorphy on M, then Ω is a pseudoconvex domain. But the converse is not true in general. (2) The definition of domains of holomorphy on a complex manifold M is independent of the one of domains of holomorphy given in Definition (2.5) in O. Suzuki [10].

As for pseudoconvex domains we state the following Proposition which is due to T. Nishino [8]:

Proposition (1.8). *Let Ω be a domain on a complex manifold M and let S be a divisor on M. If $\Omega - S$ is a pseudoconvex domain, then Ω is also a pseudoconvex domain.*

Because $\underline{\Delta}$ is a domain of holomorphy, we see

Proposition (1.9). *$\Delta = \tau^{-1}(\underline{\Delta})$ is a domain of holomorphy and so is a pseudoconvex domain on M.*

By using these notations we state our main results of this paper.

Theorem I. *Suppose that $\underline{M}$ and $\underline{H}$ satisfy the condition (A). If $\underline{\Delta}$ is a simple domain which is a domain of holomorphy, then Δ is a pseudoconvex normal conoid.*

Theorem II. *Let $\Delta \subset M - H$ be a pseudoconvex normal conoid. Then there exists a proper modification of M, (M^*, μ, M) with the following properties*: (1) *Let $\Delta_* = \overline{(\mu^{-1}(\Delta))}^0$. Then Δ_* is a weakly* 1-*complete manifold, where $\bar{E}$ denotes the closure of E and E^0 denotes the open kernel of E.* (2) *Let $\Sigma = \mu^{-1}(A)$. Then $[\Sigma]^{-n}$ is positive for some n.*

As for the definition of weakly 1-complete manifolds and notations on complex line bundles, see § 3 in O. Suzuki [10].

Remark. Because Δ_* is a relatively compact domain on M^*, so we may assume that $[\Sigma]^{-n} \otimes K_{M^*}^{-1}$ is also positive on Δ_*, where K_{M^*} denotes the canonical line bundle of M^*.

Here we consider the algebra of holomorphic functions on Δ_*. A topological trivial line bundle E is called of finite order (resp. infinite order) if $E \otimes E \otimes \cdots \otimes E$ (k-times tensor product) is analytically trivial (not analytically trivial) with some k ($k \neq 0$) (resp. for any k ($k \neq 0$)).

Theorem III. (1) *If E_M is of finite order, then $\mathcal{O}(\Delta_*)$ is a Stein algebra.* (2) *If E_M is of infinite order, then $\mathcal{O}(\Delta_*)$ is not a Stein* *algebra.*

For the definition of Stein algebras, see Definition (3.15) in O. Suzuki [10]. Finally we can prove the following theorem, which also proves our Main Theorem.

Theorem IV. *If $\underline{\Delta}$ is a domain of holomorphy, then E_M is of infinite order. Moreover, $\mathcal{O}(\underline{\Delta})$ is not a Stein algebra.*

Remark. In the case where E_M is of finite order, for any $\Phi: \underline{M} \to \mathbf{C}^3$, $\underline{\Delta}$ is a domain of holomorphy in the sense of H. Grauert and R. Remmert [5], i.e., the Φ-hulle of $\underline{\Delta}$ is identical with $\underline{\Delta}$ (for the definition, see H. Kerner [7]). But $\underline{\Delta}$ is not a domain of holomorphy in the sense of H. Kerner [7].

§2. Normal Conoids Which Are Domains of Holomorphy

In this section we consider a Stein space $\underline{M}$ with the condition (A) and an analytic set $\overline{H}$ in $\underline{M}$ which is defined in Introduction. Moreover, we assume that E_M is of infinite order. The purposes of this section are to give some characterizations of normal conoids which are domains of holomorphy and to give their examples. Only in this section, domains are assumed to have C^∞-boundaries of real submanifolds of codimension one except p_0, or intersections of such a kind of domains.

First we give some examples of normal conoids. By Theorem I we see that simple domains which are domains of holomorphy are normal.

Definition (2.1). (1) *$\underline{K} \subset \underline{\Delta}$ is called a $*$-compact if $\underline{K} - \underline{V}_\delta(p_0)$ is relatively compact in $\underline{\Delta}$ for any δ.* (2) *As for two domains $\underline{\Delta}_1, \underline{\Delta}_2$ on $\underline{M}$, $\underline{\Delta}_1 \Subset_* \underline{\Delta}_2$ means that $\underline{\Delta}_1$ is $*$-compact in $\underline{\Delta}_2$,* (3) *$\underline{\Delta} \subset \underline{M}$ has a $*$-compact exhaustion $\{\underline{\Delta}_j\}$ $(j=1, 2,\dots)$ if (i) $\underline{\Delta}_j \Subset_* \underline{\Delta}_{j+1}$ and $\bigcup_{j=1}^{\infty} \underline{\Delta}_j = \underline{\Delta}$.*

The following proposition is easy.

Proposition (2.2). (1) *Suppose that $\underline{\Delta}_j$ $(j=1, 2,\dots, r)$ is normal,* *then* $\bigcap_{j=1}^{r}\underline{\Delta}_j$ *is also normal.* (2) *Let* $\underline{\Delta}$ *be a domain which admits a* $*$*-compact exhaustion* $\{\underline{\Delta}_j\}$, *where* $\underline{\Delta}_j$ *is normal for each* j, *then* $\underline{\Delta}$ *is also normal.*

Definition (2.3). (1) *We say* $\underline{f}\in\mathscr{M}_{l,k}$, *if there exists a holomorphic section* $\varphi=\{\varphi_\lambda(z_\lambda)\}\in H^0(A, \mathcal{O}(F^{-l}\otimes G^k))$ *such that* $f=\tau^*\underline{f}$ *is expressed as* $f=\varphi_\lambda(z_\lambda)\zeta_\lambda^l\eta_\lambda^{-k}$. (2) *We say* $\underline{f}\in\mathscr{P}_{l,m}$, *if there exists a positive semi-definite metric* $\{a_\lambda(z_\lambda)\}$ *of* $F^{-l}\otimes G^m$ *such that* $f=\tau^*\underline{f}$ *is expressed as* $f=a_\lambda(z_\lambda)|\zeta_\lambda^2|^l\cdot|\eta_\lambda^2|^{-m}$.

We set $\mathscr{M}=\bigcup_{l,k}\mathscr{M}_{l,k}$ and $\mathscr{P}=\bigcup_{l,m}\mathscr{P}_{l,m}$.

Definition (2.4). *For* $\underline{f}_1, \underline{f}_2,\dots,\underline{f}_r\in\mathscr{M}$ (*or,* $\mathscr{P}$) *and for positive constants* $\varepsilon_1, \varepsilon_2,\dots, \varepsilon_r$, *we define*

$$\underline{\Delta}(\underline{f}_1, \underline{f}_2,\dots,\underline{f}_r\colon \varepsilon_1, \varepsilon_2,\dots, \varepsilon_r) = \{p\in\underline{M}\colon |\underline{f}_1|<\varepsilon_1, |\underline{f}_2|<\varepsilon_2,\dots, |\underline{f}_r|<\varepsilon_r\} \tag{2.5}$$

$$\underline{\Delta}_\delta(\underline{f}_1,\dots,\underline{f}_r\colon \varepsilon_1,\dots, \varepsilon_r)=\underline{\Delta}(\underline{f}_1,\dots,\underline{f}_r\colon \varepsilon_1,\dots, \varepsilon_r)\cap \underline{V}_\delta(p_0)\,. \tag{2.6}$$

In the following, for simplicity we write $\underline{\Delta}(\underline{f}\colon \varepsilon)$ or $\underline{\Delta}_\delta(\underline{f}\colon \varepsilon)$ for (2.5) or (2.6) respectively.

Proposition (2.7). *For* $\underline{f}_1, \underline{f}_2,\dots,\underline{f}_r\in\mathscr{M}$ (*or* $\mathscr{P}$), $\underline{\Delta}_\delta(f\colon\varepsilon)$ *is a normal conoid for each* ε *and* δ.

Proof. By Proposition (2.2), we may prove only the following: For $\underline{f}\in\mathscr{M}$ (or $\mathscr{P}$), we set

$$\underline{\Delta}(\underline{f}\colon \varepsilon)=\{\underline{f}<\varepsilon\} \quad\text{and}\quad \underline{\Delta}_\delta(\underline{f}\colon \varepsilon)=\underline{\Delta}(\underline{f}\colon \varepsilon)\cap \underline{V}_\delta(p_0)\,.$$

Then $\underline{\Delta}_\delta(f\colon\varepsilon)$ is a normal conoid for every ε and δ. We prove the assertion only for $\underline{f}\in\mathscr{M}$. We assume that $\underline{f}\in\mathscr{M}_{k,l}$. We write $\tilde{f}=\varphi_\lambda(z_\lambda)\zeta_\lambda^k\eta_\lambda^{-l}$, where $\tilde{f}=\tau^*\underline{f}$. Then by Proposition (4.1), we see that

$$kf\geqq gl. \tag{2.8}$$

Let ϕ^* be the characteristic function of M. Then we have

$$|\tilde{f}|^2=|\varphi_\lambda(z_\lambda)|^2|\zeta_\lambda^2|^k\cdot|\eta_\lambda^2|^{-l}$$

and

$$\phi^*=|\zeta_\lambda^2|^{k_0}\cdot|\eta_\lambda^2|^{-l_0}.$$

We define $\rho(\rho>0)$ by $l=l_0\rho$. By (2.8) and by the choice of k_0 and l_0, we find a non-negative constant h such that $fk=f\rho k_0+hf$. First we consider the case $h>0$. Making δ' smaller, we may assume

$$\sup_\lambda|\varphi_\lambda(z_\lambda)|^2|\zeta_\lambda^2|^h<1 \quad \text{on} \quad V_{\delta'}(A).$$

Then we see that

$$\{\phi^*<\varepsilon'\}\cap V_{\delta'}(A)\subset\Delta_\delta(\tilde{f}:\varepsilon) \quad \text{where} \quad \varepsilon'=\varepsilon^{\frac{1}{\rho}} \quad \text{and} \quad \Delta_\delta(\tilde{f}:\varepsilon)=\tau^{-1}\underline{\Delta}_\delta(f:\varepsilon),$$

which implies (i) in Definition (1.4). Let $p\in\Delta_\delta(\tilde{f}:\varepsilon)$ and let $\phi^*(p)=\sigma$. Choose a point $q\in\Gamma_p^\delta$ (see Definition (1.4)). Then

$$|\tilde{f}|^2(q)=|\varphi_\lambda(z_\lambda)|^2\cdot|\zeta_\lambda^2|^h\sigma^\rho.$$

Making δ_0 smaller, we may assume that

$$\sup_\lambda|\varphi_\lambda(z_\lambda)|^2\cdot|\zeta_\lambda^2|^h<\varepsilon/\sigma^\rho \quad \text{on} \quad V_{\delta_0}(A).$$

This implies that $|\tilde{f}|^2(q)<\varepsilon$ for $q\in\Gamma_p^{\delta,\delta_0}$. This proves (ii). (iii) is easy. Next we consider the remained case, i.e., $h=0$. Then we find that $fk=gl$. Therefore, we can find m satisfying $k=mk_0$ and $l=ml_0$. Then we see that $\{\varphi_\lambda(z_\lambda)\}\in H^0(A,\mathcal{O}(F^{-k_0}\otimes G^{l_0})^m)$. Because E_M is of infinite order, we see that $\varphi_\lambda(z_\lambda)=0$ everywhere. This implies that $f=0$. So the assertion is trivial in this case.

Remark. For $\underline{f}\in\mathscr{P}$, we can prove the assertion in the same manner as in Proposition (2.7) when $h>0$. When $h=0$, we see that $f=c(\phi^*)^m$ with some positive constant c. So the assertion is also proved in this case.

Corollary (2.9). *If $\underline{\Delta}$ admits a $*$-compact exhaustion $\{\underline{\Delta}_j\}$, where $\underline{\Delta}_j$ is a polyhedron defined as in* (2.6), *then $\underline{\Delta}$ is also a normal conoid.*

Now we shall give some characterizations of normal conoids to be domains of holomorphy.

Definition (2.10). *$\underline{\Delta}\subset\underline{M}$ is called holomorphically $*$-convex if for any compact set $K\subset\underline{\Delta}$, the convex hull of K, $\hat{K}$ is also $*$-compact.*

Definition (2.11). (1) *A real valued function φ on $\underline{\Delta}$ is called $*$-complete if $\underline{\Delta}_c=\{\varphi<c\}$ is $*$-compact for each c.* (2) *$\underline{\Delta}$ is called $*$-complete domain if there exists a $*$-complete pseudoconvex function of C^∞-class on $\underline{\Delta}$.*

Now we consider a normal conoid $\underline{\Delta}$ on $\underline{M}$. We set $\Delta=\tau^{-1}(\underline{\Delta})$. Let $\mu: M^*\to M$ be the proper modification described in Theorem II. Then Δ_* is a B-resolution of $\underline{\Delta}$ and that $\Sigma=\varpi^{-1}(\Gamma)$ holds where $\Gamma=\{p_0\}$ (as for the definitions of B-resolutions and π and Γ, see § 1 in O. Suzuki [10] and also see Proposition (6.9)). Then we have the following

Theorem (2.12). *Suppose that $\underline{\Delta}$ is a normal conoid. Then the followings are equivalent:*

(i) *$\underline{\Delta}$ is a domain of holomorphy.*

(ii) *$\underline{\Delta}$ is a $*$-complete domain.*

(iii) *$\underline{\Delta}$ is a holomorphically $*$-convex domain.*

(iv) *$\underline{\Delta}$ is an H-manifold with respect to Δ_*, $[\Sigma]^{-n}$ and Σ (see Definition* (3.3) *in O. Suzuki* [10]).

(v) *$\underline{\Delta}$ is an* L-*manifold with respect to Δ_* and $[\Sigma]^{-n}$ (see Definition* (3.11) *in O. Suzuki* [10]).

Proof. (v)⇒(iv). This is a direct consequence of Theorem 5 in O. Suzuki [10]. (iv)⇒(iii). Setting $\sigma=\tau\circ\mu$, we see that $\sigma(\Sigma)=p_0$. Let $\underline{K}$ be a compact set in $\underline{\Delta}$. In view of $\Delta_*-\Sigma\cong\underline{\Delta}$, $K=\sigma^{-1}(\underline{K})$ is also a compact set in Δ_*. By $\mathcal{O}_{(\Sigma,n)}(\Delta_*)$ we denote the subalgebra of holomorphic functions which are obtained from holomorphic sections $H^0(\Delta_*, \mathcal{O}([\Sigma]^{-n}))$ by multiplying the defining equations of $[\Sigma]^n$. The convex hull of K with respect to this subalgebra is denoted by $\hat{K}_{(n)}$. Then we see that $\hat{K}_{(n)}-\Sigma\Subset\Delta_*$ (see (3.12) in O. Suzuki [10]). For $f\in\mathcal{O}_{(\Sigma,n)}(\Delta_*)$,

$\sigma^{-1*}(f)\in\mathcal{O}(\underline{\Delta})$ and $\{\sigma^{-1*}(f)\}$ make a certain subalgebra of $\mathcal{O}(\underline{\Delta})$. The convex hull of K with respect to this subalgebra is nothing but $\sigma(\hat{K}_{(n)})$, which is $*$-compact. Hence $\underline{\hat{K}}$ is also $*$-compact. (iii)⇒(ii). Let $\underline{K}$ be a compact set in $\underline{\Delta}$. For $q\in\underline{\Delta}-\underline{\hat{K}}$, there exists a holomorphic function $f\in\mathcal{O}(\underline{\Delta})$ such that $|f(q)|>\|f\|_K$. This holds on a small neighborhood of q. Now fix an arbitrary constant δ ($\delta>0$). Then there exist holomorphic functions $\underline{f}_1,\underline{f}_2,\dots,\underline{f}_r$ on $\underline{\Delta}$ satisfying the following condition: Let

$$\underline{\tilde{\Delta}}=\{|\underline{f}_1|<\varepsilon_1,\ |\underline{f}_2|<\varepsilon_2,\dots,\ |\underline{f}_r|<\varepsilon_r\}.$$

Then we have

$$\underline{\hat{K}}-\underline{V}_\delta(p_0)\subset\underline{\tilde{\Delta}}-\underline{V}(p_0)\Subset\underline{\Delta}$$

$$\text{and}\quad \|f_j\|_K<1\quad\text{and}\quad \varepsilon_j>1\qquad (j=1,2,\dots,r).$$

Now take positive constants ε and l and compact sets K and $\tilde{K}$. Then we can find holomorphic functions $\underline{g}_1,\underline{g}_2,\dots,\underline{g}_s$ so that

$$\varphi=\sum_{j=1}^{s}|g_j|^2 \tag{2.13}$$

satisfies

$$\|\varphi\|_K<\varepsilon\quad\text{and}\quad\varphi(q)>1\qquad\text{for}\quad q\in\tilde{K}-\underline{\tilde{\Delta}}. \tag{2.14}$$

Let $\{\delta_\nu\}$ be a sequence of positive numbers with $\delta_\nu\to 0$ and let $\underline{K}_1$ be a compact set in $\underline{\Delta}$. We make an analytic polyhedron $\underline{\tilde{\Delta}}_1$ for $(\underline{K}_1, \underline{V}_{\delta_1}(p_0))$. Let $\underline{\Delta}_1^*=\underline{\tilde{\Delta}}_1-\underline{V}_{\delta_1}(p_0)$ and let $\underline{K}_2$ be a compact set with $\underline{\Delta}_1^*\subset\underline{K}_2$. Next we make $\underline{\tilde{\Delta}}_2$ for $(\underline{K}_2, \underline{V}_{\delta_2}(p_0))$. By repeating this process we have a compact exhaustion $\{\underline{\Delta}_j^*\}$ of $\underline{\Delta}$. Choose ε_j and l_j ($j=1,2,\dots$) with $\sum\varepsilon_j<\infty$ and $l_j\to\infty$ ($j\to\infty$). For each j we construct φ_j as in (2.13) which satisfies (2.14) for $K=K_j$, $\tilde{K}=\underline{\Delta}_{j+1}^*$, $\varepsilon=\varepsilon_j$ and $l=l_j$. Then

$$\varphi=\sum_{j=1}^{\infty}\varphi_j$$

is a $*$-complete pseudoconvex function on $\underline{\Delta}$. (ii)⇒(i). By Theorems II, III, IV, it is sufficient to show that $\Delta=\tau^{-1}(\underline{\Delta})$ is a pseudoconvex domain. For any point $q\in\partial\Delta-A$ and for any sequence $\{q_n\}$ with $q_n\to q$

we see that $\tau^*\varphi(q_n)\to\infty$. So Δ is pseudoconvex at $\partial\Delta - A$. Now we prove that Δ is pseudoconvex at every boundary point on A. Take a point $q\in A$. We assume that $q\in V_\lambda$. By assumption, $\Delta\cap H=\emptyset$. Choose a small neighborhood $U(q)$ and consider the following pseudoconvex function

$$\phi=1/|\eta_\lambda|^2+\tau^*\varphi \quad \text{on} \quad U(q).$$

Then we see that for any point $p\in U(q)\cap A$ and any sequence $\{p_n\}$ with $p_n\in\Delta$ and $p_n\to p$, we see that $\phi(p_n)\to\infty$, which implies that Δ is pseudoconvex at p. Finally we show that (i) implies (v). If $\underline{\Delta}$ is a domain of holomorphy, then Δ is pseudoconvex by Proposition (1.9). Then by Theorems II, III, IV, we see that $\underline{\Delta}$ is an L-manifold.

We finish this section with giving some examples of L-manifolds. The first ones are as follows:

Proposition (2.15). *Every* $\underline{\Delta}_\delta(f,\varepsilon)$ *is an* L-*manifold and is a domain of holomorphy for every pair of* δ *and* ε.

Proof. By the construction of $\underline{\Delta}_\delta(f,\varepsilon)$, we see that it is a weakly $*$-complete domain. Also by (2.7), it is a normal conoid. Then by Theorem (2.12), we prove the assertion.

Secondly we are concerned with some examples of simple domains which are explicitly expressed on $\underline{M}$. For this we assume that $\underline{M}$ is nothing but the Remmert reduction of M, i.e., $\underline{M}=\operatorname{Spec}\mathcal{O}(M)$. Then τ is expressed as follows:

$$\tau=(f_1, f_2,\dots, f_N)\colon M\longrightarrow \underline{M}\subset\mathbf{C}^N,$$

where $$f_j=\varphi_\lambda^{(j)}(z_\lambda)\zeta_\lambda^{n_j}\eta_\lambda^{m_j} \qquad (j=1, 2,\dots, N)$$

and n_j and m_j are non-negative integers. We assume that $n_j\neq 0$ for $j=1, 2,\dots, r$. For a sufficiently large n, H is expressed as

$$H\colon \{h_j=0\},$$

where $$h_j=\phi_\lambda^{(j)}(z_\lambda)\eta_\lambda^n \qquad (j=1, 2,\dots, s).$$

We define $\underline{h}_j$ by $\underline{h}_j=\tau^*(\underline{h}_j)$. Let $\varphi_1(x_1, x_2,\dots, x_N)$ and $\varphi_2(y_1, y_2,\dots, y_s)$

be two polynomials with positive coefficients satisfying the following conditions: (1) $\varphi_1(0, 0,\dots, 0)=0$ and (2) φ_2 is a homogeneous polynomial. Letting

$$\underline{\Phi}=\varphi_1(|z_1|^2, |z_2|^2,\dots, |z_N|^2)/\varphi_2(|\underline{h}_1|^2, |\underline{h}_2|^2,\dots, |\underline{h}_s|^2),$$

we set

$$\underline{\Delta}_{\delta,\varepsilon}=\{\underline{\Phi}<\varepsilon\} \cap \underline{V}_\delta.$$

The following proposition shows that the concept of simple domains contains many examples:

Proposition (2.16). *If $p_0 \in \partial\underline{\Delta}_{\delta,\varepsilon}$, $\underline{\Delta}_{\delta,\varepsilon}$ is a simple domain.*

Proof is easy and may be omitted. Choosing suitable φ_1, φ_2, we can construct simple domains which are domains of holomorphy.

Proposition (2.17). *If $\tau^*(\underline{\Phi})=\Phi$ is a pseudoconvex function on $M-H$, then $\underline{\Delta}_{\delta,\varepsilon}$ is a domain of holomorphy if it is a normal conoid.*

This follows from Theorem (2.12), (ii).

By using this proposition, we construct simple domains which are domains of holomorphy with the explicit representations:

Proposition (2.18). *Let d (resp. e) be the* l.c.m *of n_j (resp. m_j). Set $\underline{\Phi}=\varphi_1/\varphi_2$ with*

$$\varphi_1=\varphi_1'(|z_1^2|^{d_1}, |z_2^2|^{d_2},\dots, |z_r^2|^{d_r})^m$$
$$+\varphi_1''(|z_{r+1}^2|^{e_{r+1}},\dots, |z_N^2|^{e_N})^k,$$

where φ_1' and φ_1'' are homogeneous polynomials of degree $\tilde{d}$ and $\tilde{e}$ respectively and $d_j=d/n_j$ and $e_j=e/m_j$. Then simple domains defined by Φ are domains of holomorphy for a sufficiently large m and k.

Proof. Let

$$\tau^*\varphi_1'=a_\lambda|\zeta_\lambda^2|^{d\cdot\tilde{d}} \quad \text{and} \quad \tau^*\varphi_1''=b_\lambda|\eta_\lambda^2|^{e\cdot\tilde{e}}.$$

Then we get negative metrics $\{a_\lambda\}$ and $\{b_\lambda\}$ of $[S]^d$ and $[H]^e$ respectively, where $S=\{\zeta_\lambda=0\}$. Then making m and k large, we see that $\tau^*\underline{\Phi}$ is a pseudoconvex function on $M-H$. We can prove that $\underline{\Delta}_{\delta,\varepsilon}$ is a normal conoid as in the proof of Proposition (2.7).

§3. Lemmas on Pseudoconvex Domains on a Complex Manifold

In this section, $\underline{M}$ is assumed to be a Stein space with an isolated singularity and the condition (A) is not assumed on $\underline{M}$ without mentioning it. Every domain $\underline{\Delta}$ on $\underline{M}$ is assumed to be a domain of holomorphy. The main difficulties in the proofs of Theorems stated in §1 lie in the proof of (1) in Theorem II. The purpose of this section is to prepare a lemma (see Lemma (3.5)) which is an essential step in its proof. First we state the following lemma concerning pseudoconvex domains on a complex manifold, which is due to A. Takeuchi [11].

Lemma (3.1). *Let M be a complex manifold with a real analytic kähler metric and let D be a pseudoconvex domain on M. For two points p and q, we denote the distance between p and q by $d(p, q)$ and we set*

$$d(p)=\inf_{q\in\partial D} d(p, q) \quad \text{and} \quad \varphi(p)=-\log d(p) \qquad \text{for} \quad p\in D.$$

Then $\varphi(p)$ is a complete function and the infimum of the eigen values of the Hessian of φ on D is bounded below by a real constant ρ, where ρ is determined only by D.

The isolated singularity of $\underline{M}$ is denoted by p_0. Then we have the following

Proposition (3.2). *If $p_0 \notin \partial\underline{\Delta}$, then $\underline{\Delta}$ is Stein.*

Proof. We consider only the case where $p_0 \notin \underline{\Delta}$. The proof for the case where $p_0\in\underline{\Delta}$ is similar. An imbedding of $\underline{M}$ in $\mathbf{C}^N$ is given, which is denoted by $\iota: \underline{M}\to\mathbf{C}^N$. Restricting the canonical kähler metric of $\mathbf{C}^N$ to $\underline{M}-\{p_0\}$, we get a real analytic kähler metric on $\underline{M}-\{p_0\}$. As (1.1) we get an s-pseudoconvex function ω on $\underline{M}$. Because $\underline{\Delta}$ is an domain of holomorphy and $p_0 \not\in \underline{\Delta}$, $\underline{\Delta}$ is pseudoconvex. Then by Lemma (3.1) we get φ. Choosing a suitable convex increasing function χ, we see that $\varphi+\chi(\omega)$ becomes a complete s-pseudoconvex function on $\underline{\Delta}$. By Theorem of A. Andreotti and R. Narasimhan [1], we see that $\underline{\Delta}$ is Stein.

Corollary (3.3). *If a domain of holomorphy $\underline{\Delta}$ is not Stein, then p_0 must be contained in $\partial\underline{\Delta}$.*

Here we consider the Δ on M treated in §1. By Proposition (1.9), Δ is a pseudoconvex domain. Because $[S]^{-1}$ is positive, we can choose a real analytic kähler metric on M. By using Lemma (3.1) we can prove the following

Proposition (3.4). *For any neighborhood V of S, there exists a function φ on Δ such that* (1) *φ is of C^∞-class,* (2) *φ is s-pseudoconvex on $\Delta-V$ and for any boundary point $q\in\partial\Delta-V$ and for any sequence $\{q_n\}$, $q_n\in\Delta$ with $q_n\to q$, $\varphi(q_n)\to\infty$.*

Here we remark on the analytic set $\underline{H}$:

Remark. If $\underline{\Delta}$ is contained in $\underline{M}-\underline{H}$, where $\underline{H}$ is a Cartier divisor in $\underline{M}$ through p_0, then $\underline{\Delta}$ is Stein.

The rest of this section is devoted to show the following lemma:

Lemma (3.5). *Let M and S be a complex manifold with a real analytic kähler metric and a divisor on M respectively. Let D be a domain in M. Suppose that D satisfies the following three conditions. Then we see that D is a weakly* 1*-complete manifold.*

(i) *$D\cap S\neq\emptyset$ holds and $[S]^{-1}$ is positive on M.*

(ii) *D is a pseudoconvex domain whose boundary is a real one codimensional submanifold of C^∞-class except $S\cap\partial D$.*

(iii) *There exists an open set Ω $(\Omega\supsetneqq D)$ such that* (1) *$D\cap S=\Omega\cap S$ and* (2) *there exist a pseudoconvex function η on Ω and a neighborhood V of $S\cap\bar{D}$ which satisfy the following: For every point $p\in\partial\Omega\cap V$ and for any $p_n\in\Omega\cap V$ with $p_n\to p$, $\eta(p_n)\to\infty$.*

The proof of this lemma is very complicated. So we separate the proof into four steps.

(The first step). We fix a local coordinate covering $\mathfrak{U}=\{U_\lambda\}$ of M and by $\zeta_\lambda=0$ we denote the defining equation of S on U_λ with $U_\lambda \cap S \neq \emptyset$. In terms of $\mathfrak{U}$, we denote the positive metric of $[S]^{-1}$ by $\{a_\lambda\}$. Then we obtain a pseudoconvex function on M which is an s-pseudoconvex function on $M-S$ and a neighborhood system of S in $\bar{D}$:

$$h=a_\lambda^{-1}\cdot|\zeta_\lambda|^2,$$

$$V_\varepsilon \quad \text{where} \quad V_\varepsilon=\{h<\varepsilon\}.$$

Then we have the following

Proposition (3.6). *Let K be an arbitrary compact set in D. Then there exists a domain Δ in D with the following properties:* (1) *$K\subset\Delta\Subset D$ and* (2) *Δ is a weakly* 1*-complete manifold.*

Proof. First we choose a positive constant R sufficiently large so that $K\subset\{\eta<R\}$. Next making ε so small that we may assume that any connected component of $\{\eta<R\}\cap V_\varepsilon$ contains exactly one of the connected components of $\{\eta<R\}\cap S$ and

$$\{\eta=R\}\cap\partial D\cap V_\varepsilon=\emptyset.$$

If we choose $\tilde{\varepsilon}$ with $\tilde{\varepsilon}<\varepsilon$, the above condition is also satisfied for $\tilde{\varepsilon}$. For such a fixed ε, we choose ε' and ε'' with $0<\varepsilon''<\varepsilon'<\varepsilon$ and choose a C^∞-function ρ_ε $(0\leqq\rho_\varepsilon\leqq 1)$ with the following property:

$$\rho_\varepsilon=\begin{cases}1 & \text{on } M-V_{\varepsilon'}\\ 0 & \text{on } V_{\varepsilon''}.\end{cases}$$

The function on D which is obtained by Lemma (3.1) is denoted by φ. We set

$$\varphi'_\varepsilon=\varphi\cdot\rho_\varepsilon.$$

We may assume that φ'_ε is a function of C^∞-class. By Lemma (3.1) there exists a real constant c such that the infimum of the eigen values of the hessian of φ'_ε on $D \cap \{\eta<R\}$ is bounded below by c. We write this implication as $W(\varphi'_\varepsilon) \geqq c$ on $D \cap \{\eta<R\}$. So by choosing a suitable convex increasing function χ and referring to $\varphi'_\varepsilon=0$ on $V_{\varepsilon''}$, we may assume that

$$(3.7) \qquad \varphi_\varepsilon=\varphi'_\varepsilon+\chi(h)$$

is pseudoconvex on $D \cap \{\eta<R\}$ and s-pseudoconvex except S. We fix such a χ. We note that (3.7) is satisfied for $\tilde{\chi}(t)$ with $\tilde{\chi}(t) \gg \chi(t)$, where $\tilde{\chi}(t) \gg \chi(t)$ implies that $\tilde{\chi}(t) \geqq \chi(t)$, $\tilde{\chi}'(t) \geqq \chi'(t)$ and $\tilde{\chi}''(t) \geqq \chi''(t)$. Because $D \subsetneqq \Omega$, we see that $\{\eta=R\} \cap \partial D \neq \emptyset$ for a large R. Moreover, making R^* sufficiently large, we have

$$(3.8) \qquad \{\varphi_\varepsilon=R^*\} \cap \{\eta=R\} \cap V_\varepsilon=\emptyset .$$

This holds for $\tilde{R}^*$ with $\tilde{R}^*>R^*$. So choose R^* with $R^*>\bar{R}^*$, where $\bar{R}^*=\max_{p \in K} \varphi_\varepsilon(p)$ and define

$$\Delta=\{\eta<R\} \cap \{\varphi_\varepsilon<R^*\},$$

$$\phi=1/(1-\varphi_\varepsilon/R^*)+1/(1-\eta/R).$$

Then (1) and (2) are satisfied for Δ and ϕ.

(The second step). We shall construct a special compact exhaustion $\{\Delta_j\}$ of D, where Δ_j is of the form as constructed in the first step. Take a compact set K_1. As in Proposition (3.6) choosing ε_1, ρ_1, χ_1, R_1 and R^*_1, we make Δ_1. Next we choose a compact set K_2 with $\Delta_1 \Subset K_2$. In the same manner we have Δ_2. Repeating this process we make Δ_ν by choosing ε_ν, ρ_ν, χ_ν, R_ν and R^*_ν. We may assume that $R_\nu<R_{\nu+1}$ and $R_\nu \to \infty$ $(\nu \to \infty)$. In the following we write V_ν, φ'_ν and φ_ν for V_{ε_ν}, $\varphi'_{\varepsilon_\nu}$ and $\varphi_{\varepsilon_\nu}$ respectively. We may assume that (1) $\varepsilon_\nu>\varepsilon_{\nu+1}$ and $\varepsilon_\nu \to 0$ $(\nu \to \infty)$, (2) $\chi_\nu \ll \chi_{\nu+1}$, (3) $\rho_\nu<\rho_{\nu+1}$ and (4) $\varphi_\nu<\varphi_{\nu+1}$. We define $\sigma_{\nu+1}$ by

$$(3.9) \qquad \sigma_{\nu+1}=\sup_{p \in D}\{\chi_{\nu+1}(h)-\chi_{\nu-1}(h)\} + \mathrm{Max}\Big[\sup_{p \in \Delta_{\nu-1}}\{\varphi'_{\nu+1}-\varphi'_{\nu-1}\},\, 0\Big].$$

Also we may assume that $R^*_\nu<R^*_{\nu+1}$ and $R^*_\nu \to \infty$ $(\nu \to \infty)$ and

$$(3.10) \qquad R^*_\nu>R^*_{\nu-1}+\sigma_{\nu+1} \qquad (\nu \geqq 2).$$

We fix such an exhaustion in the following.

(The third step). We set

$$\partial \Delta_\nu = \delta_\nu \cup \delta'_\nu,$$

where $\delta_\nu = \{\eta = R_\nu\} \cap \{\varphi_\nu \leqq R^*_\nu\}$ and $\delta'_\nu = \{\varphi_\nu = R^*_\nu\} \cap \{\eta \leqq R_\nu\}$. Let $\Delta(\delta_\nu)$ be open sets in M with $\partial \Delta(\delta_\nu) \cap \{\eta = R_\nu\} = \delta_\nu$. By these domains we set

$$\Omega_\nu = (\Delta_\nu - \bar{\Delta}_{\nu-1}) \cap \Delta(\delta_\nu) \qquad (\nu \geqq 2).$$

Also we set

$$\mu_{\nu+1} = \sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \varphi_{\nu+1} \qquad (\nu \geqq 2). \tag{3.11}$$

Then we have

Proposition (3.12). $\mu_{\nu+1} \leqq R^*_{\nu-1} + \sigma_{\nu+1} < R^*_\nu$.

Proof. In view of $\varphi_\nu = \varphi'_\nu + \chi_\nu$, (3.9) and (3.11), the assertion is easily proved.

Now we prove the following

Proposition (3.13). *For each* $\nu+1$, *there exist positive convex increasing functions* $\Psi^{(1)}_{\nu+1}$ *and* $\Psi^{(2)}_{\nu+1}$ *such that*

$$\Phi_{\nu+1} = \Psi^{(1)}_{\nu+1}(1/(1-\varphi_{\nu+1}/R^*_{\nu+1})) + \Psi^{(2)}_{\nu+1}(1/(1-\eta/R_{\nu+1}))$$

satisfies the following condition: Let

$$m_{\nu+1} = \inf_{p \in \Delta_{\nu+1} - \Delta_\nu} \Phi_{\nu+1}(p) \quad \text{and} \quad M_{\nu+1} = \sup_{p \in \Delta_{\nu-1}} \Phi_{\nu+1}(p),$$

then we have

$$m_{\nu+1} > M_{\nu+1}.$$

Corollary (3.14). *Let*

$$\Delta_{\nu+1}(m) = \{\Phi_{\nu+1} < m\}.$$

Then

$$\Delta_{\nu-1} \subset \Delta_{\nu+1}(M_{\nu+1}) \Subset \Delta_{\nu+1}(m_{\nu+1}) \subset \Delta_{\nu} \Subset \Delta_{\nu+1}.$$

This corollary is a direct consequence of Proposition (3.13).

Proof. We prove in the following steps:

(i) First we choose $\Psi^{(2)}_{\nu+1}$ such that

$$\inf_{p \in \{\eta > R_\nu\}} \Phi_{\nu+1}(p) > \sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \Phi_{\nu+1}(p).$$

(ii) Next we choose $\Psi^{(1)}_{\nu+1}$ such that

$$\inf_{p \in \{\varphi_\nu > R^\nu{}_*\}} \Phi_{\nu+1}(p) > \sup_{p \in \Delta_{\nu-1} - \Omega_{\nu-1}} \Phi_{\nu+1}(p).$$

(iii) Finally we prove $m_{\nu+1} > M_{\nu+1}$.

In what follows we set

$$\tilde{\Psi}^{(1)}_{\nu+1}(t) = \Psi^{(1)}_{\nu+1}(1/(1-t/R^*_{\nu+1})) \quad \text{and} \quad \tilde{\Psi}^{(2)}_{\nu+1}(t) = \Psi^{(2)}_{\nu+1}(1/(1-t/R_{\nu+1})).$$

Proof of (i). We choose $\Psi^{(1)}_{\nu+1}$ with the following condition:

$$\tilde{\Psi}^{(1)}_{\nu+1}(t) = 1/(1-t/R^*_{\nu+1}) \qquad \text{for} \quad t \leqq \mu_{\nu+1}. \tag{3.15}$$

In what follows $\Psi^{(1)}_{\nu+1}$ is assumed to satisfy (3.15).

By using $\tilde{\Psi}^{(1)}_{\nu+1}(t) \geqq 0$, we have

$$\inf_{p \in \{\eta > R_\nu\}} \Phi_{\nu+1} \geqq \tilde{\Psi}^{(2)}_{\nu+1}(R_\nu). \tag{3.16}$$

On the other hand, we have

$$\sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \Phi_{\nu+1} \leqq \sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \tilde{\Psi}^{(1)}_{\nu+1}(\varphi_{\nu+1}) + \sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \tilde{\Psi}^{(2)}_{\nu+1}(\eta).$$

Because $\eta(p) < R_{\nu-1}$ holds on $\Delta_{\nu-1} \cap \Omega_{\nu-1}$, so the second term can be estimated as follows:

$$\sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \tilde{\Psi}^{(2)}_{\nu+1}(\eta) \leqq \tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1}).$$

By (3.11), we have

$$\sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \tilde{\Psi}^{(1)}_{\nu+1}(\varphi_{\nu+1}) \leqq K_{\nu+1},$$

where

(3.17) $$K_{\nu+1}=1/(1-\mu_{\nu+1}/R^*_{\nu+1}).$$

We note that $K_{\nu+1}$ does not depend on a choice of $\Psi^{(1)}_{\nu+1}$. Then we have

(3.18) $$\sup_{p\in\Delta_{\nu-1}\cap\Omega_{\nu-1}}\Phi_{\nu+1}\leqq K_{\nu+1}+\tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1}).$$

We choose a constant $M'_{\nu+1}$ with

(3.19) $$M'_{\nu+1}>K_{\nu+1}.$$

For the proof of (i), from (3.16), (3.18) and (3.19), it is sufficient to choose $\Psi^{(2)}_{\nu+1}$ such that

(3.20) $$\tilde{\Psi}^{(2)}_{\nu+1}(R_\nu)>\tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1})+K_{\nu+1}+2M'_{\nu+1}.$$

This is always possible. Moreover, we can choose it with the following additional condition:

(3.21) $$\tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1})<M'_{\nu+1}.$$

We choose such $\Psi^{(2)}_{\nu+1}$ and fix in the following.

Proof of (ii). Referring to $\tilde{\Psi}^{(2)}_{\nu+1}(\eta)\geqq 0$, we see that

$$\inf_{p\in\{\varphi_\nu<R^*_\nu\}}\Phi_{\nu+1}\geqq\inf_{p\in\{\varphi_\nu>R^*_\nu\}}\tilde{\Psi}^{(1)}_{\nu+1}(\varphi_{\nu+1}).$$

Owing to $\varphi_{\nu+1}>\varphi_\nu$, we have $\tilde{\Psi}^{(1)}_{\nu+1}(\varphi_{\nu+1})>\tilde{\Psi}^{(1)}_{\nu+1}(\varphi_\nu)$. Because

$$\inf_{p\in\{\varphi_\nu>R^*_\nu\}}\tilde{\Psi}^{(1)}_{\nu+1}(\varphi_\nu)\geqq\tilde{\Psi}^{(1)}_{\nu+1}(R^*_\nu),$$

we have

(3.22) $$\inf_{p\in\{\varphi_\nu>R^*_\nu\}}\Phi_{\nu+1}\geqq\tilde{\Psi}^{(1)}_{\nu+1}(R^*_\nu).$$

On the other hand we see that

(3.23) $$\sup_{p\in\Delta_{\nu-1}-\Omega_{\nu-1}}\Phi_{\nu+1}\leqq\sup_{p\in\Delta_{\nu-1}-\Omega_{\nu-1}}\tilde{\Psi}^{(1)}_{\nu+1}(\varphi_{\nu+1})+\sup_{p\in\Delta_{\nu-1}-\Omega_{\nu-1}}\tilde{\Psi}^{(2)}_{\nu+1}(\eta).$$

Because $\eta(p)<R_{\nu-1}$ for $p\in\Delta_{\nu-1}-\Omega_{\nu-1}$, the second term in the right-

hand-side of (3.23) is bounded above by $\tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1})$. Since $\varphi_{\nu+1} = \varphi_{\nu-1} + \chi_{\nu+1} - \chi_{\nu-1} + \varphi'_{\nu+1} - \varphi'_{\nu-1}$, the first term in the right-hand side of (3.23) is bounded above by $\tilde{\Psi}^{(1)}_{\nu+1}(\varphi_{\nu-1} + \sigma_{\nu+1})$ on $\Delta_{\nu-1} - \Omega_{\nu-1}$, where we use (3.9). Because $\sup_{p \in \Delta_{\nu-1} - \Omega_{\nu-1}} \varphi_{\nu-1} = R^*_{\nu-1}$, we obtain from (3.23)

$$\sup_{p \in \Delta_{\nu-1} - \Omega_{\nu-1}} \Phi_{\nu+1} \leqq \tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu-1} + \sigma_{\nu+1}) + \tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1}) \,. \tag{3.24}$$

For the proof of (ii), from (3.22) and (3.24), it is sufficient to choose $\Psi^{(1)}_{\nu+1}$ such that

$$\tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu-1} + \sigma_{\nu+1}) + \tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1}) < \tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu}) \,. \tag{3.25}$$

From (3.21), it is sufficient to choose $\Psi^{(1)}_{\nu+1}$ such that

$$\tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu-1} + \sigma_{\nu+1}) + M'_{\nu+1} < \tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu}). \tag{3.26}$$

This is always possible. Moreover, by Proposition (3.12) we can choose it with the following additional condition:

$$K_{\nu+1} < \tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu-1} + \sigma_{\nu+1}) \leqq M'_{\nu+1} \,, \tag{3.27}$$

where we use (3.17) and (3.19). This completes the proof of (ii).

Proof of (iii). For this it is sufficient to show that $\Phi_{\nu+1}$ which is chosen as above satisfies the following two conditions:

$$\inf_{p \in \{\varphi_\nu > R^*_\nu\}} \Phi_{\nu+1} > \sup_{p \in \Delta_{\nu-1} \cap \Omega_{\nu-1}} \Phi_{\nu+1} \tag{3.28}$$

and

$$\inf_{p \in \{\eta > R_\nu\}} \Phi_{\nu+1} > \sup_{p \in \Delta_{\nu-1} - \Omega_{\nu-1}} \Phi_{\nu+1}. \tag{3.29}$$

First we show (3.28). By (3.22) and (3.18), it is sufficient to show that

$$\tilde{\Psi}^{(1)}_{\nu+1}(R^*_\nu) > K_{\nu+1} + \tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1}) \,.$$

Referring to (3.21), it is sufficient to show that

$$\tilde{\Psi}^{(1)}_{\nu+1}(R^*_\nu) > K_{\nu+1} + M'_{\nu+1}.$$

This follows from (3.26) and (3.27). Next we show (3.29). By (3.16) and (3.24) it is sufficient to show that

$$\tilde{\Psi}^{(2)}_{\nu+1}(R_\nu) > \tilde{\Psi}^{(1)}_{\nu+1}(R^*_{\nu-1}+\sigma_{\nu+1}) + \tilde{\Psi}^{(2)}_{\nu+1}(R_{\nu-1})$$

By (3.21) and (3.27), it is sufficient to show that

$$\tilde{\Psi}^{(2)}_{\nu+1}(R_\nu) > 2M'_{\nu+1}.$$

This follows from (3.20), which proves (iii). Thus we complete the proof of Proposition (3.13).

(The fourth step). Finally we prove that D is a weakly 1-complete manifold. We choose n such that both line bundles $[S]^{-n}$ and $[S]^{-n}\otimes K_M^{-1}$ are positive on $\bar{D}$. We set for $\varphi=\{\varphi_\lambda\}\in H^0(D, \mathcal{O}([S]^{-n}))$

(3.30) $\quad \|\varphi\|^2(p)=a_\lambda|\varphi_\lambda|^2$ and $\|\varphi\|_\Delta=\sup_{p\in\Delta}\|\varphi\|(p)$ for $\Delta\subset D$,

where $\{a_\lambda\}$ denotes the positive metric of $[S]^{-n}$.

Let $\{\Delta_\nu\}$ be a compact exhaustion of D which is constructed in the third step.

Proposition (3.31). *Let $\{\delta_\nu\}$ and $\{l_\nu\}$ be two sequences of positive numbers with $\sum\delta_\nu<+\infty$ and $l_\nu<l_{\nu+1}$, $l_\nu\to\infty$ $(\nu\to\infty)$. Let $\{\Omega_\nu\}$ be a compact exhaustion of D such that $\bar{\Delta}_{\nu-1}\subset\Omega_\nu$. If there exists a system of functions $\{\varphi_\nu\}$ satisfying the following conditions* (1) *and* (2), *then D is a weakly* 1*-complete manifold*:

(1) *φ_ν is a non-negative pseudoconvex function of* C^∞*-class on D,*

(2) $\|\varphi_{\nu+1}\|_{\Delta_{\nu-1}}<\delta_\nu$ *and* $\varphi_{\nu+1}(p)\geqq l_\nu$ *for* $p\in\Omega_{\nu+1}-\Omega_\nu$.

The proof is easy.

Now we construct such $\{\varphi_\nu\}$. For this we prepare the following

Proposition (3.32). (1) *For each ν, Δ_ν is $[S]^{-n}$-convex except S (see Definition* (3.2) *in O. Suzuki* [10]). (2) *Let $\hat{\Delta}_\nu$ denotes the convex-hull of Δ_ν in $\Delta_{\nu+k}$ $(k\geqq 1)$ and let $\hat{\Delta}^*_\nu=\overline{\hat{\Delta}_\nu-S}$. Then*

$$\hat{\Delta}^*_\nu\subseteqq\Delta_{\nu+k}(m^{(\nu)}_{\nu+k}),\quad \textit{where}\quad m^{(\nu)}_{\nu+k}=\sup_{p\in\Delta_\nu}\Phi_{\nu+k}.$$

This is a direct consequence of Theorem 5 in O. Suzuki [10].

Let $\{\gamma_\nu\}$ and $\{\omega_\nu\}$ be two positive sequences such that

$$\Delta_{\nu+1}(M_{\nu+1}+\gamma_{\nu+1}) \Subset \Delta_{\nu+1}(m_{\nu+1}-\omega_{\nu+1}) \qquad (\nu \geqq 2).$$

Set

$$\Omega_\nu = \bigcup_{\mu=1}^{\nu} \Omega'_{\mu+1} \cup \Delta_2,$$

where

$$\Omega'_{\mu+1} = \Delta_{\mu+2}(m_{\mu+2}-\omega_{\mu+2}) - \Delta_{\mu+1}(M_{\mu+1}+\gamma_{\mu+1}).$$

Then $\{\Omega_\nu\}$ satisfies the condition in Proposition (3.31). First we show that for a fixed δ the following holds: For a point $p \in \bar{\Omega}'_{\nu+1} - V_\delta$, we can find a holomorphic section $\varphi_{\nu+1} \in H^0(D, \mathcal{O}([S]^{-n}))$ such that

$$\text{(3.33)} \qquad \|\varphi_{\nu+1}\|_{\Delta_{\nu-1}} < \delta_{\nu+1}/C_0 \quad \text{and} \quad \|\varphi_{\nu+1}\|(p) \geqq l_{\nu+1} D^{(\delta)}_{0,\nu},$$

where $C_0 = \sup_{p \in D} h$ and $D^{(\delta)}_{0,\nu} = \inf_{p \in \bar{\Omega}'_{\nu+1} - V_\delta} h$.

This can be proved as follows: By using $\Omega'_{\nu+1} \cap \Delta_{\nu-1} = \emptyset$ and by Proposition (3.32), we can find a section $\varphi'_{\nu+1} \in H^0(\Delta_{\nu+1}, \mathcal{O}([S]^{-n}))$ for a sufficiently small constant ε such that

$$\|\varphi'_{\nu+1}\|_{\Delta_{\nu-1}} < \delta_{\nu+1}/C_0 - \varepsilon \quad \text{and} \quad \|\varphi'_{\nu+1}\|(p) \geqq l_{\nu+1} D^{(\delta)}_{0,\nu} + \varepsilon.$$

By Corollary (3.14), we see that

$$\Delta_\nu \subset \Delta_{\nu+2}(M_{\nu+2}) \Subset \Delta_{\nu+2}(m_{\nu+2}-\omega_{\nu+2}) \subset \Delta_{\nu+1} \Subset \Delta_{\nu+2}.$$

Then by the Theorem of H. Kazama [6], for any $\varepsilon_{\nu+1}$ we can find $\varphi'_{\nu+2} \in H^0(\Delta_{\nu+2}, \mathcal{O}([S]^{-n}))$ satisfying

$$\|\varphi'_{\nu+2} - \varphi'_{\nu+1}\|_{\overline{\Delta_{\nu+2}(m_{\nu+2}-\omega_{\nu+2})}} < \varepsilon_{\nu+1}.$$

By Corollary (3.14) again, we see that

$$\Delta_{\nu+1} \subset \Delta_{\nu+3}(M_{\nu+3}) \Subset \Delta_{\nu+3}(m_{\nu+3}-\omega_{\nu+3}) \Subset \Delta_{\nu+2} \Subset \Delta_{\nu+3}.$$

Then by the Theorem of H. Kazama again, for any $\varepsilon_{\nu+2}$ we have a section $\varphi'_{\nu+3} \in H^0(\Delta_{\nu+3}, \mathcal{O}([S]^{-n}))$ such that

$$\|\varphi'_{\nu+3} - \varphi'_{\nu+2}\|_{\overline{\Delta_{\nu+3}(m_{\nu+3}-\omega_{\nu+3})}} < \varepsilon_{\nu+2}.$$

Repeating this process, we can find $\varphi_{\nu+k} \in H^0(\Delta_{\nu+k}, \mathcal{O}([S]^{-n}))$ such that

$$\|\varphi'_{v+k}-\varphi'_{v+k-1}\|_{\overline{\Delta_{v+k}(m_{v+k}-\omega_{v+k})}}<\varepsilon_{v+k-1} \qquad (k=2, 3,\dots).$$

Choose $\{\varepsilon_v\}$ with $\sum\varepsilon_v<\varepsilon$ and set

$$\varphi_{v+1}=\lim_{k\to\infty}\varphi'_{v+k}.$$

Then we have a holomorphic section $\varphi_{v+1}\in H^0(D, \mathcal{O}([S]^{-n}))$ satisfying

$$\|\varphi_{v+1}-\varphi'_{v+1}\|_{\overline{\Delta_{v+1}(m_{v+1}-\omega_{v+1})}}<\varepsilon.$$

Therefore φ_{v+1} satisfies (3.33). Referring to (3.30), we see that

$$\|\varphi_\lambda^{(v+1)}\cdot\zeta_\lambda^n\|_{\Delta_{v-1}}<\delta_v \quad\text{and}\quad |\varphi_\lambda^{(v+1)}\zeta_\lambda^n|(p)\geqq l_v,$$

where $\varphi_{v+1}=\{\varphi_\lambda^{(v+1)}\}$.

Thus choosing sections of $[S]^{-n}$ on D, $\varphi_{v+1}^{(1)}, \varphi_{v+1}^{(2)},\dots, \varphi_{v+1}^{(r_{v+1})}$ and sufficiently large m, we see that

$$\phi_{v+1}=\sum_{l=1}^{r_{v+1}}|\varphi_{v+1}^{(l)}\cdot\zeta_\lambda^n|^{2m}$$

satisfies

$$\text{(3.34)} \qquad \|\phi_{v+1}\|_{\Delta_{v-1}}<\delta_v \quad\text{and}\quad \phi_{v+1}(p)\geqq l_v \qquad\text{for}\quad p\in\Omega'_{v+1}-V_\delta.$$

By using the construction of $\{\Delta_v\}$, we can find a real analytic function $\Xi_{v+1}(t)$ for a small δ and for every v with the following properties:

$$\|\Xi_{v+1}(\eta)\|_{\Delta_{v-1}}<\delta_v \quad\text{and}\quad \Xi_{v+1}(\eta)(p)\geqq l_v \qquad\text{for}\quad p\in\Omega'_{v+1}\cap V_\delta.$$

This can be proved as follows: Set

$$\eta_m=\inf_{p\in\Omega'_{v+1}\cap S}\eta \text{ and } \eta_M=\sup_{p\in\Delta_{v-1}\cap S}\eta.$$

By using $\Phi_{v+1}|_S=1+\tilde{\Psi}_{v+1}^{(2)}(\eta)$ and noting that $\tilde{\Psi}_{v+1}^{(2)}$ is a convex increasing function and that $\Omega'_{v+1}\cap\Delta_{v+1}(M_{v+1})=\emptyset$, we see that $\eta_m>\eta_M$. So choosing a real analytic convex increasing function $\Xi_{v+1}(t)$, we can satisfy

$$\Xi_{v+1}(t)\geqq l_v \quad\text{for}\quad t\geqq\eta_m \quad\text{and}\quad \Xi_{v+1}(t)\leqq\delta_v \quad\text{for}\quad t\leqq\eta_M.$$

Because $\Delta_{v-1}\subset\{\eta<\eta_M\}$, we see that $\Xi_{v+1}(\eta)\leqq\delta_v$ for $p\in\Delta_{v-1}$. Then making δ smaller, we may assume that

$$\Xi_{\nu+1}(\eta) \geqq l_\nu \qquad \text{for} \quad p \in \Omega'_{\nu+1} \cap V_\delta,$$

$$\Xi_{\nu+1}(\eta) \leqq \delta_\nu \qquad \text{for} \quad p \in \Delta_{\nu-1}.$$

Thus for this δ we make $\phi_{\nu+1}$ and set

$$\phi^*_{\nu+1} = \frac{1}{2}(\phi_{\nu+1} + \Xi_{\nu+1}(\eta)).$$

Then $\{\phi^*_{\nu+1}\}$ satisfy the conditions (1) and (2) in Proposition (3.31). This proves that D is a weakly 1-complete manifold.

§4. Some Propositions Concerning Monoidal Transforms

In §5 we will use monoidal transforms repeatedly to resolve singularities of indeterminancy of characteristic functions (see §1). There we will consider the signs of certain kinds of line bundles in detail. For this we have to fix local coordinates on a complex manifold which is obtained by forming monoidal transforms.

Let M be a 3-dimensional complex manifold with the condition (A) (see Introduction) and let S and H be non-singular divisors in M which are defined by $S = \{\zeta_\lambda = 0\}$ and $H = \{\eta_\lambda = 0\}$ respectively. The condition (A) is stated as follows:

$(A)_{(0)}$: Let $\zeta_\lambda = f_{\lambda\mu}\zeta_\mu$ and $\eta_\lambda = g_{\lambda\mu}\eta_\mu$. Then

$$f_{\lambda\mu} = f_{\lambda\mu}(z_\mu) \quad \text{and} \quad g_{\lambda\mu} = g_{\lambda\mu}(z_\mu).$$

$(B)_{(0)}$: $[S]_{|A}$ and $[H]_{|A}$ are negative complex line bundles.

We write

$$c_1([S]^{-1}_{|A}) = f \quad \text{and} \quad c_1([H]^{-1}_{|A}) = g.$$

Then f and g are positive integers. The following proposition is easy:

Proposition (4.1). (1) *If and only if $l \cdot f \geqq g$ holds, then $[S]^{-1} \otimes [H]_{|A}$ admits a positive semi-definite metric $\{a_\lambda\}$ and*

(4.2) $$\varphi_{l,1}=a_\lambda(z_\lambda)|\zeta_\lambda^1|^2/|\eta_\lambda|^2$$

is a pseudoconvex function on $M-H$. (2) *If and only if $f\geqq k\cdot g$ holds, $[S]^{-1}\otimes[H]_{|A}^k$ admits a positive semi-definite metric $\{b_\lambda\}$ and*

(4.3) $$\varphi_{1,k}=b_\lambda(z_\lambda)|\zeta_\lambda|^2/|\eta_\lambda^k|^2$$

is a pseudoconvex function on $M-H$.

From this we make the following

Definition (4.4). (*i*) *The smallest integer σ ($\sigma\geqq 1$) satisfying $\sigma\cdot f\geqq g$ is called a σ-characteristic number of* (*S*, *H*). *The obtained pseudoconvex function* (4.2) *is called a σ-characteristic function. Moreover, if $\sigma\cdot f=g$ holds, then* (*S*, *H*) *is called a σ-complete pair. If not so, it is called a σ-incomplete pair.* (*ii*) *The largest integer τ ($\tau\geqq 0$) satisfying $f\geqq\tau\cdot g$ is called τ-characteristic number of* (*S*, *H*). *The obtained pseudoconvex function* (4.3) *is called a τ-characteristic function. Also τ-complete pairs and τ-incomplete pairs are defined.*

In what follows we use the following notations:
For a complex line bundle E, we write $E>0$ (resp. $E<0$ or $E=0$) if $c_1(E)>0$ (resp. $c_1(E)<0$ or $c_1(E)=0$).

Now we form a monoidal transform $Q_{(1)}: M_{(1)}\to M$ with center A. Let $\mathbf{P}$ be a rational curve and $V^{(1)}$ and $V^{(2)}$ be a canonical covering of $\mathbf{P}$ whose inhomogeneous coordinates are denoted by $u^{(1)}$ and $u^{(2)}$ respectively. We define a negative line bundle $\tau: F\to\mathbf{P}$ by $\xi^{(1)}=u^{(2)}\cdot\xi^{(2)}$, where $\tau^{-1}(V^{(i)})=\{(u^{(i)},\xi^{(i)}): |\xi^{(i)}|<+\infty\}$ $(i=1, 2)$. For each λ, we prepare a copy of F, which is denoted by F_λ whose local coordinates are denoted by $u_{\lambda|1}^{(i)}, \xi_{\lambda|1}^{(i)}$ $(i=1, 2)$. Then $M_{(1)}$ is identical with a complex manifold which is obtained by the following identification between $\{U_{\lambda|1}^{(i)}\}$, where $U_{\lambda|1}^{(i)}=\tau^{-1}(V_{\lambda|1}^{(i)})\times\{|z_\lambda|<\rho\}$ $(i=1, 2)$:

(4.5) $$\begin{cases} \xi_{\lambda|1}^{(1)}=f_{\lambda\mu}\xi_{\mu|1}^{(1)} & \\ u_{\lambda|1}^{(1)}=g_{\lambda\mu}f_{\lambda\mu}^{-1}\cdot u_{\mu|1}^{(1)} & \text{on } U_{\lambda|1}^{(1)}\cap U_{\mu|1}^{(1)}, \\ \xi_{\lambda|1}^{(2)}=g_{\lambda\mu}\xi_{\mu|1}^{(2)} & \\ u_{\lambda|1}^{(2)}=f_{\lambda\mu}g_{\lambda\mu}^{-1}u_{\mu|1}^{(2)} & \text{on } U_{\lambda|1}^{(2)}\cap U_{\mu|1}^{(2)}, \end{cases}$$

$$\begin{cases} \xi^{(1)}_{\lambda|1} = u^{(2)}_{\lambda|1}\xi^{(2)}_{\lambda|1} \\ u^{(1)}_{\lambda|1} = u^{(2)-1}_{\lambda|1} \end{cases} \quad \text{on } U^{(1)}_{\lambda|1} \cap U^{(2)}_{\lambda|1}.$$

Then $Q_{(1)}$ is expressed as

$$\text{(4.6)} \qquad \begin{cases} \zeta_\lambda = \xi^{(1)}_{\lambda|1} \\ \eta_\lambda = u^{(1)}_{\lambda|1}\cdot\xi^{(1)}_{\lambda|1} \end{cases} \quad \text{on } U^{(1)}_{\lambda|1} \text{ and } \begin{cases} \zeta_\lambda = u^{(2)}_{\lambda|1}\xi^{(2)}_{\lambda|1} \\ \eta_\lambda = \xi^{(2)}_{\lambda|1} \end{cases} \quad \text{on } U^{(2)}_{\lambda|1}.$$

Let $\Sigma_{(1)} = \{\xi^{(1)}_{\lambda|1}=0\} \cup \{\xi^{(2)}_{\lambda|1}=0\}$. Then we see that $Q^{-1}_{(1)}(A)=\Sigma_{(1)}$ and $Q_{(1)}\colon M_{(1)}-\Sigma_{(1)}\to M-A$ gives a biholomorphic mapping.

By using this expression, we consider resolutions of singularities of indeterminancy of functions in the following form:

$$\varphi_{k,l} = a_\lambda(z_\lambda)\,|\zeta^k_\lambda|^2/|\eta^l_\lambda|^2.$$

First we consider

(I) The resolution of singularities of indeterminancy of $\varphi_{m,1}$.

We form the monoidal transform with center A, $Q_{(1)}\colon M_{(1)}\to M$. Choosing a local coordinate covering $\{U^{(i)}_{\lambda|1}\}$ on $M_{(1)}$ as above, we see that

$$Q^*_{(1)}(\varphi_{m,1}) = Q^*_{(1)}(a_\lambda)\,|\,\xi^{(1)(m-1)}_{\lambda|1}\,|^2/\,|\,u^{(1)}_{\lambda|1}\,|^2 \qquad \text{on } U^{(1)}_{\lambda|1}.$$

When $m\geqq 2$, there remain singularities of indeterminancy on $A_{(1)}$, where $A_{(1)}=\{\xi^{(1)}_{\lambda|1}=0\}\cap\{u^{(1)}_{\lambda|1}=0\}$. We form monoidal transform with center $A_{(1)}$ again, which is denoted by $Q_{(2)}\colon M_{(2)}\to M_{(1)}$. Replacing ζ_λ and η_λ by $\xi^{(1)}_{\lambda|1}$ and $u^{(1)}_{\lambda|1}$ respectively in the previous construction, we make a local coordinate covering $\{U^{(i)}_{\lambda|2}\}$ of $M_{(2)}$. Local coordinates z_λ, $\xi^{(i)}_{\lambda|2}$, $u^{(i)}_{\lambda|2}$ are determined so that $Q_{(2)}$ is expressed as $\xi^{(1)}_{\lambda|1}=\xi^{(1)}_{\lambda|2}$ and $u^{(1)}_{\lambda|1}= \xi^{(1)}_{\lambda|2}u^{(1)}_{\lambda|2}$ on $U^{(1)}_{\lambda|2}$. Then the identification rule is given in the similar manner as (4.5). We see that

$$Q^{(2)*}(a_\lambda)\,|\,\xi^{(1)(m-1)}_{\lambda|2}\,|^2/\,|\,u^{(1)}_{\lambda|2}\,|^2 \qquad \text{on } U^{(1)}_{\lambda|2},$$

where $Q^{(2)}=Q_{(2)}\circ Q_{(1)}$.

Repeating this process j-times, we obtain a complex manifold $M_{(j)}$ and a local coordinate covering $\{U^{(i)}_{\lambda|j}\}$ of $M_{(j)}$ whose local coordinates are denoted by z_λ, $\xi^{(i)}_{\lambda|j}$, $u^{(i)}_{\lambda|j}$ in the similar manner. The identification rule is given as follows:

$$(4.7)\qquad \begin{cases} \xi^{(1)}_{\lambda|j}=f_{\lambda\mu}\xi^{(1)}_{\mu|j} \\ u^{(1)}_{\lambda|j}=g_{\lambda\mu}f^{-j}_{\lambda\mu}u^{(1)}_{\mu|j} \end{cases} \text{on } U^{(1)}_{\lambda|j}\cap U^{(1)}_{\mu|j},\quad \begin{cases} \xi^{(2)}_{\lambda|j}=g_{\lambda\mu}f^{-(j-1)}_{\lambda\mu}\xi^{(2)}_{\mu|j} \\ u^{(2)}_{\lambda|j}=f^{j}_{\lambda\mu}g^{-1}_{\lambda\mu}u^{(2)}_{\mu|j} \end{cases} \text{on } U^{(2)}_{\lambda|j}\cap U^{(2)}_{\mu|j}.\quad \begin{cases} u^{(1)}_{\lambda|j}=u^{(2)-1}_{\lambda|j} \\ \xi^{(1)}_{\lambda|j}=u^{(2)}_{\lambda|j}\cdot\xi^{(2)}_{\lambda|j} \end{cases} \text{on } U^{(1)}_{\lambda|j}\cap U^{(2)}_{\lambda|j}$$

Let $Q^{(j)}=Q_{(j)}\circ Q_{(j-1)}\circ\cdots\circ Q_{(1)}$. Then we see that

$$\begin{cases} \zeta_\lambda=\xi^{(1)}_{\lambda|j} \\ \eta_\lambda=u^{(1)}_{\lambda|j}(\xi^{(1)}_{\lambda|j})^j \end{cases} \quad \text{on } U^{(1)}_{\lambda|j}$$

and

$$Q^{(j)*}(\varphi_{m,1})=Q^{(j)*}(a_\lambda)\,|\xi^{(1)(m-j)}_{\lambda|j}|^2/|u^{(1)}_{\lambda|j}|^2 \qquad \text{on } U^{(1)}_{\lambda|j}.$$

Finally when $j=m$, we obtain

$$(4.8)\qquad Q^{(m)*}(\varphi_{m,1})=Q^{(m)*}(a_\lambda)\,|u^{(1)}_{\lambda|m}|^{-2} \qquad \text{on } U^{(1)}_{\lambda|m},$$

which is a desired resolution.

Set

$$L_0=\{u^{(2)}_{\lambda|1}=0\},$$

$$L_j=\{\xi^{(2)}_{\lambda|j}=0\}\cup\{u^{(2)}_{\lambda|j+1}=0\} \qquad (j=1,2,\dots,m-1),$$

$$L_m=\{\xi^{(1)}_{\lambda|m}=0\}\cup\{\xi^{(2)}_{\lambda|m}=0\},$$

$$\Sigma_{(m)}=L_1\cup L_2\cup\cdots\cup L_m.$$

Then $Q^{(m)^{-1}}(A)=\Sigma_{(m)}$. Consider a complex line bundle $[L_{j-1}]$ on $L_j\cap L_{j-1}$, which is denoted by E_{j-1}. Then

$$(4.9)\qquad E_{j-1}\cong[H]^{-1}\otimes[S]^{j}_{|A}.$$

Also we consider another complex line bundle $[L_j]$ on $L_{j-1}\cap L_j$, which is denoted by F_j. Then

$$(4.10)\qquad F_j\cong[S]^{-(j-1)}\otimes[H]_{|A}.$$

$\Sigma_{(m)}$ is expressed on $Q^{(m)^{-1}}(\pi^{-1}(p))$ as follows, where $p \in A$ and $\pi: M \to A$:

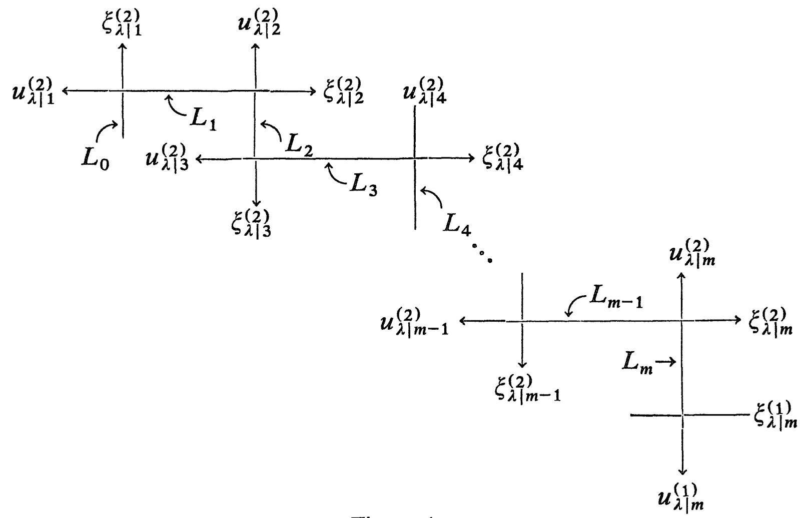


Figure 1

We have the following

Proposition (4.11). (i) *If* (S, H) *is a* σ*-complete pair, then* (1) *if* $j \geqq \sigma+2$, $F_j > 0$ *and* $E_{j-1} < 0$, (2) *if* $j = \sigma+1$, $F_j = 0$ *and* $E_{j-1} < 0$, (3) $j = \sigma$, $F_j < 0$ *and* $E_{j-1} = 0$ *and* (4) *if* $j \leqq \sigma-1$, $F_j < 0$ *and* $E_{j-1} > 0$.

(ii) *If* (S, H) *is a* σ*-incomplete pair, then* (1) *if* $j \geqq \sigma+1$, $F_j > 0$ *and* $E_{j-1} < 0$, (2) *if* $j = \sigma$, $F_j < 0$ *and* $E_{j-1} < 0$ *and* (3) *if* $j \leqq \sigma-1$, $F_j < 0$ *and* $E_{j-1} > 0$.

(II) The resolution of singularities of indeterminancy of

$$\varphi_{1,l} = b_\lambda(z_\lambda)\, |\zeta_\lambda|^2 / |\eta_\lambda^l|^2.$$

Let $Q_{(1)}: M_{(1)} \to M$ be a monoidal transform with center A. We choose a local coordinate covering of $M_{(1)}$, $\{U^{(i)}_{\lambda|1}\}$ by (4.5). Then by (4.6),

$$Q^*_{(1)}(\varphi_{1,l}) = Q^*_{(1)}(b_\lambda)\, |u^{(2)}_{\lambda|1}|^2 / |\xi^{(2)(l-1)}_{\lambda|1}|^2.$$

In the case $l \geqq 2$, there remain singularities on $A_{(1)} = \{u^{(2)}_{\lambda|1} = \xi^{(2)}_{\lambda|1} = 0\}$. Here we form $Q_{(2)}: M_{(2)} \to M_{(1)}$ with center $A_{(1)}$ again. Replacing ζ_λ and η_λ by $u^{(2)}_{\lambda|1}$ and $\xi^{(2)}_{\lambda|1}$ respectively, we choose local coordinates on $\{U^{(i)}_{\lambda|1}\}$ so that $Q^{(2)}$ is expressed as $u^{(2)}_{\lambda|1} = \xi^{(1)}_{\lambda|2}$ and $\xi^{(2)}_{\lambda|1} = u^{(1)}_{\lambda|2}\, \xi^{(1)}_{\lambda|2}$ on

$U_{\lambda|2}^{(1)}$. Repeating this process j-times, we obtain $Q_{(j)}: M_{(j)} \to M_{(j-1)}$. We choose local coordinates z_λ, $\xi_{\lambda|j}^{(i)}$, $u_{\lambda|j}^{(i)}$ on $U_{\lambda|j}^{(i)}$ by the following conditions:

$$(4.12)\quad \begin{cases} \xi_{\lambda|j}^{(1)} = f_{\lambda\mu} g_{\lambda\mu}^{-(j-1)} \xi_{\mu|j}^{(1)} \\ u_{\lambda|j}^{(1)} = f_{\lambda\mu}^{-1} g_{\lambda\mu}^{j} u_{\mu|j}^{(1)} \end{cases} \text{ on } U_{\lambda|j}^{(1)} \cap U_{\mu|j}^{(1)}, \qquad \begin{cases} \xi_{\lambda|j}^{(2)} = g_{\lambda\mu} \xi_{\mu|j}^{(2)} \\ u_{\lambda|j}^{(2)} = f_{\lambda\mu} g_{\lambda\mu}^{-j} u_{\mu|j}^{(2)} \end{cases} \text{ on } U_{\lambda|j}^{(2)} \cap U_{\mu|j}^{(2)}.$$

Then

$$Q^{(j)*}(\varphi_{1,l}) = Q^{(j)*}(b_\lambda)\,|u_{\lambda|j}^{(2)}|^2 / |\xi_{\lambda|j}^{(2)(l-j)}|^2,$$

where

$$Q^{(j)} = Q_{(j)} \circ Q_{(j-1)} \circ \cdots \circ Q_{(1)}.$$

Especially when $j=l$, $Q^{(l)*}(\varphi_{1,l})$ gives a desired resolution. $Q^{(l)}(A)$ on $Q^{(l)-1}(\pi^{-1}(p))$ is expressed as follows:

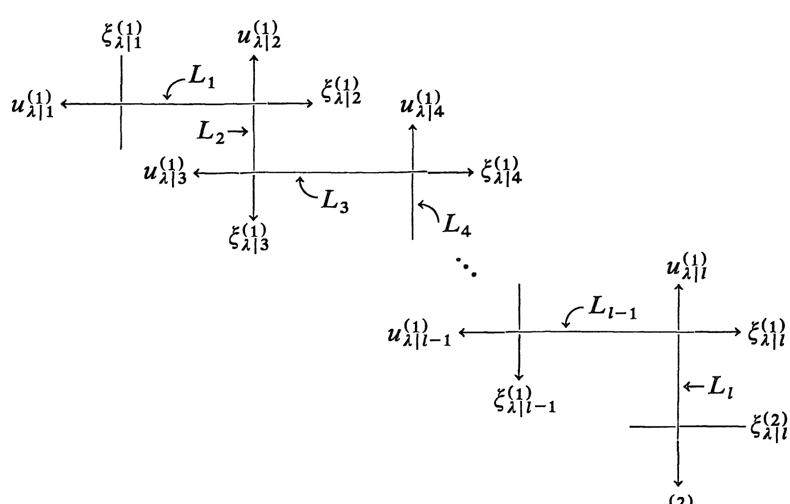


Figure 2

Set

$$L_j = \{\xi_{\lambda|j}^{(1)} = 0\} \cup \{u_{\lambda|j+1}^{(1)} = 0\} \qquad (j = 1, 2, \ldots, l-1),$$

$$L_l=\{\xi_{\lambda|l}^{(1)}=0\}\cup\{\xi_{\lambda|l}^{(2)}=0\},$$

$$\Sigma_{(1)}=L_1\cup L_2\cup\cdots\cup L_l.$$

Then we see that $Q^{(l)^{-1}}(A)=\Sigma_{(l)}$. Let $[L_k]|_{L_{k-1}\cap L_k}=F_k$. Then by (4.12) we see that

$$F_k\cong[S]\otimes[H]^{-(k-1)}|_A.$$

Let $[L_{k-1}]|_{L_{k-1}\cap L_k}=E_{k-1}$. Then we have by (4.12)

$$E_{k-1}\cong[S]^{-1}\otimes[H]^k|_A.$$

We have the following

Proposition (4.13). (i) *Let (S, H) be a τ-complete pair with $\tau\geqq 1$. Then* (1) *if $k\geqq\tau+2$, $E_{k-1}<0$ and $F_k>0$,* (2) *if $k=\tau+1$, $E_{k-1}<0$ and $F_k=0$,* (3) *if $k=\tau$, $E_{k-1}=0$ and $F_k<0$ and* (4) *$k<\tau$, $E_{k-1}>0$ and $F_k<0$.* (ii) *Let (S, H) be a τ-incomplete pair with $\tau\geqq 1$. Then* (1) *if $k\geqq\tau+2$, $E_{k-1}<0$ and $F_k>0$,* (2) *if $k=\tau+1$, $E_{k-1}<0$ and $F_k<0$ and* (3) *if $k\leqq\tau$, $E_{k-1}>0$ and $F_k<0$.*

In the same manner, as for the conoid function $\varphi_{1,l}$ we have

Proposition (4.14). (i) *Let (S, H) be a σ-complete pair. Then* (1) *if $l>2$ and $\sigma\geqq 1$, then $E_{l-1}<0$ and $F_l>0$,* (2) *if $l=2$ and $\sigma>1$, then $E_{l-1}<0$ and $F_l>0$ and* (3) *if $l=2$ and $\sigma=1$, $E_{l-1}<0$ and $F_l=0$.* (ii) *Let (S, H) be a σ-incomplete pair. Then* (1) *if $\sigma\geqq 2$ and $l\geqq 2$, then $E_{l-1}<0$ and $F_l>0$ and* (2) *if $\sigma\geqq 1$ and $l=1$, then $E_{l-1}<0$ and $F_l<0$.*

§5. Lemmas on Extension of Holomorphic Functions

The purpose of this section is to prove the following theorem, which is a part of Theorem I:

Theorem (5.1). *Suppose that M satisfies the conditions $(A)_{(0)}$ and $(B)_{(0)}$ (see the beginning of* §4) *and Δ is a simple domain along A (see Definition* (1.4)) *which is a domain of holomorphy on M. Then there exist ε and δ satisfying*

$$\{\phi^* < \varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A),$$

where ϕ^* *is the characteristic function of* M (*see Definition* (1.6)).

Let Δ be a domain on M such that there exist a conoid function $\varphi_{k,l}$ (see Definition (1.4)) and δ, ε satisfying

$$\{\varphi_{k,l} < \varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A). \tag{5.2}$$

Then (5.2) is also satisfied for a conoid function $\varphi_{1,l}$ for a sufficiently large l, i.e.,

$(C)_{(0)}$: There exist a sufficiently large l and a conoid function such that for some ε and δ,

$$\{\varphi_{1,l} < \varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A).$$

It is easily seen that if M satisfies the condition (A) and Δ is a simple domain along A (see Definition (1.4)), then M and Δ satisfy $(A)_{(0)}$, $(B)_{(0)}$ and $(C)_{(0)}$.

Lemma (5.3). *Suppose that* M *satisfies* $(A)_{(0)}$, $(B)_{(0)}$ *and* Δ *satisfies* $(C)_{(0)}$. *If* Δ *is a domain of holomorphy, then we have*

(*I*) *In the case where* $g \geqq f$, *we have*

(*i*) *if* (S, H) *is a* σ*-complete pair, then there exist* ε *and* δ *such that*

$$\{\varphi_{\sigma,1} < \varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A),$$

where $\varphi_{\sigma,1}$ *is the* σ*-characteristic function of* (S, H),

(*ii*) *if* (S, H) *is a* σ*-incomplete pair and* $\sigma \geqq 2$, *then there exist* ε *and* δ *such that*

$$\{\varphi_{\sigma-1,1} < \varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A),$$

where $\varphi_{\sigma-1,1}$ *is a conoid function.*

(*II*) *In the case where* $f \geqq g$, *we have*

(*i*) *if* (S, H) *is a* τ*-complete pair, then there exist* ε *and* δ *such that*

$$\{\varphi_{1,\tau} < \varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A),$$

where $\varphi_{1,\tau}$ *is the* τ*-characteristic function,* (*ii*) *if* (S, H) *is a* τ-*incomplete pair, then there exist* ε *and* δ *such that*

$$\{\varphi_{1,\tau+1}<\varepsilon\} \cap V_\delta(A) \subset \Delta \cap V_\delta(A),$$

where $\varphi_{1,\tau+1}$ *is a conoid function.*

By taking account that the characteristic function ϕ^* is identical with the σ-(resp. τ-)characteristic function in the case where (S, H) is a σ-(resp. τ-)complete pair (see (1.5)), we have

Corollary (5.4). *In the case of* (*i*) *in* (*I*) *or* (*i*) *in* (*II*), *Theorem* (5.1) *holds.*

For the proof of Lemma (5.3), we prepare furthermore two lemmas:

Lemma (5.5). *Let* A *be a compact Riemann surface and let* F *and* G *be complex line bundles such that* $F<0$ *and* $G\geqq 0$. *With respect to a fine covering* $\{V_\lambda\}$ *of* A, *the fibre coordinates of* F *and* G *are denoted by* ζ_λ *and* η_λ *respectively. We set* $V=F\oplus G$ *and denote a divisor which is defined by* $\{\eta_\lambda=0\}$ *by* S. *Let* W *be a small neighborhood of the zero section of* V *and let* Ω *be a neighborhood of* $(S-A)\cap W$. *Then every holomorphic function on* Ω *can be extended to a holomorphic function on some neighborhood* $\tilde{\Omega}$ *of the zero section. Moreover,* $\tilde{\Omega}$ *is determined only by* Ω.

Proof. Take metrics of F and G, $\{a_\lambda\}$, $\{b_\lambda\}$ respectively and set $h_1=a_\lambda|\zeta_\lambda|^2$ and $h_2=b_\lambda|\eta_\lambda|^2$. Then there exist ε_1, ε_2 $(\varepsilon_1<\varepsilon_2)$ and ε_3 such that $\Delta_\varepsilon=\{p\in V: \varepsilon_1<h_1<\varepsilon_2 \text{ and } h_2<\varepsilon_3\}$ is contained in Ω. Take a holomorphic function f on Ω. Then $f\in\mathcal{O}(\Delta_\varepsilon)$. Thus f is expressed on Δ_ε as follows:

$$f=\sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} a_\lambda^{(l,m)}(z_\lambda)\zeta_\lambda^l\eta_\lambda^m,$$

where $\{a_\lambda^{(l,m)}\}\in H^0(A, \mathcal{O}(F^{-l}\otimes G^{-m}))$.

By assumption, $F^{-l}\otimes G^{-m}<0$ for $l<0$. Therefore, $\{a_\lambda^{(l,m)}\}=0$ for $l<0$. Then f can be extended to $\Omega\cup\tilde{\Delta}_\varepsilon$, where $\tilde{\Delta}_\varepsilon=\{p\in V: h_1<\varepsilon_2 \text{ and } h_2<\varepsilon_3\}$.

Lemma (5.6). *Let M be a 3-dimensional complex manifold and let $L\subset M$ be a 2-dimensional compact complex submanifold which is isomorphic to the compactification of a positive complex line bundle F on a compact Riemann surface A. The zero section and the infinite section are denoted by A_0 and A_∞ respectively. Let Δ be a pseudoconvex domain with*

$$A_0\subset\Delta \quad \textit{and} \quad \partial\Delta\cap L\neq\emptyset.$$

Then we have

$$\partial\Delta\cap L=A_\infty.$$

Proof. Assume that $\partial\Delta\cap L\neq A_\infty$. In view of $F>0$, there exists a strongly pseudoconcave neighborhood system $\{V_\varepsilon\}$ such that $V_\varepsilon=\{p\in L-A_0\colon h(p)>\varepsilon\}$. Then letting $\varepsilon_0=\inf_{p\in\Delta} h(p)$, we see that $\varepsilon_0>0$ and $V_{\varepsilon_0}\subset\Delta\cap L$. Take a point $p_0\in\partial(\Delta\cap L)\cap V_{\varepsilon_0}$. By assumption $\Delta\cap L$ is pseudoconvex and V_{ε_0} is s-pseudoconcave at p_0. Choose a Stein neighborhood U of p_0 in L. Then $U\cap\Delta$ is a Stein manifold. So there exists a holomorphic function f on $U\cap\Delta$ which cannot be continued across p_0. On the other hand, restricting f to $U\cap V_{\varepsilon_0}$, we see that f is continued across p_0, which is a contradiction.

Proof of Lemma (5.3). We prove (i) in (I). For this we first prove that

$$\text{(5.7)} \qquad \text{if } \{\varphi_{1,l}<\varepsilon\}\subset\Delta, \text{ then } \{\varphi_{1,1}<\varepsilon'\}\subset\Delta$$

on a small neighborhood of A. We may assume that $l\geqq 2$. Now we consider the resolution of singularities of indeterminancy of $\varphi_{1,l}$, $Q^{(l)}\colon M_{(l)}\to M$. We choose a local coordinate covering of $M_{(l)}$ as fixed in (II) in §4. We set

$$\Delta_{(l)}=(\overline{Q^{(l)^{-1}}(\Delta)})^0, \qquad \Omega_{(l)}=(\overline{Q^{(l)^{-1}}(\varphi_{1,l}<\varepsilon)})^0$$

$$\Omega_{(1)}=(\overline{Q^{(l)^{-1}}(\varphi_{1,1}<\varepsilon')})^0 \quad \text{and} \quad \Sigma_{(l)}=\bigcup_{j=1}^{l} L_j.$$

Then $\Omega_{(l)}$ and $\Omega_{(1)}$ are drawn in Figure 2 in §4 respectively as follows:

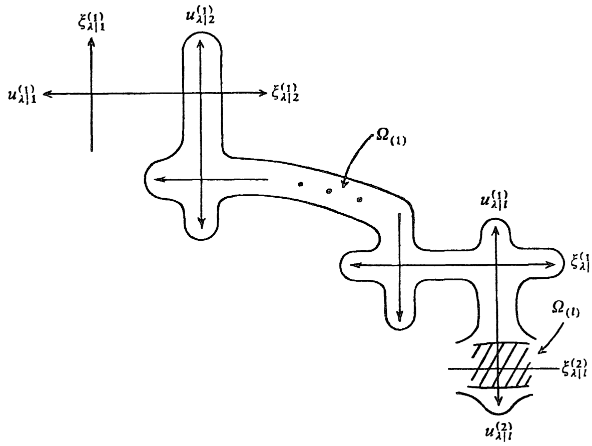


Figure 3

First we see that $\Delta_{(l)}$ is a pseudoconvex domain by Proposition (1.8). We prove that $\Delta_{(l)}$ contains a neighborhood of $L_l \cap L_{l-1}$. E_{l-1} is negative by (i) in Proposition (4.14). Take a complex line bundle which is defined by $\{u^{(2)}_{\lambda|l}=0\}$ and restrict it to $L_l \cap \{u^{(2)}_{\lambda|l}=0\}$. Then we have a positive line bundle. Considering the case $j=l$ in (4.12), we see that the line bundle defined by $\{\xi^{(2)}_{\lambda|l}=0\}$ is negative on $L_l \cap \{u^{(2)}_{\lambda|l}=0\}$. Hence by using Lemma (5.5), every holomorphic function on $\Delta_{(l)}-\Sigma_{(l)}$ can be extended at least to $\{\xi^{(2)}_{\lambda|l}=0\}$. If $\partial\Delta_{(l)} \cap L_l=\emptyset$, then $L_l \cap L_{l-1} \subset \Delta_{(l)}$. So for the proof of (5.7), we may assume that $\partial\Delta_{(l)} \cap L_l \neq \emptyset$. Then by Lemma (5.6) we see that $\partial\Delta_{(l)} \cap L_l = L_l \cap L_{l-1}$. Next we consider F_l. Then by (i) in Proposition (4.14) F_l is non-negative and F_l is flat if and only if $l=2$ and $\sigma=1$. Since Δ is a domain of holomorphy on M, $\Delta_{(l)}$ must contain a neighborhood of $L_l \cap L_{l-1}$ by Lemma (5.5). From this we see that every holomorphic function on $\Delta_{(l)}-\Sigma_{(l)}$ can be extended to $L_l \cup \{u^{(1)}_{\lambda|l}=0\}$. In the case where $l=2$, the assertion (5.7) is hereby proved. Assume that $l>2$. In this case we see that F_l is positive. By the same discussion as above, we find that $L_{l-2} \cap L_{l-1} \subset \Delta_{(l)}$. Repeating this process we can prove (5.7).

Next we prove that

(5.8) $$\text{if } \{\varphi_{1,1}<\varepsilon\} \subset \Delta, \text{ then } \{\varphi_{\sigma,1}<\varepsilon\} \subset \Delta$$

on a small neighborhood of the zero section.

We may assume that $\sigma \geqq 2$. Let $Q^{(\sigma)}$ be the resolution of singularities of $\varphi_{\sigma,1}$. We choose a local coordinate covering of $M_{(\sigma)}$ as expressed in (I) in §4. Set

$$\Delta_{(\sigma)} = (\overline{Q^{(\sigma)-1}(\Delta)^0}), \quad \Omega_{(\sigma)} = (\overline{Q^{(\sigma)-1}(\varphi_{\sigma,1} < \varepsilon)^0}),$$

$$\Omega_{(1)} = (\overline{Q^{(\sigma)-1}(\varphi_{1,1} < \varepsilon')^0}) \text{ and } \Sigma_{(\sigma)} = \bigcup_{j=1}^{\sigma} L_j.$$

Then $\Omega_{(\sigma)}$ and $\Omega_{(1)}$ are expressed in Figure 1 in §4 respectively as follows:

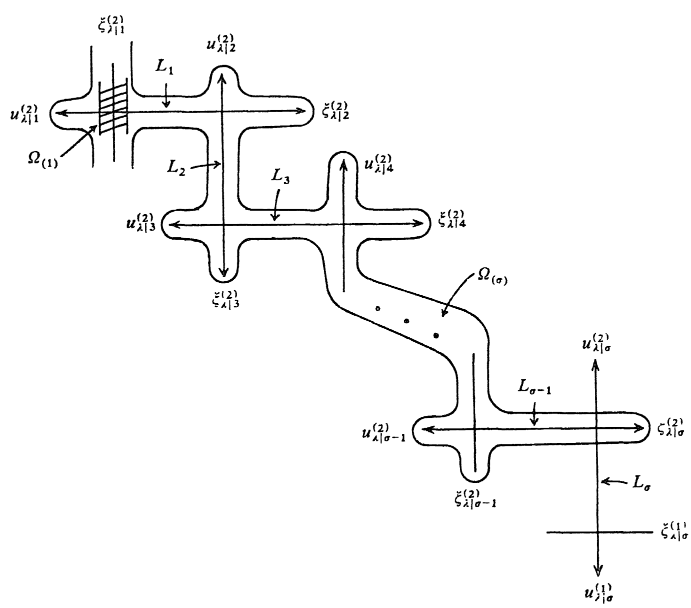


Figure 4

We see that $\Delta_{(\sigma)}$ is a pseudoconvex domain by Proposition (1.8). Since $\sigma \geqq 2$, E_0 is positive and F_1 is negative by (4), (i) in Proposition (4.11). So from Lemma (5.5), every holomorphic function on $\Delta_{(\sigma)} - \Sigma_{(\sigma)}$ can be extended to $\{\zeta^{(2)}_{\lambda|l} = 0\}$. As in the proof of (5.7), first we prove that $L_1 \cap L_2 \subset \Delta_{(\sigma)}$. So we may assume that $\partial\Delta_{(\sigma)} \cap L_1 \neq \emptyset$. By Lemma (5.5) we see that $\partial\Delta_{(\sigma)} \cap L_1 = L_1 \cap L_2$. By (3) or (4) in Proposition (4.11), F_2 is negative and E_1 is non-negative and E_1 is flat if and only if $\sigma = 2$. In the case where $\sigma = 2$, the assertion is proved. If $\sigma > 2$, then E_1 is positive. And by Proposition (4.11) F_3 is negative. Then we see that $\partial\Delta_{(\sigma)} \cap L_2 = L_2 \cap L_3$ by Lemma (5.6). So we see that $\Delta_{(\sigma)}$ contains a neighborhood of $L_2 \cap L_3$. Repeating this process, we prove the assertion (5.8). Here we also prove that $\mathcal{O}(\Delta_{(\sigma)} - \Sigma_{(\sigma)}) \cong \mathcal{O}(\Delta_{(\sigma)})$. So we complete the proof of (i) in (I). Proofs in the other cases can be done in the same manner by using (4.11) and (4.14). We omit them.

Remark. If (S, H) is a σ-complete pair, then the characteristic function ϕ^* is identical with the σ-characteristic function $\varphi_{\sigma,1}$ and

$$Q^{(\sigma)*}(\varphi_{\sigma,1}) = |u_{\lambda|\sigma}^{(2)}|^2$$

on a neighborhood of $L_{\sigma-1} \cap L_\sigma$ is a pseudoconvex function on $M_{(\sigma)} - \{u_{\lambda|\sigma}^{(1)} = 0\}$.

Proof of (II). We prove only (i) in (II). The proof of (ii) in (II) is almost the same as that of (i) in (II) and is omitted.

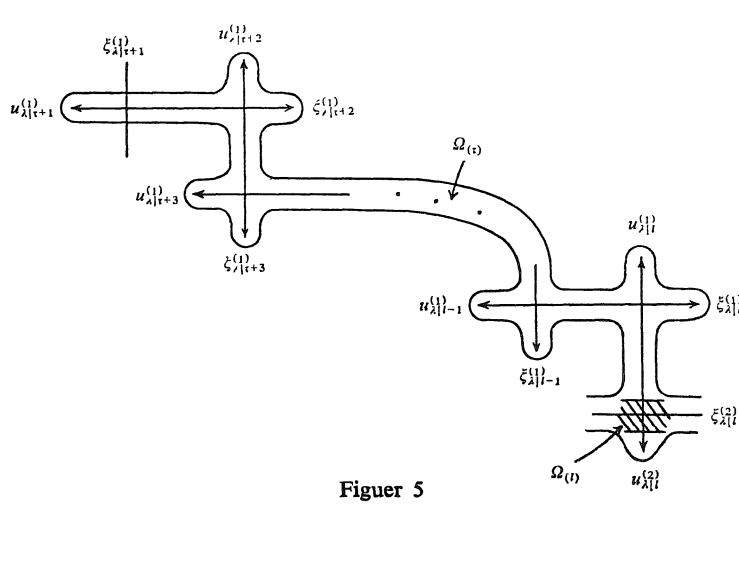
By $(C)_{(0)}$ we see that

$$\{\phi_{1,l} < \varepsilon'\} \subset \Delta \quad \text{on a small neighborhood of } A.$$

We may prove (i) only when $l > \tau$. Consider the resolution of singularities of indeterminancy of $\phi_{1,l}$ as (II) in §4, which is denoted by $Q^{(l)}$: $M_{(l)} \to M$. We set

$$\Omega_{(l)} = (\overline{\Omega^{(l)-1}(\phi_{1,l} < \varepsilon')^0}), \quad Q_{(\tau)} = (\overline{Q^{(l)-1}(\phi_{1,\tau} < \varepsilon)^0})$$

$$\Delta_{(l)} = (\overline{Q^{(l)-1}(\Delta)^0}) \text{ and } \Sigma_{(l)} = \bigcup_{j=1}^{l} L_j.$$



Figuer 5

We see that $\Delta_{(l)}$ is a pseudoconvex domain. $\Omega_{(l)}$ and $\Omega_{(\tau)}$ are expressed in Figure 2 respectively as in Figure 5.
By (1), (2) in (i) in Proposition (4.13), E_{l-1} is negative for $l \geqq \tau+1$. Then the complex line bundle which is determined by $\{u_{\lambda|l}^{(2)}=0\}$ is positive on $A_l^{(2)}=L_l \cap \{\xi_{\lambda|l}^{(2)}=0\}$ when $l \geqq \tau+1$. By Lemma (5.5) every holomorphic function on a neighborhood of $A_l^{(2)}$ except $\{\xi_{\lambda|l}^{(2)}=0\}$ can be extended to $\{\xi_{\lambda|l}^{(2)}=0\}$. Then we see that $\partial\Delta_{(l)} \cap L_l \supseteqq L_l \cap L_{l-1}$ by Lemma (5.6). Also we see that F_l is non-negative when $l \geqq \tau+1$ by (1), (2) in (i) in Proposition (4.13). Therefore we see that $\Delta_{(l)}$ contains a neighborhood of $L_{l-1} \cap L_l$. If $l=\tau+1$, the assertion is hereby proved. In the case where $l>\tau+1$, we can prove the assertion by repeating the same discussions as given in the proofs of (I). The details are omitted. Also we see that $\mathcal{O}(\Delta_{(l)}-\Sigma_{(l)}) \cong \mathcal{O}(\Delta_{(l)})$.

Remark. In the case (i), letting $Q^{(\tau)*}(\phi^*)=\phi$, we see that (1) ϕ is a pseudoconvex function on $M_{(\tau)}-\{u_{\lambda|\tau}^{(1)}=0\}$ and (2) ϕ is expressed as $\phi=|u_{\lambda|\tau}^{(2)}|^2$ near $L_\tau \cap \{u_{\lambda|\tau}^{(2)}=0\}$. Moreover, $\{\phi<\varepsilon\} \subset \Delta_{(l)}$.

Proof of Theorem (5.1). We prove our Theorem only in the case where $g \geqq f$. The proof for the other case is similar. We express the Euclidian algorithm of f and g as follows:

$$
(5.9) \qquad
\begin{aligned}
g &= \rho_1 f + r_1, \\
f &= \rho_2 r_1 + r_2, \\
r_1 &= \rho_3 r_2 + r_3, \\
&\vdots \\
r_q &= \rho_{q+2} r_{q+1}.
\end{aligned}
$$

We prove it in the following cases successively:
(Case I)$_{(1)}$. The case where (S, H) is a σ-complete pair.
We see that $\sigma=\rho_1$ and $r_1=0$ (see Definition (4.4)). Thus by Corollary (5.4) we prove the assertion.
(Case I)$_{(2)}$. (S, H) is a σ-incomplete pair.
By the definition of σ, we have $g=(\sigma-1)f+r_1$ (see Definition (4.4)). So we see that $\rho_1=\sigma-1$. First we restrict ourselves to the case where $\rho_1 \geqq 1$. By (ii) in (I) in Lemma (5.3), we see that $\{\varphi_{\rho_1,1}<\varepsilon\} \subset \Delta$ on some neigh-

borhood of A. Let $Q^{(1)}: M_{(1)} \to M$ be the resolution of $\varphi_{\rho_1,1}$. We choose a local coordinate covering of $M_{(1)}$ as (I) in §4. We set

$$H_{(1)} = u^{(1)}_{\lambda|\rho_1} = 0, \qquad S_{(1)} = \{\zeta^{(1)}_{\lambda|\rho_1} = 0\},$$

$$A_{(1)} = H_{(1)} \cap S_{(1)}, \qquad \Delta_{(1)} = (\overline{Q^{(1)-1}(\Delta)}^0).$$

Now we infer that E_{ρ_1-1} is positive by (3) in (ii) in Proposition (4.11). Then by Lemma (5.6), we may assume that $\partial\Delta_{(1)} \cap S_{(1)} = A_{(1)}$. So if Δ is a simple domain along A, then $\Delta_{(1)}$ is also a simple domain along $A_{(1)}$. Moreover, $M_{(1)}, S_{(1)}, H_{(1)}, A_{(1)}$ and $\Delta_{(1)}$ satisfy the conditions $A_{(0)}, (B)_{(0)}$ and $(C)_{(0)}$ near $A_{(1)}$, which are denoted by $(A)_{(1)}, (B)_{(1)}$ and $(C)_{(1)}$ respectively. Set

$$(5.10) \qquad \begin{matrix} \zeta^{(1)}_\lambda = \zeta^{(1)}_{\lambda|\rho_1} \\ \eta^{(1)}_\lambda = u^{(1)}_{\lambda|\rho_1} \end{matrix} \quad \text{on } U^{(1)}_{\lambda|\rho_1} \text{ and } \quad \begin{matrix} f^{(1)}_{\lambda\mu} = f_{\lambda\mu} \\ g^{(1)}_{\lambda\mu} = g_{\lambda\mu} f^{-\rho_1}_{\lambda\mu} \end{matrix} \quad \text{on } U^{(1)}_{\lambda|\rho_1} \cap U^{(1)}_{\lambda|\rho_1}.$$

We see that by (4.7)

$$\begin{cases} \zeta^{(1)}_\lambda = f^{(1)}_{\lambda\mu} \cdot \zeta^{(1)}_\mu \\ \eta^{(1)}_\lambda = g^{(1)}_{\lambda\mu} \cdot \eta^{(1)}_\mu \end{cases} \quad \text{on } U^{(1)}_{\lambda|\rho_1} \cap U^{(1)}_{\mu|\rho_1}$$

and we have

$$c_1([S_{(1)}]^{-1}_{|A_{(1)}}) = f \quad \text{and} \quad c_1([H_{(1)}]^{-1}|_{A_{(1)}}) = r_1.$$

The τ-characteristic number of $(S_{(1)}, H_{(1)})$ is denoted by τ_1. Then by $c_1([S_{(1)}]^{-1} \otimes [H_{(1)}]) = f - r_1$, we see that $\tau_1 \geqq 1$. In the case where $\rho_1 = 0$, setting $S_{(1)} = S$ and $H_{(1)} = H$, we find that the τ-characteristic number of $(S_{(1)}, H_{(1)})$ is also greater than one in this case. Hence for all ρ_1 we can choose the τ-characteristic number of $(S_{(1)}, H_{(1)})$ greater than one.

(case II)$_{(1)}$. The case where $(S_{(1)}, H_{(1)})$ is τ-complete. We see that $f = \rho_2 \cdot r_1$, $r_2 = 0$ and $\tau_1 = \rho_2$ by (5.11) and (4.6). Then we see that by (i) in (II) in Lemma (5.3) $\{\varphi_{1,\rho_2} < \varepsilon\} \subset \Delta_{(1)}$ on some neighborhood of $A_{(1)}$. The τ_1-characteristic function is

$$\varphi_{1,\rho_2} = |\zeta^{(1)}_\lambda|^2 / |\eta^{(1)\rho_2}_\lambda|^2.$$

By (I) in §4, we see that

$$(5.11)\qquad \begin{aligned} \zeta_\lambda &= \xi^{(1)}_{\lambda|\rho_1} \\ \eta_\lambda &= \xi^{(1)\rho_1}_{\lambda|\rho_1} u^{(1)}_{\lambda|\rho_1} \end{aligned} \qquad \text{on } U^{(1)}_{\lambda|\rho_1}.$$

By (5.11) we see that $k_0=\rho_1\cdot\rho_2+1$ and $l_0=\rho_2$, where k_0 and l_0 are defined in (1.7). By this we obtain

$$Q^{(1)*}(\phi^*)=\varphi_{1,\rho_2},$$

which implies $\{\phi^*<\varepsilon\}\subset\Delta$ near A. Thus we prove our Theorem in this case.

(Case II)$_{(2)}$. The case where $(S_{(1)}, H_{(1)})$ is τ-incomplete pair. By (ii) in (II) in Lemma (5.3), we find that $\{\varphi_{1,\rho_2+1}<\varepsilon\}\subset\Delta_{(1)}$ near $A_{(1)}$. We form monoidal transforms successively on $M_{(1)}$ ρ_2-times, which is denoted by $Q^{(2)}: M_{(2)}\to M_{(1)}$. The local coordinate covering of $M_{(2)}$ is chosen as in (II) in §4. Then we have

$$Q^{(2)*}(\varphi_{1,\rho_2+1})=Q^{(2)*}(b_\lambda)\,|u^{(2)}_{\lambda|\rho_2}|^2/|\xi^{(2)}_{\lambda|\rho_2}|^2.$$

By (4.12) we have

$$(5.12)\qquad \begin{aligned} \xi^{(2)}_{\lambda|\rho_2} &= g^{(1)}_{\lambda\mu}\xi^{(2)}_{\mu|\rho_2} \\ u^{(2)}_{\lambda|\rho_2} &= f^{(1)}_{\lambda\mu} g^{(1)-\rho_2}_{\lambda\mu} u^{(2)}_{\mu|\rho_2} \end{aligned} \qquad \text{near } L_{\rho_2}\cap\{u^{(2)}_{\lambda|\rho_2}=0\}.$$

Set

$$\Delta_{(2)}=(\overline{Q^{(2)-1}(\Delta_{(1)})}^0),\qquad S_{(2)}=\{u^{(2)}_{\lambda|\rho_2}=0\},$$

$$H_{(2)}=\{\xi^{(2)}_{\lambda|\rho_2}=0\}\quad\text{and}\quad A_{(2)}=S_{(2)}\cap H_{(2)}.$$

By (5.12) and (5.9), we obtain

$$c_1([H_{(2)}]^{-1}_{|A_{(2)}})=r_1\quad\text{and}\quad c_1([S_{(2)}]^{-1}_{|A_{(2)}})=r_2.$$

Then we see that $M_{(2)}, H_{(2)}, S_{(2)}, A_{(2)}$ and $\Delta_{(2)}$ satisfy the conditions $(A)_{(0)}, (B)_{(0)}$ and $(C)_{(0)}$ near $A_{(2)}$, which are denoted by $(A)_{(2)}, (B)_{(2)}$ and $(C)_{(2)}$. We set

$$\zeta_\lambda^{(2)} = u_{\lambda|\rho_2}^{(2)} \qquad \qquad f_{\lambda\mu}^{(2)} = f_{\lambda\mu}^{(1)} g_{\lambda\mu}^{(1)-\rho_2}$$
and
$$\eta_\lambda^{(2)} = \xi_{\lambda|\rho_2}^{(2)} \qquad \qquad g_{\lambda\mu}^{(2)} = g_{\lambda\mu}^{(1)}.$$

Then we see that by (5.12)

$$\zeta_\lambda^{(2)} = f_{\lambda\mu}^{(2)} \zeta_\mu^{(2)}$$

$$\eta_\lambda^{(2)} = g_{\lambda\mu}^{(2)} \eta_\mu^{(2)}.$$

Here we note that $r_1 > r_2$.

(Case III)$_{(1)}$. The case where $(S_{(2)}, H_{(2)})$ is σ-complete. We see that $r_3 = 0$ in (5.9) and $\sigma = \rho_3$. Then (i) in (I) in Lemma (5.3), we see that $\{\varphi_{\rho_3,1} < \varepsilon\} \subset \Delta_{(2)}$. By using

$$\zeta_\lambda^{(1)} = \zeta_\lambda^{(2)} \cdot \eta_\lambda^{(2)\rho_2}$$

$$\eta_\lambda^{(1)} = \eta_\lambda^{(2)},$$

We see that by (5.11)

$$\zeta_\lambda = \zeta_\lambda^{(2)} \cdot \eta_\lambda^{(2)\rho_2}$$

$$\eta_\lambda = \zeta_\lambda^{(2)\rho_1} \cdot \eta_\lambda^{(2)\rho_1 \cdot \rho_2 + 1},$$

Moreover, by (5.9) and (1.5), we obtain

$$k_0 = (\rho_2 \cdot \rho_3 + 1) \cdot \rho_1 + \rho_3 \quad \text{and} \quad l_0 = \rho_2 \cdot \rho_3 + 1.$$

Then we see that $(Q^{(2)} \circ Q^{(1)})^*(\phi^*) = \varphi_{\rho_3,1}$. Therefore we obtain $\{\phi^* < \varepsilon\} \subset \Delta$.

(Case III)$_{(2)}$. The case where $(S_{(2)}, H_{(2)})$ is σ-incomplete. As we defined $S_{(1)}$ and $H_{(1)}$ from S and H, we define $S_{(3)}$ and $H_{(3)}$ from $S_{(2)}$ and $H_{(2)}$. Then we see that

$$c_1([S_{(3)}]^{-1}) = r_2 \quad \text{and} \quad c_1([H_{(3)}])^{-1} = r_3.$$

Now we consider general cases. From (Case I) we see that by making $Q^{(1)}: M_{(1)} \to M$,

$$c_1([S_{(1)}]^{-1}) = f \quad \text{and} \quad c_1([H_{(1)}]^{-1}) = r_1$$

and $Q^{(1)}$ is the ρ_1-times composition of monoidal transforms. From

(Case II) we see that by making $Q^{(2)}: M_{(2)} \to M_{(1)}$

$$c_1([H_{(2)}]^{-1}) = r_1 \quad \text{and} \quad c_1([S_{(2)}]^{-1}) = r_2$$

and $Q^{(2)}$ is the ρ_2-times monoidal transforms. From (Case III) by making $Q^{(3)}: M_{(3)} \to M_{(2)}$, we see that

$$c_1([S_{(3)}]^{-1}) = r_2 \quad \text{and} \quad c_1([H_{(3)}{}^{-1}]) = r_3$$

and $Q^{(3)}$ is the ρ_3-times composition of monoidal transform.

Hence repeating this process $q+1$ times, we obtain a complex manifold $M_{(q+1)}$ and divisors $S_{(q+1)}$ and $H_{(q+1)}$ on $M_{(q+1)}$ satisfying

$$c_1([S_{(q+1)}]^{-1}) = r_q \text{ (or } r_{q+1}) \quad \text{and} \quad c_1([H_{(q+1)}]^{-1}) = r_{q+1} \text{ (resp. } r_q).$$

and $Q^{(q+1)}: M_{(q+1)} \to M_{(q)}$ is the composition of ρ_{q+1}-times monoidal transforms. In view of (5.9), we see that $(S_{(q+1)}, H_{(q+1)})$ is now σ (resp. τ)-complete. Then as in (Case I)$_{(1)}$ or (Case III)$_{(1)}$, we can prove the assertion. By this we complete the proof of Theorem (5.1).

Let $\mu: M_* \to M$ be the resolution of the singularities of indeterminancy of ϕ^* and let $h = \mu^*(\phi^*)$. The exceptional divisor is denoted by Σ and the divisor defined by $\{h=0\}$ is denoted by Σ'. The exceptional baum which is inserted in the final step is denoted by L_* and we write $A_* = \Sigma' \cap L_*$. Finally we set $\Delta_* = (\overline{\mu^{-1}(\Delta)})^0$. Then we have the following

Proposition (5.13). (i) *Let* $E_* = [L_*]_{|L_* \cap \Sigma'}$.
Then

$$E_* = [S]^{-k_0} \otimes [H]^{l_0}.$$

(ii) $\mathcal{O}(\Delta_* - \Sigma) \cong \mathcal{O}(\Delta_*)$ *holds.*

Proofs can be done in each step in the proof of Theorem (5.1).

§6. Proofs of Theorems I, II, III and IV

In this section we shall prove our Theorems stated in §1. We fix notations. Let μ, M_*, h, Σ, Σ', L_*, A_* and Δ_* be as described at the end of §5. Let $V_\delta(\Sigma) = \mu^{-1}(V_\delta(A))$ and $V_\varepsilon(\Sigma') = \{h < \varepsilon\}$. Then $V_\delta(\Sigma)$ and $V_\varepsilon(\Sigma')$ give neighborhood systems of Σ and Σ' respectively.

Proof of Theorem I.

For the proof of Theorem I, we prepare a lemma. Let A be a compact Riemann surface and let F be a topological trivial line bundle on A. With respect to a fine covering $\{V_\lambda\}$ of A, F is expressed as $\{f_{\lambda\mu}\}$. We may assume that $|f_{\lambda\mu}|=1$. Then $h'=|\zeta_\lambda|^2$ and $V_\varepsilon=\{h'<\varepsilon\}$ are a C^∞-function on F and a neighborhood system of the zero section respectively, where ζ_λ denotes the fibre coordinate on V_λ.

Lemma (6.1). *Let Δ be a connected pseudoconvex domain on F. Assume that* (1) *There exists a constant ε such that $\{h'<\varepsilon\}\subset\Delta$ and* (2) *for a point $p\in F$ with $p=(z_\lambda(p), \zeta_\lambda(p))$ and for a real number θ, we set $p_\theta=(z_\lambda(p), e^{i\theta}\zeta_\lambda(p))\in F$. Suppose that $p\in\Delta$ implies $p_\theta\in\Delta$ for any θ. Then there exist a constant c such that*

$$\Delta=\{h'<c\}.$$

Proof. For a point p in the zero section, we denote the Hartogs radius at p by $d(p)$. Then $\varphi(p)=-\log d(p)$ becomes a pseudoconvex function on A. Then $\varphi(p)$ is a constant function. By (2) we prove the assertion.

We infer that there exists a weakly 1-complete function η on Δ, i.e.,

$$\eta=1/(1-h'/c).$$

Now we consider the resolution manifold $\tau: M\to \underline{M}$ which is stated in Introduction and the simple conoid Δ along A (see §1). By Proposition (1.9) Δ is a domain of holomorphy on M. Then M, S, H, A and Δ satisfy the conditions $(A)_{(0)}, (B)_{(0)}$ and $(C)_{(0)}$ in §4 and 5. So by Theorem (5.1)

$$\{\phi^*<\varepsilon\}\cap V_\delta(A)\subset\Delta\cap V_\delta(A). \tag{6.2}$$

By (i) in Proposition (5.13), we see that $[L_*]_{|A_*}$ is a topologically trivial line bundle on A_*. We choose a local coordinate covering of M_* by using (I) and (II) in §4. The local coordinates near $L_*\cap\Sigma'$ are denoted by z_λ, u_λ^* and ξ_λ^*, where $\{\xi_\lambda^*=0\}=L_*$. Set $\Delta_*\cap L_*=\Delta_*(L_*)$. Then if $p\in\Delta_*(L_*)$, then $p_\theta\in\Delta_*(L_*)$ for every θ by Definition (0.1). Moreover, since Δ is a simple domain along A, $\Delta_*(L_*)$ is connected. Hence by the choice of local coordinates, we see that the absolute value of the transition functions of $[L_*]_{|\Delta_*}$ is identical with one. So we find that $h|_{L_*} = h'$. Therefore by Lemma (6.1), we see that

$$\Delta_*(L_*) = \{h < c\} \tag{6.3}$$

with some constant c. Let $\Omega_\lambda^* = \{(z_\lambda, u_\lambda^*, \xi_\lambda^*) : |u_\lambda^*| < +\infty, |\xi_\lambda^*| < +\infty\}$ and $\Omega^* = \bigcup_\lambda \Omega_\lambda^*$. Then Ω^* is the maximal domain which admits the fibre structure on A_* whose fibre is isomorphic to $\mathbf{C}^2$. The natural projection is denoted by $\rho: \Omega^* \to A_*$. We remark that $M_* - \mu^{-1}(S \cup H) \subset \Omega^*$ holds. Then we have

Lemma (6.4). $\Delta_* \cap \Omega^* \subset \{h < c\}$.

Proof. By Definition (0.1) we see that if $p = (z_\lambda(p), \xi_\lambda^*(p), u_\lambda^*(p)) \in \Delta_* \cap \Omega^*$, then $p_{\alpha\beta} = (z_\lambda(p), e^{i\alpha}\xi_\lambda^*(p), e^{i\beta}u_\lambda^*(p)) \in \Delta_* \cap \Omega^*$. This implies that $\Delta_* \cap \Omega^* \cap \rho^{-1}(p)$ is a Reinhart domain. Assume that there exists a point $p_0 \in \Delta_* \cap \Omega^*$ satisfying $h(p_0) = c^*$ with $c^* > c$. By (6.2) we have

$$\{|\xi_\lambda^*|^2 < \varepsilon, |u_\lambda^*|^2 < \varepsilon\} \subset \Delta_* \cap \Omega^* \cap \rho^{-1}(\rho(p_0))$$

for a sufficiently small ε. Thus by Abel's theorem, we see that

$$\{|\xi_\lambda^*|^2 < c^{**}, |u_\lambda^*|^2 < c^*\} \subset \Delta_* \cap \Omega^* \cap \rho^{-1}(\rho(p_0)),$$

where $c^{**} = |\xi_\lambda^*(p_0)|^2$. Restricting this domain to L_*, we have $\{|u_\lambda^*|^2 < c^*\} \subset \Delta_*(L_*)$, which leads a contradiction.

For the proof of Theorem I, it is sufficient to show (2) in (ii) in Definition (1.4) (also see Definition (1.6)). By (6.2) we get $V_\varepsilon(\Sigma') \subset \Delta_*$ with a sufficiently small ε. By the construction of μ, we see that $\Sigma \cap (\Delta_* - \Omega^*) \subset \Sigma'$ on a small neighborhood of Σ. So making δ smaller, we may assume that $(\Delta_* - \Omega^*) \cap V_\delta(\Sigma) \subset V_\varepsilon(\Sigma')$ holds. Then for a point $p \in V_\delta(\Sigma) \cap (\Delta_* - \Omega^*)$, we obtain $\Gamma_p^{*\delta} \subset \Delta_*$, where $\Gamma_p^{*\delta} = \{q \in V_\delta(\Sigma) : h(p) = h(q)\}$. Take a point $p \in \Delta_* \cap \Omega^*$. Then we see that $h(p) < c$ by Lemma (6.4). Then $\Gamma_p^{*\delta} \cap L_* \subset \Delta_*(L_*)$. Thus making δ_0 smaller, we get $\Gamma_p^{*\delta_0} \subset \Delta_*$. So we see that $\Gamma_p^{\delta,\delta_0} \subset \Delta$.

Proof of Theorem II.

For the proof of Theorem we prepare two Propositions:

Proposition (6.5). *Suppose that M satisfies the condition $(A)_{(0)}$. Let $Q: M_{(1)} \to M$ be a monoidal transform with center A. Then we obtain (i) if M satisfies the condition $(B)_{(0)}$ furthermore, then $[L]<0$, where $L=Q^{-1}(A)$.*

(ii) Suppose that a negative line bundle E is given on M. Let $L = Q^{-1}(A)$. Then we can find a positive integer n_0 such that

(6.6) $Q^*(E^n)\otimes[L]<0$ *for $n \geqq n_0$ on a small neighborhood of L.*

Proof. Proof of (i). By the conditions $(A)_{(0)}$ and $(B)_{(0)}$, we see that L is an exceptional divisor on $M_{(1)}$ and L admits a neighborhood which is isomorphic to a tubler neighborhood of the zero section of the normal bundle. Then by H. Grauert [3, Satz 1, p. 341], we see that $[L]<0$.

Proof of (ii). Choosing metrics $\{a_\lambda\}$ and $\{b_\lambda\}$ of $[S]$ and $[H]$, we set

$$h = a_\lambda |\zeta_\lambda|^2 + b_\lambda |\eta_\lambda|^2 .$$

We choose a local coordinate covering of $M_{(1)}$ as (4.5). Set

$$c_\lambda^{(1)} = a_\lambda + b_\lambda |u_{\lambda|1}^{(1)}|^2 \text{ on } U_{\lambda|1}^{(1)}, \qquad c_\lambda^{(2)} = a_\lambda |u_{\lambda|1}^{(2)}|^2 + b_\lambda \text{ on } U_{\lambda|1}^{(2)}.$$

Then $\{c_\lambda^{(i)}\}$ becomes a metric of $[L]$. Let $\{e_\lambda\}$ be a negative metric of E. Then we have a metric of $Q^*(E^n)\otimes L$, $\{\tilde{e}_\lambda^{(i)}\}$ by

$$\tilde{e}_\lambda^{(i)} = c_\lambda^{(i)} \cdot Q^{(1)*}(e_\lambda^n) \qquad (i=1, 2) .$$

For the proof of (ii), it is sufficient to show that the restriction of the above line bundle to L is negative. We write

$$\partial\bar{\partial} \log \tilde{e}_\lambda^{(i)} = a_{1,1}^{(i)} dz_\lambda \wedge d\bar{z}_\lambda + a_{1,2}^{(i)} dz_\lambda \wedge d\bar{u}_{\lambda|1}^{(i)}$$
$$+ a_{2,1}^{(i)} du_{\lambda|1}^{(i)} \wedge d\bar{z}_\lambda + a_{2,2}^{(i)} du_{\lambda|1}^{(i)} \wedge d\bar{u}_{\lambda|1}^{(i)},$$

where $e_\lambda^{(i)} = Q^*(e_\lambda)$ on $U_{\lambda|1}^{(i)} \cap L$ and

$$\partial\bar{\partial} \log c_\lambda^{(i)} = h_{1,1}^{(i)} dz_\lambda \wedge d\bar{z}_\lambda + h_{1,2}^{(i)} dz_\lambda \wedge d\bar{u}_{\lambda|1}^{(i)}$$

$$+h^{(i)}_{2,1}du^{(i)}_{\lambda|1}\wedge d\bar{z}_\lambda+h^{(i)}_{2,2}du^{(i)}_{\lambda|1}\wedge d\bar{u}^{(i)}_{\lambda|1}.$$

Here we note that

$$h^{(i)}_{2,2}=a_\lambda b_\lambda|c^{(i)}|^{-2}\qquad(i=1,2).$$

Since $E<0$, we can find positive constants δ and C such that

$$a^{(i)}_{1,1}\geqq\delta,\ h^{(i)}_{2,2}\geqq\delta\quad\text{and}\quad|h^{(i)}_{j,k}|\leqq C\qquad\text{for}\quad i,j,k=1,2.$$

Moreover, from $Q^*(E)\geqq 0$, we have

(6.7) $$\text{if}\quad a^{(i)}_{2,2}=0,\quad\text{then}\quad a^{(i)}_{1,2}=0\quad\text{and}\quad a^{(i)}_{2,1}=0.$$

Then we see that $na^{(i)}_{2,2}+h^{(i)}_{2,2}>0$ for $n\geqq 0$. For a sufficiently large n we have $na^{(i)}_{1,1}+h^{(i)}_{1,1}>0$. Hence for the proof of the assertion, it is sufficient to show that the following determinant is positive definite for a sufficiently large n:

$$S=\|n\cdot a^{(i)}_{k,j}+h^{(i)}_{k,j}\|.$$

We see that

$$S\geqq n^2(a^{(i)}_{1,1}\cdot a^{(i)}_{2,2}-a^{(i)}_{1,2}\cdot a^{(i)}_{2,1})+n\delta^2-nC(a^{(i)}_{2,2}+a^{(i)}_{1,2}+a^{(i)}_{2,1})-2C^2.$$

Take a positive constant ε_0. Then for any point p with $a^{(i)}_{2,2}(p)\neq 0$, we can satisfy $S\geqq\varepsilon_0$ on some neighborhood of p. If $a^{(i)}_{2,2}(p)=0$, then by (6.7) we get $S\geqq n\delta^2-2C^2$ on some neighborhood of p. So making n larger, we have $S\geqq\varepsilon_0$ on a small neighborhood of L, which proves the assertion.

Proposition (6.8). *Suppose that Δ is normal. Let Δ_* and Ω^* be as in Lemma* (6.4). *Then there exists a positive constant c such that*
(i) $\Delta_*\cap\Omega^*\subset\{h<c\}$ *and* (ii) $\Delta_*(L_*)=\{h<c\}$,
where $c=\sup_{p\in\Delta_*(L_*)} h$.

Proof. If (i) is not true, then there exists a point $p_0\in\Delta_*\cap\Omega^*$ such that $h(p_0)=c^*$ with $c^*>c$. By assumption, $\Gamma^{*\delta}_{p_0}\subset\Delta_*$ for some δ. This yields $\Gamma^{*\delta}_{p_0}\cap L_*\subset\Delta_*(L_*)$, which contradicts the definition of c. (ii) follows from the condition (iii) in Definition (1.4).

First we show (2) in Theorem (II). Note that μ is the composition of monoidal transforms, i.e., $\mu=Q_{(m)}\circ Q_{(m-1)}\circ\cdots\circ Q_{(1)}$. Then by $(A)_{(0)}$, $(B)_{(0)}$ and (i) in Proposition (6.5), we see that $[L]$ is a negative line bundle, where $L=Q_{(1)}^{-1}(A)$. By (ii) in the same proposition, $Q_{(2)}^{*}([L]^{n_1})\otimes[L_{(2)}]$ is negative on $M_{(2)}$ for some n_1, where $M_{(2)}=Q_{(1)}^{-1}(M_{(1)})$ and $L_{(2)}=Q_{(2)}^{-1}(A_{(1)})$. Repeating this process, we obtain a negative line bundle $[\Sigma]^n$ on M_* (for implication, see §3 in O. Suzuki [10]). Next we prove that Δ_* is weakly 1-complete. For this we may prove that Δ_* satisfies (i), (ii) and (iii) in Lemma (3.5). (i) is satisfied by $[\Sigma]^n$. (ii) is satisfied by Proposition (1.8). Now we will check (iii):

$$\Gamma'_c=\{q\in\bar{\Delta}_*: h(q)=c\},$$

Γ_c=the connected component of Γ'_c which contains $\{h=c\}\cap L_*$.

Since Δ_* is a relatively compact domain in M_*, we can choose neighborhoods Ω_1 and Ω_2 of Γ_c so that $\Omega_1\Subset\Omega_2$ and $h(p)<c$ for $p\in(\Omega_2-\Omega_1)\cap\bar{\Delta}_*$. For δ_1 and δ_2, we set

$$U_{\delta_1}=(\Delta_*\cap V_{\delta_1}(A))\cup\Omega_1 \quad\text{and}\quad U_{\delta_2}=(\Delta_*\cap V_{\delta_2}(A))\cup\Omega_2.$$

We choose δ_1 and δ_2 so that $U_{\delta_1}\Subset U_{\delta_2}$ holds. Choose a C^∞-function $\alpha(p)$ $(0\leqq\alpha(p)\leqq 1)$ on M such that

$$\alpha(p)=\begin{cases}1 & \text{on } U_{\delta_1}\\ 0 & \text{on } U_{\delta_2}^c\end{cases}$$

and set $\eta'=\alpha\cdot h$. Then $\eta'(p)<c$ holds for $p\in\Delta_*$ and η' is a pseudoconvex function on U_{δ_1}. Set

$$\eta=1/(1-\eta'/c)+\chi(\omega^*),$$

where $\omega^*=\mu^*\circ\tau^*(\omega')$ (see (1.1)) and χ is a convex increasing function. Choosing a suitable χ, we get a pseudoconvex function η on $\Delta_*\cup U_{\delta_2}^c$.

Now replacing D, Ω, S and V in Lemma (3.5) by Δ_*, $\Delta_*\cup(U_{\delta_2}^c-\{h\leqq c\})$, Σ and U_{δ_1} respectively, we see that (iii) is satisfied. So we find that Δ_* is a weakly 1-complete manifold.

Proof of Theorem III.

Proof of (i). Because $E_M=[S]^{-k_0}\otimes[H]^{l_0}$ is of finite order, there exists a positive integer r such that r-times tensor product of E_M is analytically trivial. So

$$f=\zeta_\lambda^{k_0 r}/\eta_\lambda^{l_0 r}$$

is a meromorphic function on M. Referring to $|f|^2=\phi^{*r}, f^*=\mu^*(f)$ is a holomorphic function on Δ_* and $g=f^*|_{\Delta_*(L_*)}: \Delta_*(L_*)\to\mathbf{D}$ is a proper mapping, where $\mathbf{D}$ is the 1-dimensional disk. Every fibre of g is connected. We see easily that for any pair of two fibres of g there exists a holomorphic function on Δ_* which separates their values at the given fibres. Also by O. Suzuki [10] (see Theorem 5), Δ_* is $[\Sigma]^{-n}$-convex except Σ. So we see that each fibre of $\varpi: \Delta_*\to\operatorname{Spec}\mathcal{O}(\Delta_*)$ is compact and connected. So by O. Suzuki [10] (see Proposition (2.3)) $\underline{\Delta}_*=\operatorname{Spec}\mathcal{O}(\Delta_*)$ admits the structure of complex space. Also we see that $\underline{\Delta}_*$ is a weakly 1-complete manifold. Then by a well known Theorem of A. Andreotti and R. Narasimhan [1], we see that $\underline{\Delta}_*$ is a Stein space.

Proof of (ii). Consider $\varpi: \underline{\Delta}_*\to\operatorname{Spec}\mathcal{O}(\underline{\Delta}_*)$. By (i) in Proposition (5.13) every holomorphic function on $\underline{\Delta}_*$ is constant on Σ. Because $\underline{\Delta}_*$ is $[\Sigma]^{-n}$-convex except Σ (see Theorem 5 in O. Suzuki [10]), we see that $\varpi^{-1}(\varpi(\Sigma))=\Sigma$. Let $h=g_\lambda|\phi_\lambda^{n'}|^2$, where ϕ_λ denotes the minimal defining equation of Σ'. Then $[\Sigma']^{-n'}$ is a non-negative complex line bundle. In (2) in Theorem 6 in O. Suzuki [10], replacing D, S and A by Σ' Σ and L_*, we prove the assertion.

Proof of Theorem IV.

For the proof of Theorem IV, it is sufficient to show the following

Proposition (6.9). (*i*) *If E_M is of finite order, then $\underline{\Delta}_*=\operatorname{Spec}\mathcal{O}(\Delta_*)$ is the K-convex hull of $\underline{\Delta}$ (see Definition* (2.5) *in O. Suzuki* [10]). (*ii*) *If E_M is of infinite order, then $\underline{\Delta}$ is an* L-*manifold.*

Proof. Proof of (i). Let $\rho=\tau\circ\mu: \Delta_*\to\underline{\Delta}$. Then in view of $\Sigma=\rho^{-1}(p_0)$, $\Delta_*-\Sigma\cong\underline{\Delta}$. Take an arbitrary fibre discrete holomorphic mapping $\underline{\Phi}=(\underline{f}_1,\underline{f}_2,\underline{f}_3): \underline{\Delta}\to\mathbf{C}^3$. Since $\Delta_*-\Sigma$ is holomorphically separable, $\lambda=\omega\circ\rho^{-1}: \underline{\Delta}\to\underline{\Delta}_*$ is injective. We infer that $\mathcal{O}(\underline{\Delta})\cong\mathcal{O}(\underline{\Delta}_*)$ by (ii) in Proposition (5.13). So there exists a fibre discrete mapping $\underline{\Phi}^*=(\underline{f}_1^*,\underline{f}_2^*,\underline{f}_3^*): \underline{\Delta}_*\to\mathbf{C}^3$ such that $\underline{\Phi}=\underline{\Phi}^*\circ\lambda$. This implies that the K-convex hull of $\underline{\Delta}$ is contained in $\underline{\Delta}_*$. Because $\underline{\Delta}_*$ is a Stein space, we prove the assertion. Proof of (ii). In the same manner as in the proof of Proposition (4.4) in O. Suzuki [10], we see that Γ is nothing but $\varpi(\Sigma)$, where Γ denotes the closed set which never admit the structure of a complex space (see Introduction in O. Suzuki [10]). Then Δ_* is a B-resolution of $\underline{\Delta}$, which proves that $\underline{\Delta}$ is an L-manifold (see Introduction in O. Suzuki [10]).

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