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On an Application of the Averaging Method for Nonlinear Systems of Integro Differential Equations

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Summary The present paper justifies a variant of the averaging method for a system of integro differential equations of a standard type, and finds an estimation for proximity of the solutions of the considered system and its averaged system.

In paper [1] the averaging method for a system of ordinary differential equations of a standard type is justified. An estimation for proximity of the solutions of the initial and the averaged system is found.

In the present paper this method is applied to a nonlinear system of integro-differential equations of a standard type. An estimation for proximity of the solutions of the initial and its corresponding averaged system is found, using two of the schemes for averaging proposed in [2].

Consider the equation

$$\dot{x} = \varepsilon X\left(t, x, \int_0^t \varphi(t, s, x)\, ds\right) \tag{1}$$

with initial condition

$$x(0) = x_0 \tag{2}$$

where $x, X, \varphi \in \boldsymbol{R}^n$, and $\varepsilon > 0$ is a small parameter.

Let the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X\left(t, x, \int_0^t \varphi(t, s, x)\, ds\right) dt = X_0(x) \tag{3}$$

exist.

An averaged equation corresponding to (1) will be called the equation

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(4) $$\dot{\xi} = \varepsilon X_0(\xi)$$

with initial condition

(5) $$\xi(0) = x_0$$

The following theorem holds:

Theorem 1. *Let the functions $X(t, x, y)$ and $\varphi(t, s, x)$ be defined and continuous in the domain $Q\{t, s \geqq 0, x \in \mathcal{D} \subset \boldsymbol{R}^n, y \in \boldsymbol{R}^n\}$, where the domain $\mathcal{D}$ is assumed to be open, and let, in this domain, the following conditions be satisfied:*

1. *There exists a constant M, such that $\|X(t, x, y)\| \leqq M$.*

2. *The functions $X(t, x, y)$ and $\varphi(t, s, x)$ satisfy the Lipschitz condition*

$$\|X(t, x', y') - X(t, x'', y'')\| \leqq \lambda\{\|x' - x''\| + \|y' + y''\|\}, \ \lambda = \text{const.},$$

$$\|\varphi(t, s, x') - \varphi(t, s, x'')\| \leqq \mu(t, s)\|x' - x''\|.$$

3. $\dfrac{1}{t}\displaystyle\int_0^t d\tau \int_0^\tau \mu(\tau, s)\, ds \to 0, \quad t \to \infty.$

4. *The limit* (3) *exists uniformly with respect to $x \in \mathcal{D}$.*

5. *The solution $\xi = \xi(t)$, $\xi(0) = x_0 \in \mathcal{D}$ of the Cauchy problem* (4), (5) *is defined for every $t \geqq 0$ and lies in $\mathcal{D}$ with some of its ρ-neighbourhoods.*

Then, for each arbitrarily chosen, sufficiently large positive number $L > 0$ there can be found such a number $\varepsilon_0 > 0$, that for $\varepsilon \in (0, \varepsilon_0]$ on the interval $0 \leqq t \leqq L\varepsilon^{-1}$ the following inequality would be satisfied:

$$\|x(t) - \xi(t)\| \leqq e^{\lambda[L+\delta(\varepsilon)]}\{\lambda ML\delta(\varepsilon) + 2\psi(\varepsilon) + 2\sqrt{2\lambda ML\psi(\varepsilon)(L + \delta(\varepsilon))}\}$$

where

(6) $$\psi(\varepsilon) = \sup_{0 \leqq \tau \leqq L}\left\{\sup_{\xi \in \mathcal{D}}\left\| \int_0^{\tau/\varepsilon}\left[X\left(t, \xi, \int_0^t \varphi(t, s, \xi)\, ds\right) - X_0(\xi)\right]dt\right\|\right\},$$

(7) $$\delta(\varepsilon) = \sup_{0 \leqq \tau \leqq L} \tau \bar{\mu}_0\left(\frac{\tau}{\varepsilon}\right),$$

(8) $$\bar{\mu}_0(t) = \frac{1}{t}\int_0^t d\tau \int_0^\tau \mu(\tau, s)\, ds$$

Proof. We assume that $x(t) \in \mathcal{D}$ when $0 \leqq t \leqq L\varepsilon^{-1}$. For the difference $x(t) - \xi(t)$ there holds the integral representation

$$x(t) - \xi(t) = \varepsilon \int_0^t \Big[X\Big(\tau, x(\tau), \int_0^\tau \varphi(\tau, s, x(s))\, ds\Big) - X\Big(\tau, \xi(\tau), \int_0^\tau \varphi(\tau, s, \xi(s))\, ds\Big) \Big] d\tau + \varepsilon \int_0^t \Big[X\Big(\tau, \xi(\tau), \int_0^\tau \varphi(\tau, s, \xi(s))\, ds\Big) - X\Big(\tau, \xi(\tau), \int_0^\tau \varphi(\tau, s, \xi(\tau))\, ds\Big) \Big] d\tau + \varepsilon \int_0^t \Big[X\Big(\tau, \xi(\tau), \int_0^\tau \varphi(\tau, s, \xi(\tau))\, ds\Big) - X_0(\xi(\tau)) \Big] d\tau,$$

whence the following estimation follows

$$\|x(t) - \xi(t)\| \leqq \varepsilon\lambda \int_0^t \Big\{ \|x(\tau) - \xi(\tau)\| + \int_0^\tau \mu(\tau, s) \|x(s) - \xi(s)\| ds \Big\} d\tau + \varepsilon\lambda \int_0^t d\tau \int_0^\tau \mu(\tau, s) \|\xi(s) - \xi(\tau)\| ds + \varepsilon \Big\| \int_0^t X_1(\tau, \xi(\tau))\, d\tau \Big\|. \tag{9}$$

Here

$$X_1(\tau, \xi(\tau)) = X\Big(\tau, \xi(\tau), \int_0^\tau \varphi(\tau, s, \xi(\tau))\, ds\Big) - X_0(\xi(\tau)).$$

The function $X_1(\tau, \xi(\tau))$ satisfies the Lipschitz condition. Indeed,

$$\begin{aligned} \|X_1(\tau, \xi') - X_1(\tau, \xi'')\| &\leqq \Big\| X\Big(\tau, \xi', \int_0^\tau \varphi(\tau, s, \xi')\, ds\Big) - X\Big(\tau, \xi'', \int_0^\tau \varphi(\tau, s, \xi'')\, ds\Big) \Big\| + \|X_0(\xi') - X_0(\xi'')\| \\ &\leqq \lambda \|\xi' - \xi''\| + \lambda \int_0^\tau \mu(\tau, s) \|\xi' - \xi''\| ds + \lim_{T \to \infty} \frac{1}{T} \int_0^T \Big\| X\Big(\tau, \xi', \int_0^\tau \varphi(\tau, s, \xi')\, ds\Big) - X\Big(\tau, \xi'', \int_0^\tau \varphi(\tau, s, \xi'')\, ds\Big) \Big\| d\tau \\ &\leqq \lambda \|\xi' - \xi''\| + \lambda \int_0^\tau \mu(\tau, s) \|\xi' - \xi''\| ds + \lambda \|\xi' - \xi''\| + \lambda \|\xi' - \xi''\| \lim_{T \to \infty} \frac{1}{T} \int_0^T d\tau \int_0^\tau \mu(\tau, s)\, ds \\ &= [2\lambda + \lambda\mu_0(\tau)] \|\xi' - \xi''\|. \end{aligned}$$

Here the notation $\mu_0(\tau) = \int_0^\tau \mu(\tau, s)\, ds$ is introduced.

We estimate the last summand of (9) on the interval $0\leqq t\leqq L\varepsilon^{-1}$. For this purpose we divide the interval into m parts with the help of the points $t_0=0,\ t_1,\cdots,t_{m-1},\ t_m=L\varepsilon^{-1}$ and we find

$$(10)\qquad \left\|\varepsilon\int_0^t X_1(\tau,\xi(\tau))d\tau\right\|\leqq\left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}[X_1(\tau,\xi(\tau))-X_1(\tau,\xi(t_i))]d\tau\right\| + \left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}X_1(\tau,\xi(t_i))d\tau\right\|.$$

For the first summand on the right hand side of (10) we obtain the estimation

$$(11)\qquad \left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}[X_1(\tau,\xi(\tau))-X_1(\tau,\xi(t_i))]d\tau\right\|\leqq\frac{\lambda ML^2}{m}+\frac{\lambda ML}{m}\delta(\varepsilon)$$

where $\delta(\varepsilon)$ is determined by (7) and $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$.

From condition 4 of the theorem there follows the existence of the function

$$\Phi(t)=\sup_{\xi\in\mathscr{D}}\left\|\frac{1}{t}\int_0^t X_1(\tau,\xi)d\tau\right\|,\qquad \Phi(t)\to 0\quad\text{as}\quad t\to\infty.$$

Then

$$\varepsilon\left\|\int_0^t X_1(\tau,\xi)d\tau\right\|\leqq\varepsilon t\Phi(t)\leqq\sup_{0\leqq\tau\leqq L}\tau\Phi\left(\frac{\tau}{\varepsilon}\right)=\psi(\varepsilon)\qquad \psi(\varepsilon)\to 0\quad\text{as}\quad\varepsilon\to 0$$

For the second summand of (10) we get

$$(12)\qquad \left\|\varepsilon\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}}\varphi(\tau,\xi(t_i))d\tau\right\|\leqq 2m\psi(\varepsilon).$$

From (10), (11), (12) there follows the estimation

$$(13)\qquad \left\|\varepsilon\int_0^t X_1(\tau,\xi(\tau))d\tau\right\|\leqq\frac{\lambda ML^2}{m}+\frac{\lambda ML}{m}\delta(\varepsilon)+2m\psi(\varepsilon).$$

From (9) and (13) we obtain

$$\|x(t)-\xi(t)\|\leqq\varepsilon\lambda\int_0^t\left\{\|x(\tau)-\xi(\tau)\|+\int_0^\tau\mu(\tau,s)\|x(s)-\xi(s)\|ds\right\}d\tau + \lambda ML\delta(\varepsilon)+\frac{\lambda ML^2}{m}+\frac{\lambda ML}{m}\delta(\varepsilon)+2m\psi(\varepsilon)$$

or

$$\|x(t)-\xi(t)\| \leqq \left[\lambda ML\delta(\varepsilon)+\frac{\lambda ML^2}{m}+\frac{\lambda ML}{m}\delta(\varepsilon)+2m\psi(\varepsilon)\right]e^{\lambda[L+\delta(\varepsilon)]}$$

whence we get the estimation

$$\|x(t)-\xi(t)\| \leqq e^{\lambda[L+\delta(\varepsilon)]}[\lambda ML\delta(\varepsilon)+2\psi(\varepsilon)+2\sqrt{2\lambda ML\psi(\varepsilon)[L+\delta(\varepsilon)]}].$$

The proof of the fact that $x(t)\in\mathcal{D}$ when $t\in[0, L\varepsilon^{-1}]$ is trivial. In this way Theorem 1 is proved.

Suppose that the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X\left(t, x, \int_0^\infty \varphi(t, s, x)\,ds\right)dt = X_0(x) \tag{14}$$

exists.

Then the following theorem holds:

Theorem 2. *Let the functions $X(t, x, y)$ and $\varphi(t, s, x)$ be defined and continuous in the domain $Q\{t, s\geqq 0, x\in\mathcal{D}\subset\boldsymbol{R}^n, y\in\boldsymbol{R}^n\}$ and let the following conditions be satisfied in this domain:*

1. $\|X(t, x, y)\|\leqq M, \quad M=\text{const}.$
2. $\|X(t, x', y')-X(t, x'', y'')\|\leqq\lambda\{\|x'-x''\|+\|y'-y''\|\}$

   $\|\varphi(t, s, x')-\varphi(t, s, x'')\|\leqq\mu(t, s)\|x'-x''\|$
3. $\dfrac{1}{t}\displaystyle\int_0^t d\tau\int_0^\tau \mu(\tau, s)\,ds\to 0, \quad t\to\infty;\ \lambda=\text{const}.$
4. *The limit* (14) *exists uniformly with respect to* $x\in\mathcal{D}$.
5. *The solution* $\xi=\xi(t)$, $\xi(0)=x(0)\in\mathcal{D}$ *of the averaged equation is defined for every* $t\geqq 0$ *and lies in* $\mathcal{D}$ *with some of its* ρ*-neighbourhoods.*
6. $\displaystyle\lim_{t\to\infty}\frac{1}{t}\int_0^t\left\|\int_0^\infty\varphi(\tau, s, \xi(\tau))\,ds\right\|d\tau=0$
7. $\displaystyle\left\|\int_0^\infty\varphi(t, s, x')\,ds-\int_0^\infty\varphi(t, s, x'')\,ds\right\|\leqq\nu\|x'-x''\|, \quad \nu=\text{const}.$

Then, for every $L>0$ there exists $\varepsilon_0>0$, such that when $0<\varepsilon\leqq\varepsilon_0$ on the interval $0\leqq t\leqq L\varepsilon^{-1}$ the following inequality is fulfilled:

$$\|x(t) - \xi(t)\| \leqq e^{\lambda[L+\delta(\varepsilon)]}\{\lambda\gamma(\varepsilon) + \lambda ML\delta(\varepsilon) + 2\psi_1(\varepsilon) + 2\sqrt{2\lambda ML^2\psi_1(\varepsilon)(1+\nu)}\}$$

where

$$\psi_1(\varepsilon) \sup_{0\leqq\tau\leqq L}\left\{\sup_{\xi\in\mathscr{D}}\left\| \int_0^{\tau/\varepsilon}\left[X\left(t, \xi, \int_0^\infty \varphi(t, s, \xi)\,ds\right) - X_0(\xi)\right]dt\right\|\right\},$$

$$\gamma(\varepsilon) = \sup_{0\leqq\tau\leqq L} \tau F\left(\frac{\tau}{\varepsilon}\right), \quad F(t) = \frac{1}{t}\int_0^t\left\|\int_\tau^\infty \varphi(\tau, s, \xi(\tau))\,ds\right\|d\tau.$$

The proof of Theorem 2 is analogous to that of Theorem 1.

References

[1] Besjes, J. G., On the asymptotic methods for non-linear differential equations, *J. Mécanique,* 8 (1969), N 3.

[2] Filatov, A. N., *Metodi usrednenjia v diferentsialnih i integro-diferentsialnih uravnenijah.* Izd. "FAN", Tashkent, 1971.