Publ. RIMS, Kyoto Univ.
13 (1977), 627-680

Mod p Decomposition of Compact Lie Groups

By

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§ 0. Introduction

Let p be a prime number. A simply connected CW-complex X is called mod p *decomposable into r spaces* if there exist simply connected CW-complexes X_i $(1\leq i\leq r)$ such that $\tilde{H}^*(X_i; Z_p)\neq 0$, and if there exists a p-equivalence $f: \prod_{1\leq i\leq r} X_i\to X$. A mod p decomposition $\prod_{1\leq i\leq r} X_i\to X$ is *irreducible* if each X_i is not mod p decomposable.

In the present paper we shall consider the mod p decomposition of simply connected simple Lie groups. For Lie groups (more generally, for finite H-complexes) there is a well-known rational decomposition

$$G\underset{0}{\simeq}\prod S^{2n_i-1}$$

into the product of spheres. J.-P. Serre has shown a similar decomposition into a product of spheres for primes greater than a fixed prime depending on G using the class theory. Then our main theorem is stated as follows. If a compact Lie group has no p-torsion, then as is well known $H^*(G; Z_p)\cong\Lambda(x_1, \cdots, x_l)$ is the exterior algebra with $\deg x_i=2n_i-1$. We define an integer $r(G)$ to be the number of n_i's which are distinct in Z_{p-1}.

Main Theorem. *Let G be a simply connected, simple Lie group without p-torsion. Then if $G\neq Spin(2n)$, G is irreducibly* mod p *decomposable into $r(G)$ spaces and $Spin(2n)$ is irreducibly* mod p *decomposable into $r(G)+1$ spaces.*

Communicated by N. Shimada, October 25, 1976.
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For more concrete expression of our results see Theorems 3.5, 4.1 and 8.1 (cf. Theorem 4.2 of [6]).

The factors of the mod p decomposition of G are, what we call, mod p *Stiefel complexes* $B_m{}^k(p)$ having the following properties:

(1) $H^*(B_m{}^k(p); Z_p) \cong \Lambda(x_{2m+1}, x_{2m+1+q}, \cdots, x_{2m+1+(k-1)q})$ with $q=2(p-1)$,

(2) there exists a map

$$f: B_m{}^k(p) \to W_{m+s,s}, \quad s=1+(k-1)(p-1)$$

inducing an epimorphism of Z_p-cohomology.

The proof of the main theorem for the classical groups is quite different from that for the exceptional groups.

Among classical groups, $SU(n)$ is particularly important and mod p decompositions of $Sp(n)$ and $Spin(n)$ follow from that of $SU(n)$ by the result of Harris [6]. Our mod p decomposition of $SU(n)$ is an unstable version of the mod p decomposition of p-adic complete K-theory (hence the decomposition of $BU_p{}^\wedge$) by Sullivan [21, 22], and the localization technique is used to make the decomposition in the category of finite complexes.

For the exceptional groups, first we introduce a spectral sequence which is quite useful to compute the homotopy groups of a certain complex, especially a complex whose cohomology mod p is an exterior algebra. Then we construct $B_m{}^k(p)$ and embed them (in the mod p sense) into G by making use of the obstruction theory, after calculating $\pi_*(G:p)$, the p-component of $\pi_*(G)$, by the above spectral sequence.

The paper is organized as follows:

Chapter III The exceptional cases.

§ 8. Mod p decomposition of p-torsion free exceptional groups,

§ 9. Mod 5 decomposition of E_7 and E_7/G_2,

§ 10. Mod 7 decomposition of E_7 and E_8.

In § 1 the localization theory is summarized. Details of Sullivan's construction (an unstable version of the Adams operation) are given in § 2. Then mod p decompositions of the classical Lie groups are proved in § 3 and § 4. The complex $B_m^k(p)$, which is a factor in the mod p decomposition, is constructed in § 5. A spectral sequence converging to meta-stable homotopy groups of a space is constructed in § 6. The complexes $B_m^k(p)$ are characterized for particular k, m, p in § 7. The section 8 is to discuss the mod p decomposition of the p-torsion free exceptional groups and to state the main theorem for them. In § 9 the mod 5 decomposition of E_7 and in § 10 the mod 7 decomposition of E_7 and E_8 are proved.

Unless otherwise stated, the coefficient Z of the integral (co)homology shall be omitted.

The present paper is the revised version of the two mimeographed notes [12] and [17] circulated in 1970 and 1971 respectively. In fact, the note [17] corresponds to Chapter I which was written by G. Nishida. The note [12] corresponds to Chapter II and Chapter III, although entirely rewritten, and they were written by H. Toda and M. Mimura.

Chapter I The Classical Cases

§ 1. Localization of CW-Complexes

Let P be a set of prime numbers and let Q_p denote the ring of fractions whose denominators are, in the lowest term, prime to p for any $p \in P$. If P is the void set, $Q_P \cong Q$ is denoted by $Q_{(0)}$.

The notion of localization of CW-complexes at P is defined by Bousfield-Kan [5], Mimura-Nishida-Toda [16], Sullivan [21, 22] and others. According to Sullivan, we define the localization of a CW-complex as follows. A CW-complex Y is called *P-local* if $\pi_*(Y)$ is a Q_P-module. A continuous map

$$l: X \to X_P$$

from a 1-connected CW-complex X to a P-local space X_P is called *the localization* if for any map $f: X \to Y$, Y P-local, there exists a map $g: X_P \to Y$, unique up to homotopy such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ l\ } & X_P \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & Y \end{array}$$

is homotopy commutative. Then for 1-connected CW-complexes, the localization theorem is stated as follows.

Theorem 1.1 ([21]). *Let X and X_P be 1-connected CW-complexes and let $l: X \to X_P$ be a map. Then the following conditions are equivalent*:

(i) *l is a localization,*

(ii) *there is an isomorphism $\pi_*(X_P) \cong \pi_*(X) \otimes Q_P$ which makes the following diagram commutative*

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\ l_*\ } & \pi_*(X_P) \\ & \searrow^{1 \otimes j} & \downarrow{\scriptstyle \cong} \\ & & \pi_*(X) \otimes Q_P . \end{array}$$

(iii) *there is an isomorphism $H_*(X_P) \cong H_*(X) \otimes Q_P$ which makes the following diagram commutative*

$$\begin{array}{ccc} H_*(X) & \xrightarrow{\ l_*\ } & H_*(X_P) \\ & \searrow^{1 \otimes j} & \downarrow{\scriptstyle \cong} \\ & & H_*(X) \otimes Q_P , \end{array}$$

where $j: Z \to Q_P$ is the canonical injection.

Theorem 1.2 ([21]). *In the homotopy category of 1-connected CW-complexes, there exists a covariant functor L and a natural transformation $\phi: Id \to L$ such that, for any complex X, $\phi_X: X \to L(X) = X_P$ is a localization.*

Now we recall the notion of P-equivalence and P-universality [13, 15]. If p is a prime or 0, we denote by Z_p the prime field of characteristic p. Then a map $f: X \to Y$ is called a *P-equivalence* if

$$f_*: H_*(X; Z_p) \to H_*(Y; Z_p)$$

is an isomorphism for any $p \in P$ and $p = 0$. It is known that P-equivalence is an equivalence relation in the category of P-universal spaces. Then we have

Theorem 1.3 ([16]). *Let X and Y be 1-connected CW-complexes of finite type. Then a map $f: X \to Y$ is a P-equivalence if and only if $f_P: X_P \to Y_P$ is a homotopy equivalence.*

Now a countable 1-connected CW-complex Y is called *finite P-local* if $H_*(X)$ is a finitely generated Q_P-module.

Proposition 1.4. *Let Y be a* mod 0 *H-space. Then Y is finite P-local if and only if there exists a 1-connected finite complex X and $Y \simeq X_P$. Furthermore such a complex X is unique up to P-equivalence.*

Proof. The "if part" is obvious. So assume that Y is finite P-local. Let

$$l: Y \to Y_{(0)}$$

be the localization at 0. By assumption, $H_*(Y_{(0)}) \cong H_*(Y) \otimes Q$ is a finitely generated Q-module. Since Y is a mod 0 H space, we have a homotopy equivalence:

$$Y_{(0)} \simeq \prod K(Q, 2n_i - 1).$$

Then we see that

$$Y_{(0)} \simeq (\prod S^{2n_i-1})_{(0)} .$$

Consider the diagram

$$\begin{array}{ccc} & & (\prod S^{2n_i-1})_{\bar{P}} \\ & & \downarrow l \\ Y & \xrightarrow{l} & Y_{(0)} \simeq (\prod S^{2n_i-1})_{(0)} \end{array}$$

where $\bar{P}$ is the complementary set of P. Let $X = Y \times_{Y_0} (\prod S^{2n_i-1})_{\bar{P}}$ be the pull-back. Then we have a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & (\prod S^{2n_i-1})_{\bar{P}} \\ f \downarrow & & \downarrow l \\ Y & \xrightarrow{l} & Y_{(0)} . \end{array}$$

Let F be the fibre of $l: (\prod S^{2n_i-1})_{\bar{P}} \to Y_{(0)}$ which is also the fibre of $f: X \to Y$. Since l is the localization, we see easily from Theorem 1. 1 that $\tilde{H}_*(F)$ is a $\bar{P}$-torsion group. Consider the Q_P-coefficient homology spectral sequence associated with the fibring:

$$F \to X \to Y .$$

Since $\tilde{H}_*(F; Q_P) = 0$, we see that

$$f_*; H_*(X) \otimes Q_P \to H_*(Y) \otimes Q_P \cong H_*(Y)$$

is an isomorphism. This shows that f is the localization. Similarly we can see that $H_*(X)$ is a finitely generated abelian group. Hence we may take X as a finite complex. Meanwhile Y is a mod 0 H-space and hence P-universal ([13]). Then uniqueness up to P-type of a complex X follows from Theorem 5. 3 of [16]. Q.E.D.

Lemma 1. 5. *If $p \in P$ or $p = 0$, then $l^*: H^*(X_P; Z_p) \to H^*(X; Z_p)$ is an isomorphism. If $p \notin P$, then $\tilde{H}^*(X_P; Z_p) = 0$.*

Proof is easy using the universal coefficient theorem.

§ 2. A Construction of Sullivan

In this section we shall state the Sullivan's construction of unstable Adams operations for the classifying spaces of $U(n)$ and $SU(n)$, and give some easy consequences of the construction.

Theorem 2.1 (Sullivan [21]). *Let n be an integer. Let q be a prime $>n$, then there exists a map $\psi^q: BG \to BG$, $G=U(n)$ or $SU(n)$, such that $(\psi^q)^* c_i = q^i c_i$ where $c_i \in H^{2i}(BG; Z)$ is the i-th Chern class.*

We give an outline of the proof. Let $G_{n,k}(C)$ be the Grassmannian variety. It is shown (Theorem 5.2 [21]) that the "complete etale homotopy type" of $BU(n) = \varinjlim_k G_{n,k}(C)$ is equivalent to the profinite completion $BU(n)^{\wedge}$ of the classical homotopy of $BU(n)$. Since the algebraic variety $G_{n,k}(C)$ is defined over Q, the natural action of the Galois group $\mathrm{Gal}(\tilde{Q}/Q)$ on the etale homotopy type of $G_{n,k}(C)$ defines the action of Gal $(\tilde{Q}/Q)$ on $G_{n,k}(C)^{\wedge}$ and $BU(n)^{\wedge}$, where $\tilde{Q}$ denotes the field of algebraic numbers. The action on cohomology is given as follows (Cor. 5.5, [21]). It is known that there is a canonical epimorphism

$$A: \mathrm{Gal}\ (\tilde{Q}/Q) \to \hat{Z}^*$$

where $\hat{Z}^*$ is the group of units of the profinite completion of Z, and $\mathrm{Ker}\ A \cong [\mathrm{Gal}(\tilde{Q}/Q), \mathrm{Gal}(\tilde{Q}/Q)]$. Let $\sigma \in \mathrm{Gal}(\tilde{Q}/Q)$ and let $A(\sigma) = \alpha$. Then

$$\sigma^*(c_i) = \alpha^i c_i,$$

where $c_i \in H^{2i}(BU(n)^{\wedge}; Z) \cong Z[c_1, \cdots, c_n]$ is the Chern class.

Now let q be a prime and let $[q]$ denote the set of all primes except q. Let

$$\bar{q} = \{q, q, \cdots, q, 1, q, q, \cdots\} \in \prod Z_p^* = Z^*,$$

where 1 is the coordinate of $\hat{Z}_q^*$. Let $\sigma \in \mathrm{Gal}(\tilde{Q}/Q)$ be such that $A(\sigma) = \bar{q}$. Next let $\tilde{q} = \{1, \cdots, 1, q, 1, \cdots\} \in \hat{Z}$ such that $\tilde{q} \cdot \bar{q} = q \in Z \subset \hat{Z}$. We shall show that if $q > n$, then there exists a map

$$\lambda: BU(n)^{\wedge} \to BU(n)^{\wedge}$$

such that

$$\lambda^* = \tilde{q}^k \cdot id: H^{2k}(BU(n)^{\wedge}; Z) \to H^{2k}(BU(n)^{\wedge}; Z).$$

Let $T^n \subset U(n)$ be a maximal torus and let $N \subset U(n)$ be the normalizer of T^n. Then the Weyl group is $N/T^n \cong \Sigma_n$ and hence we may identify BN with $E\Sigma_n \times BT^n/\Sigma_n$, where $E\Sigma_n$ is a universal Σ_n-space. If $q>n$, then from the fibring

$$BT^n \xrightarrow{i} BN \rightarrow B\Sigma_n$$

we see that

$$i^*: H^*(BN; Z_q) \rightarrow H^*(BT^n; Z_q)$$

is injective and $\operatorname{Im} i^* \cong H^*(BT^n; Z_q)^{\Sigma_n}$, the invariant subgroup of Σ_n. Hence we see that the induced homomorphism

$$H^*(BU(n); Z_q) \rightarrow H^*(BN; Z_q)$$

is an isomorphism. Since BN has a bad fundamental group, namely $\pi_1 = \Sigma_n$, we consider the canonical projection

$$BN = E\Sigma_n \times BT^n/\Sigma_n \xrightarrow{\pi} BT^n/\Sigma_n .$$

Then by the Leray spectral sequence of the above map, we see that

$$\pi^*: H^*(BT^n/\Sigma_n; Z_q) \rightarrow H^*(BN; Z_q)$$

is an isomorphism.

Lemma 2.2. *Let X be a* 1-*connected CW-complex. Then the symmetric product $SP^n(X) = X^n/\Sigma_n$ is* 1-*connected.*

Proof. Let K be an s.s. complex and let $|K|$ be the geometric realization of K. Then it is known that there exists a canonical weak homotopy equivalence

$$|SP^n(K)| \rightarrow SP^n|K|.$$

Let $S(X)$ be the singular complex of X. As is well known $S(X)$ has a minimal complex K as a deformation retract. Then if X is 1-connected, K has unique 0 and 1 simplexes. Hence so does $SP^n(K)$ and we see that $|SP^n(K)|$ has no 1-cell. Therefore we see that $SP^n(X)$ is 1-connected. Q.E.D.

Then by the obstruction theory we obtain a homotopy equivalence:

$$h: BU(n)_q{}^{\wedge} \to (BT^n/\Sigma_n)_q{}^{\wedge}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} (BT^n)_q{}^{\wedge} & \xrightarrow{\ i^{\wedge}\ } & BU(n)_q{}^{\wedge} \\ & \searrow^{\pi} & \downarrow \\ & & (BT^n/\Sigma_n)_q{}^{\wedge}\,. \end{array}$$

Let $f: BT \to BT$ be a map of degree q on $H^2(BT; Z)$. Then $f^n: BT^n \to BT^n$ defines a map on BT^n/Σ_n and hence defines a map $g: BU(n)_q{}^{\wedge} \to BU(n)_q{}^{\wedge}$ such that $g \cdot i^{\wedge} \simeq i^{\wedge}(f^n)^{\wedge}$. Then by the homotopy equivalence $BU(n)^{\wedge} \to BU(n)_q{}^{\wedge} \times BU(n)^{\wedge}_{[q]}$, we define

$$\lambda = g \times id: BU(n)^{\wedge} \to BU(n)^{\wedge}\,.$$

Note that $H^*(BU(n)^{\wedge}; \hat{Z}) \cong H^*(BU(n)_q{}^{\wedge}; \hat{Z}_q) \otimes H^*(BU(n)^{\wedge}_{[q]}; Z_{[q]})$ by the Künneth formula and by the fact $H^*(X_q{}^{\wedge}; \hat{Z}_p) = 0$ if $q \neq p$ and if X is 1-connected and of finite type. Then comparing λ with $(f^n)_q{}^{\wedge} \times id: (BT^n)^{\wedge} \to (BT^n)^{\wedge}$, we have

$$\lambda^* = \tilde{q}^k \cdot id: \ H^{2k}(BU(n)^{\wedge}; \hat{Z}) \to H^{2k}(BU(n)^{\wedge}; \hat{Z})\,.$$

Now let us consider the composition

$$\lambda\sigma: \ BU(n)^{\wedge} \to BU(n)^{\wedge}\,,$$

then $(\lambda\sigma)^* = q^k \cdot id$ on $H^{2k}(BU(n)^{\wedge}; Z)$.

Since the rational type of $BU(n)$ is homotopy equivalent to $\prod_{k=1}^{n} K(Q, 2k)$, we can define easily a map

$$r: \ BU(n)_{(0)} \to BU(n)_{(0)}$$

such that $r^* = q^k \cdot id$ on $H^{2k}(BU(n)_{(0)}; Q)$.

Finally to get a map on the ordinary homotopy type $BU(n)$ from maps on the profinite type and on the rational type, we must check the coherence condition ([22]). Here the coherence map is a canonical homotopy equivalence

$$c: (BU(n)^{\wedge})_{(0)} \to (BU(n)_{(0)})_f{}^{\wedge}\,,$$

where $X_f{}^{\wedge}$ denotes the formal completion ([21]). The coherence condition requires that the map $(\lambda\sigma)_{(0)}$ on $(BU(n)^{\wedge})_{(0)}$ is homotopic to $(r)_f{}^{\wedge}$ on

$(BU(n)_{(0)})_f\hat{}$ after identifying by c.

Note that we have a homotopy equivalence

$$(BU(n)_{(0)})_f\hat{}\simeq\prod_{k=1}^{n} K(Q\otimes\hat{Z}, 2k).$$

But it is clear that $(\lambda\sigma)_{(0)}$ and $(r)_f\hat{}$ are homotopic after this further identification. Therefore by the pull-back of $\lambda\sigma$ and r we obtain a map (in general not unique up to homotopy)

$$\psi^q : BU(n)\to BU(n)$$

satisfying the required property. Now $BSU(n)$ is the fibre of the map $BU(n)\to BU(1)=K(Z,2)$ corresponding to the first Chern class. Hence ψ^q restricts to $\psi^q: BSU(n)\to BSU(n)$. Q.E.D.

§ 3. Mod p Decomposition of $SU(n)$

Let n be a positive integer and $q>n$ a prime. Let $\psi^q: BSU(n)\to BSU(n)$ be a map defined in § 2. By applying the loop functor, we obtain a map

$$\Omega\psi^q : SU(n)\to SU(n).$$

Recall that $H^*(SU(n);Z)\cong\Lambda(h_2,\cdots,h_n)$ is the exterior algebra generated by the universal transgressive generators h_i with $\deg h_i=2i-1$. Since $(\psi^q)^*x=q^k x$ for any $x\in H^{2k}(BSU(n);Z)$ by Theorem 2.1, we have

$$(\Omega\psi^q)^*h_i=q^i h_i.$$

Let $k_r: SU(n)\to SU(n)$ be the map defined by $k_r(x)=x^{-q^r}$ for $x\in SU(n)$. We define a map

$$\lambda_{q,r}=\Omega\psi^q\cdot k_r : SU(n)\to SU(n)$$

as the composite

$$SU(n)\xrightarrow{d}SU(n)\times SU(n)\xrightarrow{\Omega\psi^q\times k_r}SU(n)\times SU(n)\xrightarrow{\mu}SU(n),$$

where d is the diagonal map and μ is the multiplication of $SU(n)$.

Lemma 3.1. $(\lambda_{q,r})^*(h_i)=(q^i-q^r)h_i$.

Proof. Note that h_i is primitive. Then as is easily seen, $k_r{}^*(h_i) = -q^r h_i$ and

$$\begin{aligned}(\lambda_{q,r})^*(h_i) &= d^*(\Omega\psi^q \times k_r)^*\mu^*(h_i)\\ &= d^*(\Omega\psi^q \times k_r)^*(h_i\otimes 1 + 1\otimes h_i)\\ &= (q^i - q^r)h_i .\end{aligned}$$

Q.E.D.

Lemma 3.2. *Let n be a positive integer and p a prime. Then there exists a prime $q>n$ which is a primitive root* mod p.

Proof. Let k be a primitive root mod p. Then so is $k+pt$ for any positive integer t. Since $(k,p)=1$, there exist infinitely many primes of this form by the classical theorem of Dirichlet. This proves the lemma. Q.E.D.

Proposition 3.3. *Let n and p be as in Lemma* 3.2. *Then for each m, $2\le m\le \min(n,p)$, there exists a* 1*-connected finite complex $X_m(n)$ and there exists a map $f_m\colon SU(n)\to X_m(n)$ satisfying*

i) $$H^*(X_m(n);Z_p)\cong \Lambda(x_m, x_{m+p-1}, \cdots, x_{m+s(p-1)}),$$

where $\deg x_i = 2i-1$ and $s=\left[\dfrac{n-m}{p-1}\right]$ is the largest integer $\le \dfrac{n-m}{p-1}$,

ii) $$f_m{}^*(x_i) = h_i .$$

Proof. We choose a prime as in Lemma 3.2. Let

$$g_m : SU(n)\to SU(n)$$

be the composition of $\lambda_{q,i}$, $2\le i\le n$, $i\not\equiv m \pmod{p-1}$. Let $q_1, q_2, \cdots$ be all primes except p and let $d_k = q_1{}^k\cdots q_k{}^k$. Let $r_k(x) = x^{d_k}$, $x\in SU(n)$. Consider the sequence

$$SU(n)\xrightarrow{r_1}SU(n)\xrightarrow{g_m}SU(n)\xrightarrow{r_2}SU(n)\xrightarrow{g_m}SU(n)\xrightarrow{r_3}\cdots .$$

Let $\widetilde{X}_m(n)$ be the telescope of the sequence. As is easily seen,

$$r_k{}^*(h_i) = d_k h_i$$

and by Theorem 2.1 and by Lemma 3.1

$$g_m{}^*(h_i) = \prod_{\substack{2\leq t\leq n \\ t\not\equiv m \pmod{p-1}}} (q^i - q^t)\, h_i\,.$$

If $i \not\equiv m \pmod{p-1}$, $c = \prod_{\substack{2\leq t\leq n \\ t\not\equiv m \pmod{p-1}}} (q^i - q^t) = 0$ and if $i \equiv m \pmod{p-1}$ then $c \not\equiv 0 \pmod{p}$ by Lemma 3. 2. From this we see easily that

$$H^*(\widetilde{X}_m(n); Z_p) \cong \Lambda(y_m, y_{m+p-1}, \cdots, y_{m+s(p-1)})$$

and

$$H^*(\widetilde{X}_m(n); Q) \cong \Lambda(y'_m, y'_{m+p-1}, \cdots, y'_{m+s(p-1)}),$$

where $\deg(y_i) = \deg(y_i') = 2i-1$ and $s = \left[\dfrac{n-m}{p-1}\right]$. Since $X_m(n)$ is 1-connected, we see that $\widetilde{X}_m(n)$ is finite p-local and is a mod 0 H-space. Then by Proposition 1. 4, there exists a finite complex $X_m(n)$ (unique up to p-equivalence) such that

$$X_m(n)_p \simeq \widetilde{X}_m(n).$$

Let $j: SU(n) \to \widetilde{X}_m(n)$ be the canonical inclusion map and let $l: X_m(n) \to X_m(n)_p \simeq \widetilde{X}_m(n)$ be the localization map. Note that $\pi_i(X_m(n)_p, X_m(n))$ is a torsion group without elements of order p. Then by the obstruction theory, we see that the map j can be compressed into $X_m(n)$ after composing $k_l: SU(n) \to SU(n)$ for some l, i.e., there is a map

$$f_m: SU(n) \to X_m(n)$$

such that the diagram

$$\begin{array}{ccc} SU(n) & \xrightarrow{j} & \widetilde{X}_m(n) \\ \uparrow{\scriptstyle k_l} & & \uparrow{\scriptstyle l} \\ SU(n) & \xrightarrow{f_m} & X_m(n) \end{array}$$

is homotopy commutative. Note that k_l is a p-equivalence and $(k_l)^*(h_i) = d_l h_i$. Then by Lemma 1. 6,

$$H^*(X_m(n); Z_p) \cong \Lambda(x_m, x_{m+p-1}, \cdots, x_{m+s(p-1)})$$

and

$$f_m{}^*(x_i) = c h_i, \quad c \not\equiv 0 \pmod{p}. \qquad \text{Q.E.D.}$$

Although the mod p splitting of $SU(n)$ follows immediately from the above proposition, we state it in a slightly different form. Put

$$f=\prod_{m=2}^{p} f_m : SU(n) \to \prod_{m=2}^{p} X_m(n),$$

where $X_m(n)=*$ if $m>n$. By the above proposition f is a p-equivalence. Since $SU(n)$ is p-universal, there exists a converse p-equivalence

$$g : \prod_{m=2}^{p} X_m(n) \to SU(n)$$

by [13]. Let g_m denote the composite $X_m(n) \xrightarrow{\subset} \prod_{i=2}^{p} X_i(n) \xrightarrow{g} SU(n)$. The construction of $X_m(n)$ depends on a choice of the map ψ^q. $BSU(n) \to BSU(n)$. (Note that ψ^q is not uniquely determined). However we have

Proposition 3.4. *Let $Y_m(n)$ be a finite complex ($2\leq m\leq p$) and let $a_m: Y_m(n) \to SU(n)$ be a map such that $H^*(Y_m(n); Z_p) \cong \Lambda(y_m, y_{m+p-1}, \cdots, y_{m+s(p-1)})$, $s=\left[\dfrac{n-m}{p-1}\right]$ and a_m^*: $H^*(SU(n); Z_p) \to H^*(Y_m(n); Z_p)$ is an epimorphism. Then*

i) *$Y_m(n)$ is p-equivalent to $X_m(n)$,*

ii) *if $m+s(p-1)\leq n\leq n'<m+(s+1)(p-1)$, then $Y_m(n)$ is p-equivalent to $Y_m(n')$,*

iii) *there exists a sequence*

$$Y_m(m+s(p-1)) \to Y_m(m+(s+1)(p-1)) \to S^{2m+2(s+1)(p-1)-1}$$

which is p-equivalent to a fibring.

Proof. Let $m+s(p-1)\leq n<m+(s+1)(p-1)$. Let $g_m': X_m(m+s(p-1)) \xrightarrow{g_m} SU(m+s(p-1)) \to SU(n)$ be the composite and let

$$g': X_m(m+s(p-1)) \times \prod_{i\neq m} X_i(n) \xrightarrow{g_m'\times\prod_{i\neq m} g_i} \prod SU(n) \xrightarrow{\mu} SU(n),$$

where μ is the multiplication. Similarly consider

$$g'': Y_m(n) \times \prod_{i\neq m} X_i(n) \xrightarrow{a_m\times\prod_{i\neq m} g_i} \prod SU(n) \xrightarrow{\mu} SU(n).$$

Then clearly g' and g'' are p-equivalences. Since all spaces in the above are p-universal, we see easily that $Y_m(n)$ is p-equivalent to $X_m(m+s(p-1))$. This proves i) and ii).

Next let

$$SU(m+p-2+s(p-1)) \xrightarrow{i} SU(m+(s+1)(p-1)) \xrightarrow{\pi} S^{2m+2(s+1)(p-1)-1}$$

be the usual fibring. Let π' be the composite

$$X_m(m+(s+1)(p-1)) \xrightarrow{g_m} SU(m+(s+1)(p-1)) \xrightarrow{\pi} S^{2m+2(s+1)(p-1)-1}$$

and let F be the fibre of π'. Then we have a homotopy commutative diagram:

$$\begin{array}{ccccc} SU(m+p-2+s(p-1)) & \xrightarrow{i} & SU(m+(s+1)(p-1)) & \xrightarrow{\pi} & S^{2m+2(s+1)(p-1)-1} \\ \uparrow a & & \uparrow g_m & & \| \\ F & \xrightarrow{i'} & X_m(m+(s+1)(p-1)) & \xrightarrow{\pi'} & S^{2m+2(s+1)(p-1)-1} \end{array}$$

where a is an induced map. Then easily we see that $a:F\to SU(m+p-2+s(p-1))$ satisfies the condition of the proposition and by i) and ii) we have

$$F \underset{p}{\simeq} X_m(m+s(p-1)).$$

This completes the proof. Q.E.D.

Now by the above proposition, if $m+s(p-1)<n<m+(s+1)(p-1)$, $X_m(n)$ uniquely determines a mod p homotopy type which we denote by $B_{m-1}^{s+1}(p)$ by an abuse of notation (cf. § 5). Then from the above argument, we obtain

Theorem 3.5. *Let p be a prime and let $1\leq m<p$ be an integer. Then for any positive integer k, there exists a space $B_m{}^k(p)$ and there exists a p-equivalence*

$$\prod_{1\leq m<p} B_m^{k(n,m)}(p) \to SU(n),$$

where $k(n,m)=\left[\dfrac{n-m-1}{p-1}\right]$.

Corollary 3.6. *Let p, m and k be as above. Then the space $B_n{}^k(p)$ is a* mod p *H-space.*

Remark 3.7. In § 5 there is given a slightly different definition of $B_m^k(p)$ for general m. The difference with this definition is to require the existence of a map to $SU(m+1+(k-1)(p-1))/SU(m)$ instead of $SU(n)$. In this weaker definition, it is not known in general if $B_m^k(p)$ is unique up to p-equivalence.

§ 4. Mod p Decomposition of the Other Classical Groups

In this section, p denotes always an odd prime. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. A map $s: B \to E$ is called a *cross-section* mod p if $p \circ s$ is a p-equivalence. Let E be an H-space mod p with the mod p multiplication μ. Suppose $p: E \to B$ admits a cross section mod p. Now if F, E and B are 1-connected finite CW-complexes, then $F \times B \xrightarrow{i \times s} E \times E \xrightarrow{\mu} E$ gives a p-equivalence by Serre's class theory.

Consider the canonical bundles associated with the classical groups:

$$Sp(n) \to SU(2n) \to SU(2n)/Sp(n),$$

$$Spin(2n+1) \to SU(2n+1) \to SU(2n+1)/Spin(2n+1),$$

$$Spin(2n-1) \to Spin(2n) \to S^{2n-1}.$$

B. Harris [6] has shown that such bundles have cross-section mod p for odd p. Hence we have p-equivalences:

$$Sp(n) \times (SU(2n)/Sp(n)) \underset{p}{\simeq} SU(2n),$$

$$Spin(2n+1) \times (SU(2n+1)/Spin(2n+1)) \underset{p}{\simeq} SU(2n+1),$$

$$Spin(2n-1) \times S^{2n-1} \underset{p}{\simeq} Spin(2n).$$

It is also shown in [6] that $Sp(n) \underset{p}{\simeq} Spin(2n+1)$.

Theorem 4.1. *Let p be an odd prime. Let $k_{a,b} = \left[\dfrac{2(a-b)}{p-1}\right] + 1$. Then there exist the following p-equivalences*:

$$Sp(n) \underset{p}{\simeq} Spin(2n+1) \simeq \prod_{m=1}^{(p-1)/2} B_{2m-1}^{k_{n,m}}(p),$$

$$Spin(2n) \underset{p}{\simeq} S^{2n-1} \times \prod_{m=1}^{(p-1)/2} B_{2m-1}^{k_{n-1,m}}(p),$$

$$SU(2n)/Sp(n) \underset{p}{\simeq} \prod_{m=1}^{(p-1)/2} B_{2m}^{k_{n-1,m}}(p),$$

$$SU(2n+1)/Spin(2n+1) \underset{p}{\simeq} \prod_{m=1}^{(p-1)/2} B_{2m}^{k_n,m}(p).$$

Proof is straightforward from Theorem 3.5 by virtue of the above formula and will be left to the reader.

Theorem 4.2. *$SU(n)$ has no* mod p *decomposition into p spaces. Let p be odd, then $Sp(n)$ and $Spin(2n+1)$ have no* mod p *decomposition into $\frac{p+1}{2}$ spaces. $Sp(n)$ has no* mod 2 *decomposition into 2 spaces.*

Proof. Assume that $SU(n)$ is mod p decomposable into p spaces, i.e., $\prod_{i=1}^{p} X_i \underset{p}{\simeq} SU(n)$. It is easy to see that $H^*(X_i; Z_p)$ is an exterior algebra and hence there exists a number t such that the degree of the lowest generator of $H^*(X_t; Z_p)$ is greater than $2p+1$. Let x be such a generator and let $k=\deg x$. Then clearly the mod p Hurewicz homomorphism $h: \pi_k(\prod X_i) \otimes Z_p \to H_k(\prod X_i; Z_p)$ is non-trivial. Hence so is $h: \pi_k(SU(n)) \otimes Z_p \to H_k(SU(n); Z_p)$. But since $k \geq 2p+1$, this is a contradiction. For $Sp(n)$ and $Spin(2n+1)$, the proof is quite similar. Q.E.D.

Chapter II Mod p Stiefel Complex $B_m^k(p)$

§ 5. Existence of $B_m^k(p)$

Throughout this chapter p will be an odd prime and we use the notation $q=2(p-1)$.

Definition 5.1. i) A map $f: X \to Y$ is called a mod p *injection* (resp. a mod p *surjection*) if f induces an epimorphism (resp. a monomorphism) $f^*: H^*(Y; Z_p) \to H^*(X; Z_p)$.

ii) A complex B is called to be *of p-type* $(n_1, n_2, \cdots, n_l)$ and indicated by $B=B(n_1, n_2, \cdots, n_l)$ if

$$H^*(B; Z_p) = \Lambda(x_{n_1}, \cdots, x_{n_l}),$$

where $\deg x_{n_i} = n_i$, $i=1, \cdots, l$, and each cell of B represents an additive base of $H^*(B; Z_p)$.

Note that

(5. 1) *if* X *is* 1-*connected and* $H^*(X; Z_p) = \Lambda(x_{n_1}, \cdots, x_{n_l})$ *for* $\dim \leq n_1 + \cdots + n_l$, *then there is a complex* B *of* p-*type* $(n_1, \cdots, n_l)$ *with a* mod p *injection* $f: B \to X$.

Definition 5. 2. We call a complex $B_m^k(p)$ of p-type $(2m+1, 2m+1+q, \cdots, 2m+1+(k-1)q)$, $q=2(p-1)$, a mod p *Stiefel complex* if there exists a mod p injection

$$f: B_m^k(p) \to SU(m+1+(k-1)(p-1))/SU(m) = W_{m+s,s},$$

where $s = 1+(k-1)(p-1)$.

Note that

(5. 2) $H^*(W_{m+s,s}; Z) = \Lambda(x_{2m+1}, x_{2(m+1)+1}, \cdots, x_{2(m+s-1)+1})$ *and in* Z_p-*coefficient we can choose the generators* $x_{2(m+j)+1}$ *such that* $f^*(x_{2(m+j)+1}) = 0$ *for* $j \not\equiv 0 \pmod{p-1}$ *and* $f^*(x_{2m+1+iq}) = x_{2m+1+iq}$ *for* $0 \leq i < k$.

Example 5. 3. 1. $B_m^1(p) = S^{2m+1}$.

Example 5. 3. 2. $B_m^2(p) = S^{2m+1} \times S^{2m+2p-1}$ if $m \equiv 0 \pmod p$ and $B_m^2(p) = B_m(p)$ if $m \not\equiv 0 \pmod p$, where $B_m(p)$ is a S^{2m+1}-bundle over $S^{2m+2p-1}$ given in [11, 18].

Example 5. 3. 3. $B_{2n+1}^k(3) = X_{n+k,k} = Sp(n+k)/Sp(n)$ with the natural map $f: B_{2n+1}^k(3) \to SU(2n+2k)/SU(2n+1)$.

Example 5. 3. 4. By Theorem 3. 5, for $1 \leq m \leq p$ there exists $B_m^k(p)$ with

$$f: B_m^k(p) \to SU(m+1+(k-1)(p-1)) \to SU(m+1+(k-1)(p-1))/SU(m).$$

In meta-stable ranges we have the following theorems.

Theorem 5.4. *If $k(2m+1+(k-1)(p-1))<2(m+1)p-2$, there exists $B_m{}^k(p)$ uniquely up to p-equivalence.*

Theorem 5.5. *If*

(α) $(k-1)(2m+1+k(p-1))<2(m+2)p-2$

or

(β) *m is odd and* $(k-1)(2m+1+k(p-1))<2(m+3)p-2$,

there exists a $B_m{}^k(p)$ which is a S^{2m+1}-bundle over $B^{k-1}_{m+p-1}(p)$.

First we recall James' work [8]. For each positive integer s, there associates a James number $b=b_s$ such that for all positive integers n and N, there are maps

$$j_i : S^{2Nb}W_{n+i,i}\rightarrow W_{n+i+Nb,i} \quad \text{for} \quad i\leq s$$

satisfying the commutativity of the diagram

$$\begin{array}{ccccc}
S^{2Nb}W_{n+i-1,i-1} & \xrightarrow{S^{2Nb}i} & S^{2Nb}W_{n+i,i} & \xrightarrow{S^{2Nb}p} & S^{2(n+i+Nb)-1} \\
\downarrow j_{i-1} & & \downarrow j_i & & \downarrow j_1=id \\
W_{n+i+Nb-1,i-1} & \xrightarrow{i} & W_{n+i+Nb,i} & \xrightarrow{p} & S^{2(n+i+Nb)-1}
\end{array}$$

where $W_{t+i,i}=SU(t+i)/SU(t)$ and $\xrightarrow{i}$, $\xrightarrow{p}$ are the natural fiberings.

We put

$$K_{t+i,i}=S(CP^{t+i-1}/CP^{t-1}).$$

According to Yokota [25], $K_{t+i,i}$ is embedded in $W_{t+i,i}$ such that $(W_{t+i,i}, K_{t+i,i})$ is $(4t+3)$-connected and $H^*(K_{t+i,i})$ has an additive base $\{y_{2(t+j)-1}\}$ for the restrictions $y_{2(t+j)-1}$ of the generators $x_{2(t+j)-1}$ of $H^*(W_{t+i,i})$.

So, by taking N sufficiently large, we have the following homotopy commutative diagram:

$$\begin{array}{ccccc}
S^{2Nb}W_{n+i-1,i-1} & \xrightarrow{S^{2Nb}i} & S^{2Nb}W_{n+i,i} & \xrightarrow{S^{2Nb}p} & S^{2(n+i+Nb)-1} \\
\downarrow j_{i-1} & & \downarrow j_i & & \downarrow j_1=id \\
K_{n+i+Nb-1,i-1} & \xrightarrow{i} & K_{n+i+Nb,i} & \xrightarrow{p} & S^{2(n+i+Nb)-1}
\end{array}$$

in which the lower sequence is a cofibering.

Using (5. 2) and the above diagram inductively we have

(5. 3) *$j_s: S^{2Nb}W_{n+s,s}\to K_{n+s+Nb,s}$ and the adjoint map $J_s: W_{n+s,s}\to \Omega^{2Nb}K_{n+s+Nb,s}$ satisfy $j_s^*(y_{2(n+i+Nb)+1})=S^{2Nb}x_{2(n+i)+1}$ and $J_s^*(\sigma^{2Nb}y_{2(n+i-Nb)+1})=x_{2(n+i)+1}$. Thus j_s is a* mod p *surjection and J_s is a* mod p *injection.*

By Corollary 9. 5 of [16]

(5. 4) *we have a p-equivalence*

$$g': K_{n+s+Nb,s}\to\bigvee_{i=0}^{p-2} L_i\,,$$

where $L_i=S^M\cup e^{M+q}\cup\cdots\cup e^{M+hq}$ for $M=M_i=2(n+i+Nb)+1$ and $h=h_i=\left[\dfrac{s-i-1}{p-1}\right]$.

From (5. 2), (5. 3) and (5. 4) we remark that

(5. 5) *there is a* mod p *surjection $S^{2N}B_m{}^k(p)\to L_0$ for sufficiently large N, given the existence of $B_m{}^k(p)$, where $L_0=S^{2m+2N+1}\cup e^{2m+2N+1+q}\cup\cdots\cup e^{2m+2N+1+(k-1)q}$.*

An easy calculation, using Corollary 3. 3 of [24], shows

(5. 6) *$H^*(\Omega^{2Nb}L_i; Z_p)=\Lambda(x_{2n+2i+1}, x_{2n+2i+1+q}, \cdots, x_{2n+2i+1+hq})$ for* $\dim<2(n+i+1)p-2$,

and for the composite map

$$g=g'\circ J_s\colon W_{n+s,s}\to\Omega^{2Nb}(\bigvee_{i=0}^{p-2}L_i),$$

$$(5.7)\qquad g^*: H^*(\Omega^{2Nb}(\bigvee_{i=0}^{p-2}L_i); Z_p)\to H^*(W_{n+s,s}; Z_p)$$

are isomorphisms for $\dim<2(n+1)p-2$.

Then it follows from (5. 1) that

(5.8) *if* $(h+1)(2n+2i+1+h(p-1))<2(n+i+1)p-2$ *there exist a complex* B_i *of type* $(2n+2i+1, 2n+2i+1+q, \cdots, 2n+2i+1+hq)$ *and a* mod p *injection*

$$f_i : B_i \to \Omega^{2Nb} L_i .$$

Proof of Theorem 5.4. Let $n=m$, $s=1+(k-1)(p-1)$ and consider the following diagram:

$$\begin{array}{ccc} W_{m+s,s} & \xrightarrow{\ g\ } & \Omega^{2Nb}(\bigvee_{i=0}^{p-2} L_i) \\ \uparrow f & & \uparrow j_0 \\ B_0 & \overset{h}{\cdots\cdots\to} & B_0 \end{array}$$

where j_0 is the composite map of $f_0 : B_0 \to \Omega^{2Nb} L_0$ and the inclusion $\Omega^{2Nb} L_0 \to \Omega^{2Nb}(\bigvee_i L_i)$. By the assumption, $\dim B_0 < 2(m+1)p-2$. By Corollary 1.4 of [13], B_0 is p-universal. Applying Theorem 2.1 of [15], we have that there exist a p-equivalence h and a mod p injection f such that the above diagram homotopy commutes. Thus $B_0 = B_m^{\ k}(p)$.

Given a $B_m^{\ k}(p)$ with a mod p injection $f: B_m^{\ k}(p) \to W_{m+s,s}$, consider the composition

$$\bar{g} : B_m^{\ k}(p) \xrightarrow{\ f\ } W_{m+s,s} \xrightarrow{\ g\ } \Omega^{2Nb}(\bigvee_i L_i) \xrightarrow{\text{proj}} \Omega^{2Nb} L_0 .$$

It is easy to see that $\bar{g}^* : H^*(\Omega^{2Nb} L_0; Z_p) \to H^*(B_m^{\ k}(p); Z_p)$ are isomorphisms for $\dim < 2(m+1)p-2$. Apply the above dicussion to $\bar{g}$, in place of g, then we get a mod p injection $\bar{f}: B_0 \to B_m^{\ k}(p)$ which is a p-equivalence.

Q.E.D.

Proof of Theorem 5.5. First consider the case (α). Let $n=m+1$, $s=(k-1)(p-1)$ and apply Theorem 2.1 of [15] to the diagram

$$\begin{array}{ccc} W_{m+1+s,s} & \xrightarrow{\ g\ } & \Omega^{2Nb}(\bigvee_i L_i) \\ \uparrow f' & & \uparrow j_{p-2} \\ B_{p-2} & \overset{h'}{\cdots\cdots\to} & B_{p-2} . \end{array}$$

The condition (α) is equivalent to $\dim B_{p-2} < 2(m+2)p-2$. So, we get a mod p injection $f' : B_{p-2} \to W_{m+1+s,s}$. By composing the projection $W_{m+1+s,s} \to W_{m+1+s,s-p+2}$, we see that $B_{p-2} = B_{m+p-1}^{k-1}(p)$. Consider the sphere bundle

$S^{2m+1} \to W_{m+1+s,1+s} \to W_{m+1+s,s}$ and let B be the total space of the sphere bundle induced by f'. Then B is an S^{2m+1}-bundle over $B^{k-1}_{m+p-1}(p)$ with the induced map $f: B \to W_{m+1+s,1+s}$. Obviously f is a mod p injection and $B = B_m{}^k(p)$.

For the case (β), let $s' = s/2$, $m' = (m+1)/2$ and use the composition

$$g' : X_{m'+s',s'} = Sp(m'+s')/Sp(m') \to W_{m+1+s,s}$$
$$\xrightarrow{g} \Omega^{2Nb}(\bigvee_{i=0}^{p-2} L_i) \xrightarrow{\text{proj}} \Omega^{2Nb}(\bigvee_{j=1}^{(p-1)/2} L_{2j-1})$$

in place of g. We see that g'^* are isomorphisms of $H^*(\ ; Z_p)$ for dim $< 2(m+3)p-2$. Then the case (β) is proved similarly. Q.E.D.

A slight generalization of Theorems 5. 4, 5. 5 may be obtained in unstable ranges, as will be seen in the proof of Proposition 5. 6.

For small values of k we have

Proposition 5. 6. (i) *$B_m{}^k(p)$ exists if $k=1$, $k=2$, $k=3$.*
(ii) *$B_m{}^4(p)$ exists if $p>3$ or if $p=3$ and m is odd.*

Proof. By Proposition 3. 4, $B_m{}^k(p)$ exists for $m<p$. So, we may assume that $m \geq p$.

For $k=3$, the condition (α) of Theorem 5. 5 is equivalent to $m > \frac{p-1}{p-2}$ which is satisfied for $m \geq p \geq 3$. Thus $B_m{}^3(p)$ exists. The existence of $B_m{}^1(p)$ and $B_m{}^2(p)$ is easily proved.

For $k=4$, the conditions (α) and (β) of Theorem 5. 5 are equivalent to $m > \frac{4p-3}{p-3}$ and $m > \frac{3p-3}{p-3}$ respectively. Thus $B_m{}^4(p)$ exists for $p>5$ and for $p=5$ and $m \neq 5, 6, 8$. For odd m, Example 5. 3. 3 shows the existence of $B_m{}^4(3)$.

Consider the construction of $B_8{}^4(5)$ along the proof of Theorem 5. 5. Then the only difficulty is to extend a map f of $B^3_{12}(5)$ into $W_{21,12} = SU(21)/SU(9)$ over the top cell of $B^3_{12}(5)$, and the obstruction lies in the kernel of $S^{2Nb}: \pi_{98}(S^{19}) \to \pi_{98+2Nb}(S^{19+2Nb})$. By [24], this S^{2Nb} is a mod 5 injection. Thus $B_8{}^4(5)$ can be constructed. $B_5{}^4(5)$ is similarly constructed by using $X_{9,6} = Sp(9)/Sp(3)$.

Finally, the obstruction to construct $B_6{}^4(5)$ lies in the kernel of

$g_*: \pi_{86}(SU(9)/SU(7)) \to \pi_{86}(\Omega^{2Nb}(S^{15+2Nb} \vee S^{17+2Nb}))$. Here $SU(9)/SU(7)$ is p-equivalent to $S^{15} \times S^{17}$ and g_* is equivalent to $S^{2Nb} \times S^{2Nb}$ on $\pi_{86}(S^{15}) \times \pi_{86}(S^{17})$ which is mod 5 injective by [24]. Thus $B_6^4(5)$ can be constructed.

Q.E.D.

§ 6. A Spectral Sequence for Meta-stable Homotopy

In this section all homotopy groups $\pi_i(X)$ are localized at p and considered on 1-connected spaces of finite type. We use the notations

$$\pi_i(X: p) = \pi_i(X) \otimes Z_{(p)},$$

$$(G_i: p) = G_i \otimes Z_{(p)} \quad \text{for} \quad G_i = \lim_n \pi_{i+n}(S^n),$$

where

$$Z_{(p)} = \left\{ \frac{n}{m} \,\middle|\, m, n \in Z;\ (m, p) = 1 \right\}.$$

Let n be an odd integer ≥ 3. We shall give a functor which associates a spectral sequence $\{E^r_{s,t}\}$ for each n-connected map $f: B \to X$, satisfying the following properties.

(6. 1) (i) *$E^1_{s,t} = 0$ for $s < n$, $E^1_{n,t} = \pi_{n+t}(B: p)$ and $E^1_{s,t} = \pi_{n+t}(B_s: p)$ for $s > n$, where $B_s = \vee S^n$ is a bouquet of n-spheres.*

(ii) *With respect to the differential $d^r: E^r_{s,t} \to E^r_{s-r,t+r-1}$*

$$d^r d^r = 0 \quad \textit{and} \quad H(E^r_{s,t}) \cong E^{r+1}_{s,t}.$$

(iii) *$E^r_{s,t} = E^\infty_{s,t}$ for $r > \mathrm{Max}\,(s-n, t+1)$. $E^\infty_{s,t} = D_{s,t}/D_{s-1,t+1}$ and $D_{n,t} = E^\infty_{n,t} = \mathrm{Im}\,(f_*: \pi_{n+t}(B: p) \to \pi_{n+t}(X: p))$ for a filtration*

$$\pi_{s+t}(X: p) = D_{s+t,0} \supset \cdots \supset D_{s,t} \supset D_{s-1,t+1} \supset \cdots \supset D_{n,s+t-n}.$$

(iv) *$d^1(E^1_{s,t}) \subset p \cdot E^1_{s-1,t}$ for $s-1 > n$.*

(v) *For each $\gamma \in \pi_{n+u}(S^n)$, there associates a map (composition) $\cdot \gamma: E^r_{s,t} \to E^r_{s,t+u}$ of the spectral sequences such that $\beta \cdot \gamma = \beta \circ (S^t \gamma)$ for $\beta \in \pi_{n+t}(B_s: p) = E^1_{s,t}$ and similar for $\beta \in D_{s,t} \subset \pi_{s+t}(X: p)$.*

(vi) *A spectral sequence converging to $\pi_{s+t}(X, B: p)$ is obtained by taking $E^r_{n,t} = D_{n,t} = 0$.*

Let $N = \{n_1, n_2, \cdots\}$ be a strictly increasing sequence of positive integers $n_i>3$, and let

$$n=n_1-1 \textit{ if } n_1 \textit{ is even and } n=n_1-2 \textit{ if } n_1 \textit{ is odd.}$$

Theorem 6.1. *For the map $f: B\to X$ assume that*

$$H^*(X; Z_p)=B^*\otimes\Lambda(x_{n_i}: n_i \text{ is odd})\otimes Z_p[x_{n_i}: n_i \text{ is even}]$$

for $\dim<M+1\leq p(n+1)$, $f^*(x_{n_i})=0$ *and* $f^*|B^*$ *is an isomorphism of* B^* *onto* $H^*(B; Z_p)$. *Then the above spectral sequence* $\{E^r_{s,t}\}$ *satisfies the following properties.*

(i) *Let* $s<M$, *then* $B_s=S^n$ $(s\in N)$ *and* $B_s=*$ $(s\notin N)$.

(ii) *Let* $\beta\in E^r_{s,t}$ *for* $1\leq r\leq q$, $s<M$ *and* $s, s-r\in N$. *If* $r=1$, $d^1(\beta)=a\beta$ *for an integer* $a\equiv 0 \pmod p$ *satisfying* $(\delta/a)x_{s-1}=x_s$ (mod *decomp.*) *for* $a\neq 0$. $d^r(\beta)=0$ *if* $1<r<q$. *If* $r=q$, $d^q(\beta)=b\alpha_1\cdot\beta$ *for some* $b\in Z_p$ *satisfying* $\mathcal{P}^1x_{s-q}=bx_s$ (mod *decomp.*), *where* $\alpha_1\in\pi_{n+q-1}(S^n: p)$ *is detected by* $\mathcal{P}^1$.

In the following (iii) *and* (iv), *we assume that* $s, s-r\in N$, $s<M$, $\alpha\circ S^{r-1}\beta=\beta\circ S^t\gamma=0$ *for* $\alpha\in\pi_{n+r-1}(S^n: p)=E^1_{s-r,r-1}$, $\beta\in\pi_{n+t}(S^n: p)=E^1_{s,t}$, $\gamma\in\pi_{n+u}(S^n)$, $d^r(\iota)=\alpha$ *for the identity class* $\iota\in\pi_n(S^n: p)=E^1_{s,0}$ *and* ξ *indicates a suitably chosen element of*

$$\{\alpha, S^{r-1}\beta, S^{r+t-1}\gamma\}\subset\pi_{n+t+r+u}(S^n)=E^1_{s-r,t+r+u}.$$

Moreover we assume $E^1_{s-i,t+i}=0$ *for* $1\leq i<r$.

(iii) *If* $[\beta]\in\pi_{s+t}(X: p)$ *is represented by* $\beta\in E^\infty_{s,t}$, *then* $[\beta]\cdot\gamma\in\pi_{s+t+u}(X: p)$ *is represented by* $\xi\in E^\infty_{s-r,t+r+u}$.

(iv) *If* $d^{r'}(\delta)=\beta$ *for some* $\delta\in E^{r'}_{s+r',t-r'+1}$, *then* $d^{r+r'}(\delta\cdot\gamma)=\xi\in E^{r+r'}_{s-r,t+r+u}$.

(v) *If* $B=S^{2m+1}$, *by shifting* $E^r_{n,t}$ *to* $E^r_{2m+1,t}$, *the above* (ii), (iii), (iv) *are satisfied for* $s-r=2m+1$.

Corollary 6.2. *If* $\pi_k(X, B: p)\neq 0$ *for some* $k<M-1$, *then* $k=s+t$ *for some* $s\in N$ *and* t *with* $(G_t: p)\neq 0$.

The construction of the spectral sequence and the proofs of (6.1) and Theorem 6.1 will be given at the end of this section.

We shall show some applications. Recall the following results on the stable homotopy groups of spheres from [23].

(6.2) *For* $t<(2p+1)q-2$, *the groups* $(G_t:p)$ *have the following generators*:

$$(G_0:p)=Z_{(p)}\langle\iota\rangle, \qquad \iota \text{ is the class of the identity},$$
$$(G_{rq-1}:p)=Z_p\langle\alpha_r\rangle \qquad \text{for } 0<r<2p \text{ and } r\neq p,$$
$$(G_{spq-1}:p)=Z_{p^2}\langle\alpha'_{sp}\rangle \qquad \text{for } s=1,2,\ p\alpha'_{sp}=\alpha_{sp},$$
$$(G_{spq-2s}:p)=Z_p\langle\beta_1^s\rangle \qquad \text{for } s=1,\ 2,$$
$$(G_{(sp+1)q-2s-1}:p)=Z_p\langle\alpha_1\beta_1^s\rangle \text{ for } s=1,\ 2.$$

(6.3) $\alpha_1\alpha_r=\alpha_1\alpha'_{sp}=0$ *and* $\{\alpha_1, \alpha_r, p\iota\}$ *contains* α_r/r $(=\alpha'_{sp}/s$ *for* $r=sp)$.

A typical case of X is the following one.

(6.4) $H^*(X;Z_p)=\Lambda(x_{2m+1}, \mathcal{P}^1x_{2m+1}, \cdots, \mathcal{P}^{k-1}x_{2m+1}), \quad 1<k\leq p.$

Proposition 6.3. *For a 2-connected* X *with the property* (6.4), $\pi_t(X:p)=0$ *for* $t<\mathrm{Min}(2(m+1)p-3, 2m+(2p+1)q-1)$ *except for the following:*

$$\pi_{2m+1+jq}(X:p)=Z_{(p)}\langle[p^j\iota(2m+1+jq)]\rangle \qquad \text{for } 0\leq j<k,$$
$$\pi_{2m+jq}(X:p)=Z_{p^k}\langle[\alpha_{j-k+1}(2m+1+(k-1)q)]\rangle$$
$$\text{for } k\leq j<p \text{ and } p+k\leq j<2p,$$
$$\pi_{2m+jq}(X:p)=Z_{p^{k+1}}\langle[\alpha_{j-k+1}(2m+1+(k-1)q)]\rangle \text{ for } p\leq j<p+k-1,$$
$$\pi_{2m+(p+k-1)q}(X:p)=Z_{p^{k+1}}\langle[\alpha_p'(2m+1+(k-1)q)]\rangle,$$

with the exception

$$\pi_{2m+pq}(X:p)=Z_{p^p}\langle[\alpha_2(2m+1+(p-2)q)]\rangle \qquad \text{for } k=p,$$

and, in addition for $k<p$,

$$\pi_{2m+1+spq-2s}(X:p)=Z_p\langle[\beta_1^s(2m+1)]\rangle \qquad \text{for } s=1,\ 2,$$
$$\pi_{2m+(p+k)q-2}(X:p)=Z_p\langle[\alpha_1\beta_1(2m+1+2(k-1)q)]\rangle.$$

Here $\gamma(s)\in E^r_{s,t}$ *indicates the element whose stable class is* γ, *and* $[\gamma(s)]\in\pi_{s+t}(X:p)$ *is represented by the permanent cycle* $\gamma(s)$ $\in E^\infty_{s,t}$.

Proof. For $t<2(m+1)p-3$, $\pi_t(S^{2m+1}: p)$ is stable. We use the spectral sequence $\{E^r_{s,t}\}$ converging to $\pi_t(X: p)$ with $E^1_{s,t}=(G_t: p)$ for $s=2m+1+iq$ $(0\leq i<k)$. The first non-trivial differential is d^q and it is computed by (ii) of Theorem 6.1 and by (6.3), (6.4), and $E^{q+1}_{s,t}$ has the following generators:

$$p\iota(2m+1+iq) \quad \text{for} \quad 0\leq i<k,$$

$$\alpha_r(2m+1+iq) \quad \text{for} \quad 0\leq i<k,\ 2\leq r,\ i\not\equiv 0 (\mathrm{mod}\ p) \quad \text{and} \quad i+r\leq 2p,$$

$$\alpha'_{sp}(2m+1+iq) \quad \text{for} \quad 0\leq i<k,\ s=1,\ 2 \quad \text{and} \quad sp+i\leq 2p,$$

$$\alpha_1(2m+1+(k-1)q),\quad \beta_1(2m+1),\quad \alpha_1\beta_1(2m+1+(k-1)q)$$

and $\beta_1^2(2m+1)$.

Next we have

(6.5) $d^{fq}(p^{f-1}\iota(2m+1+iq))=\alpha_f(2m+1+(i-f)q)$ *for* $1\leq f\leq i$ *up to non-zero coefficients.*

This is true for $f=1$ by (ii) of Theorem 6.1 and proved inductively by use of (iv) of Theorem 6.1 and (6.3). For dimensional reasons and by the following lemma, the other non-trivial differentials are $d^{(p-1)q}(\alpha_1\beta_1^{s-1}(2m+1+(p-1)q))=\beta_1^s(2m+1)$, $s=1,\ 2$, of the case $k=p$. By use of (ii) of Theorem 6.1 and (6.3), we see that the groups $\pi_{2m+jq}(X: p)$ are cyclic. Consequently we obtain the required results. Q.E.D.

In the above proof, the only differentials in question are

$$d^{iq}(\alpha_{p-i}(2m+1+iq))=a\cdot\beta_1(2m+1) \qquad \text{for some } a\in Z_p.$$

The element $\beta_1(2m+1)$ is stable only if $m\geq p$.

Lemma 6.4. *Let* $m\geq p$ *and* $1\leq i\leq k$. *Then*

$$d^{iq}(\alpha_{p-i}(2m+1+iq))=0 \qquad \textit{for } i<p-1$$

and $\quad d^{(p-1)q}(\alpha_1\beta_1^{s-1}(2m+1+(p-1)q))=\beta_1^s(2m+1)$ *for* $k=p$

up to non-zero coefficients.

Proof. We shall consider a complex

$$L_m^k = S^{2m+1} \cup e^{2m+1+q} \cup \cdots \cup e^{2m+1+(k-1)q} \quad \text{with} \quad \mathcal{P}^{k-1} \neq 0$$

for $k \leq p \leq m$. Denote by L_m^j the $2m+1+(j-1)q$ skeleton of L_m^k, by $i_j: L_m^j \to L_m^{j+1}$ the inclusion and by

$$\gamma_k \in \pi_{2m+(k-1)q}(L_m^{k-1})$$

the attaching class of the top cell $e^{2m+1+(k-1)q}$. First we prove

$(6.6)_k$. *For $2 \leq k \leq p$, there exists a complex L_m^k such that $\gamma_k = \gamma_{k,k-1}$, $p \cdot \gamma_{k,j+1} = i_{j*}\gamma_{k,j} (1 \leq j < k-1)$ and $\gamma_{k,1} = \alpha_{k-1}(2m+1)$ up to non-zero coefficients, for a series of elements $\gamma_{k,j} \in \pi_{2m+(k-1)q}(L_m^k)$.*

This is obvious for $k=2$, $L_m^2 = S^{2m+1} \cup e^{2m+1+q}$ the mapping cone of $\alpha_1(2m+1)$. Assume that $(6.6)_{k-1}$ holds for $k>2$ and consider L_{m+q-1}^{k-1}. We define L_m^k as the mapping cone of a map $f: SL_{m+q-1}^{k-1} \to S^{2m+1}$ such that $f|SL_{m+q-1}^1 = f|S^{2m+2q}$ represents $\alpha_1(2m+1)$. The existence of such a map f follows from $\pi_{2m+jq-1}(S^{2m+1}:p) = (G_{jq-2}:p) = 0$ for $1<j<p$. The attaching class γ_k is given by a coextension of the attaching class $S\gamma_{k-1}$ for SL_{m+q-1}^{k-1}. If $\gamma_{k,j+1}$ is given by a coextension of $S\gamma_{k-1,j}$, $p \cdot \gamma_{k,j+1}$ is a coextension of $pS\gamma_{k-1,j} = Si_{j-1*}S\gamma_{k-1,j-1}$. Then $p \cdot \gamma_{k,j+1} = i_{j*}\gamma_{k,j}$ for a coextension $\gamma_{k,j}$ of $S\gamma_{k-1,j-1}$. Since $\gamma_{k,2}$ is a coextension of $S\gamma_{k-1,1} = \gamma_{k-2}$ in the mapping cone L_m^2 of $\alpha_1(2m+1)$, $p \cdot \gamma_{k,2} = i_{1*}\{\alpha_1(2m+1), \alpha_{k-2}, p\iota\} = i_{1*}(\alpha_{k-1}(2m+1))$ by (6.3). Thus $(6.6)_k$ is proved by induction on k.

Next we show

(6.7) *Let $j_k: S^{2m+1} \to L_m^k$ be the inclusion. Then $j_{p*}(\beta_1(2m+1)) = 0$ and $j_{k*}(\alpha_p'(2m+1))$ is divisible by p^{k-1} if $k<p$.*

Consider the map $f: SL_{m+q-1}^{p-1} \to S^{2m+1}$ and try to extend f over SL_{m+q-1}^p. The obstruction to the extension is

$$f_*(S\gamma_p) \in \pi_{2m+pq-1}(S^{2m+1}:p) = Z_p\langle \beta_1(2m+1) \rangle.$$

Assume that $f_*(S\gamma_p) = 0$, then f can be extended over $\bar{f}: SL_{m+q-1}^p \to S^{2m+1}$. Then in the mapping cone of $\bar{f}$ we have $\mathcal{P}^1\mathcal{P}^{p-1} \neq 0$ which contradicts to the Adem relation $\mathcal{P}^1\mathcal{P}^{p-1} = 0$. Thus $f_*(S\gamma_p) \neq 0$, and in the mapping cone L_m^p of f, $j_{p*}(\beta_1(2m+1)) = 0$.

It follows from $(6.6)_p$ that $f_*(S\gamma_{p,k-1}) = p^{p-k}i_{1*}(\beta_1(2m+1)) = 0$ for

$k<p$. Thus there exists a coextension $\xi_k \in \pi_{2m+pq}(L_m^k)$ of $S\gamma_{p,k-1}$. Then $p^{k-2}\xi_k = i_*'\eta$ for a coextension η of $\gamma_{p,1} = \alpha_{p-1}$ and for the inclusion $i' : L_m^2 \to L_m^k$. By (6. 3), $p\cdot\eta = i_{1*}\{\alpha_1(2m+1), \alpha_{p-1}, p\iota\} = i_{1*}(\alpha_p'(2m+1))$. Thus $p^{k-1}\xi_k = i_*'(p\cdot\eta) = j_{k*}(\alpha_p'(2m+1))$, which completes the proof of (6. 7).

Now we go back to the proof of the lemma. A mod p injection $S^{2m+1} = L_m^1 \to X$ can be extended over a map $g: L_m^k \to X$ since $\pi_{2m+iq}(X:p) = 0$, $1 \le i < k$, as is seen in the proof of Proposition 6. 3. Since $\alpha_p'(2m+1)$ is divisible by p^{k-1}, so is it in X, and $\pi_{2m+pq}(X:p)$ contains a cyclic group of order p^{k+1}. By counting the orders of generators in $E^1_{s,2m+pq}$, we have that $\alpha_{p-i}(2m+1+iq)$, $i<p-1$, are permanent cycles for $k<p$.

In the case $k=p$, $\beta_1(2m+1)$ vanishes in L_m^p, so does it in X and E^∞, and $d^{iq}(\alpha_{p-i}(2m+1+iq)) = \beta_1(2m+1)$ for some i. If $i<p-1$, $\alpha_1(2m+1+(p-1)q)$ is a permanent cycle. Then by use of (iii) of Theorem 6. 1 and (6. 3), we see that $\alpha_j(2m+1+(p-j)q)$ are permanent cycles successively for $j=1, 2, \cdots, p-1$. This contradicts to the above result for $j=p-i$. Thus $d^{(p-1)q}(\alpha_1(2m+1+(p-1)q)) = \beta_1(2m+1)$. By (iv) of Theorem 6. 1, $d^{(p-1)q}(\alpha_1\beta_1(2m+1+(p-1)q)) = \beta_1^2(2m+1)$. Q.E.D.

Remark 6. 5. For the case $H^*(X; Z_p) = Z_p[x_{2m+2}, \mathcal{P}^1 x_{2m+2}, \cdots, \mathcal{P}^{k-1} x_{2m+2}]$ for $\dim < 2p(m+1)$, a result parallel to Proposition 6. 3 holds by considering SL_m^k or ΩX. In general $H^*(X; Z_p)$ may be the tensor product of some subalgebras of the above type and the type of (6. 4), and the discussions of differentials in the proof of Proposition 6. 3 are valid. In particular, Lemma 6. 4 may be applicable provided there exists a mod p injection $g: L_m^k \to X$.

If the connectedness is lower, such as $B_1^k(p)$, Proposition 6. 3 is not so useful. In such a case it is convenient to consider the fiber F_k of the mod p injection $B_1^k(p) \to B_1^\infty(p)$ because

(6. 8) $\pi_t(B_1^\infty(p):p) = Z_{(p)}$ *for* $t \equiv 3 \pmod q$ *and* $=0$ *for* $t \not\equiv 3 \pmod q$.

Then we have

(6. 9) $H^*(F_k; Z_p) = Z_p[x_{2+kq}, x_{2+(k+1)q}, x_{2+(k+2)q}, \cdots]$ *for* $\dim < p(2+kq) - 2$ *and* $\mathcal{P}^1 x_{2+jq} = (j-1)x_{2+(j+1)q}$.

Computing $\pi_t(F_k:p)$ and applying the exact sequence

$$\cdots\to\pi_{t+1}(B_1^{\infty}(p))\to\pi_t(F_k)\to\pi_t(B_1^k(p))\to\pi_t(B_1^{\infty}(p))\to\cdots$$

and also using the fact that $\pi_t(B_1^k(p))$ is finite for $t>3+(k-1)q$, we have the following

Proposition 6.6. *Let* $2\leq k\leq p+1$. For $t<\mathrm{Min}((kp+1)q-1, (2p+k+1)q)$ *the group* $\pi_t(B_1^k(p):p)$ *vanishes except the following values of* t:

$$t=iq+3 \qquad \text{for } 0\leq i<k,$$

$$t=iq+2 \qquad \text{for } i\geq k,$$

$$t=iq+1 \qquad \text{for } i\geq p+2,$$

$$t=(k+p)q,\ (2p+2)q-1,\ (2p+2)q,\ (k+2p)q-2,$$

$$(3p+2)q-3,\ (3p+2)q-2,\ (3p+2)q-1.$$

For the group structure of non-trivial $\pi_t(B_1^k(p):p)$, see [7].

We shall construct the spectral sequence $\{E^r_{s,t}\}$ for a given n-connected map $f:B\to X$. For the sake of simplicity, every space will be localized at p; $\pi_*(\ :p)=\pi_*(\ )$. We define a sequence of fiberings

$$(6.10)_s \qquad F_s\xrightarrow{i}B_s\xrightarrow{f}F_{s-1} \quad (s=n, n+1, n+2, \cdots)$$

by giving maps $f=f_s$ inductively. For $s=n$, $f_n=f:B_n=B\to X=F_{n-1}$, then F_n is $(n-1)$-connected. For $s>n$, given an $(n-1)$-connected F_{s-1}, there exists a bouquet $B_s=\bigvee S^n$ of n-spheres with a map $f_s:B_s\to F_{s-1}$ such that $f_s{}^*:H^n(F_{s-1};Z_p)\cong H^n(B_s;Z_p)$, then the fibre F_s is $(n-1)$-connected.

Note that B_s and f_s are unique up to p-equivalence.

Put

$$A^1_{s,t}=\begin{cases}\pi_{n+t}(F_s) & \text{for } s\geq n,\\ \pi_{s+t+1}(X) & \text{for } s<n,\end{cases} \qquad E^1_{s,t}=\begin{cases}\pi_{n+t}(B_s) & \text{for } s\geq n,\\ 0 & \text{for } s<n.\end{cases}$$

The following exact sequences are those of homotopy groups for the fiberings $(6.10)_s$, $s\geq n$, and trivial ones ($\partial_1=id$) for $s<n$:

$$\cdots\to A^1_{s-1,t+1}\xrightarrow{\partial_1}A^1_{s,t}\xrightarrow{i_1}E^1_{s,t}\xrightarrow{f_1}A^1_{s-1,t}\xrightarrow{\partial_1}A^1_{s,t-1}\to\cdots.$$

So, we have an exact couple ($r=1$) and derived couples ($r=2, 3, \cdots$) $(A^r, E^r; \partial_r, i_r, f_r)$, $A^r=\sum A^r_{s,t}$, $E^r=\sum E^r_{s,t}$ with exact sequences

$$\cdots\to A^r_{s+r-1,t-r+1}\xrightarrow{i_r}E^r_{s,t}\xrightarrow{f_r}A^r_{s-1,t}\xrightarrow{\partial_r}A^r_{s,t-1}\xrightarrow{i_r}\cdots,$$

where $A^{r+1}_{s,t}=\partial_r(A^r_{s-1,t+1})=(\partial_1)^r A^1_{s-r,t+r}$, $\partial_{r+1}=\partial_1|A^{r+1}$, $E^{r+1}_{s,t}\cong H(E^r_{s,t})$ with respect to the differential

$$d^r=i_r\circ f_r : E^r_{s,t}\to E^r_{s-r,t+r-1},$$

f_{r+1} is induced by f_r and i_{r+1} by $i_r\partial_r^{-1}$.

Proof of (6. 1). The properties (i) and (ii) are obvious.

Put $D_{s,t}=\mathrm{Ker}((\partial_1)^{s-n+1}: \pi_{s+t}(X)=A^1_{n-1,s+t-n}\to A^1_{s,t-1})$, then

$$\pi_{s+t}(X)/D_{s,t}\cong A^{s-n+1}_{s,t-1}=A^r_{s,t-1} \quad \text{for} \quad r\geq s-n+1.$$

Since F_s ($s\geq n$) is $(n-1)$-connected, $A^r_{s,t}=0$ if $s\geq n$ and $t<0$. So, we get a short exact sequence

$$0\to E^r_{s,t}\xrightarrow{f_r}A^r_{s-1,t}\xrightarrow{\partial_r}A^r_{s,t-1}\to 0 \quad \text{for} \quad r>\mathrm{Max}(t+1, s-n).$$

From these results (iii) follows easily by putting $E^\infty_{s,t}=E^r_{s,t}$ for $r>\mathrm{Max}(t+1, s-n)$.

For $g=i\circ f: B_s\to F_{s-1}\to B_{s-1}$, $g^*=0: H^n(B_{s-1}; Z_p)\to H^n(B_s; Z_p)$. So, there exists a map $g': B_s\to B_{s-1}$ such that g is homotopic to $g'\circ(p\cdot id)$. Then (iv) follows.

The compositions $\beta\cdot\gamma=\beta\circ(S^t\gamma)$, $\beta\in\pi_{n+t}(\ )$, define a map $\cdot\gamma$: $(A^1_{s,t}, E^1_{s,t})\to(A^1_{s,t+u}, E^1_{s,t+u})$ of the exact couple. Then (v) follows.

From the fibering $(6. 10)_n$, $\pi_{s+t}(X, B)\cong\pi_{s+t-1}(F_n)$. Use $f: B_{n+1}\to F_n$ in place of $f: B\to X$, then we get a spectral sequence $\{'E^r_{s,t}\}$ converging to $\pi_*(F_n)$ with $'E^1_{s,t}=E^1_{s,t}$ for $s\neq n$ and $'E^1_{n,t}=0$. Then (iv) follows.

Q.E.D.

For homogeneous elements x_α, $A(x_\alpha)$ denotes the free commutative algebra generated by $\{x_\alpha\}$, i.e.,

$$A(x_\alpha)=\Lambda(x_\alpha; \deg x_\alpha \text{ is odd})\otimes Z_p[x_\alpha; \deg x_\alpha \text{ is even}].$$

Lemma 6. 7. *Let $F\to B\xrightarrow{f}X$ be a fibering, $x_\alpha\in\mathrm{Ker}(f^*: H^{n_\alpha}(X; Z_p)\to H^{n_\alpha}(B; Z_p))$ and let B^* be a submodule of $H^*(X; Z_p)$ such that*

$f^*|B^*: B^*\to H^*(B; Z_p)$ is bijective. Assume that the natural map, defined by cup product,

$$B^*\otimes A(x_\alpha)\to H^*(X; Z_p)$$

is bijective for $\dim<M$ *and injective for* $\dim=M$, *then, for the cohomology suspensions* $\sigma(x_\alpha)\in H^{n_\alpha-1}(F; Z_p)$, *the natural map*

$$A(\sigma(x_\alpha))\to H^*(F; Z_p)$$

is bijective for $\dim<\mathrm{Min}(M-1, p(m-1))$ *and injective for* $\dim=\mathrm{Min}(M-1, p(m-1))$, *where* $m=\mathrm{Min}(\deg x_\alpha=n_\alpha\colon odd)$.

This is proved by use of the comparison theorem (cf. Cor. 3. 3 of [24]).

Proof of Theorem 6. 1. Applying Lemma 6. 7 successively we have

(6. 11) *the natural map* $A(\sigma^{s-n+1}x_{n_i}; n_i\in N, s<n_i<M)\to H^*(F_s; Z_p)$ *is bijective for* $\dim<M-s+n-1$ *and injective for* $\dim=M-s+n-1$.

Then (i) follows immediately.

Let $s, s-1\in N$. Then $d^1(\iota)=a\iota$ for the degree a of $g=i\circ f\colon S^n=B_s\to F_{s-1}\to B_{s-1}=S^n$. $a\equiv 0 \pmod p$ by (iv) of (6. 1). Then $(\delta/a)(\sigma^{s-n-1}x_{s-1})=\sigma^{s-n-1}x_s$ in $H^*(F_{s-2}; Z_p)=\{\sigma^{s-n-1}x_{s-1}, \sigma^{s-n-1}x_s, \cdots\}$ if $a\neq 0$, whence $(\delta/a)x_{s-1}=x_s$ (mod decomp.). By (v) of (6. 1), $d^1(\beta)=d^1(\iota\cdot\beta)=a\iota\cdot\beta=a\beta$, and the first half of (ii) is proved.

Let $s\in N$, $\bar{g}\colon S^{n+t}\to F_s$ be a map, put $g=i\circ\bar{g}\colon S^{n+t}\to B_s$ and let $L=S^n\cup e^{n+t+1}$ be the mapping cone of g. The following (6. 12) is proved by constructing maps.

(6. 12) *There exists an extension* $h\colon L\to F_{s-1}$ *of* $f\colon B_s\to F_{s-1}$ *such that* $\partial_1(h_*\tilde{\zeta})=\bar{g}_*(\zeta)$ *holds for any coextension* $\tilde{\zeta}\in\pi_{n+t+u+1}(L)$ *of* $\zeta\in\pi_{n+t+u}(S^{n+t})$.

Apply this to the case $t=0$, $\bar{g}=f\colon S^n\to F_s$ and $L=S^n\cup e^{n+1}$. In the homotopy exact sequence of the fibering $(6.10)_{s-i}$:

$$[S^iL, F_{s-i-1}]\overset{\partial_*}{\to}[S^{i-1}L, F_{s-i}]\overset{i_*}{\to}[S^{i-1}L, B_{s-i}]$$

the last term vanishes for $1\leq i<r-1$ ($<q$) by (6. 2). Thus there exists an element $\eta\in[S^{r-2}L, F_{s-r+1}]$ such that $(\partial_*)^{r-2}\eta=\{h\}$. $d^1(\beta)=g_*(\beta)=0$ for $\beta\in E^1_{s,t}$. So there exists a coextension $\tilde{\beta}\in\pi_{n+t+1}(L)$ of $\beta\in E^1_{s,t}=\pi_{n+t}(S^n)$. From

$$f_1(\beta)=\bar{g}_*(\beta)=\partial_1(\{h\}\circ\tilde{\beta})=\partial_1((\partial_*)^{r-2}\eta\circ\tilde{\beta})=(\partial_1)^{r-1}(\eta\circ S^{r-2}\beta)$$

we have that $f_1(\eta\circ S^{r-2}\tilde{\beta})=i_*(\eta)\circ S^{r-2}\tilde{\beta}$ represents $d^r(\beta)$, where $i: F_{s-r}\to B_{s-r}=S^n$. If $1<r<q$, $i_*(\eta)\in[S^{r-2}L, S^n]=0$ by (6. 2), and $d^r(\beta)=0$. If $r=q$, $i_*(\eta)=\pi^*(b\cdot\alpha_1)$ for the projection $\pi:S^{q-2}L\to S^{n+q-1}$ and for some $b\in Z_p$. Then $d^q(\beta)$ is represented by

$$i_*(\eta)\circ S^{q-2}\tilde{\beta}=(b\cdot\alpha_1)\circ\pi_*(S^{q-2}\tilde{\beta})=(b\cdot\alpha_1)\circ S^{q-1}\beta=b\alpha_1\cdot\beta\,.$$

Let $K=S^n\cup C(S^{q-2}L)=(S^n\vee S^{q-1}B_{s-1})\cup e^{n+q}$ be the mapping cone of a representative of η. Then we can construct a map of K into F_{s-r-1} inducing isomorphisms of H^n and H^{n+q}. Then the relation $\mathcal{P}^1x_{s-q}=bx_s$ (mod decomp.) follows from (6. 11), completing the proof of (ii).

Next consider the property (iii). From the assumption $d^r(\iota)=\alpha$, $f_1(\iota)=(\partial_1)^{r-1}\bar{\alpha}$ and $i_1(\bar{\alpha})=\alpha$ for some $\bar{\alpha}\in A^1_{s-r,r-1}$. Apply (6. 12) to a representative $\bar{g}$ of $\bar{\alpha}$, then $L=S^n\cup e^{n+r}$ is the mapping cone of α and $h:L\to F_{s-r-1}$ satisfies $h|S^n=f$ and $\partial_1(h_*\tilde{\zeta})=\bar{g}_*(\zeta)=\bar{\alpha}\cdot\beta$ for $\zeta=S^{r-1}\beta$. Then $f_1(\beta)=(\partial_1)^{r-1}\bar{\alpha}\cdot\beta=(\partial_1)^r(h_*\tilde{\zeta})$. Since β represents $[\beta]$, $f_1(\beta)=(\partial_1)^r(\partial_1)^{s-r-n}[\beta]$. From the assumption $E^1_{s+t,t-i}=0$, $1\leq i<r$, it follows that $(\partial_1)^r: A^1_{s-r-1,r+t}\to A^1_{s-1,t}$ is injective. So, $(\partial_1)^{s-r-n}[\beta]=h_*\tilde{\zeta}$. Since $\tilde{\zeta}$ is a coextension of $S^{r-1}\beta$, $\tilde{\zeta}\circ S^{t+r}\gamma=j_*(\xi)$ for a ξ of $\{\alpha, S^{r-1}\beta, S^{r+t-1}\gamma\}$, where $j: S^n\to L$ is the inclusion. Then the equality

$$(\partial_1)^{s-r-n}([\beta]\cdot\gamma)=h_*\tilde{\zeta}\circ S^{r+t}\gamma=h_*j_*\xi=f_1(\xi)$$

shows that ξ represents $[\beta]\cdot\gamma$, and (iii) is proved.

Consider the property (iv). The equality $d^{r'}(\delta)=\beta$ means that $f_1(\delta)=(\partial_1)^{r'-1}\bar{\beta}$ and $i_1(\bar{\beta})=\beta$ for some $\bar{\beta}\in A^1_{s,t}$. Apply (6. 12) to the representative $\bar{g}$ of $\bar{\beta}$, then $L=S^n\cup e^{n+t+1}$ is the mapping cone of β and $h: L\to F_{s-1}$ satisfies $h|S^n=f$ and $\partial_1(h_*\tilde{\zeta})=\bar{g}_*\zeta=\bar{\beta}\cdot\gamma$ for $\zeta=S^t\gamma$. Let $j: S^n\to L$ be the inclusion and $\eta_1\in[L, F_{s-1}]$ be the class of h, then $j^*(\eta_1)=f_1(\iota)=(\partial_1)^{r-1}\bar{\alpha}$ and $i_1(\bar{\alpha})=\alpha$. In the exact and commutative diagram

$$\begin{array}{ccccccc}
[S^{n+t+i}, B_{s-i}] & & & & & & \\
\uparrow \beta^* & & & & & & \\
[S^{n+i}, B_{s-i}] & \xrightarrow{f_1} & [S^{n+i}, F_{s-i-1}] & \xrightarrow{\partial_1} & [S^{n+i-1}, F_{s-i}] & \xrightarrow{i_1} & [S^{n+i-1}, B_{s-i}] \\
\uparrow j^* & & \uparrow j^* & & \uparrow j^* & & \uparrow j^* \\
[S^iL, B_{s-i}] & \xrightarrow{f_1} & [S^iL, F_{s-i-1}] & \xrightarrow{\partial_1} & [S^{i-1}L, F_{s-i}] & \xrightarrow{i_1} & [S^{i-1}L, B_{s-i}] \\
 & & & & & & \uparrow \pi^* \\
 & & & & & & [S^{n+t+i}, B_{s-i}],
\end{array}$$

$E^1_{s-i,t+i}=[S^{n+t+i}, B_{s-i}]=0$ for $1\leq i<r$ by the assumption. By diagram chasing, we get a sequence of elements $\eta_i\in[S^{i-1}L, B_{s-i}]$, $i=2, 3, \cdots, r$, such that $\partial_*(\eta_i)=\eta_{i-1}$ and $j^*(\eta_i)=(\partial_1)^{r-i}\bar{\alpha}$. Since $j^*i_*(\eta_r)=i_1j^*(\eta_r)=i_1(\bar{\alpha})=\alpha$, $i_*\eta_r\in[S^{r-1}L, S^n]$ is an extension of α. Since $\tilde{\zeta}$ is a coextension of $S^t\gamma$, $i_*(\eta_r\circ S^{r-1}\tilde{\zeta})=\xi\in\{\alpha, S^{r-1}\beta, S^{r+t-1}\gamma\}$. This and the equality

$$\begin{aligned} f_1(\delta\cdot\gamma) &= (\partial_1)^{r'-1}(\bar{\beta}\cdot\gamma)=(\partial_1)^{r'}(h_*\zeta)=(\partial_1)^{r'}(\eta_1\circ\tilde{\zeta}) \\ &= (\partial_1)^{r+r'-1}(\eta_r\circ S^{r-1}\tilde{\zeta}) \end{aligned}$$

show that $d^{r+r'}(\delta\cdot\gamma)=\xi$, proving (iv).

When $B=S^{2m+1}$, we define the spectral sequence by putting

$$A^1_{s,t}=\begin{cases}\pi_{n+t}(F_s) & \text{for } s\geq n\,,\\ \pi_{s+t}(F_n) & \text{for } 2m+1\leq s<n\,,\\ \pi_{s+t+1}(X) & \text{for } s<2m+1\,,\end{cases}$$

$$E^1_{s,t}=\begin{cases}\pi_{n+t}(B_s) & \text{for } s\geq n\,,\\ \pi_{s+t}(S^{2m+1}) & \text{for } s=2m+1\,,\\ 0 & \text{for } s<n\,,\quad s\neq 2m+1\,.\end{cases}$$

Then we get the required spectral sequence in (v). Q.E.D.

§ 7. Characterization of Some $B_m^k(p)$

We shall try to characterize $B_m{}^k(p)$ of some type by its cohomological structure.

Proposition 7.1. *Assume that* $m\not\equiv 0 \pmod p$, $m\neq 2p^2-3p-1$ *and* $m<2p^2-2p-1$. *Let B be a complex of p-type* $(2m+1, 2m+2p-1)$ *such* *that* $H^*(B; Z_p)=\Lambda(x_{2m+1}, \mathcal{P}^1x_{2m+1})$. *Then B is p-equivalent to* $B_m^2(p)$.

Proof. $B=S^{2m+1}\cup e^{2m+2p-1}\cup e^{4m+2p}$. From $m\not\equiv 0 \pmod p$ we have $H^*(B_m^2(p); Z_p)=\Lambda(x_{2m+1}, \mathcal{P}^1x_{2m+1})$. We consider an extension $B\to B_m^2(p)$ of a mod p injection $S^{2m+1}\to B_m^2(p)$. The obstruction to the extension lies in $\pi_{2m+2p-2}(B_m^2(p):p)$ and $\pi_{4m+2p-1}(B_m^2(p):p)$. These homotopy groups are in the meta-stable range and they are computed in Proposition 6. 3 when $4m+2p-1<2m+1+2(2p+1)(p-1)-2$, i.e., $m<2p^2-2p-1$. By Proposition 6. 3, $\pi_{2m+2p-2}(B_m^2(p):p)=0$ and the case $\pi_{4m+2p-1}(B_m^2(p):p)\neq 0$ may occur only if $4m+2p-1=2m+1+2p(p-1)-2$ or $=2m+1+4p(p-1)-4$, that is, $m=p^2$ or $m=2p^2-3p-1$ which are excluded by the assumption. Thus we have an extension $B\to B_m^2(p)$ which is a p-equivalence by the naturality of $\mathcal{P}^1$. Q.E.D.

Proposition 7. 2. *Assume that* $p>3$, $m\not\equiv 0, 1 \pmod p$, $m<p^2-2p$ *and* $m\neq p-1, \frac{p(p-1)}{2}-1, p^2-4p+2$. *Let B be a complex of p-type* $(2m+1, 2m+2p-1, 2m+4p-3)$ *such that* $H^*(B; Z_p)=\Lambda(x_{2m+1}, \mathcal{P}^1x_{2m+1}, \mathcal{P}^2x_{2m+1})$. *Then B is p-equivalent to* $B_m^3(p)$.

Proof. B consists of cells of dimensions $i+1$ for

$$i=2m,\ 2m+2p-2,\ 2m+4p-4,\ 4m+2p-1,\ 2m+4p-3,$$
$$4m+6p-5,\ 6m+6p-4.$$

As in the previous proof it is sufficient to show that $\pi_i(B_m^3(p):p)=0$ for the above values of i.

Let $m<p$. By Proposition 6. 6, we see that the possibility of $\pi_i(B_m^3(p):p)\neq 0$ is $i=4m+2p-1=2m+4p-3$ or $i=2m+1+2k(p-1)-1=6m+6p-4$, that is, $m=p-1$ or $2m=(k-1)(p-1)-1$. The first case is excluded and the second one does not occur since $p-1$ is even.

Let $m\geq p$. Then the homotopy groups are in the meta-stable range and computed in Proposition 6. 3 for $i<2m+1+2(2p+1)(p-1)-2$, that is, $m<p^2-2p$ for $i=6m+6p-4$. Then an obstruction may appear in the following cases:

$$2m+1+2p(p-1)-2,$$

$$2m+1+4p(p-1)-4=4m+2p-1\,,\ 4m+4p-3\,,\ 4m+6p-5\,,$$

$$6m+6p-4=2m+1+2k(p-1)-1\,,\ 2m+4p-3+2(p+1)(p-1)-3\,.$$

The cases $2m+1+4p(p-1)-4=\cdots$ do not occur since $p-1$ is even. The remaining cases are excluded by assumption. Q.E.D.

Proposition 7.3. *Assume that* $p>5$, $m\not\equiv 0,\ 1,\ 2\ (\mathrm{mod}\, p)$, $\frac{5p-3}{p-4}<m<\frac{(2p-5)(p-1)}{3}-1$, $3m\neq(p-6)(p-1)-2,\ 2(p-3)(p-1)-3,\ 2m\not\equiv -2\ (\mathrm{mod}\, p-1)$ *and* $m\not\equiv -1,\ -2\ (\mathrm{mod}\, p-1)$. *Let* B *be a complex of* p*-type* $(2m+1, 2m+2p-1, 2m+4p-3, 2m+6p-5)$ *such that* $H^*(B; Z_p)=\Lambda(x_{2m+1}, \mathcal{P}^1x_{2m+1}, \mathcal{P}^2x_{2m+1}, \mathcal{P}^3x_{2m+1})$. *Then* B *is* p*-equivalent to* $B_m{}^4(p)$.

Proof. As before, it suffices to show $\pi_{2m+1+j}(B_m{}^4(p):p)=0$ for $j=2m+2k(p-1)$ $(k=1, 2, 3, 4, 5)$, $j=4m+2k(p-1)+1$ $(k=3, 4, 5, 6)$ and for $j=6m+12(p-1)+2$. These j are not of the form $2h(p-1)-1$. So, we need to exclude the cases $j=2p(p-1)-2$, $j=6(p-1)+2(p+1)\cdot(p-1)-3$ and $j=4p(p-1)-4$. And, we see that the assumptions on m are sufficient to construct a p-equivalence $B\to B_m{}^4(p)$. Q.E.D.

Proposition 7.4. *Let* B *be a complex of* p*-type* $(3, 2p+1, \cdots, 3+(k-1)q)$, $k=3,\ 4,\ 5$, *and let* Φ *be a secondary operation which detects* α_2.

(i) *If* $k=3$ *and* $H^*(B; Z_p)=\Lambda(x_3, \mathcal{P}^1x_3, \Phi x_3)$, *then* B *is* p*-equivalent to* $B_1{}^3(p)$.

(ii) *If* $k=4$, $p>3$ *and* $H^*(B; Z_p)=\Lambda(x_3, \mathcal{P}^1x_3, \Phi x_3, \mathcal{P}^1\Phi x_3)$, *then* B *is* p*-equivalent to* $B_1{}^4(p)$.

(iii) *If* $k=5$, $p>7$ *and* $H^*(B; Z_p)=\Lambda(x_3, \mathcal{P}^1x_3, \Phi x_3, \mathcal{P}^1\Phi x_3, \mathcal{P}^2\Phi x_3)$, *then* B *is* p*-equivalent to* $B_1{}^5(p)$.

The proof is given by use of Proposition 6.6 and omitted.

In the sequel of this section we consider the complexes $B_{11}^4(7)$, $B_9{}^3(5)$ and $B_1{}^5(5)$ which have been not characterized by the previous propositions and will be used in the next chapter. Note that these three complexes are unique up to p-equivalence by Theorem 5. 4 and Proposition 3. 4.

First we see that the only obstruction to construct a p-equivalence $B=B(23, 35, 47, 59)\rightarrow B_{11}^4(7)$ lies in $\pi_{105}(B_{11}^4(7):7)$ which is generated by $[\beta_1(23)]$. This obstruction can be removed by use of the following Lemma 7. 5.

Let $B_{11}^4(7)^{(k)}$ denote the k-skeleton of $B_{11}^4(7)$. Let $\xi\in\pi_{105}(B_{11}^4(7)^{(105)})$ be the class of the attaching map of the top cell e^{106} of $B_{11}^4(7)^{(106)}=B_{11}^4(7)^{(105)}\cup e^{106}$ which represents $x_{47}x_{59}$. $\beta_1(23)$ is a generator of the 7-component of $\pi_{105}(S^{23})$, and we denote by the same symbol $\beta_1(23)$ its image $[\beta_1(23)]$ in $\pi_{105}(B_{11}^4(7)^{(105)})$ by the injection.

Lemma 7. 5. *For each $a\in Z_7$, there exists a map $h:B_{11}^4(7)^{(105)}\rightarrow B_{11}^4(7)^{(105)}$ such that $h|S^{23}$ is a* mod 7 *injection and $h_*(\xi)=n\cdot\xi-a\cdot\beta_1(23)$, $n\not\equiv 0 (\text{mod } 7)$.*

Proof. For the sake of simplicity, we put $B=B_{11}^4(7)$ and $B^k=B_{11}^4(7)^{(k)}$. $B^{70}=B^{59}\cup e^{70}$ for a cell e^{70} representing $x_{23}x_{47}$. Let $\psi:B^{70}\rightarrow B^{70}\vee S^{70}$ be the map pinching an equator of e^{70} and let $A:S^{70}\rightarrow B^{59}$ be a representative of $[\alpha_1(59)]\in\pi_{70}(B:7)$. Consider the composition

$$h_0:\ B^{70}\xrightarrow{\psi}B^{70}\vee S^{70}\xrightarrow{id\vee A}B^{70}\subset B^{105}.$$

Since $\beta_1(23)$ is the only obstruction, h_0 can be extended to $h_1:B^{105}\rightarrow B^{105}$, and

$$h_{1*}(\xi)=m\cdot\xi+b\cdot\beta_1(23) \text{ for some } b\in Z_7 \text{ and } m\not\equiv 0 \pmod 7.$$

Now we assume $b=0$ and deduce a contradiction. From $b=0$, we have an extension $h_2:B^{106}\rightarrow B^{106}$ of h_1. As in § 5, we have a mod 7 injection (M: large)

$$j:\ B^{106}\rightarrow\Omega^{2M}L\,,\quad L=S^{2M+23}\cup e^{2M+35}\cup e^{2M+47}\cup e^{2M+59}.$$

It follows that $\mathcal{P}^3\neq 0$ in L. By the loop-multiplication we have a map

$$k=j^{-1}\cdot(j\circ h_2):B^{106}\rightarrow\Omega^{2M}L\,.$$

Since $h_2|B^{59}=id$, $k|B^{59}$ is homotopic to zero and k is factored as $B^{106}\xrightarrow{\text{proj}}B^{106}/B^{59}\xrightarrow{\bar{k}}\Omega^{2M}L$. Consider the adjoint map $\overline{K}:S^{2M}(B^{106}/B^{59})\rightarrow L$ and let $C=L\cup CS^{2M}(B^{106}/B^{59})$ be its mapping cone. Then $\mathcal{P}^1(e^{2M+59})=CS^{2M}(x_{27}x_{47})$. By Cartan's formula $\mathcal{P}^3(x_{27}x_{47})=2x_{47}x_{59}$ in B^{106}/B^{59}. Thus we have $\mathcal{P}^3\mathcal{P}^1\mathcal{P}^3(e^{2M+23})=2CS^{2M}(x_{47}x_{59})\neq 0$ which contradicts to Adem's relation $\mathcal{P}^3\mathcal{P}^1\mathcal{P}^3=\mathcal{P}^3\mathcal{P}^4=0$. Consequently we have proved that $b\neq 0$.

Let h be the t-fold iteration of h_1. Then $h_*(\xi)=m^t\cdot\xi+b(1+m(t-1))\beta_1(23)$. Since m, $b\not\equiv 0 \pmod 7$, $-a\equiv b(1+m(t-1)) \pmod 7$ for suitable t, and the map h satisfies the condition of the lemma. Q.E.D.

If the obstruction to construct a p-equivalence $B=B(23, 35, 47, 59)\to B_{11}^4(7)$ is $a\cdot\beta_1(23)$, the obstruction vanishes by changing the constructed map $B^{(105)}\to B_{11}^4(7)^{(105)}$ by the composition with the map h of the lemma. So, we have

Proposition 7.6. *If $\mathcal{P}^3x_{23}=x_{59}$ in a complex B of 7-type (23, 35, 47, 59), B is 7-equivalent to $B_{11}^4(7)$.*

Apply Proposition 6.3 to $B_9^3(5)$, then we have

(7.1) *$\pi_i(B_9^3(5):5)$ for $i<97$ has the generators $\iota(19)$, $[5\cdot\iota(27)]$, $[25\cdot\iota(35)]$, $[\alpha_1(35)]$, $[\alpha_2(35)]$, $\beta_1(19)$, $[\alpha_3(35)]$, $[\alpha_4(35)]$, $[\alpha_5'(35)]$, $[\alpha_1\beta_1(35)]$, $[\alpha_6(35)]$, $[\alpha_7(35)]$ and $\beta_1^2(19)$ of dimensions $i=19$, 27, 35, 42, 50, 57, 58, 66, 74, 80, 82, 90 and 95 respectively.*

Consider $B=B(19, 27, 35)$ with $\mathcal{P}^2\neq 0$. Since the dimensions of the cells of B are 0, 19, 27, 35, 46, 54, 62, 81, the only obstruction to construct a 5-equivalence $B\to B_9^3(5)$ lies in $H^{81}(B;\pi_{80}(B_9^3(5):5))$. Thus we obtain a 5-equivalence of the 62-skeletons: $B^{(62)}\to B_9^3(5)^{(62)}$. Let $\xi\in\pi_{80}(B_9^3(5)^{(62)})$ be the attaching class of the 81-cell of $B_9^3(5)=B_9^3(5)^{(62)}\cup e^{81}$. By the exact sequence

$$0\to\pi_{81}(B_9^3(5), B_9^3(5)^{(62)}:5)\xrightarrow{\partial}\pi_{80}(B_9^3(5)^{(62)}:5)\xrightarrow{i_*}\pi_{80}(B_9^3(5):5)\to 0,$$

there exists an element $\gamma'\in\pi_{80}(B_9^3(5):5)$ with $i_*\gamma'=[\alpha_1\beta_1(35)]$ and $5\cdot\gamma'=m\cdot\xi$ for some $m\in Z$. If $m\not\equiv 0 \pmod 5$, $B_9^3(5)$ is 5-equivalent to the mapping cone C of $m\cdot\xi$, and $x_{19}x_{27}x_{35}\neq 0$ in C. But, since $m\cdot\xi=5\cdot\gamma'$. $x_{19}x_{27}x_{35}=0$ in C contradicting the above. Thus $m=5m'$, $m'\in Z$, and by putting $\gamma=\gamma'-m'\cdot\xi$ we have

(7.2) $\pi_{80}(B_9^3(5)^{(62)}:5)=Z_{(5)}\langle\xi\rangle+Z_5\langle\gamma\rangle$ *with* $i_*\xi=0$ *and* $i_*\gamma=[\alpha_1\beta_1(35)]$.

Then we have easily:

(7.3) *Let* $B_9^3(5;a)=B_9^3(5)^{(62)}\cup e^{81}$ *be the mapping cone of* $\xi+a\cdot\gamma$. *Then any complex of* 5*-type* (19, 27, 35) *with* $\mathcal{P}^2\neq 0$ *is* 5*-equivalent to* $B_9^3(5;a)$ *for some* $a\in Z_5$.

Proposition 7.7. *Let* B *be a complex of* 5*-type* (19, 27, 35) *with* $H^*(B;Z_5)=\Lambda(x_{19},\mathcal{P}^1x_{19},\mathcal{P}^2x_{19})$. *Then the following three conditions are equivalent.*

(i) B *is* 5*-equivalent to* $B_9^3(5)$.

(ii) *There exists a map* $\mu:(S^3\cup e^{19})\times B\to B$ *such that* $\mu^*(x_{19})=x_{19}\otimes 1+1\otimes x_{19}$ *up to non-zero coefficients in* Z_5.

(iii) *There exists a* mod 5 *surjection* $\rho: B\to S^{35}$.

Proof. (i)⇒(ii). It is sufficient to prove (ii) for $B=B_9^3(5)$. Let

$$g':W_{18,9}\to\Omega^{2M}(\vee L_i)\xrightarrow{\text{proj}}\Omega^{2M}L_0,\quad L_0=S^{2M}(S^{19}\cup e^{27}\cup e^{35}),$$

be the map considered in § 5 (M: large) such that $\bar g=g'\circ f: B_9^3(5)\to W_{18,9}\to\Omega^{2M}L_0$ induces isomorphisms of cohomology mod 5 for dim<98. Moreover $\bar g_*:\pi_t(B_9^3(5):5)\cong\pi_t(\Omega^{2M}L_0:5)$ for $t<104$, $t\neq 97$, as is seen in the proof of Proposition 5.6. Then the composite map

$$B_1^5(5)\times B_9^3(5)\xrightarrow{f\times f}SU(18)\times W_{18,9}\xrightarrow{\text{action}}W_{18,9}\xrightarrow{g'}\Omega^{2M}L_0$$

restricted on $B_1^5(5)^{(19)}\times B_9^3(5)$ is factored through $\bar g$, in the mod 5 sense, and we obtain a mod 5 surjection $\mu': B_1^5(5)^{(19)}\times B_9^3(5)\to B_9^3(5)$. Let $S^3\cup e^{4p-1}$ be the mapping cone of α_2, then we have that

(7.4) *there exists a* mod p *injection* $i: S^3\cup e^{4p-1}\to B_1^k(p)$, $k\geq 3$.

For, the inclusion $S^3\to B_1^k(p)$ is extendable over e^{4p-1} since $\pi_{4p-2}(B_1^k(p):p)=0$. Then $i^*(\Phi x_3)=\Phi i^*(x_3)\neq 0$, and (7.4) follows.

The composite map $\mu=\mu'\circ(i\times id)$ satisfies (ii) for $B=B_9^3(5)$.

(ii) $\Rightarrow$ (iii). By (7. 3) we may assume $B^{(62)}=B_9^3(5)^{(62)}$. Put $P=(S^3 \cup e^{19})\times B^{(62)}$, $\mu_1=\mu|P$, $\mu_2=\mu|P^{(73)}:P^{(73)}\to B^{(62)}$ and let $\eta\in\pi_{80}(P^{(73)})$ be the attaching class of the top cell $e^{81}=P-P^{(73)}$. From the assumption, $\mu_1{}^*(x_{19}x_{27}x_{35})=x_{19}\otimes x_{27}x_{35}$. Thus $\mu_1:P\to B_9^3(5)$ has a non-zero degree on the top cell, and it defines a 5-equivalence of the mapping cone $B^{(62)}\cup e^{81}$ of $\mu_{2*}(\eta)$ onto B. So, we may assume that B is the mapping cone of $\mu_{2*}(\eta)$. Consider the composite map

$$P^{(73)}\xrightarrow{\mu_2}B^{(62)}\xrightarrow{\bar g}\Omega^{2M}L_0\xrightarrow{\bar\pi}\Omega^{2M}S^{2M+35},$$

where $\bar\pi$ is induced by the projection $\pi: L_0\to S^{2M+35}=L_0/L_0{}^{2M+27}$.

We shall prove

$$(*)\qquad \bar\pi_*\bar g_*\mu_{2*}(\eta)=0\,.$$

Then $\bar\pi\circ\bar g$ is extended over B, and the extended map gives the required mod 5 surjection ρ since the inclusion $S^{35}\to\Omega^{2M}S^{2M+35}$ induces isomorphisms of cohomology mod 5 for dim<178.

As is well-known the suspension $S(A\times B)$ is homotopy equivalent to the one point union of SA, SB and $S(A\wedge B)$. Put

$$K_2=(S^3\cup e^{19})\wedge B^{(46)}\,,\quad K_1=(S^3\cup e^{19})\wedge B^{(54)}\,,$$

$$K_0=K_1\cup(S^3\wedge B^{(62)})\quad\text{and}\quad K=(S^3\cup e^{19})\wedge B^{(62)}=K_0\cup e^{81}\,,$$

then there is a map $S^{2M}K\to S^{2M}P$ having degree 1 on the top cell. Thus, to prove (*) it is sufficient to show that the composite map

$$S^{2M}K_0\xrightarrow{G}L_0\xrightarrow{\pi}S^{2M+35}$$

is mod 5 trivial for any map G. $K_0=K_1\cup e^{65}$, $K_1=K_2\cup e^{57}\cup e^{73}$ and K_2 consists of cells of dimensions 0, 22, 30, 38, 46, 49, 54, 65. $\pi_{2M+t}(L_0:5)\cong\pi_t(\Omega^{2M}L_0:5)\cong\pi_t(B_9^3(5):5)$ for $t<97$. So, by (7. 1), $G|K_2$ is mod 5 trivial, and G is factored to

$$S^{2M}K_0\xrightarrow{\text{proj}}S^{2M}(K_0/K_2)\xrightarrow{G_0}L_0\,,$$

where $S^{2M}(K_0/K_2)=(S^{2M+57}\cup e^{2M+73})\cup e^{2M+65}$. Since $\pi_{57}(B_9^3(5):5)=\langle\beta_1(19)\rangle$, $G_0|S^{2M+57}$ is homotopic to a map $G_0{}':S^{2M+57}\to S^{2M+19}$. Now we can extend the map $G_0{}'$ to $G':S^{2M}(K_0/K_2)\to L_0{}^{2M+27}$ since $\pi_{2M+72}(S^{2M+19}:5)=0$ and $\beta_1\cdot\pi_{2M+64}(S^{2M+57}:5)=\langle\beta_1\circ\alpha_1\rangle$ is trivial in $L_0{}^{2M+27}$, the mapping cone of α_1. The difference of G and G' is trivial since $\pi_{2M+73}(L_0:5)=\pi_{2M+65}(L_0:5)=0$. Thus G is homotopic to G' and $\pi\circ G$ is homotopic to the constant map $\pi\circ G'$, and (*) has been proved.

(iii) $\Rightarrow$ (i). By Theorem 5.5 we have the following composition of fibre bundles

$$\rho_1: B_9^3(5) \to B_{13}^2(5) \to B_{17}^1(5) = S^{35}$$

which is a mod 5 surjection. By (7.3), we may put $B = B_9^3(5; a)$, then $B^{(k)} = B_9^3(5)^{(k)}$ for $k<81$. Compare mod 5 surjections ρ and ρ_1 on $B^{(62)}$. We may assume that $\rho|B^{(35)}$ is homotopic to $\rho_1|B^{(35)}$, by composing 5-equivalences $S^{35}\to S^{35}$. By (7.1), $H^n(B; \pi_n(B_9^3(5):5))=0$ for $35<n<81$, and the homotopy can by extended to one between $\rho|B^{(62)}$ and $\rho_1|B^{(62)}$. So, we may put $\rho' = \rho|B^{(62)} = \rho_1|B^{(62)}$. The existence of a mod 5 surjection onto S^{35} implies

$$\rho_*'(\xi) = 0 \quad \text{and} \quad a\cdot\rho_*'(\gamma) = \rho_*'(\xi + a\cdot\gamma) = 0\,.$$

It is easy to see that $\rho_*'(\gamma) = \alpha_1\beta_1(35)$ generates $\pi_{80}(S^{35}:5)\cong Z_5$. Thus $a=0$ and $B = B_9^3(5;0) = B_9^3(5)$. Q.E.D.

In the proof we see that

(7.5) *$B_9^3(5;a)$ is 5-equivalent to $B_9^3(5)$ if and only if $a=0$.*

Next from Proposition 6.6, $\pi_t(B_1^5(5):5) = 0$ except for $t=3$, 11, 19, 27, 35, 43, 50, 57, 58, 65, 66, 73, 74, 80, 81, 89, 90, 95, 96, $\cdots$.

$B = B(3, 11, 19, 27, 35)$ consists of cells of dimensions 0, 3, 11, 14, 19, 22, 27, 30, 30, 33, 35, 38, 41, 46, 46, 49, 49, 54, 57, 57, 60, 62, 65, 65, 68, 73, 76, 81, 84, 92, 95. Thus $H^n(B; \pi_{n-1}(B_1^5(5):5)) = 0$ if $n\neq 81$, and we have the following:

(7.6) *Let B be a complex of 5-type $(3, 11, 19, 27, 35)$ with $\mathcal{P}^1 x_3\neq 0$ and $\mathcal{P}^2\Phi x_3\neq 0$. Then $B^{(76)}$ is 5-equivalent to $B_5^1(5)^{(76)}$. Further if $B^{(81)}$ is 5-equivalent to $B_1^5(5)^{(81)}$, then B is 5-equivalent to $B_1^5(5)$.*

Proposition 7.8. *Let B be a complex of 5-type $(3, 11, 19, 27, 35)$ with $H^*(B; Z_5) = \Lambda(x_3, \mathcal{P}^1 x_3, \Phi x_3, \mathcal{P}^1\Phi x_3, \mathcal{P}^2\Phi x_3)$. Then B is 5-equivalent to $B_1^5(5)$ if and only if there exists a mod 5 surjection $\pi: B \to B_9^3(5)$.*

Proof. As is seen in the first part of the proof of Proposition 7. 7, the composite map

$$B_1^5(5) \xrightarrow{f} SU(18) \xrightarrow{\text{proj}} W_{18,9} \xrightarrow{g'} \Omega^{2M} L_0$$

is factored to $B_1^5(5) \xrightarrow{\pi_1} B_9^3(5) \xrightarrow{\bar{g}} \Omega^{2M} L_0$ since $\dim B_1^5(5) = 95 < 98$. Thus we have a mod 5 surjection

$$\pi_1 : B_1^5(5) \to B_9^3(5),$$

and the only if part of the proposition follows.

Let F be the fibre of π_1, then we have easily that $H^*(F; Z_5) = \Lambda(x_3, \mathcal{P}^1 x_3)$. By Proposition 6. 6, $\pi_t(F: 5) = 0$ for $t = 80$, 81. It follows that $\pi_{1*} : \pi_{80}(B_1^5(5): 5) \to \pi_{80}(B_9^3(5): 5)$ is an isomorphism. Since $\pi_1^*(x_{19}x_{27}x_{35}) \neq 0$, the following (7. 7) is obtained as in (7. 2). Let ξ be the attaching class of the cell $e^{81} = B_1^5(5)^{(81)} - B_1^5(5)^{(76)}$.

(7. 7) $\pi_{80}(B_1^5(5)^{(76)}: 5) = Z_{(5)}\langle\xi\rangle + Z_5\langle\gamma\rangle$ *with* $\pi_{1*}(\xi) = 0$ *and* $\pi_{1*}(\gamma) = [\alpha_1\beta_1(35)]$.

Now assume the existence of a mod 5 surjection $\pi: B \to B_9^3(5)$. By (7. 6) we may assume that $B^{(76)} = B_1^5(5)^{(76)}$ and by (7. 7) that $B^{(81)} = B^{(76)} \cup e^{81}$ is (5-equivalent to) the mapping cone of $\xi + a\cdot\gamma$ for some $a \in Z_5$. Obviously $\pi_*(\xi + a\cdot\gamma) = 0$. We shall prove

(*) $\rho_* \pi_{1*}(\xi + a\cdot\gamma) = 0$ *for the* mod 5 *surjection* $\rho: B_9^3(5) \to S^{35}$.

From (*) and (7. 7), we have easily $a\cdot\alpha_1\beta_1(35) = 0$, so $a = 0$ and thus $B^{(81)}$ is 5-equivalent to $B_1^5(5)^{(81)}$. Then, by (7. 6), B is 5-equivalent to $B_1^5(5)$, completing the proof of the proposition.

To prove (*), we may replace S^{35} by $\Omega^{2M} S^{2M+35}$ and (*) by a condition in adjoint maps. Then it is sufficient to prove the following:

(7. 8) *Put* $\eta = S^{2M}(\xi + a\cdot\gamma)$. *For any* mod p *surjections* $\bar{\pi}$, $\bar{\pi}_1 : S^{2M} B^{(76)} \to L_0 = S^{2M}(S^{19} \cup e^{27} \cup e^{35})$ *and for the projection* $\bar{\rho}: L_0 \to S^{2M+35}$, $\bar{\rho}_* \bar{\pi}_*(\eta) = 0$ *implies* $\bar{\rho}_* \bar{\pi}_{1*}(\eta) = 0$.

Here is an important remark.

(7. 9) *η is in the image of the injection homomorphism* $\pi_{2M+80}(S^{2M}B^{(65)}) \to \pi_{2M+80}(S^{2M}B^{(76)})$.

For, if (7. 9) is not true, then $x_{19}x_{27}x_{35} \in \mathcal{P}^1 H^{73}(B; Z_5)$ since the non-trivial elements of $\pi_{2M+80}(S^{2M}(B^{(76)}/B^{(65)}))$ are detected by $\mathcal{P}^1$. But $\mathcal{P}^1 H^{73}(B; Z_5) = \langle \mathcal{P}^1(x_{11}x_{27}x_{35}) \rangle = 0$. So, we have (7. 9).

In (7. 8) we may replace $\bar{\pi}$ and $\bar{\pi}_1$ by the compositions with 5-equivalences of L_0. For example, we may assume that $\bar{\pi}|S^{2M}B^{(19)}$ is homotopic to $\bar{\pi}_1|S^{2M}B^{(19)}$. The primary obstruction to extend this homotopy is in $H^{2M+27}(S^{2M}B; \pi_{2M+27}(L_0 : 5)) \cong \pi_{27}(B_9^3(5) : 5) = \langle [5 \cdot \iota(27)] \rangle$. Let the obstruction be represented by $g: S^{2M+27} \to L_0$, and construct a 5-equivalence $h: L_0 \to L_0$ such that $h|S^{2M+19} = id$ and the difference $id - h$ is represented by an extension of g. Then $h \circ \bar{\pi}|S^{2M}B^{(27)}$ is homotopic to $\bar{\pi}_1|S^{2M}B^{(35)}$. Similarly, by a suitable change of $\bar{\pi}$, we have a homotopy between $\bar{\pi}|S^{2M}B^{(35)}$ and $\bar{\pi}_1|S^{2M}B^{(35)}$. Since $H^n(B; \pi_n(B_9^5(5) : 5)) = 0$ for $35 < n < 57$, the homotopy is extended over $S^{2M}B^{(54)}$. Then the difference $\bar{\pi} - \bar{\pi}_1$ is represented by the composite map

$$S^{2M}B^{(76)} \xrightarrow{p} S^{2M}(B^{(76)}/B^{(54)}) \xrightarrow{G} L_0 .$$

As in the second part of the proof of Proposition 7. 7, $G \circ P|S^{2M}B^{(65)}$ is homotopic to a map $G': S^{2M}B^{(65)} \to L_0^{2M+27}$. Then by (7. 9),

$$\bar{\rho}_* \bar{\pi}_*(\eta) - \bar{\rho}_* \bar{\pi}_{1*}(\eta) = \rho_* G_* p_*(\eta) = \rho_* G_*'(\eta) = 0$$

since $\rho(L_0^{2M+27}) = *$. This shows (7. 8). Q.E.D.

Chapter III The Exceptional Cases

§ 8. Mod p Decomposition of p-torsion Free Exceptional Groups

By [11],

(8. 1) *except the cases*

$$(G, p) = (G_2, 3),\ (E_7, 5),\ (E_7, 7),\ (E_8, 7),$$

each p-torsion free exceptional group G is quasi-p-regular, that is, *G is p-equivalent to a product of spheres and* mod p *Stiefel complexes* $B_m^2(p)$.

For the case $(G, p) = (G_2, 3)$, G_2 is not mod 3 decomposable since

$$H^*(G_2; Z_3) = \Lambda(x_3, \Phi x_3).$$

G_2 is not 3-equivalent to our complex $B_m^k(p)$, but it is 3-equivalent to an analogous complex:

(8. 2) *G_2 is 3-equivalent to a complex B of 3-type* $(3, 11)$ *with a* mod 3 *injection* $f: B \to SU(6) = W_{6,5}$ (cf. Definition 5. 2).

The main purpose of this chapter is to give a mod p decomposition of the remaining three cases in (8. 1).

Theorem 8. 1. (i) $E_7 \underset{5}{\simeq} B_1^5(5) \times B_7^2(5)$,

(ii) $E_7 \underset{7}{\simeq} B_1^3(7) \times B_5^3(7) \times S^{19}$.

(iii) $E_8 \underset{7}{\simeq} B_1^4(7) \times B_{11}^4(7)$.

Theorem 8. 2. $E_7/F_4 \underset{5}{\simeq} B_9^3(5)$ *and* $E_7/G_2 \underset{5}{\simeq} B_7^2(5) \times B_9^3(5)$.

By Proposition 3. 4 and Theorem 5. 4, the complexes $B_m^k(p)$ in the above theorems are characterized, up to p-equivalence, by their p-types $(2m+1, 2m+1+q, \cdots, 2m+1+(k-1)q)$ and the existence of mod p injections

(8. 3) $f: B_m^k(p) \to SU(m+1+(k-1)(p-1))/SU(m)$.

Propositions 7. 1, 7. 2, 7. 3, 7. 4 and 7. 6 show that the complexes

$$B_7^2(5),\quad B_1^3(7),\quad B_5^3(7),\quad B_1^4(7) \quad \text{and} \quad B_{11}^4(7)$$

are characterized by their cohomology rings with the operations $\mathcal{P}^i$ ($i = 1, 2, 3,$) and Φ.

To characterize the complexes $B_9^3(5)$ and $B_1^5(5)$, we need more properties as in Propositions 7. 7 and 7. 8.

Remark 8. 3. If we weaken the existence of the mod p injection (8. 3) to that of a mod p injection

$$(8.3)' \qquad f': B_m^k(p) \to SU(m+N)/SU(m), \ N\text{: large},$$

then $B_9^3(5)$ and $B_1^5(5)$ are not characterized. In fact, $B_9^3(5; a)$, $a \neq 0$, is not 5-equivalent to $B_9^3(5)$ but a mod 5 injection $f': B_9^3(5; a) \to SU(22)/SU(9)$ exists. A complex B of 5-type (3, 11, 19, 27, 35) has a mod 5 injection $B \to SU(22)$ if and only if it is 5-equivalent to a complex $B_1^5(5; a)$ for some $a \in Z_5$, where $B_1^5(5; a)$ is characterized by $H^*(B_1^5(5; \mathrm{a}); Z_5) = \Lambda(x_3, \mathcal{P}^1 x_3, \Phi x_3, \mathcal{P}^1 \Phi x_3, \mathcal{P}^2 \Phi x_3)$ and a mod 5 surjection $B_1^5(5; a) \to B_9^3(5; a)$. $B_1^5(5; a)$ is not 5-equivalent to $B_1^5(5)$ if $a \neq 0$.

The existence of a mod 5 injection $f': B_9^3(5; a) \to SU(22)/SU(9)$ follows from that $i_*'[\alpha_1\beta_1(35)] = 0$ for the inclusion $i': B_9^3(5) \to B_9^4(5)$. In the next section $B_1^5(5)$ is constructed as the total space B of the principal G_2-bundle over $B_9^3(5)$ induced by a mod 5 injection $B_9^3(5) \to E_7/G_2$. Let $g: B_9^3(5) \to BG_2$ be a map which induces B. Consider $B_9^3(5; a) = B_9^3(5)^{(62)} \cup e^{81}$ and $g_*[\alpha_1\beta_1(35)] \in \pi_{80}(BG_2 : 5) \cong \pi_{79}(G_2 : 5)$. Since G_2 is 5-equivalent to $B_1^2(5)$, $\pi_{79}(G_2 : 5) = 0$ by Proposition 6. 6. Thus $g | B_9^3(5)^{(62)}$ can be extended over $g': B_9^3(5; a) \to BG_2$. Then $B_1^5(5; a)$ is realized as the total space of the G_2-bundle induced by g' (cf. (5. 1)).

The proof of Proposition 7. 8 yields that there exists a mod 5 surjection $B_1^5(5; a) \to S^{35}$ if and only if $a = 0$. Thus $B_1^5(5; a)$ is not 5-equivalent to $B_1^5(5)$ if $a \neq 0$. On the other hand, the only obstruction to construct a 5-equivalent $B_1^5(5; a) \to B_1^5(5)$ is the element γ of (7. 7) and it vanishes in $B_1^6(5)$, by Proposition 6. 6. Thus there exists a mod 5 injection $B_1^5(5; a) \to B_1^6(5) \to SU(22)$. We have seen that a mod p injection of the type (8. 3)′ does not necessarily characterize the complex $B_m^k(p)$.

The statement (8. 2) can be generalized as follows.

Proposition 8. 4. *There exists a complex B of p-type* $(3, 4p-1)$ *satisfying the following properties.*

(i) $H^*(B; Z_p) = \Lambda(x_3, \Phi x_3)$.

(ii) *There exists a* mod p *injection* $B \to SU(2p)$.

(iii) *There exists a* mod p *fibering* $B \to B_1^3(p) \to S^{2p+1}$.

(iv) *B is a* mod p *H-space.*

One of the properties (i), (ii) *and* (iii) *characterizes the complex B of type* $(3, 4p-1)$ *up to p-equivalence.*

Proof. For $p=3$ this is true by (8. 2), and also true even for $p=2$ by putting $B=Sp(2)$, $B_1^3(2)=SU(4)$. Let $p\geq 5$. Let $g: B_1^3(p)^{(2p+1)}=S^3 \cup e^{2p+1}\to S^{2p+1}$ be a map of degree 1, and extend it over $B_1^3(p)$. The obstructions are in $H^n(B_1^3(p); \pi_{n-1}(S^{2p+1}: p))$ for $n=2p+4$, $4p-1$, $4p+2$, $6p$, $6p+3$. So, the only obstruction is in $H^{4p-1}(B_1^3(p); \pi_{4p-2}(S^{2p+1}: p)) \cong \pi_{4p-2}(S^{2p+1}:p)=Z_p\langle\alpha_1\rangle$. But this obstruction is trivial since $\mathcal{P}^1 x_{2p+1}=0$ in $B_1^3(p)$. Thus we have a mod p surjection $g: B_1^3(p)\to S^{2p+1}$, and let F be its fibre. Then $H^*(F; Z_p)\cong\Lambda(x_3, \Phi x_3)$, and we obtain B by use of (5. 1) as a complex of type $(3, 4p-1)$ p-equivalent to F. Obviously B satisfies (i), (iii) and (ii) : $B\to B_1^3(p)\to SU(2p)$.

(iv) is proved by constructing a multiplication $B\times B\to B$ directly, where the obstructions are in $H^n(B\times B, B\vee B; \pi_{n-1}(B: p))$. By the fibering (iii) we have an exact sequence

$$\pi_n(S^{2p+1}: p)\to\pi_{n-1}(B: p)\to\pi_{n-1}(B_1^3(p): p)\xrightarrow{g_*}\pi_{n-1}(S^{2p+1}: p).$$

Then it follows from Proposition 6. 6 and (6. 2) that $\pi_{n-1}(B: p)=0$ except for

$$n=4,\ 2p+1,\ 4p-2,\ 4p,\ 6p-4,\ 6p-3,\ 8p-6,\ 8p-5,$$
$$10p-8,\ 10p-7,\ \cdots.$$

Thus $H^n(B\times B, B\vee B; \pi_{n-1}(B: p))=0$, and (iv) is proved.

The proof of the last statement is easy and omitted. Q.E.D.

§ 9. Mod 5 Decomposition of E_7 and E_7/G_2

For $p\geq 5$, the simply connected compact exceptional groups G_2, F_4, E_7 are p-torsion free and they have the following cohomology rings:

$$(9.1)\qquad H^*(G_2; Z_p)=\Lambda(x_3, x_{11}),$$
$$H^*(F_4; Z_p)=\Lambda(x_3, x_{11}, x_{15}, x_{23})$$

and

$$H^*(E_7; Z_p) = \Lambda(x_3, x_{11}, x_{15}, x_{19}, x_{23}, x_{27}, x_{35}),$$

where x_i's are universally transgressive elements of degree i.

As is well-known, we have a sequence of injective homomorphisms

$$G_2 \to Spin(7) \to Spin(9) \to F_4$$

with $Spin(7)/G_2 = S^7$, $Spin(9)/Spin(7) = S^{15}$ and $F_4/Spin(9) = \Pi = S^8 \cup e^{16}$. Thus

(9. 2) *we have an inclusion $i: G_2 \to F_4$ such that $i^*: H^*(F_4) \cong H^*(G_2)$ for* $\dim < 6$.

Next, in [2], we have injective homomorphisms

$$F_4 \to E_6 \to E_7$$

such that for the quotient spaces E_6/F_4 and E_7/E_6,

(9. 3) (i) $H^*(E_6/F_4) = \Lambda(x_9, x_{17})$

and

(ii) $H^*(\Omega(E_7/E_6)) = \Lambda(u_9, u_{17}) \otimes Z[u_{18}, u_{26}]$ *for* $\dim \leq 26$.

Proposition 9. 1. *Let $p \geq 5$. With respect to the above injections $G_2 \subset F_4 \subset E_7$, F_4 (resp. G_2) is totally non-homologous to zero* mod p *in E_7 (resp. F_4), and*

$$H^*(F_4/G_2; Z_p) = \Lambda(x_{15}, x_{23}),$$

$$H^*(E_7/F_4; Z_p) = \Lambda(x_{19}, x_{27}, x_{35})$$

and

$$H^*(E_7/G_2; Z_p) = \Lambda(x_{15}, x_{19}, x_{23}, x_{27}, x_{35}).$$

Proof. Consider the injection homomorphism $i^*: H^*(E_7; Z_p) \to H^*(F_4; Z_p)$. Applying (9. 3) to the fibering $\Omega(E_7/E_6) \to E_6/F_4 \to E_7/F_4$, we see that E_7/F_4 is 8-connected, and by (9. 1) that $i^*: H^*(E_7; Z_p) \cong H^*(F_4; Z_p)$ for $\dim < 11$ and $H^*(E_7/F_4; Z_p) = 0$ for $\dim < 10$. Again by (9. 3) and the above fibering we have $H^*(E_7/F_4; Z_p) = 0$ for $\dim < 17$. And, by (9.1),

$$i^*: H^*(E_7; Z_p) \cong H^*(F_4; Z_p) \quad \text{for } \dim < 19.$$

Next, consider the mod p spectral sequence associated with the fibering

$$E_7 \to E_7/F_4 \to BF_4,$$

in which $E_2^{**} \cong H^*(BF_4; Z_p) \otimes H^*(E_7; Z_p) = Z_p[x_4, x_{12}, x_{16}, x_{24}] \otimes \Lambda(x_3, x_{11}, x_{15}, x_{19}, x_{23}, x_{27}, x_{35})$. By the above discussion, the transgression τ satisfies $\tau(x_i) = x_{i+1}$ for $i=3$, 11, 15, up to non-zero coefficients. Then $E_{17}^{**} \cong Z_p[x_{24}] \otimes \Lambda(x_{19}, x_{23}, x_{27}, x_{35})$. For dimensional reasons, $\tau(x_i)=0$ for $i=19$, 27, 35. Since E_7/F_4 is finite dimensional, $\tau(x_{23})$ must be non-trivial. Thus $E_\infty^{**} \cong \Lambda(x_{19}, x_{27}, x_{35}) \cong H^*(E_7/F_4; Z_p)$. Then, by (9.1), we see that the spectral sequence associated with the fibering $F_4 \to E_7 \to E_7/F_4$ collapses, and F_4 is totally non-homologous to zero in E_7.

Similarly but more easily, from (9.2) and (9.1) we have that G_2 is totally non-homologous to zero mod p in F_4 and $H^*(F_4/G_2; Z_p) \cong \Lambda(x_{15}, x_{23})$. The last statement also follows easily. Q.E.D.

Note that Proposition 9.1 is valid for $p=2$, 3 ([1, 2]):

(9.4) *For $p=2$, 3, F_4 (resp. G_2) is totally non-homologous to zero* mod p *in E_7 (resp. F_4).*

Now we consider the case $p=5$. By Theorem 4.2 of [11]

(9.5) $$\mathcal{P}^1 x_3 = x_{11},\ \mathcal{P}^1 x_{11} = 0 \text{ in } G_2,\ F_4,\ E_7,$$
$$\mathcal{P}^1 x_{15} = x_{23},\ \mathcal{P}^1 x_{23} = 0 \text{ in } F_4,\ E_7,$$

and

$$\mathcal{P}^1 x_{19} = x_{27},\ \mathcal{P}^1 x_{27} = x_{35} \text{ and } \mathcal{P}^1 x_{35} = 0 \text{ in } E_7.$$

Then it follows from Proposition 9.1 that

(9.6) $$H^*(F_4/G_2; Z_5) = \Lambda(x_{15}, \mathcal{P}^1 x_{15}),$$
$$H^*(E_7/F_4; Z_5) = \Lambda(x_{19}, \mathcal{P}^1 x_{19}, \mathcal{P}^2 x_{19})$$

and

$$H^*(E_7/G_2;\, Z_5) = \Lambda(x_{15}, x_{19}, \mathcal{P}^1 x_{15}, \mathcal{P}^1 x_{19}, \mathcal{P}^2 x_{19}).$$

By Theorem 2.3 of [14]

(9.7) $\quad \Phi x_3 = x_{19} \quad in \ H^*(E_7;\, Z_5).$

Proposition 9.2. *E_7/F_4 is 5-equivalent to $B_9^3(5)$.*

Proof. By (5.1), we may assume that E_7/F_4 is a complex of 5-type (19, 27, 35). Apply Theorem 6.1 to the inclusion $S^3\ (\to G_2) \to E_7$. Since $\mathcal{P}^1 x_{11} = 0$, we have $\pi_{19}(E_7, S^3 : 5) = Z_{(5)}\langle[\iota(19)]\rangle$. By use of a representative $(E^{19}, S^{18}) \to (E_7, S^3)$ of $[\iota(19)]$, we get a complex $S^3 \cup e^{19}$ and a mod 5 injection

$$g : S^3 \cup e^{19} \to E_7, \quad g^*(x_{19}) \neq 0.$$

Then it is easy to see that the composite map

$$\mu : (S^3 \cup e^{19}) \times (E_7/F_4) \xrightarrow{g \times id} E_7 \times (E_7/F_4) \xrightarrow{\text{action}} E_7/F_4$$

satisfies the condition (ii) of Proposition 6.6. Thus the proposition is proved by (9.6) and Proposition 6.6. Q.E.D.

Let

$$\Psi : \operatorname{Ker} \mathcal{P}^2 (\subset H^{15}(X;\, Z_5)) \to H^{54}(X;\, Z_5)/\mathcal{P}^3 H^{30}(X;\, Z_5)$$

be the secondary operation associated with the Adem relation

$$\mathcal{P}^3 \mathcal{P}^2 = 0.$$

As is well-known (cf. § 6), Ψ detects the generator $\beta_1(15)$ of $\pi_{53}(S^{15}:5) \cong Z_5$.

Lemma 9.3. *Let $X = E_7$ and E_7/G_2. The secondary operation Ψ is defined on the generator x_{15} of $H^{15}(X;\, Z_5)$ and $\Psi(x_{15}) = 0$ with the trivial indeterminacy.*

Proof. We consider the secondary operation Ψ for $X = E_7$, E_7/G_2 and $E_7 \times E_7$. For these spaces X, $H^{31}(X;\, Z_5) = 0$. Thus Ψ is defined on the whole of $H^{15}(X; Z_5)$. We see also $H^{30}(E_7; Z_5)=Z_5\langle x_3x_{27}, x_{11}x_{19}\rangle$, $H^{30}(E_7/G_2; Z_5)=0$ and $H^{30}(E_7\times E_7; Z_5)=Z_5\langle 1\otimes x_3x_{27}, 1\otimes x_{11}x_{19}, x_3\otimes x_{27}, x_{11}\otimes x_{19}, x_{15}\otimes x_{15}, \cdots, x_{11}x_{19}\otimes 1\rangle$. By Cartan's formula and (9.5), we have that $\mathcal{P}^3H^{30}(X; Z_5)=0$ and that the secondary operation

$$\Psi: H^{15}(X; Z_5)\to H^{54}(X; Z_5)$$

is a well-defined, single valued and natural homomorphism for $X=E_7$, E_7/G_2 and $E_7\times E_7$.

For the multiplication $\mu: E_7\times E_7\to E_7$ and the projections $p_1, p_2: E_7\times E_7\to E_7$ to each factor, we have

$$\begin{aligned}\mu^*(\Psi(x_{15})) &= \Psi(\mu^*(x_{15}))=\Psi(x_{15}\otimes 1+1\otimes x_{15})\\ &=\Psi(p_1^*(x_{15})+p_2^*(x_{15}))=p_1^*(\Psi(x_{15}))+p_2^*(\Psi(x_{15}))\\ &=\Psi(x_{15})\otimes 1+1\otimes\Psi(x_{15}).\end{aligned}$$

This shows that $\Psi(x_{15})\in H^{54}(E_7; Z_5)=Z_5\langle x_{19}x_{35}\rangle$ is primitive, while $x_{19}x_{35}$ is not primitive. Thus $\Psi(x_{15})=0$ in $H^{54}(E_7; Z_5)$. Let $\pi: E_7\to E_7/G_2$ be the projection. By Proposition 9.1, $\pi^*: H^*(E_7/G_2; Z_5)\to H^*(E_7; Z_5)$ is injective. Then the naturality $\pi^*\Psi=\Psi\pi^*$ implies that $\Psi(x_{15})=0$ in $H^{54}(E_7/G_2; Z_5)$. Q.E.D.

Lemma 9.4. *There exists a* mod 5 *injection* $f: B_9^3(5)\to E_7/G_2$.

Proof. Let $f': B_9^3(5)\to E_7/F_4$ be a 5-equivalence given by Proposition 9.2 and let $\pi: E_7/G_2\to E_7/F_4$ be the bundle map with the fibre F_4/G_2. By (9.6) and Proposition 7.1, $B_7^2(5)$ is 5-equivalent to F_4/G_2.

We consider to lift the map f' to a map $f: B_9^3(5)\to E_7/G_2$ such that $\pi\circ f$ is homotopic to $f'\circ h$ for a 5-equivalence $h: B_9^3(5)\to B_9^3(5)$. The obstruction to the lifting is in $H^n(B_9^3(5); \pi_{n-1}(F_4/G_2: 5))$. The homotopy groups $\pi_*(F_4/G_2: 5)\cong\pi_*(B_7^2(5): 5)$ are computed by applying Theorem 6.1 to a mod 5 injection $S^{15}\to F_4/G_2$ or by use of the homotopy exact sequence for the fibering $S^{15}\to B_7^2(5)\to S^{23}$ (cf. [12]). Then we have

(9.8) $\pi_i(F_4/G_2: 5)=0$ *except for* $i=15, 23, 30, 38, 46, 53, 54, 62, 68, 70, 77, 78, 84, \cdots$, *where* $\pi_{53}(F_4/G_2: 5)\cong Z_5$ *is generated by the injection image of* $\beta_1(15)\in\pi_{53}(S^{15}: 5)$.

Note that $\pi_i(S^{15}:5)$ is unstable for $i\geqq 77$. Then the only obstruction is in $H^{54}(B_9^3(5);\pi_{53}(F_4/G_2:5))$. So, there is a partial lifting $f_1: B_9^3(5)^{(45)}\to E_7/G_2$. Let $\xi\in\pi_{54}(B_9^3(5)^{(45)})$ be the attaching class of the cell $e^{54}=B_9^3(5)^{(54)}-B_9^3(5)^{(45)}$, then

$$f_{1*}(\xi)=i_*(a\cdot\beta_1(15))$$

for some $a\in Z_5$ and for a mod 5 injection $i: S^{15}\to F_4/G_2\to E_7/G_2$. The obstruction vanishes if and only if $a=0$, whence the required lifting f exists.

Construct a complex

$$K=(B_9^3(5)^{(45)}\vee S^{15})\cup e^{54}$$

by attaching a 54-cell e^{54} by $i_{1*}(\xi)+i_{2*}(-a\cdot\beta_1(15))$, where i_1 and i_2 are the inclusions of $B_9^3(5)^{(45)}$ and S^{15} into $B_9^3(5)^{(45)}\vee S^{15}$ respectively. Then the map $f_1\vee i: B_9^3(5)\vee S^{15}\to E_7/G_2$ can be extended over a map $h_2: K\to E_7/G_2$ such that

$$h_2^*: H^i(E_7/G_2; Z_5)\cong H^i(K; Z_5) \text{ for } i=15,\ 54\,.$$

As before we see that the secondary operation $\Psi: H^{15}(K; Z_5)\to H^{54}(K; Z_5)$ is well-defined and single valued. By the naturality $\Psi h_2^*=h_2^*\Psi$, $\Psi H^{15}(K; Z_5)=0$. Next let

$$p: K\to L=K/B_9^3(5)^{(45)}=S^{15}\cup e^{54}$$

be the map smashing $B_9^3(5)^{(45)}$. Then $p^*: H^i(L; Z_5)\cong H^i(K; Z_5)$ for $i=15$, 54, and $\Psi H^{15}(L; Z_5)=0$. Since L is a mapping cone of $a\cdot\beta_1(15)$ and since Ψ detects $\beta_1(15)$, we have that $a=0$, and the existence of the lifting f is proved. Q.E.D.

Proof of Theorem 8.2. Proposition 9.2 shows the first assertion. By Theorem 4.5 of [11], there is a 5-equivalence $B_1^2(5)\times B_7^2(5)\to F_4$. So, we have a mod 5 injection

$$g: B_7^2(5)\to F_4\to E_7\,,$$

by Proposition 9.1. Then it is easy to see that the composite map

$$B_7^2(5)\times B_9^3(5)\xrightarrow{g\times f}E_7\times(E_7/G_2)\xrightarrow{\text{action}}E_7/G_2$$

is a 5-equivalence, where f is the mod 5 injection given in Proposition

9.4. Q.E.D.

Proof of (i) *of Theorem* 8.1. The mod 5 injection $f:B_9^3(5)\to E_7/G_2$ of Lemma 9.4 induces a map $\bar{f}: B\to E_7$ of principal G_2-bundles, where B is the induced G_2-bundle over $B_9^3(5)$. It follows from Proposition 9.1 that G_2 is totally non-homologous to zero mod 5 in B. Then we have easily that $\bar{f}$ is a mod 5 injection, the projection $\pi:B\to B_9^3(5)$ is a mod 5 surjection and

$$H^*(B;Z_5)=\Lambda(x_3,\mathcal{P}^1x_3,\Phi x_3,\mathcal{P}^1\Phi x_3,\mathcal{P}^2\Phi x_3)$$

by (9.5) and (9.7). By (5.1), we may assume that B is a complex of 5-type (3, 11, 19, 27, 35). Then it follows from Proposition 7.5 that B is 5-equivalent to $B_1^5(5)$. Thus we have a mod 5 injection $f: B_1^5(5)\to E_7$. Then the composite map

$$B_1^5(5)\times B_7^2(5)\xrightarrow{f\times g}E_7\times E_7\xrightarrow{\mu}E_7$$

is a 5-equivalence. Q.E.D.

§10. Mod 7 Decomposition of E_7 and E_8

By Theorem 4.2 of [11] and Theorem 2.3 of [14],

(10.1) *in* $H^*(E_7;Z_7)=\Lambda(x_3,x_{11},x_{15},x_{19},x_{23},x_{27},x_{35})$,

$\mathcal{P}^1x_i=x_{i+12}$ *for* $i=3,\ 11,\ 23$,

$\mathcal{P}^1x_i=0$ *for* $i=15,\ 19,\ 27,\ 35$

and $\Phi x_3=x_{27}$;

(10.2) *in* $H^*(E_8;Z_7)=\Lambda(x_3,x_{15},x_{23},x_{27},x_{35},x_{39},x_{47},x_{59})$,

$\mathcal{P}^1x_i=x_{i+12}$ *for* $i=3,\ 23,\ 27,\ 35,\ 47$,

$\mathcal{P}^1x_i=0$ *for* $i=15,\ 39,\ 59$

and $\Phi x_3=x_{27}$.

Theorem 2.3 of [14] is based on the existence of homomorphisms

$$\lambda: E_7\to U(56)\quad\text{and}\quad\mu: E_8\to U(240)$$

such that λ^* and μ^* are isomorphisms of $H^3(\ ;Z_7)$. By use of Theorem 3.5, we have mod 7 surjections

$$\pi_7: U(56) \to B_1^{10}(7) \quad \text{and} \quad \pi_8: U(240) \to B_1^{40}(7).$$

Let F_7 and F_8 be the fibres of the compositions $p_7=\pi_7\circ\lambda$ and $p_8=\pi_8\circ\mu$ respectively. Thus we have fiberings

$$(10.3)\qquad \begin{aligned} &F_7 \xrightarrow{i_7} E_7 \xrightarrow{p_7} B_1^{10}(7), \\ &F_8 \xrightarrow{i_8} E_8 \xrightarrow{p_8} B_1^{40}(7). \end{aligned}$$

Lemma 10.1. (i) $H^*(F_7;\ Z_7) = \Lambda(x_{11},\ x_{19},\ x_{23},\ x_{35}) \otimes Z_7[x_{38+12j};\ j=0, 1, 2, \cdots, 6)$ *for* $\dim<264$, *where* $\mathcal{P}^1 x_i = x_{i+12}$ *for* $i=11,\ 27,\ 38+12j$ $(0\leq j<5)$.

(ii) $H^*(F_8;\ Z_7) = \Lambda(x_{23}, x_{35}, x_{47}, x_{59}) \otimes Z_7[x_{50+12j};\ j=0, 1, 2, \cdots]$ *for* $\dim<348$, *where* $\mathcal{P}^1 x_i = x_{i+12}$ *for* $i=23,\ 35,\ 47$ *and for* $i=50+12j$, $j\not\equiv 4 \pmod 7$.

Proof. We shall prove (i). The proof of (ii) is similar, and left to the reader. As is well-known,

$$H^*(U(56);Z_7) = \Lambda(y_{2i+1};\ i=0, 1, 2, \cdots, 55)$$

for the suspension image y_{2i+1} of the $(i+1)$-th Chern class c_{i+1}. It follows from the Wu formula

$$(10.4)\qquad \mathcal{P}^t y_{2i+1} = \binom{i}{t} y_{2i+12t+1}.$$

We shall show

(10.5) $H^*(B_1^{10}(7);\ Z_7) = \Lambda(x'_{3+12j};\ j=0,\ 1,\ 2,\ \cdots,\ 9)$ *for generators* $\{x'_{3+12j}\}$ *satisfying* $p_7^*(x'_{3+12j}) = x_{3+12j}$ $(i=0, 1, 2)$, $p_7^*(x'_{3+12j}) = 0$ $(i=3, 4, \cdots, 9)$ *and* $\mathcal{P}^t x'_{27} = x'_{27+12t}$ $(t=1, 2, \cdots, 7)$.

Since $p_7^*: H^3(B_1^{10}(7);Z_7) \cong H^3(E_7;Z_7)$, we put $x_3' = p_7^{*-1}(x_3)$, $x'_{15} = \mathcal{P}^1 x_3'$, $x'_{27} = \Phi x_3'$ and $x'_{27+12t} = \mathcal{P}^t x'_{27}$ for $1\leq t\leq 7$. By the naturality, $p_7^*(x_k') = x_k$ for $k=3,\ 15,\ 27$. By (10.4), $\pi_7^*(x_k')$ are indecomposable. For $1\leq t\leq 7$, $p_7^*(x'_{27+12t}) = \mathcal{P}^t x_{27}$ is primitive and it vanishes since the generators $\{x_i\}$ of (10. 1) span the set of the primitive elements in $H^*(E_7; Z_7)$. Thus (10. 5) is proved.

Let $\{E_r^{**}\}$ be the mod 7 cohomology spectral sequence associated with the first fibering of (10. 3): $E_2^{**}=H^*(B_1^{10}(7); Z_7)\otimes H^*(F_7; Z_7)$. We construct a formal spectral sequence $\{'E_r^{**}\}$ by putting

$$'E_2^{**}=H^*(B_1^{10}(7); Z_7)\otimes F^*$$

with $$F^*=\Lambda(x_{11}, x_{19}, x_{23}, x_{35})\otimes Z_7[x_{38+12j}; 0\leq j\leq 6],$$

and by giving derivative differentials d_r by

$$d_r(H^*(B_1^{10}(7); Z_7)\otimes 1)=0,$$

$$d_r(1\otimes x_i)=0 \qquad \text{for } i=11,\ 19,\ 23,\ 35,$$

$$d_r(1\otimes x_{38+12j})=0 \ (r\leq 38+12j) \quad =x_{39+12j}\otimes 1 \ (r=39+12j).$$

Then it is directly verified that

$$'E_\infty^{**}\cong\Lambda(x_3, x_{15}, x_{27})\otimes(x_{11}, x_{19}, x_{23}, x_{35}) \text{ for } \dim<265.$$

The natural map defines a map f_2: $'E_2^{**}\rightarrow E_2^{**}$. The differential d_r in E_r^{**} satisfies the properties corresponding to those in $'E_r^{**}$. Thus f_2 induces a map f_r: $'E_r^{**}\rightarrow E_r^{**}$ of spectral sequences such that f_∞ is bijective for $\dim<265$. By virtue of the comparison theorem, it follows from f_2: $'E_2^{*,0}\cong E_2^{*,0}$ that f_2: $'E_2^{0,*}=F^*\rightarrow E_2^{0,*}=H^*(F_4; Z_7)$ is bijective for $\dim<264$. Thus (i) is proved. Q.E.D.

Now apply Theorem 6.1 to F_7 and F_8. Then the follwing results are computed as in § 6.

Lemma 10. 2. (i) $\pi_i(F_7:7)=0$ *except for* $i=11$, 19, 23, 30, 35, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 93, 94, $\cdots$. *For* $i=11$, 19, *there are* mod 7 *injections of* S^i *into* F_7.

(ii) $\pi_i(F_8:7)=0$ *except for* $i=23$, 35, 47, 50, 59, 62, 70, 74, 82, 86, 94, 98, 105, 106, 109, 110, 118, 121, 122, 130, 132, 133, 134, 142, 145, 146, 152, 154, 157, 158, 165, 166, $\cdots$. *There is a* mod 7 *injection* i: $S^{23}\rightarrow F_8$ *such that* $i_*(\beta_1(23))$ *generates* $\pi_{105}(F_8:7)\cong Z_7$.

Proof of (ii) *of Theorem* 8. 1. By (i) of Lemma 10. 2, we have mod 7 injections i_1': $S^{11}\to F_7$ and i_2: $S^{19}\to F_7$. Also we have $H^n(B_5^3(7); \pi_{n-1}(F_7: 7))=0$. Thus i_1' can be extended, in the mod 7 sense, over a mod 7 injection i_1: $B_5^3(7)\to F_7$.

For the inclusion f': $B_1^3(7)\to B_1^{10}(7)$, consider a lifting f: $B_1^3(7)\to E_7$ such that $p_7\circ f\simeq f'\circ h$ for a 7-equivalence h: $B_1^3(7)\to B_1^3(7)$. The obstructions to the lifting are trivial since $H^n(B_1^3(7); \pi_{n-1}(F_7: 7))=0$ by (i) of Lemma 10.1. Then f exists and it is a mod 7 injection. The composite map

$$B_1^3(7)\times B_5^3(7)\times S^{19}\xrightarrow{f\times i_1\times i_2} E_7\times F_7\times F_7$$
$$\xrightarrow{id\times i_7\times i_7} E_7\times E_7\times E_7\xrightarrow{\text{multip}} E_7$$

is a 7-equivalence since it induces an epimorphism, thus an isomorphism, of the mod 7 cohomology. Q.E.D.

Proof of (iii) *of Theorem* 8.1. The proof is done similarly to that of (ii) by using (ii) of Lemma 10.2. The only difficulty is that $H^n(B_{11}^4(7); \pi_{n-1}(F_8: 7))\neq 0$ if $n=106$. This obstruction is avoided by use of Lemma 7.5 as in proving Proposition 7.6. Thus (iii) of Theorem 8.1 is proved. Q.E.D.

References

[1] Araki, S., On the non-commutativity of Pontrjagin rings of mod 3 of some compact exceptional groups, *Nagoya Math. J.*, **17** (1960), 225-260.
[2] ———, Cohomology modulo 2 of the compact exceptional groups E_6 and E_7, *J. Math. Osaka City Univ.*, **12** (1961), 43-65.
[3] Artin, M. and Mazur, B., Etale homotopy, *Lecture Notes in Math.*, **100** Springer (1969).
[4] Borel, A., Sous groupes commutatifs et torsion des groupes de Lie compacts connexes, *Tôhoku Math. J.*, **13** (1961), 216-240.
[5] Bousfield, A. K. and Kan, D. M., Homotopy limits, completions and localizations, *Lecture Notes in Math.*, **304** Springer (1972).
[6] Harris, B., On the homotopy groups of classical groups, *Ann. Math.*, **74** (1961), 407-413.
[7] Imanishi, H., Unstable homotopy groups of classical groups (odd primary components), *J. Math. Kyoto Univ.*, **7** (1967), 221-243.
[8] James, I. M., Cross-sections of Stiefel manifolds, *Proc. London Math. Soc.*, 8 (1958), 536-547.
[9] Lubkin, S., On a conjecture of Weil, *Amer. J. Math.*, **89** (1967), 443-548.
[10] Massey, W. S., Exact couples in algebraic topology. I, II, III, IV, V, *Ann. Math.*, **56** (1952), 363-396; **57** (1953), 248-286.
[11] Mimura, M. and Toda, H., Cohomology operations and the homotopy of compact

Lie groups, I, *Topology*, **9** (1970), 317-336.

[12] ———, Cohomology operations and the homotopy of compact Lie groups, II, (*mimeographed note*) (1970).

[13] ———, On p-equivalences and p-universal spaces, *Comment. Math. Helv.*, **46** (1971), 87-97.

[14] ———, A representation and the quasi p-regularity of the compact Lie groups, *Japanese J. Math.*, **1** (1975), 101-109.

[15] Mimura, M., O'Neill, R. C. and Toda, H., On p-equivalence in the sense of Serre, *Japanese J. Math.*, **40** (1971), 1-10.

[16] Mimura, M., Nishida, G. and Toda, H., Localization of CW-complexes and its applications, *J. Math. Soc. Japan*, **23** (1971), 593-624.

[17] Nishida, G., On a result of Sullivan and the mod p decomposition of Lie groups, (*mimeographed note*) (1971).

[18] Oka, S., On the homotopy groups of sphere bundles over spheres, *J. Sci. Hiroshima Univ.*, **33** (1966), 161-195.

[19] Peterson, F. P., The mod p homotopy type of BSO and F/PL, *Bol. Soc. Math. Mexicana*, **14** (1969), 22-27.

[20] Serre, J.-P., Groupes d'homotopie et classes des groupes abeliens, *Ann. Math.*, **58** (1953), 258-294.

[21] Sullivan, D., Geometric topology, part I, Localization, periodicity and Galois symmetry, *Notes, M. I. T.* (1970).

[22] ———, Genetics of homotopy theory, *Ann. Math.*, **100** (1975), 1-79.

[23] Toda, H., p-primary components of homotopy groups, IV, *Mem. Coll. Sci. Kyoto Univ.*, **32** (1959), 297-332.

[24] ———, On iterated suspensions, I, II, III, *J. Math. Kyoto Univ.*, **5** (1965), 82-142; **5** (1966), 209-250; **8** (1968), 101-130.

[25] Yokota, I., On the homology of classical Lie groups, *J. Inst. Polytech. Osaka City Univ.*, **8** (1957), 93-120.

[26] Zeeman, E. C., A proof of the comparison theorem for spectral sequences, *Proc. Cambridge Phil. Soc.*, **53** (1957), 57-62.