Publ. RIMS, Kyoto Univ.
13 (1977), 723-753

A Finite Element Method for Solving the Two Phase Stefan Problem in One Space Dimension[1)]

By

Masatake MORI*

Abstract

A finite element method based on time dependent basis functions is presented for solving a two phase Stefan problem for the heat equation in one space dimension. The stability and the convergence of the method are studied, and a numerical example is given.

§ 1. A Two Phase Stefan Problem in One Space Dimension

This paper is concerned with a finite element solution of the following two phase Stefan problem in one space dimension for the heat equation. Given the data $g_1(t)$, $g_2(t)$, $f_1(x)$, $f_2(x)$ and b, find functions $u_1(x,t)$, $u_2(x,t)$ and $x=s(t)$ in $0<x<L$, $0<t\leqq T$, such that

$$(1.1)\qquad \begin{cases} \dfrac{\partial u_1}{\partial t}=\sigma_1\dfrac{\partial^2 u_1}{\partial x^2}, & 0<x<s(t),\ 0<t\leqq T, \\[2ex] \dfrac{\partial u_2}{\partial t}=\sigma_2\dfrac{\partial^2 u_2}{\partial x^2}, & s(t)<x<L,\ 0<t\leqq T, \end{cases}$$

$$(1.2)\qquad \begin{cases} u_1(0,t)=g_1(t),\ u_1(s(t),t)=0, & 0<t\leqq T, \\ u_2(L,t)=g_2(t),\ u_2(s(t),t)=0, & 0<t\leqq T, \end{cases}$$

$$(1.3)\qquad s(0)=b,\quad 0<b<L,$$

$$(1.4)\qquad \begin{cases} u_1(x,0)=f_1(x), & 0\leqq x\leqq b, \\ u_2(x,0)=f_2(x), & b\leqq x\leqq L, \end{cases}$$

Received November 11, 1976.

* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

1) Part of this work was reported in the Japan-France Seminar on Numerical Analysis and Functional Analysis, Tokyo and Kyoto, 1976, September [9].

$$(1.5)\qquad \frac{ds}{dt}=-\kappa_1\frac{\partial u_1}{\partial x}(s(t),t)+\kappa_2\frac{\partial u_2}{\partial x}(s(t),t),\ 0<t\leqq T,$$

where σ_1, σ_2, κ_1, κ_2 and L are positive constants, and T is an arbitrarily fixed positive number. The last equation (1.5) expresses the heat balance and is called the Stefan condition. This condition (1.5) gives the speed of propagation of the free boundary. Specifically, the functions u_1 and u_2 may be interpreted as the temperature of the water existing in $0<x<s(t)$ and that of the ice existing in $s(t)<x<L$, respectively, which contact at the front $x=s(t)$ with each other.

For the moment we assume an appropriate smoothness for g_1, g_2, f_1 and f_2, and we make the following four assumptions for these Stefan data. First we assume that the initial data are bounded by quadratic functions both from above and from below:

Assumption A. There exist positive constants A_1', A_2', B_1' and B_2' such that

$$(1.6)\qquad \begin{cases} B_1'\left\{\left(\dfrac{x}{b}-2\right)^2-1\right\}\leqq f_1(x)\leqq A_1'\left\{1-\dfrac{x^2}{b^2}\right\},\ 0\leqq x\leqq b,\\ -A_2'\left\{1-\dfrac{(L-x)^2}{(L-b)^2}\right\}\leqq f_2(x)\leqq -B_2'\left\{\left(\dfrac{L-x}{L-b}-2\right)^2-1\right\},\\ \hfill b\leqq x\leqq L.\end{cases}$$

For the boundary data we make

Assumption B. There exist positive constants A_1'', B_1'', A_2'' and B_2'' such that

$$(1.7)\qquad \begin{cases} 0<3B_1''\leqq g_1(t)\leqq A_1'',\quad 0\leqq t\leqq T,\\ -A_2''\leqq g_2(t)\leqq -3B_2''<0,\quad 0\leqq t\leqq T.\end{cases}$$

Set

$$(1.8)\qquad \begin{cases} A_1=\max(A_1',A_1''),\quad B_1=\min(B_1',B_1''),\\ A_2=\max(A_2',A_2''),\quad B_2=\min(B_2',B_2''),\end{cases}$$

and define

(1.9) $$b_m=\frac{\kappa_1 B_1}{\kappa_2 A_2+\kappa_1 B_1}L, \quad b_M=\frac{\kappa_1 A_1}{\kappa_1 A_1+\kappa_2 B_2}L.$$

We shall show later that b_m and b_M are a lower bound and an upper bound of $s(t)$, respectively. For the initial position b of the free boundary we make

Assumption C.

(1.10) $$b_m \leqq b \leqq b_M.$$

Next we define

(1.11) $$\begin{cases} \gamma_+=\dfrac{2}{L}(\kappa_2 A_2+\kappa_1 B_1)\left(\dfrac{A_1}{B_1}-\dfrac{B_2}{A_2}\right), \\ \gamma_-=\dfrac{2}{L}(\kappa_1 A_1+\kappa_2 B_2)\left(\dfrac{A_2}{B_2}-\dfrac{B_1}{A_1}\right), \end{cases}$$

and set

(1.12) $$\gamma=\max(\gamma_+, \gamma_-)$$

and

(1.13) $$\eta=\max\left(\frac{b_M}{\sigma_1}, \frac{L-b_m}{\sigma_2}\right).$$

Then finally we make

Assumption D.

(1.14) $$\gamma\eta\leqq 1.$$

From Assumptions A and B it is obvious that

(1.15) $$3B_1\leqq A_1, \quad 3B_2\leqq A_2,$$

so that from (1.11) we have $\gamma_+>0$ and $\gamma_->0$. As we will see later the physical meaning of γ_+ and γ_- would be more evident if we rewrite (1.11) equivalently as

(1.16) $$\gamma_+=\frac{2\kappa_1 A_1}{b_m}-\frac{2\kappa_2 B_2}{L-b_m}, \quad \gamma_-=\frac{2\kappa_2 A_2}{L-b_M}-\frac{2\kappa_1 B_1}{b_M}.$$

In the preceding paper [7] we proposed a finite element scheme for solving the one phase problem and discussed the stability and the convergence of the scheme assuming that the initial data is bounded by a linear function from above. In the one phase problem the free boundary function $s(t)$ is monotone with respect to t, while in the two phase problem $s(t)$ is not monotone in general, so that we need quadratic or some other kind of functions as the bounding functions of the initial data in order that the maximum principle holds with our scheme. Cannon and Primicerio [2, 3] proved the existence and the uniqueness of the solution of (1. 1)-(1. 5) assuming that the initial data are bounded by exponential functions under a similar assumption as Assumption D.

Various numerical methods have been presented for solving the one phase Stefan problem in one space dimension. See, for example, Douglas and Gallie [4], Meyer [6], Bonnerot and Jamet [1], Nogi [10], Kawarada and Natori [5] and Mori [7]. See also Mori [8] for the numerical solution of the Stefan problem in two space dimension.

The purpose of the present paper is to present a finite element method for solving (1. 1)-(1. 5), and to discuss the stability and the convergence of the method.

Before proceeding to the numerical method, we shall give here some remarks on the quantities b_m, b_M, γ_+ and γ_-. First we claim that under Assumptions A, B, C and D the solutions u_1 and u_2 of (1. 1)-(1. 5) are always bounded by quadratic functions as follows (see Fig. 1) provided that $|ds/dt| \leqq \gamma$:

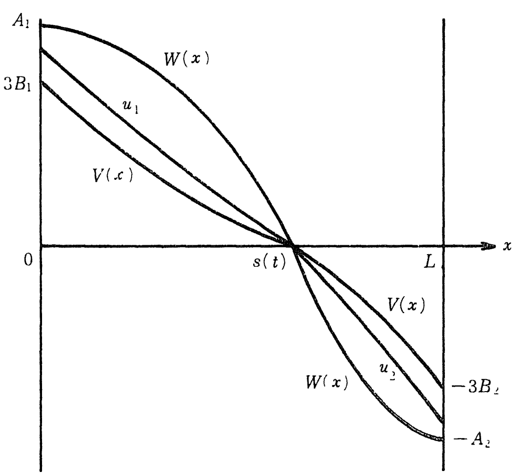

Fig. 1.

$$
(1.17) \quad \begin{cases} V(x) \leqq u_1(x,t) \leqq W(x), & 0 \leqq x \leqq (t),\ 0 \leqq t \leqq T, \\ W(x) \leqq u_2(x,t) \leqq V(x), & s(t) \leqq x \leqq L,\ 0 \leqq t \leqq T, \end{cases}
$$

where

$$(1.18)\qquad W(x)=\begin{cases} A_1\left\{1-\dfrac{x^2}{s^2(t)}\right\}, & 0\leqq x\leqq s(t) \\ -A_2\left\{1-\dfrac{(L-x)^2}{(L-s(t))^2}\right\}, & s(t)\leqq x\leqq L \end{cases}$$

$$(1.19)\qquad V(x)=\begin{cases} B_1\left\{\left(\dfrac{x}{s(t)}-2\right)^2-1\right\}, & 0\leqq x\leqq s(t) \\ -B_2\left\{\left(\dfrac{L-x}{L-s(t)}-2\right)^2-1\right\}, & s(t)\leqq x\leqq L. \end{cases}$$

The mathematical tool which can be used in order to prove these inequalities is the maximum principle. We shall show here only the inequality

$$(1.20)\qquad W(x)=A_1\left\{1-\frac{x^2}{s^2(t)}\right\}\geqq u_1(x,t),\quad 0\leqq x\leqq s(t).$$

Define

$$(1.21)\qquad d_1(x,t)\equiv A_1\left\{1-\frac{x^2}{s^2(t)}\right\}-u_1(x,t),$$

then, if $|ds/dt|\leqq\gamma$, we have

$$(1.22)\qquad \left(\frac{\partial}{\partial t}-\sigma_1\frac{\partial^2}{\partial x^2}\right)d_1(x,t)=-A_1\left(\frac{\partial}{\partial t}-\sigma_1\frac{\partial^2}{\partial x^2}\right)\frac{x^2}{s^2(t)}$$

$$=\frac{2A_1\sigma_1}{s^2(t)}\left(\frac{x^2}{\sigma_1 s(t)}\times\frac{ds}{dt}+1\right)$$

$$\geqq\frac{2A_1\sigma_1}{s^2(t)}\left(1-\frac{s(t)}{\sigma_1}\left|\frac{ds}{dt}\right|\right)\geqq\frac{2A_1\sigma_1}{s^2(t)}(1-\eta\gamma)$$

by (1.12) and (1.13), so that from Assumption D

$$(1.23)\qquad \left(\frac{\partial}{\partial t}-\sigma_1\frac{\partial^2}{\partial x^2}\right)d_1(x,t)\geqq 0.$$

Therefore, we obtain (1.20) using the maximum principle in view of Assumption A, the boundary condition (1.2) and Assumption B. The other three inequalities in (1.17) can be proved in a similar way.

If we assume that the derivatives of u_1 and u_2 at $x=s(t)$ exist (cf. [2]), then from (1.17) we have

$$(1.24)\qquad \begin{cases} -\dfrac{2A_1}{s(t)} \leqq \dfrac{\partial u_1}{\partial x}(s(t), t) \leqq -\dfrac{2B_1}{s(t)} \\ -\dfrac{2A_2}{L-s(t)} \leqq \dfrac{\partial u_2}{\partial x}(s(t), t) \leqq -\dfrac{2B_2}{L-s(t)}, \end{cases}$$

so that from (1.5)

$$(1.25)\qquad \frac{2\kappa_1 A_1}{s(t)} - \frac{2\kappa_2 B_2}{L-s(t)} \geqq \frac{ds}{dt} \geqq -\left(\frac{2\kappa_2 A_2}{L-s(t)} - \frac{2\kappa_1 B_1}{s(t)}\right).$$

The bounding function $2\kappa_1 A_1/s(t) - 2\kappa_2 B_2/(L-s(t))$ which bounds $\frac{ds}{dt}$ from above is nothing but the difference between the gradient of $W(x)$ at $x=s(t)-0$ and that of $V(x)$ at $x=s(t)+0$, and it decreases as the free boundary moves to the right, i.e. as $s(t)$ increases. It vanishes when the free boundary reaches the point $s(t)=b_M$ defined by (1.9), and hence we see that the free boundary $s(t)$ can never go rightwards beyond the point $x=b_M$. Similarly $s(t)$ can never go leftwards beyond the point $x=b_m$ defined by (1.9). Hence we have

$$(1.26)\qquad b_m \leqq s(t) \leqq b_M.$$

On the other hand, the bounding function $2\kappa_1 A_1/s(t) - 2\kappa_2 B_2/(L-s(t))$ bounding $\frac{ds}{dt}$ from above attains its maximum value γ_+ defined by (1.16) at $s(t)=b_m$, and the function $-\{2\kappa_2 A_2/(L-s(t)) - 2\kappa_1 B_1/s(t)\}$ bounding $\frac{ds}{dt}$ from below attains its minimum value $-\gamma_-$ defined by (1.16) at $s(t)=b_M$, so that, as long as we assume (1.17), we have

$$(1.27)\qquad -\gamma_- \leqq \frac{ds}{dt} \leqq \gamma_+ \quad \text{or} \quad \left|\frac{ds}{dt}\right| \leqq \gamma.$$

In § 3 we shall show in a more consistent way that our finite element solutions also satisfy inequalities similar to (1.17) and (1.27).

§ 2. Application of the Finite Element Method

In this section we shall give a finite element scheme for solving (1.1)-(1.5) approximately. We write the approximate boundary function as $s_n(t)$ in order to show explicitly that it is an approximation of $s(t)$.

First we fix time $t>0$, and divide the whole domain $0\leqq x\leqq L$ into two subdomains D_1 and D_2 separated by the free boundary $x=s_n(t)$:

$$(2.1)\qquad \begin{cases} D_1=\{x \mid 0\leqq x\leqq s_n(t)\}, \\ D_2=\{x \mid s_n(t)\leqq x\leqq L\}. \end{cases}$$

Then we partition D_1 and D_2 into n_1 equal and n_2 equal subintervals, respectively, and hence the free boundary $x=s_n(t)$ always coincides with a mesh point. The numbers of partition n_1 and n_2 are fixed throughout computation. Although each of D_1 and D_2 might be partitioned using a non-uniform mesh, we employed the equi-distant partition for simplicity. We denote each mesh point as x_j.

$$(2.2)\qquad \begin{cases} x_j=jh_1(t),\ j=0,1,\cdots,n_1;\ h_1(t)=\dfrac{1}{n_1}s_n(t), \\ x_j=s_n(t)+(j-n_1)h_2(t),\ j=n_1,\cdots,n_1+n_2; \\ \qquad\qquad h_2(t)=\dfrac{1}{n_2}(L-s_n(t)). \end{cases}$$

In the usual finite element method the mesh points are always fixed as t varies, while in the present problem they change with time t.

We construct piecewise linear basis functions $\{\phi_j(x,t)\}$ as shown in Fig. 2:

$$(2.3)\qquad \phi_j(x,t)=\begin{cases} \dfrac{x-x_{j-1}}{x_j-x_{j-1}}; & x_{j-1}<x\leqq x_j \\ \dfrac{x_{j+1}-x}{x_{j+1}-x_j}; & x_j<x\leqq x_{j+1} \\ 0; & \text{otherwise}. \end{cases}$$

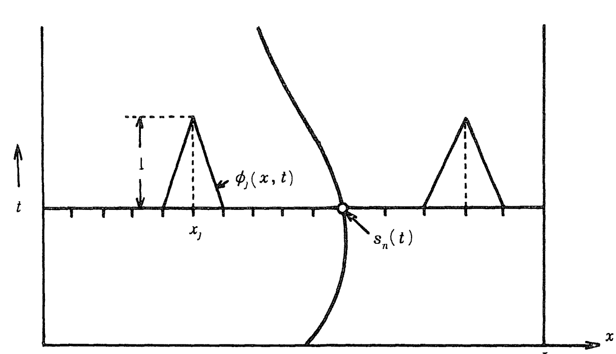

Fig. 2.

For ϕ_0 and $\phi_{n_1+n_2}$ we employ the components of (2. 3) in $0\leqq x\leqq x_1$ and $x_{n_1+n_2-1}<x\leqq L$, respectively. Note that the basis function $\phi_j(x,t)$ is time-dependent because x_j depends on time t. See [7] for the derivatives of $\phi_j(x,t)$ with respect to x and t.

Now we apply the Galerkin method based on the basis functions constructed above. We expand the approximate solutions $\tilde{u}_1$ and $\tilde{u}_2$ of (1. 1)-(1. 5) in terms of linear combinations of $\phi_j(x,t)$'s:

$$
\begin{cases} \tilde{u}_1(x,t)=\sum_{j=0}^{n_1} a_j(t)\phi_j(x,t) \\ \tilde{u}_2(x,t)=\sum_{j=n_1}^{n_1+n_2} a_j(t)\phi_j(x,t), \end{cases} \tag{2. 4}
$$

where from (1. 2)

$$
\begin{cases} a_0(t)=g_1(t), \quad a_{n_1+n_2}(t)=g_2(t) \\ a_{n_1}(t)=0. \end{cases} \tag{2. 5}
$$

If we substitute (2. 4) into (1. 1), multiply $\phi_i(x,t)$, $i=1,2,\cdots,n_1-1$, $n_1+1,\cdots,n_1+n_2-1$, and integrate over $0\leqq x\leqq L$, then we have the following system of ordinary differential equations:

$$
\begin{cases} M_1\dfrac{d\boldsymbol{a}_1}{dt}=-(\sigma_1K_1+N_1)\boldsymbol{a}_1 \\ M_2\dfrac{d\boldsymbol{a}_2}{dt}=-(\sigma_2K_2+N_2)\boldsymbol{a}_2, \end{cases} \tag{2. 6}
$$

where

$$
\boldsymbol{a}_1=\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n_1} \end{pmatrix}, \quad \boldsymbol{a}_2=\begin{pmatrix} a_{n_1} \\ a_{n_1+1} \\ \vdots \\ a_{n_1+n_2} \end{pmatrix}, \tag{2. 7}
$$

and M_1, M_2 are mass matrices, K_1, K_2 are stiffness matrices, and N_1, N_2 are velocity matrices [7]. We note that the matrices M_1, K_1, N_1 are of $(n_1-1)\times(n_1+1)$ and M_2, K_2, N_2 are of $(n_2-1)\times(n_2+1)$.

If we use the basis functions

$$
\psi_j(x,t)=\begin{cases} 1; \ \frac{1}{2}(x_{j-1}+x_j)<x\leqq\frac{1}{2}(x_j+x_{j+1}), \\ 0; \ \text{otherwise} \end{cases} \tag{2. 8}
$$

instead of $\phi_j(x,t)$ in the computation of the mass matrices, we have equations for the lumped mass system, while the equations obtained using $\phi_j(x,t)$'s are for the consistent mass system.

The explicit forms of M_ν, K_ν, N_ν $(\nu=1,2)$ are as follows:

Lumped Mass System:

$$(M_\nu)_{ij}=\begin{cases} h_\nu; \ j=i, \ \nu=1,2, \\ 0; \ \text{otherwise}, \end{cases} \tag{2.9}$$

$$(K_\nu)_{ij}=\begin{cases} -\dfrac{1}{h_\nu}; \ j=i\pm 1, \ \nu=1,2, \\ \dfrac{2}{h_\nu}; \ j=i, \ \nu=1,2, \\ 0; \ \text{otherwise}, \end{cases} \tag{2.10}$$

$$(N_1)_{ij}=\begin{cases} \mp\dfrac{1}{6n_1}(3i\pm 1)\dfrac{ds_n}{dt}; \ j=i\pm 1, \\ \dfrac{1}{3n_1}\times\dfrac{ds_n}{dt}; \ j=i, \\ 0; \ \text{otherwise}, \end{cases} \tag{2.11}$$

$$(N_2)_{ij}=\begin{cases} \mp\dfrac{1}{6n_2}\{3(n_1+n_2-i)\mp 1\}\dfrac{ds_n}{dt}; \ j=i\pm 1, \\ -\dfrac{1}{3n_2}\times\dfrac{ds_n}{dt}; \ j=i, \\ 0; \ \text{otherwise}. \end{cases} \tag{2.12}$$

Consistent Mass System:

We have only to replace the mass matrices by the following ones:

$$(M_\nu)_{ij}=\begin{cases} \dfrac{1}{6}h_\nu; \ j=i\pm 1, \ \nu=1,2, \\ \dfrac{2}{3}h_\nu; \ j=i, \ \nu=1,2, \\ 0; \ \text{otherwise}. \end{cases} \tag{2.13}$$

In the next step we discretize the time t using an equal time mesh Δt:

$$t = k\Delta t,\ k=0, 1, \cdots, m;\ t = \frac{T}{m}, \tag{2.14}$$

and replace the time derivatives of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ by the time differences:

$$\frac{d\boldsymbol{a}_\nu}{dt} \doteqdot \frac{\boldsymbol{a}_\nu(k\Delta t) - \boldsymbol{a}_\nu((k-1)\Delta t)}{\Delta t},\ \nu = 1, 2. \tag{2.15}$$

Then we have simultaneous algebraic linear equations from (2.6). Similarly we make an approximation

$$\frac{ds_n}{dt} \doteqdot \frac{\Delta s_n}{\Delta t}, \tag{2.16}$$

where Δs_n is the increment of $s_n(t)$ from $t=(k-1)\Delta t$ to $t=k\Delta t$. We compute Δs_n by replacing the gradients of u_1 and u_2 at $x=s(t)$ by those of $\tilde{u}_1$ and $\tilde{u}_2$ at $x=s_n(t)$ in the right hand side of (1.5). Although the functions $\{\tilde{u}_1, \tilde{u}_2, s_n(t)\}$ should be computed simultanously and consistently, we employ an approximation such that we compute $\{\tilde{u}_1, \tilde{u}_2\}$ and $s_n(t)$ alternatively.

We summarize here the whole procedure obtained. We introduce a parameter θ with $0 \leqq \theta \leqq 1$, which denotes the mixing ratio of the forward difference $(\theta=0)$ and the backward one $(\theta=1)$ in the discretization of the time derivative.

Initial Routine:

$$\begin{cases} a_j(0) = f_1(x_j);\ j=0, 1, \cdots, n_1; \\ a_j(0) = f_2(x_j);\ j=n_1+1, \cdots, n_1+n_2, \end{cases} \tag{2.17}$$

$$s_n(0) = b. \tag{2.18}$$

General Routine:

Repeat the following process for $k=1, 2, \cdots, m$.

(i) Compute $\Delta s_n(k\Delta t)$ and $s_n(k\Delta t)$ using $\boldsymbol{a}_\nu((k-1)\Delta t)$ and $s_n((k-1)\Delta t)$ by means of

$$\Delta s_n(k\Delta t) = \left\{ \frac{\kappa_1 n_1 a_{n_1-1}((k-1)\Delta t)}{s_n((k-1)\Delta t)} + \frac{\kappa_2 n_2 a_{n_1+1}((k-1)\Delta t)}{L - s_n((k-1)\Delta t)} \right\} \Delta t, \tag{2.19}$$

$$s_n(k\Delta t) = s_n((k-1)\Delta t) + \Delta s_n(k\Delta t). \tag{2.20}$$

(ii) Compute $M_\nu, K_\nu, N_\nu, \nu=1, 2$ using $\Delta s_n(k\Delta t)$ and $s_n(k\Delta t)$.

(iii) Solve the following linear equations for $\boldsymbol{a}_1(k\Delta t)$ and $\boldsymbol{a}_2(k\Delta t)$:

$$(2.21)\qquad \{M_\nu+\theta\Delta t(\sigma_\nu K_\nu+N_\nu)\}\boldsymbol{a}_\nu(k\Delta t)$$
$$=\{M_\nu-(1-\theta)\Delta t(\sigma_\nu K_\nu+N_\nu)\}\boldsymbol{a}_\nu((k-1)\Delta t),\ \nu=1, 2.$$

§ 3. Stability

In this section we consider the stability of the scheme (2.17)-(2.21) obtained in § 2. Here we shall confine ourselves to the case of the lumped mass system, while the stability of the scheme of the consistent mass system will be referred to at the end of this section.

In order to simplify the description we introduce

$$(3.1)\qquad \lambda_1(k\Delta t)\equiv\frac{\sigma_1 n_1^2\Delta t}{s_n^2(k\Delta t)},\quad \lambda_2(k\Delta t)\equiv\frac{\sigma_2 n_2^2\Delta t}{(L-s_n(k\Delta t))^2}$$

and

$$(3.2)\qquad \beta_1(k\Delta t)\equiv\frac{\Delta s_n(k\Delta t)}{2s_n(k\Delta t)},\quad \beta_2(k\Delta t)\equiv\frac{\Delta s_n(k\Delta t)}{2(L-s_n(k\Delta t))}.$$

When the argument is $k\Delta t$, it may be omitted. λ_1 and λ_2 correspond to the parameter $\lambda=\sigma\Delta t/h$, where h is the space mesh size, appearing in the finite difference or in the finite element method for the usual heat equation. In the present case, since h_1 and h_2 are time dependent, λ_1 and λ_2 also depend on t.

We define here two linear discrete operators P_1 and P_2:

$$(3.3)\quad P_1(k, j)w_k^j\equiv-\theta\left\{\lambda_1-\left(j-\frac{1}{3}\right)\beta_1\right\}w_{j-1}^k+\left\{1+2\theta\left(\lambda_1+\frac{1}{3}\beta_1\right)\right\}w_j^k$$
$$-\theta\left\{\lambda_1+\left(j+\frac{1}{3}\right)\beta_1\right\}w_{j+1}^k-(1-\theta)\left\{\lambda_1-\left(j-\frac{1}{3}\right)\beta_1\right\}w_{j-1}^{k-1}$$
$$-\left\{1-2(1-\theta)\left(\lambda_1+\frac{1}{3}\beta_1\right)\right\}w_j^{k-1}$$
$$-(1-\theta)\left\{\lambda_1+\left(j+\frac{1}{3}\right)\beta_1\right\}w_{j+1}^{k-1}$$
$$:j=1, 2, \cdots, n_1-1;\ k=1, 2, \cdots, m,$$

$$(3.4)\quad \begin{aligned} P_2(k,j)\,w_j^k \equiv & -\theta\left\{\lambda_2-\left(n_1+n_2-j+\frac{1}{3}\right)\beta_2\right\}w_{j-1}^k \\ & +\left\{1+2\theta\left(\lambda_2-\frac{1}{3}\beta_2\right)\right\}w_j^k \\ & -\theta\left\{\lambda_2+\left(n_1+n_2-j-\frac{1}{3}\right)\beta_2\right\}w_{j+1}^k \\ & -(1-\theta)\left\{\lambda_2-\left(n_1+n_2-j+\frac{1}{3}\right)\beta_2\right\}w_{j-1}^{k-1} \\ & -\left\{1-2(1-\theta)\left(\lambda_2-\frac{1}{3}\beta_2\right)\right\}w_j^{k-1} \\ & -(1-\theta)\left\{\lambda_2+\left(n_1+n_2-j-\frac{1}{3}\right)\beta_2\right\}w_{j+1}^{k-1} \\ & :j=n_1+1,\cdots,n_1+n_2-1;\ k=1,2,\cdots,m. \end{aligned}$$

If we write

$$(3.5)\qquad a_j^k=a_j(k\Delta t),$$

then the scheme (2.21) is expressed simply as

$$(3.6)\qquad \begin{cases} P_1(k,j)\,a_j^k=0,\ j=1,2,\cdots,n_1-1,\ k=1,2,\cdots,m, \\ P_2(k,j)\,a_j^k=0,\ j=n_1+1,\cdots,n_1+n_2-1,\ k=1,2,\cdots,m. \end{cases}$$

In addition to the assumptions for the Stefan data given in § 1, we make two assumptions for the choice of the parameters n_1, n_2 and Δt. We set

$$(3.7)\qquad \begin{cases} K_1\equiv\max(\kappa_1A_1+\kappa_2B_2,\ \kappa_2A_2+\kappa_1B_1), \\ K_2\equiv\min(A_1B_2,\ A_2B_1). \end{cases}$$

Assumption E.

$$(3.8)\qquad \frac{K_1^3}{\kappa_1\kappa_2K_2}\Delta t\leqq L^2.$$

Next writing

$$(3.9)\qquad \lambda_1^M\equiv\frac{\sigma_1n_1^2\Delta t}{b_m^2}\quad\text{and}\quad\lambda_2^M\equiv\frac{\sigma_2n_2^2\Delta t}{(L-b_M)^2},$$

we make

Assumption F (lumped mass system).

$$\frac{1}{2(1-\theta)} \geq \max\left(\lambda_1^M\left(1+\frac{1}{6n_1^2}\right),\ \lambda_2^M\left(1+\frac{1}{6n_2^2}\right)\right). \tag{3.10}$$

We shall see later that $b_m \leq s_n(k\Delta t) \leq b_M$, so that λ_1^M and λ_2^M are the upper bounds of λ_1 and λ_2, respectively.

Note that, Assumption E becomes trivial as $\Delta t \to 0$ and $n_1, n_2 \to \infty$ while Assumption F remains essential except when $\theta = 1$.

The aim of the present section is to establish the stability theorem, which will be obtained as a byproduct of the proof of the following finite element analogue of (1.17) (see Fig. 1):

$$\begin{cases} B_1\left\{\left(\dfrac{x_j}{s_n}-2\right)^2-1\right\} \leq \tilde{u}_1(x_j, k\Delta t) \leq A_1\left\{1-\dfrac{x_j^2}{s_n^2}\right\}, \\ \qquad\qquad j=0, 1, \cdots n_1, \\ -A_2\left\{1-\dfrac{(L-x_j)^2}{(L-s_n)^2}\right\} \leq \tilde{u}_2(x_j, k\Delta t) \leq -B_2\left\{\left(\dfrac{L-x_j}{L-s_n}-2\right)^2-1\right\}, \\ \qquad\qquad j=n_1, \cdots, n_1+n_2. \end{cases} \tag{3.11}$$

We shall prove these inequalities by induction with respect to k, and for that purpose we need the following five lemmas.

Lemma 1. *If*

$$\begin{cases} 2B_1\left(1+\dfrac{1}{2n_1}\right) \leq n_1 a_{n_1-1}^{l-1} \leq 2A_1\left(1-\dfrac{1}{2n_1}\right), \\ 2B_2\left(1+\dfrac{1}{2n_2}\right) \leq -n_2 a_{n_1+1}^{l-1} \leq 2A_2\left(1-\dfrac{1}{2n_2}\right), \end{cases} \tag{3.12}$$

$$b_m \leq s_n((l-1)\Delta t) \leq b_M, \tag{3.13}$$

then

$$-\gamma_-\left(1-\frac{1}{2n_2}\right) \leq \frac{\Delta s_n(l\Delta t)}{\Delta t} \leq \gamma_+\left(1-\frac{1}{2n_1}\right). \tag{3.14}$$

Proof. Suppose we have $\Delta s_n(l\Delta t) \geq 0$ when we compute the right hand side of (2.19). Then from (2.19), (3.12) and (3.13) we have

$$\frac{\Delta s_n(l\Delta t)}{\Delta t} \leqq \frac{2\kappa_1 A_1}{s_n((l-1)\Delta t)}\left(1-\frac{1}{2n_1}\right)-\frac{2\kappa_2 B_2}{L-s_n((l-1)\Delta t)}\left(1+\frac{1}{2n_2}\right)$$

$$\leqq \frac{2\kappa_1 A_1}{b_m}\left(1-\frac{1}{2n_1}\right)-\frac{2\kappa_2 B_2}{L-b_m}\left(1+\frac{1}{2n_2}\right)$$

$$\leqq \left(\frac{2\kappa_1 A_1}{b_m}-\frac{2\kappa_2 B_2}{L-b_m}\right)\left(1-\frac{1}{2n_1}\right)\leqq \gamma_+\left(1-\frac{1}{2n_1}\right).$$

We can prove the left inequality of (3. 14) in the same way supposing that $\Delta s_n(l\Delta t)\leqq 0$. Q.E.D.

Lemma 2. *Under Assumption E,*

(3. 15) $$\left\{\frac{2\kappa_2 A_2}{(L-b_m)^2}+\frac{2\kappa_1 B_1}{b_m^2}\right\}\Delta t\leqq 1,\quad \left\{\frac{2\kappa_1 A_1}{b_M^2}+\frac{2\kappa_2 B_2}{(L-b_M)^2}\right\}\Delta t\leqq 1,$$

(3. 16) $$b_M-b_m\geqq \gamma_+\Delta t,\quad b_M-b_m\geqq \gamma_-\Delta t.$$

Proof. If we substitute the explicit forms of b_m, b_M, γ_- and γ_+ into the above four inequalities, they are expressed equivalently as follows in that order:

(3. 17) $$\frac{2(\kappa_2 A_2+\kappa_1 B_1)^3}{\kappa_1\kappa_2 A_2 B_1}\Delta t\leqq L^2,\quad \frac{2(\kappa_1 A_1+\kappa_2 B_2)^3}{\kappa_1\kappa_2 A_1 B_2}\Delta t\leqq L^2,$$

(3. 18) $$\begin{cases} \dfrac{2(\kappa_2 A_2+\kappa_1 B_1)(\kappa_1 A_1+\kappa_2 B_2)^2}{\kappa_1\kappa_2 A_1 B_2}\Delta t\leqq L^2, \\[2ex] \dfrac{2(\kappa_1 A_1+\kappa_2 B_2)(\kappa_2 A_2+\kappa_1 B_1)^2}{\kappa_1\kappa_2 A_2 B_1}\Delta t\leqq L^2. \end{cases}$$

It is evident from Assumption E that all these inequalities are valid. Q.E.D.

Lemma 3. *Under the same hypothesis of Lemma* 1 *and under Assumption E,*

(3. 19) $$b_m\leqq s_n(l\Delta t)\leqq b_M.$$

Proof. Since

(3. 20) $$-\gamma_-\Delta t\leqq \Delta s_n(l\Delta t)\leqq \gamma_+\Delta t$$

from Lemma 1, the increment of s_n never exceeds $\gamma_+ \Delta t$ or $\gamma_- \Delta t$ at each step. When $\Delta s_n(l\Delta t) \geqq 0$, we suppose $s_n((l-1)\Delta t) \leqq b_M - \delta$, $0 \leqq \delta \leqq \gamma_+ \Delta t$ in view of (3.20) and prove that $s_n(l\Delta t) \leqq b_M$, while when $\Delta s_n(l\Delta t) \leqq 0$, we suppose $b_m + \delta \leqq s_n((l-1)\Delta t)$, $0 \leqq \delta \leqq \gamma_- \Delta t$ and prove that $b_m \leqq s_n(l\Delta t)$.

First assume that $\Delta s_n(l\Delta t) \geqq 0$. Suppose that

(3.21) $$s_n((l-1)\Delta t) \leqq b_M - \delta,\ 0 \leqq \delta \leqq \gamma_+ \Delta t,$$

where $b_m \leqq b_M - \delta$ because of (3.16). Define an auxiliary function

(3.22) $$F_+(\delta) \equiv \left\{ \frac{2\kappa_1 A_1}{b_M - \delta} - \frac{2\kappa_2 B_2}{L - b_M + \delta} \right\} \Delta t.$$

Then it is obvious from (2.19) in view of (3.21) and (3.12) that

(3.23) $$0 \leqq \Delta s_n(l\Delta t) \leqq F_+(\delta),$$

so that, if $F_+(\delta) \leqq \delta$, then

(3.24) $$\begin{aligned} s_n(l\Delta t) &= s_n((l-1)\Delta t) + \Delta s_n(l\Delta t) \leqq s_n((l-1)\Delta t) + F_+(\delta) \\ &\leqq s_n((l-1)\Delta t) + \delta \leqq b_M \end{aligned}$$

from (3.21). So what we need is to show that $F_+(\delta) \leqq \delta$. It is easy to see that $F_+(\delta) \leqq \delta$ follows from

(3.25) $$\frac{d}{d\delta} F_+(\delta) \Big|_{\delta=0} \leqq 1$$

and

(3.26) $$F_+(\gamma_+ \Delta t) \leqq \gamma_+ \Delta t.$$

If we write (3.25) explicitly using (3.22), we have

(3.27) $$\left\{ \frac{2\kappa_1 A_1}{b_M^2} + \frac{2\kappa_2 B_2}{(L - b_M)^2} \right\} \Delta t \leqq 1,$$

which is guaranteed by (3.15). The inequality (3.26), on the other hand, can be shown to hold as follows using (3.16) and (1.16):

(3.28) $$\begin{aligned} F_+(\gamma_+ \Delta t) &= \left\{ \frac{2\kappa_1 A_1}{b_M - \gamma_+ \Delta t} - \frac{2\kappa_2 B_2}{L - b_M + \gamma_+ \Delta t} \right\} \Delta t \\ &\leqq \left\{ \frac{2\kappa_1 A_1}{b_m} - \frac{2\kappa_2 B_2}{L - b_m} \right\} \Delta t = \gamma_+ \Delta t, \end{aligned}$$

which verifies $F_+(\delta) \leqq \delta$.

When $\varDelta s_n(l\varDelta t)\leqq 0$, we define

$$F_-(\delta)\equiv\left\{\frac{2\kappa_2 A_2}{L-b_m-\delta}-\frac{2\kappa_1 B_1}{b_m+\delta}\right\}\varDelta t. \tag{3.29}$$

Then $b_m\leqq s_n(l\varDelta t)$ can be derived from $b_m+\delta\leqq s_n((l-1)\varDelta t)$, $0\leqq\delta\leqq\gamma_-\varDelta t$ in the same way as given above. Q.E.D.

Lemma 4 (*lumped mass system*). *Under the same hypothesis of Lemma* 1 *and under Assumptions D and F*,

$$\begin{cases}\dfrac{1}{2(1-\theta)}\geqq\lambda_1(l\varDelta t)+\dfrac{1}{3}\beta_1(l\varDelta t)\geqq n_1|\beta_1(l\varDelta t)|,\\[2ex] \dfrac{1}{2(1-\theta)}\geqq\lambda_2(l\varDelta t)-\dfrac{1}{3}\beta_2(l\varDelta t)\geqq n_2|\beta_2(l\varDelta t)|.\end{cases} \tag{3.30}$$

Proof. We begin with the first inequality.

$$\begin{aligned}\frac{1}{2(1-\theta)}-\lambda_1-\frac{1}{3}\beta_1&=\frac{1}{2(1-\theta)}-\lambda_1-\frac{\sigma_1\varDelta t}{6s_n^2}\times\frac{s_n}{\sigma_1}\times\frac{\varDelta s_n}{\varDelta t}\\ &\geqq\frac{1}{2(1-\theta)}-\lambda_1-\frac{\sigma_1\varDelta t}{6s_n^2}\eta\gamma\\ &\geqq\frac{1}{2(1-\theta)}-\lambda_1\left(1+\frac{1}{6n_1^2}\right)\quad\text{(from Assumption D)}\\ &\geqq\frac{1}{2(1-\theta)}-\lambda_M\left(1+\frac{1}{6n_1^2}\right)\geqq 0\quad\text{(from Assumption F)}.\end{aligned} \tag{3.31}$$

The second inequality can be proved as follows.

$$\begin{aligned}\lambda_1+\frac{1}{3}\beta_1-n_1|\beta_1|&\geqq\lambda_1-\left(n_1+\frac{1}{3}\right)\frac{\sigma_1\varDelta t}{2s_n^2}\times\frac{s_n}{\sigma_1}\left|\frac{\varDelta s_n}{\varDelta t}\right|\\ &\geqq\lambda_1\left\{1-\frac{(n_1+1/3)}{2n_1^2}\eta\gamma\right\}\geqq\lambda_1(1-\eta\gamma)\geqq 0.\end{aligned} \tag{3.32}$$

The other inequalities can also be proved in a similar way. Q.E.D.

Finally using Lemma 4, we have the following local maximum principle for the present scheme, the proof of which is exactly the same as that of Lemma 1 in the preceding paper [7].

Lemma 5 (*lumped mass system*). *If* (3. 30) *holds for* $l=k$ *and if*

$$p_j^k \geqq 0, \quad j=1, 2, \cdots, n_1-1, n_1+1, \cdots, n_1+n_2-1, \tag{3.33}$$

then the following maximum principle holds for the scheme

$$\begin{cases} P_1(k, j) w_j^k = p_j^k, & j=1, 2, \cdots, n_1-1, \\ P_2(k, j) w_j^k = p_j^k, & j=n_1+1, \cdots, n_1+n_2-1, \end{cases} \tag{3.34}$$

locally at k:

$$\min\{w_0^k, w_{n_1}^k, w_{\min,1}^{k-1}\} \leqq w_j^k \leqq \max\{w_0^k, w_{n_1}^k, w_{\max,1}^{k-1}\} + p_j^k, \quad j=0, 1, \cdots, n_1, \tag{3.35}$$

$$\min\{w_{n_1}^k, w_{n_1+n_2}^k, w_{\min,2}^{k-1}\} \leqq w_j^k \leqq \max\{w_{n_1}^k, w_{n_1+n_2}^k, w_{\max,2}^{k-1}\} + p_j^k, \quad j=n_1, n_1+1, \cdots, n_1+n_2, \tag{3.36}$$

where

$$\begin{cases} w_{\min,1}^{k-1} = \min\limits_{0 \leqq j \leqq n_1} w_j^{k-1}, & w_{\max,1}^{k-1} = \max\limits_{0 \leqq j \leqq n_1} w_j^{k-1}, \\ w_{\min,2}^{k-1} = \min\limits_{n_1 \leqq j \leqq n_1+n_2} w_j^{k-1}, & w_{\max,2}^{k-1} = \max\limits_{n_1 \leqq j \leqq n_1+n_2} w_j^{k-1}. \end{cases} \tag{3.37}$$

Now we are ready to verify the inequalities (3. 11). We introduce the following quantities for simplicity.

$$\widetilde{W}_j \equiv \begin{cases} A_1\left\{1-\dfrac{j^2}{n_1^2}\right\}, & j=0, 1, \cdots, n_1, \\ -A_2\left\{1-\dfrac{(n_1+n_2-j)^2}{n_2^2}\right\}, & j=n_1, \cdots, n_1+n_2. \end{cases} \tag{3.38}$$

$$\widetilde{V}_j \equiv \begin{cases} B_1\left\{\left(\dfrac{j}{n_1}-2\right)^2-1\right\}, & j=0, 1, \cdots, n_1, \\ -B_2\left\{\left(\dfrac{n_1+n_2-j}{n_2}-2\right)^2-1\right\}, & j=n_1, \cdots, n_1+n_2. \end{cases} \tag{3.39}$$

Lemma 6 (*lumped mass system*). *Under Assumptions* A, B, C, D, E *and* F, *the following inequalities hold for* $k=0, 1, \cdots, m$.

$$\widetilde{V}_j \leqq a_j^k \leqq \widetilde{W}_j, \quad j=0, \cdots, n_1, \tag{3.40}$$

(3.41) $$\widetilde{W}_j \leqq a_j^k \leqq \widetilde{V}_j, \quad j=n_1, \cdots, n_1+n_2,$$

(3.42) $$b_m \leqq s_n(k\Delta t) \leqq b_M.$$

Proof. We shall prove this lemma by induction with respect to k.

When $k=0$, (3.40) and (3.41) are nothing but (1.6) of Assumption A, and (3.42) is (1.10) of Assumption C.

Suppose that (3.40), (3.41) and (3.42) hold for $k=l-1$. If we put $j=n_1-1$ in (3.40) and $j=n_1+1$ in (3.41), we have

(3.43) $$2B_1\left(1+\frac{1}{2n_1}\right) \leqq n_1 a_{n_1-1}^{l-1} \leqq 2A_1\left(1-\frac{1}{2n_1}\right),$$

(3.44) $$2B_2\left(1+\frac{1}{2n_2}\right) \leqq -n_2 a_{n_1+1}^{l-1} \leqq 2A_2\left(1-\frac{1}{2n_2}\right).$$

From these inequalities together with (3.42), we see that the hypothesis of Lemma 1 holds. Then, since the hypothesis of Lemma 3 also holds, we immediately see that (3.42) is valid for $k=l$. Next we define the difference between $\widetilde{W}_j$ and a_j^l, i.e.

(3.45) $$d_j^l \equiv \widetilde{W}_j - a_j^l,$$

and prove that $d_j^l \geqq 0$, i.e. the second inequality in (3.40), by using Lemma 5. The hypothesis of Lemma 4 holds, so that (3.30) is valid. In order to use Lemma 5, we need to show that $P_1(l,j)d_j^l = p_j^l \geqq 0$, which corresponds to (3.33). The inequality $p_j^l \geqq 0$ can be proved as follows from Assumption D:

(3.46) $$P_1(l,j)d_j^l = P_1(l,j)(\widetilde{W}_j - a_j^l) = P_1(l,j)\widetilde{W}_j \quad (=p_j^l)$$

$$= -\left(\lambda_1 + \frac{1}{3}\beta_1\right)(\widetilde{W}_{j-1} - 2\widetilde{W}_j + \widetilde{W}_{j+1}) + j\beta_1(\widetilde{W}_{j-1} - \widetilde{W}_{j+1})$$

$$= \frac{2A_1\sigma_1\Delta t}{s_n^2}\left\{1 + \frac{1}{n_1^2}\left(j^2 + \frac{1}{6}\right)\frac{s_n\Delta s_n}{\sigma_1\Delta t}\right\}$$

$$\geqq \frac{2A_1\sigma_1\Delta t}{s_n^2}\left\{1 - \frac{s_n}{\sigma_1}\left|\frac{\Delta s_n}{\Delta t}\right|\right\} \geqq \frac{2A_1\sigma_1\Delta t}{s_n^2}(1-\eta\gamma) \geqq 0,$$

$$j=1, 2, \cdots, n_1-1.$$

In view of the boundary condition

$$\widetilde{W}_0 \geqq a_0^l, \quad \widetilde{W}_{n_1} = a_{n_1}^k = 0 \tag{3. 47}$$

from Assumption B, we have $\widetilde{W}_j \geqq a_j^l$ by Lemma 5.

The other three inequalities can also be verified in the same way with the aid of the following three inequalities:

$$\begin{cases} P_1(l, j)\,(a_j^l - \widetilde{V}_j) \geqq 0, & j=1, 2, \cdots, n_1-1, \\ P_2(l, j)\,(a_j^l - \widetilde{W}_j) \geqq 0, & j=n_1+1, \cdots, n_1+n_2-1, \\ P_2(l, j)\,(\widetilde{V}_j - a_j^l) \geqq 0, & j=n_1+1, \cdots, n_1+n_2-1. \end{cases} \tag{3. 48}$$

Q.E.D.

Lemma 6 asserts that, under Assumptions A, B, C, D, E and F, the maximum principle in the sense of Lemma 5 holds for the present scheme (3. 6) locally at each $k=1, 2, \cdots, m$, so that for stability we have

Theorem 1 (*lumped mass system*). *Under Assumptions A, B, C, D, E and F, the scheme*

$$\begin{cases} P_1(k, j)\,a_j^k = 0, & j=1, 2, \cdots, n_1-1, \\ P_2(k, j)\,a_j^k = 0, & j=n_1+1, \cdots, n_1+n_2-1, \end{cases} \tag{3. 49}$$

is stable in the sense that the following maximum principle holds locally at $k=1, 2, \cdots, m$:

$$\begin{cases} 0 \leqq a_j^k \leqq \max\,(a_0^k, a_{\max,1}^{k-1}), & j=1, 2, \cdots, n_1-1, \\ \min\,(a_{n_1+n_2}^k, a_{\min,2}^{k-1}) \leqq a_j^k \leqq 0, & j=n_1+1, \cdots, n_1+n_2-1. \end{cases} \tag{3. 50}$$

Theorem 1 can be shown to hold also for the scheme of the consistent mass system

$$\begin{aligned} &\left[1-6\theta\left\{\lambda_1-\left(j-\frac{1}{3}\right)\beta_1\right\}\right]a_{j-1}^k+\left\{4+12\theta\left(\lambda_1+\frac{1}{3}\beta_1\right)\right\}a_j^k \\ &\quad+\left[1-6\theta\left\{\lambda_1+\left(j+\frac{1}{3}\right)\beta_1\right\}\right]a_{j+1}^k \\ &\quad=\left[1+6(1-\theta)\left\{\lambda_1-\left(j-\frac{1}{3}\right)\beta_1\right\}\right]a_{j-1}^{k-1} \\ &\quad+\left\{4-12(1-\theta)\left(\lambda_1+\frac{1}{3}\beta_1\right)\right\}a_j^{k-1} \end{aligned} \tag{3. 51}$$

$$+\left[1+6(1-\theta)\left\{\lambda_1+\left(j+\frac{1}{3}\right)\beta_1\right\}\right]a_{j+1}^{k-1},$$

$$j=1,2,\cdots,n_1-1;\ k=1,2,\cdots,m,$$

(3.52)
$$\left[1-6\theta\left\{\lambda_2-\left(n_1+n_2-j+\frac{1}{3}\right)\beta_2\right\}\right]a_{j-1}^k+\left\{4+12\left(\lambda_2-\frac{1}{3}\beta_2\right)\right\}a_j^k$$

$$+\left[1-6\theta\left\{\lambda_2+\left(n_1+n_2-j-\frac{1}{3}\right)\beta_2\right\}\right]a_{j+1}^k$$

$$=\left[1+6(1-\theta)\left\{\lambda_2-\left(n_1+n_2-j+\frac{1}{3}\right)\beta_2\right\}\right]a_{j-1}^{k-1}$$

$$+\left\{4-12(1-\theta)\left(\lambda_2-\frac{1}{3}\beta_2\right)\right\}a_j^{k-1}$$

$$+\left[1+6(1-\theta)\left\{\lambda_2+\left(n_1+n_2-j-\frac{1}{3}\right)\beta_2\right\}\right]a_{j+1}^{k-1},$$

$$j=n_1+1,\cdots,n_1+n_2-1;\ k=1,2,\cdots,m$$

if we replace Assumption F by the following one. We define here

(3.53)
$$\lambda_1^m\equiv\frac{\sigma_1 n_1^2\Delta t}{b_M^2},\quad \lambda_2^m\equiv\frac{\sigma_2 n_2^2\Delta t}{(L-b_m)^2}.$$

Assumption F (consistent mass system).

(3.54)
$$\frac{1}{3(1-\theta)}\geqq\lambda_1^M\left(1+\frac{1}{6n_1^2}\right),\quad \lambda_1^m\left(1-\frac{1}{2n_1}\right)\geqq\frac{1}{6\theta},$$

(3.55)
$$\frac{1}{3(1-\theta)}\geqq\lambda_2^M\left(1+\frac{1}{6n_2^2}\right),\quad \lambda_2^m\left(1-\frac{1}{2n_2}\right)\geqq\frac{1}{6\theta}.$$

It is easy to see that for $k=0,1,\cdots,m$

(3.56)
$$\begin{cases}\lambda_1^m\leqq\lambda_1(k\Delta t)\leqq\lambda_1^M,\\ \lambda_2^m\leqq\lambda_2(k\Delta t)\leqq\lambda_2^M.\end{cases}$$

§ 4. Convergence

In this section we shall show that the approximate solution $\{\tilde{u}_1,\tilde{u}_2,s_n\}$ converges to the solution $\{u_1,u_2,s\}$ of (1.1)-(1.5) uniformly as $\Delta t\to 0$.

For simplicity we shall confine ourselves to the case of the lumped mass system with $\theta=1$, i.e.

$$
(4.1)\quad \begin{cases} -\left\{\lambda_1-\left(j-\frac{1}{3}\right)\beta_1\right\}a_{j-1}^k+\left\{1+2\left(\lambda_1+\frac{1}{3}\beta_1\right)\right\}a_j^k \\ \qquad -\left\{\lambda_1+\left(j+\frac{1}{3}\right)\beta_1\right\}a_{j+1}^k=a_j^{k-1}, \quad j=1,2,\cdots,n_1-1, \\ -\left\{\lambda_2-\left(n_1+n_2-j+\frac{1}{3}\right)\beta_2\right\}a_{j-1}^k+\left\{1+2\left(\lambda_2-\frac{1}{3}\beta_2\right)\right\}a_j^k \\ \qquad -\left\{\lambda_2+\left(n_1+n_2-j-\frac{1}{3}\right)\beta_2\right\}a_{j+1}^k=a_j^{k-1}, \\ \qquad\qquad j=n_1+1,\cdots,n_1+n_2-1. \end{cases}
$$

Now we make two assumptions for the limit $\Delta t\to 0$, $n_1, n_2\to\infty$ and for the smoothness of the initial and the boundary data.

Assumption G.

$$
(4.2)\quad \begin{cases} \lambda_1^M=\dfrac{\sigma_1 n_1^2 \Delta t}{b_m^2}=\text{constant}, \\ \lambda_2^M=\dfrac{\sigma_2 n_2^2 \Delta t}{(L-b_M)^2}=\text{constant}. \end{cases}
$$

Assumption H.

$$
(4.3)\quad f_1, f_2\in C^2(x),\ g_1, g_2\in C^1(t).
$$

$$
(4.4)\quad \begin{cases} g_1(0)=f_1(0),\ \dfrac{dg_1}{dt}(0)=\sigma_1\dfrac{d^2f_1}{dx^2}(0), \\ g_2(0)=f_2(0),\ \dfrac{dg_2}{dt}(0)=\sigma_2\dfrac{d^2f_2}{dx^2}(L). \end{cases}
$$

We extend the approximate solutions $\{\tilde{u}_1, \tilde{u}_2, s_n(t)\}$ which are defined only at the discrete points $t=k\Delta t$ to those defined also at the intermediate values of t, i.e. $k\Delta t<t\leqq(k+1)\Delta t$, as follows. First we define $s_n(t)$ by means of the linear interpolation

$$
(4.5)\quad s_n(t)=s_n(k\Delta t)+\alpha\Delta s_n((k+1)\Delta t),\ k\Delta t<t\leqq(k+1)\Delta t,
$$

where

$$\alpha = \frac{t - k\Delta t}{\Delta t}. \tag{4.6}$$

Accordingly, $\phi_j(x, t)$ is also defined for any t. Next we extend $a_j(t)$ again by means of the linear interpolation based on $a_j(k\Delta t)$ and $a_j((k+1)\Delta t)$. Once $a_j(t)$ is extended, then $\tilde{u}_1(x, t)$ and $\tilde{u}_2(x, t)$ are defined by (2.4). Note that the value of $\tilde{u}_1(x, t)$ at $jh_1 < x \leqq (j+1)h_1$ is equal to that of the linear interpolation based on $a_j(t)$ and $a_{j+1}(t)$.

In regard to the extended $\{s_n(t)\}$, we have

Lemma 7. *The functions $\{s_n(t)\}$ form an equicontinuous uniformly bounded family in $0 \leqq t \leqq T$.*

Proof. From Lemmas 6 and 3, we have $b_m \leqq s_n(k\Delta t) \leqq b_M$, $k = 0, 1, \cdots, m$, so that by the definition (4.5) of $s_n(t)$ we see that $s_n(t)$ are uniformly bounded. Similarly from Lemmas 6 and 1, we have

$$|\Delta s_n(k\Delta t)| \leqq \gamma \Delta t, \quad k = 1, 2, \cdots, m. \tag{4.7}$$

This inequality together with (4.5) implies the equicontinuity of $\{s_n(t)\}$.
Q.E.D.

According to this lemma, we can extract a subsequence from $\{s_n(t)\}$ which converges. Namely, if we write this subsequence as $\{s_n(t)\}$ anew, and if we let the limit function be $s_\infty(t)$, then for any $\varepsilon > 0$ we have

$$|s_n(t) - s_\infty(t)| < \varepsilon \tag{4.8}$$

for sufficiently large n.

In the next step we regard the boundary function $s(t)$ to be the given function $s_\infty(t)$, which is uniformly Lipschitz continuous, and let $\hat{u}_1$ be the solution in D_1 and $\hat{u}_2$ be that in D_2 of the heat equation (1.1)-(1.4) associated with the moving boundary $s_\infty(t)$. Then, if we consider the domains D_1 and D_2 separately, we can prove in the same way as in the proof of § 4 of [7] that $\tilde{u}_1$ and $\tilde{u}_2$ converge uniformly to $\hat{u}_1$ and $\hat{u}_2$, respectively, as $\Delta t \to 0$, $n_1, n_2 \to \infty$ under Assumptions A, B, C. D, E, F, G and H. The only different point is that in [7] $s_n(t)$ and $s_\infty(t)$ are monotone while in the present case they are not monotone, so that Lemma 4 in [7] does not hold. We see, however, that the conclusion of Lemma 4 of [7] is also valid with A replaced by $2A_1/b_m$ in D_1 and by $2A_2/(L-b_M)$ in D_2 if we remind of (4.7) and the discussion about the inequalities (1.17) in § 1.

What is left to be done is to show that the functions $\hat{u}_1$, $\hat{u}_2$ and $s_\infty(t)$ satisfy the Stefan condition (1.5). For that purpose we define the second difference of a_j^k:

$$(4.9)\qquad \begin{cases} c_j^k = \dfrac{a_{j-1}^k - 2a_j^k + a_{j+1}^k}{h_1^2}, & j=1,2,\cdots,n_1-1, \\[2ex] c_j^k = \dfrac{a_{j-1}^k - 2a_j^k + a_{j+1}^k}{h_2^2}, & j=n_1+1,\cdots,n_1+n_2-1. \end{cases}$$

Then, by considering D_1 and D_2 separately, we can prove the following lemma. For the proof see Lemma 5 in [7].

Lemma 8 (*lumped mass system*, $\theta=1$). *Under Assumption A, B, C, D, E, F and H,*

$$(4.10)\qquad |c_j^k| \leqq M,\ j=1,2,\cdots,n_1-1,n_1+1,\cdots,n_1+n_2-1;\ k=1,2,\cdots,m.$$

We can also show that the following inequalities are valid as in the same way as in the proof of Lemma 6 in [7].

Lemma 9 (*lumped mass system*, $\theta=1$). *Under Assumptions A, B, C, D, E, F and H,*

$$(4.11)\qquad \begin{cases} \left| \dfrac{n_1 a_{n_1-1}((k+1)\Delta t)}{s_n((k+1)\Delta t)} - \dfrac{n_1 a_{n_1-1}(k\Delta t)}{s_n(k\Delta t)} \right| \leqq M_1 \Delta t^{1/2}, \\[2ex] \left| \dfrac{n_2 a_{n_1+1}((k+1)\Delta t)}{L-s_n((k+1)\Delta t)} - \dfrac{n_2 a_{n_1+1}(k\Delta t)}{L-s_n(k\Delta t)} \right| \leqq M_2 \Delta t^{1/2}. \end{cases}$$

Now we define a piecewise constant function

$$(4.12)\qquad z_n(t) = \frac{\kappa_1 n_1 a_{n_1-1}(k\Delta t)}{s_n(k\Delta t)} + \frac{\kappa_2 n_2 a_{n_1+1}(k\Delta t)}{L-s_n(k\Delta t)}, \qquad k\Delta t \leqq t < (k+1)\Delta t.$$

Then from (4.5) we have

$$(4.13)\qquad s_n(t) = \int_0^t z_n(\tau)\, d\tau.$$

From Lemma 8, on the other hand,

$$\frac{\partial \tilde{u}_1}{\partial x}(s_n(t), t) \quad \text{and} \quad \frac{\partial \tilde{u}_2}{\partial x}(s_n(t), t)$$

exist, and are given explicitly as

$$(4.14) \quad \begin{cases} -\kappa_1 \dfrac{\partial \tilde{u}_1}{\partial x}(s_n(t), t) \\ \qquad = \kappa_1 n_1 \dfrac{a_{n_1-1}(k\Delta t) + \alpha\{a_{n_1-1}((k+1)\Delta t) - a_{n_1-1}(k\Delta t)\}}{s_n(k\Delta t) + \alpha\{s_n((k+1)\Delta t) - s_n(k\Delta t)\}} \\ \kappa_2 \dfrac{\partial \tilde{u}_2}{\partial x}(s_n(t), t) \\ \qquad = \kappa_2 n_2 \dfrac{a_{n_1+1}(k\Delta t) + \alpha\{a_{n_1+1}((k+1)\Delta t) - a_{n_1+1}(k\Delta t)\}}{(L - s_n(k\Delta t)) - \alpha\{s_n((k+1)\Delta t) - s_n(k\Delta t)\}}. \end{cases}$$

Define

$$(4.15) \qquad \zeta_n(t) = -\kappa_1 \frac{\partial \tilde{u}_1}{\partial x}(s_n(t), t) + \kappa_2 \frac{\partial \tilde{u}_2}{\partial x}(s_n(t), t),$$

then we have

Lemma 10 (*lumped mass system*, $\theta = 1$).

$$(4.16) \qquad |z_n(t) - \zeta_n(t)| \leqq M_3 \Delta t^{1/2}.$$

Proof. From $b_m \leqq s_n(t) \leqq b_M$ and Lemma 9, we have in D_1

$$(4.17) \quad n_1 \left| \frac{a_{n_1-1}(k\Delta t) + \alpha\{a_{n_1-1}((k+1)\Delta t) - a_{n_1-1}(k\Delta t)\}}{s_n(k\Delta t) + \alpha\{s_n((k+1)\Delta t) - s_n(k\Delta t)\}} - \frac{a_{n_1-1}(k\Delta t)}{s_n(k\Delta t)} \right|$$

$$= \frac{\alpha s_n((k+1)\Delta t)}{s_n(k\Delta t) + \alpha\{s_n((k+1)\Delta t) - s_n(k\Delta t)\}}$$

$$\times \left| \frac{n_1 a_{n_1-1}((k+1)\Delta t)}{s_n((k+1)\Delta t)} - \frac{n_1 a_{n_1-1}(k\Delta t)}{s_n(k\Delta t)} \right| \leqq \frac{\alpha b_M}{b_m} M_1 \Delta t^{1/2}.$$

We can verify a similar inequality in D_2, and from these inequalities in view of (4.12) and (4.15) we conclude (4.16). Q.E.D.

We write

(4. 18) $$s_n(t)=\int_0^t \zeta_n(\tau)d\tau+\int_0^t (z_n(\tau)-\zeta_n(\tau))d\tau.$$

As Δt tends to zero the second term of the right hand side vanishes according to Lemma 10, so that we have

(4. 19) $$s_\infty(t)=\int_0^t \left\{-\kappa_1\frac{\partial \hat{u}_1}{\partial x}(s_\infty(\tau),\tau)+\kappa_2\frac{\partial \hat{u}_2}{\partial x}(s_\infty(\tau),\tau)\right\}d\tau,$$

i.e.

(4. 20) $$\frac{ds_\infty(t)}{dt}=-\kappa_1\frac{\partial \hat{u}_1}{\partial x}(s_\infty(t),t)+\kappa_2\frac{\partial \hat{u}_2}{\partial x}(s_\infty(t),t).$$

This shows that $\hat{u}_1, \hat{u}_2$ and $s_\infty(t)$ satisfy (1. 5).

Finally, the assumptions for the Stefan data made by Cannon and Primicerio [2] cover the assumptions in the present paper, and hence the solution of (1. 1)-(1. 5) is unique [2], so that we have the main convergence

Theorem 2 (*lumped mass system,* $\theta=1$). *Under Assumptions A, B, C, D, E, F, G and H, the approximate solution obtained by* (2. 17)-(2. 21) *converges to the solution of the Stefan problem* (1. 1)-(1. 5) *as* $\Delta t\to 0$, $n_1, n_2\to\infty$.

This theorem also establishes the existence of the solution of (1. 1)-(1. 5) under Assumptions A, B, C, D and H.

§ 5. Improved Scheme

Although the scheme given at the end of § 2 is very simple and easy to compute, the speed of convergence has been observed to be a little slow. However, it can be remarkably improved with a slight modification of the scheme. The idea is to revise Δs_n and s_n at each step immediately after the new data are obtained. The improved scheme is as follows.

Initial Routine:

$$(5.1)\qquad \begin{cases} a_j(0)=f_1(x_j)\,;\ j=0,1,\cdots,n_1, \\ a_j(0)=f_2(x_j)\,;\ j=n_1+1,\cdots,n_1+n_2, \end{cases}$$

$$(5.2)\qquad s_n(0)=b.$$

General Routine:

Repeat the following process for $k=1,2,\cdots,m$.

(i) Compute $\Delta s_n((k-1/2)\Delta t)$ and $s_n((k-1/2)\Delta t)$ using $\boldsymbol{a}_\nu((k-1)\Delta t)$ and $s_n((k-1)\Delta t)$ by means of

$$(5.3)\qquad \Delta s_n\left(\left(k-\frac{1}{2}\right)\Delta t\right) = \left\{\frac{\kappa_1 n_1 a_{n_1-1}((k-1)\Delta t)}{s_n((k-1)\Delta t)}+\frac{\kappa_2 n_2 a_{n_1+1}((k-1)\Delta t)}{L-s_n((k-1)\Delta t)}\right\}\Delta t,$$

$$(5.4)\qquad s_n\left(\left(k-\frac{1}{2}\right)\Delta t\right)=s_n((k-1)\Delta t)+\frac{1}{2}\Delta s_n\left(\left(k-\frac{1}{2}\right)\Delta t\right).$$

(ii) Compute M_ν, K_ν, N_ν, $\nu=1,2$ using $\Delta s_n((k-\frac{1}{2})\Delta t)$ and $s_n((k-\frac{1}{2})\Delta t)$.

(iii) Solve the following linear equations for $\boldsymbol{a}_1(k\Delta t)$ and $\boldsymbol{a}_2(k\Delta t)$:

$$(5.5)\qquad \{M_\nu+\theta\Delta t(\sigma_\nu K_\nu+N_\nu)\}\boldsymbol{a}_\nu(k\Delta t) = \{M_\nu-(1-\theta)\Delta t(\sigma_\nu K_\nu+N_\nu)\}\boldsymbol{a}_\nu((k-1)\Delta t),\ \ \nu=1,2.$$

(iv) Compute $\Delta s_n(k\Delta t)$ and $s_n(k\Delta t)$ using $\boldsymbol{a}_\nu(k\Delta t)$ and $s_n((k-\frac{1}{2})\Delta t)$ by means of

$$(5.6)\qquad \Delta s_n(k\Delta t)=\left\{\frac{\kappa_1 n_1 a_{n_1-1}(k\Delta t)}{s_n((k-\frac{1}{2})\Delta t)}+\frac{\kappa_2 n_2 a_{n_1+1}(k\Delta t)}{L-s_n((k-\frac{1}{2})\Delta t)}\right\}\Delta t,$$

$$(5.7)\qquad s_n(k\Delta t)=s_n\left(\left(k-\frac{1}{2}\right)\Delta t\right)+\frac{1}{2}\Delta s_n(k\Delta t).$$

We can show in almost the same way as in that of § 3 that Theorem 1 also holds for the stability of the improved scheme under Assumption A, B, C, D, E and F. In the present case the arguments $k\Delta t$ of $\lambda_1, \lambda_2, \beta_1, \beta_2$ in (3. 3) and (3. 4) should be replaced by $(k-\frac{1}{2})\Delta t$. Then it is easy to see that we have

$$(5.8)\qquad -\gamma_-\left(1-\frac{1}{2n_2}\right)\leqq\frac{\Delta s_n((l-\frac{1}{2})\Delta t)}{\Delta t}\leqq\gamma_+\left(1-\frac{1}{2n_1}\right)$$

in place of (3. 14), and

$$b_m \leqq s_n\left(\left(l-\frac{1}{2}\right)\Delta t\right) \leqq b_M \tag{5.9}$$

in place of (3.19). In the proof of Lemma 3 we must derive $s_n\left(\left(l-\frac{1}{2}\right)\Delta t\right) \leqq b_M$ from $s_n((l-1)\Delta t) \leqq b_M-\delta$, $0 \leqq \delta \leqq \frac{1}{2}\gamma_+\Delta t$ in stead of (3.21). For this purpose we need $\frac{1}{2}F_+(\delta) \leqq \delta$ and $\frac{1}{2}F_+\left(\frac{1}{2}\gamma_+\Delta t\right) \leqq \frac{1}{2}\gamma_+\Delta t$, which, however, follow from $F_+(\delta) \leqq \delta$ and $F_+(\gamma_+\Delta t) \leqq \gamma_+\Delta t$. Therefore at each step from $(k-1)\Delta t$ to $\left(k-\frac{1}{2}\right)\Delta t$ the scheme given by (3.3) and (3.4) is stable. Then we repeat the same reasoning once more at the step from $\left(k-\frac{1}{2}\right)\Delta t$ to $k\Delta t$ replacing $a_{n_1-1}^{l-1}$ by $a_{n_1-1}^{l}$ and $a_{n_1+1}^{l-1}$ by $a_{n_1+1}^{l}$ in (3.12) and also replacing $s_n((l-1)\Delta t)$ by $s_n\left(\left(l-\frac{1}{2}\right)\Delta t\right)$ in (3.13), so that we have (3.14) and (3.19). Hence we have

Theorem 1′ (*improved scheme*). *Under Assumptions A, B, C, D, E and F the scheme* (5.1)-(5.7) *is stable in the sense that the maximum principle* (3.50) *holds.*

In order to prove the convergence of the improved scheme, we extend $s_n(k\Delta t)$ defined at discrete points to a continuous function in the following way:

$$s_n(t) = \begin{cases} s_n(k\Delta t) + \alpha\Delta s_n\left(\left(k+\frac{1}{2}\right)\Delta t\right); & k\Delta t < t \leqq \left(k+\frac{1}{2}\right)\Delta t, \\ s_n\left(\left(k+\frac{1}{2}\right)\Delta t\right) + \left(\alpha-\frac{1}{2}\right)\Delta s_n(k\Delta t); & \left(k+\frac{1}{2}\right)\Delta t < t \leqq k\Delta t, \end{cases} \tag{5.10}$$

$$\alpha = \frac{t-k\Delta t}{\Delta t}.$$

The extention of $a_j(t)$, i.e. that of $\tilde{u}_1$ and $\tilde{u}_2$, to intermediate values of t is exactly the same as is done in § 4. Then we can again extract a subsequence from $\{s_n(t)\}$ that converges.

In the definition of c_j^k of (4.9) we replace $h_\nu((k-1)\Delta t)$ by $h_\nu((k-\frac{1}{2})\Delta t)$.

$z_n(t)$ in Lemma 10 should be modified to be

$$(5.11)\qquad z_n(t)=\begin{cases}\dfrac{\kappa_1 n_1 a_{n_1-1}(k\Delta t)}{s_n(k\Delta t)}+\dfrac{\kappa_2 n_2 a_{n_1+1}(k\Delta t)}{L-s_n(k\Delta t)}; \\ \qquad\qquad k\Delta t<t\leqq\left(k+\dfrac{1}{2}\right)\Delta t, \\ \dfrac{\kappa_1 n_1 a_{n_1-1}((k+1)\Delta t)}{s_n((k+\frac{1}{2})\Delta t)}+\dfrac{\kappa_2 n_2 a_{n_1+1}((k+1)\Delta t)}{L-s_n((k+\frac{1}{2})\Delta t)}; \\ \qquad\qquad \left(k+\dfrac{1}{2}\right)\Delta t<t\leqq(k+1)\Delta t.\end{cases}$$

In regard to the extended solution $\tilde{u}_1$ we have

$$(5.12)\qquad -\kappa_1\frac{\partial\tilde{u}_1}{\partial x}(s_n(t),t)$$

$$=\begin{cases}\kappa_1 n_1\dfrac{a_{n_1-1}(k\Delta t)+\alpha\{a_{n_1-1}((k+1)\Delta t)-a_{n_1-1}(k\Delta t)\}}{s_n(k\Delta t)+2\alpha\{s_n((k+\frac{1}{2})\Delta t)-s_n(k\Delta t)\}}; \\ \qquad\qquad k\Delta t<t\leqq\left(k+\dfrac{1}{2}\right)\Delta t, \\ \kappa_1 n_1\dfrac{a_{n_1-1}(k\Delta t)+\alpha\{a_{n_1-1}((k+1)\Delta t)-a_{n_1-1}(k\Delta t)\}}{s_n((k+\frac{1}{2})\Delta t)+(2\alpha-1)\{s_n((k+1)\Delta t)-s_n((k+\frac{1}{2})\Delta t)\}}; \\ \qquad\qquad \left(k+\dfrac{1}{2}\right)\Delta t<t\leqq(k+1)\Delta t,\end{cases}$$

and hence the estimates we need for the proof of Lemma 10 become

$$(5.13)\qquad\begin{cases}n_1\left|\dfrac{a_{n_1-1}(k\Delta t)+\alpha\{a_{n_1-1}((k+1)\Delta t)-a_{n_1-1}(k\Delta t)\}}{s_n(k\Delta t)+2\alpha\{s_n((k+\frac{1}{2})\Delta t)-s_n(k\Delta t)\}}\right. \\ \qquad\qquad\left.-\dfrac{a_{n_1-1}(k\Delta t)}{s_n(k\Delta t)}\right|\leqq M_1'\Delta t^{1/2}, \\ n_1\left|\dfrac{a_{n_1-1}(k\Delta t)+\alpha\{a_{n_1-1}((k+1)\Delta t)-a_{n_1-1}(k\Delta t)\}}{s_n((k+\frac{1}{2})\Delta t)+(2\alpha-1)\{s_n((k+1)\Delta t)-s_n((k+\frac{1}{2})\Delta t)\}}\right. \\ \qquad\qquad\left.-\dfrac{a_{n_1-1}((k+1)\Delta t)}{s_n((k+\frac{1}{2})\Delta t)}\right|\leqq M_1''\Delta t^{1/2},\end{cases}$$

which are shown to be valid in a similar way as that in § 4 using (4. 11) in view of $|\Delta s_n((k+\frac{1}{2})\Delta t)|\leqq\gamma\Delta t$ and $|\Delta s_n((k+1)\Delta t)|\leqq\gamma\Delta t$. The situation is the same for $\tilde{u}_2$, so that for the convergence of the improved scheme we have

Theorem 2′ (*improved scheme, lumped mass system,* $\theta=1$). *Under*

Assumptions A, B, C, D, E, F, G and H, the approximate solution obtained by (5. 1)-(5. 7) *converges to the solution of the Stefan problem* (1. 1)-(1. 5) *as* $\Delta t \to 0$, $n_1, n_2 \to \infty$.

§ 6. Numerical Example

We applied the present method to the following problem:

$$0 \leqq x \leqq L = 2, \tag{6.1}$$

$$\sigma_1 = 5, \quad \sigma_2 = 10, \quad \kappa_1 = \frac{1}{10}, \quad \kappa_2 = \frac{3}{10}, \tag{6.2}$$

$$\begin{cases} g_1(t) = 1 \\ g_2(t) = \begin{cases} -\dfrac{3}{8}\left(\cos \pi t + \dfrac{5}{3}\right); & 0 \leqq t \leqq 1, \\ -\dfrac{1}{4} & ; \ 1 < t \leqq 5, \end{cases} \end{cases} \tag{6.3}$$

$$s(0) = b = 1, \tag{6.4}$$

$$\begin{cases} f_1(x) = 1 - x, \\ f_2(x) = 1 - x. \end{cases} \tag{6.5}$$

The computation was carried out by means of the improved scheme of the lumped mass system with $\theta = 1$. We employed two sets of parameters:

$$\Delta t = 0.1, \quad n_1 = n_2 = 5. \tag{6.6}$$

$$\Delta t = 0.001, \quad n_1 = n_2 = 10. \tag{6.7}$$

The data and the parameters satisfy all the assumptions we have made, and the computation was actually stable. Fig. 3 shows the change of $s_n(t)$ obtained using the parameters (6. 6). It can be shown theoretically that $s(t)$ approaches to 8/7 asymptotically in the present case. It turned out that, even with the coarser mesh sizes (6. 6), the accuracy of the result was good enough. In fact, the differences of $\tilde{u}_1$, $\tilde{u}_2$ and

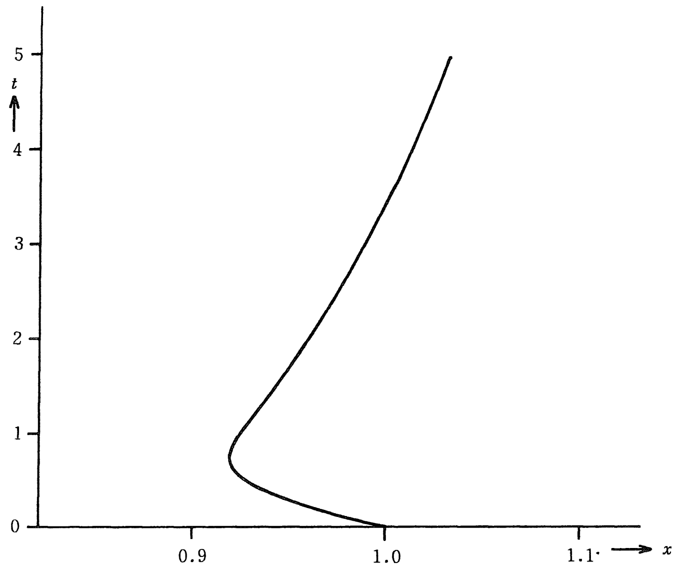

Fig. 3. The change of $s_n(t)$.

$s_n(t)$ with (6.6) and those with the finer parameters (6.7) are less than 10^{-3} at the points corresponding to the mesh points of the solution obtained with (6.6).

References

[1] Bonnerot, R. and Jamet, P., A second order finite element method for the one-dimensional Stefan problem, *Internat. J. Numer. Methods Engrg.*, **8** (1974), 811-820.

[2] Cannon, J. R. and Primicerio, M., A two phase Stefan problem with temperature boundary conditions, *Ann. Mat. Pura Appl.* (*IV*) **88** (1971), 177-191.

[3] ———, A two phase Stefan problem with flux boundary conditions, *Ann. Mat. Pura Appl.* (*IV*) **88** (1971), 193-205.

[4] Douglas, J. Jr. and Gallie, T. M., On the numerical integration of a parabolic differential equation subject to a moving boundary condition, *Duke Math. J.*, **22** (1955), 557-571.

[5] Kawarada, H. and Natori, M., On numerical solutions of Stefan problem I, Mem. Numer. Math., No. 1 (1975), 43-54.

[6] Meyer, G. H., On a free interface problem for linear ordinary differential equations and the one phase Stefan problem, *Numer. Math.*, **16** (1970), 248-267.

[7] Mori, M., Stability and convergence of a finite element method for solving the Stefan problem, *Publ. RIMS, Kyoto Univ.*, **12** (1976), 539-563.

[8] ———, A finite element method for solving moving boundary problems, *Preprints of IFIP Working Conference on Modelling of Environmental Systems*, Tokyo, April, 1976, 167-171.

[9] ——— , Stability of a finite element method for solving the Stefan problem in one space dimension, *Proceedings of the Japan-France Seminar on Numerical Analysis and Functional Analysis*, Tokyo and Kyoto, September, 1976.

[10] Nogi, T., A difference scheme for solving the Stefan problem, *Publ. RIMS, Kyoto Univ.*, **9** (1974), 543-575.