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The Squaring Operations in the Eilenberg-Moore Spectral Sequence and the Classifying Space of an Associative H-Space, I

By

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§ 0. Introduction

Let G be a compact, connected, simple Lie group. Let p be a prime. Consider $\{G; p\}$ the set of all compact, associative H-spaces X such that $H^*(X; Z_p) \cong H^*(G; Z_p)$ as Hopf algebras over the Steenrod algebra $\mathcal{A}_p$. (Remark that we do not require the existence of any map between X and G inducing the isomorphism.) As is well known, X has the the classifying space BX (see for example [8]).

The Eilenberg-Moore spectral sequence for X

$$(0.1) \qquad E_2(X) = \mathrm{Cotor}_A(Z_p, Z_p) \Rightarrow H^*(BX; Z_p), \quad \text{where } A = H^*(X; Z_p),$$

is a machinery to calculate $H^*(BX; Z_p)$. When $H_*(G; Z)$ has no p-torsion, it is quite easy to obtain $H^*(BX; Z_p)$. In fact, $\mathrm{Cotor}_A(Z_p, Z_p)$ is a polynomial algebra and the Eilenberg-Moore spectral sequence collapses. But when $H_*(G; Z)$ has p-torsion, it is, in general, difficult to obtain the structure of $H^*(BX; Z_p)$.

Let E_j be the compact, 1-connected, simple, exceptional Lie group of rank j $(j=6, 7)$. Recently, Kono-Mimura [6] and Kono-Mimura-Shimada [7] have determined the module structure of $H^*(BE_j; Z_2)$ $(j=6, 7)$. Their method was to calculate algebraically $\mathrm{Cotor}_A(Z_2, Z_2)$ and then to show the collapsing of the spectral sequence (0, 1) for E_j by making use of the properties of E_j as Lie groups.

The aim of this paper is to give a proof of the collapsing of the

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spectral sequence (0.1) independently of the properties as Lie groups, namely, to show the collapsing of the spectral sequence (0.1) for X_j of $\{E_j; 2\}$ $(j=6, 7)$. Our method is to make use of the relationship between the defferentials and the two kinds of the squaring operations in the spectral sequence, which was obtained by W. Singer [12].

We denote by $E_0H^*(BX; Z_2)$ the bigraded, associated algebra of $H^*(BX; Z_2)$ with respect to the filtration $F^pH^*(BX; Z_2)$ in the sense of Eilenberg-Moore, that is,

$$E_0^{p,q}H^*(BX; Z_2) = F^pH^{p+q}(BX; Z_2)/F^{p+1}H^{p+q}(BX; Z_2).$$

We shall use the convention to identify the elements in $E_0H^*(BX; Z_2)$ with those in $H^*(BX; Z_2)$, since $E_0H^*(BX; Z_2) \cong H^*(BX; Z_2)$ as modules.

Our results are stated as follows.

Theorem A. *For any* $X_6 \in \{E_6; 2\}$,

$$E_0H^*(BX_6; Z_2) \cong Z_2[y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{34}, y_{48}]/R,$$

as an algebra, where R *is the ideal generated by* (3.7).

Theorem B. (i) *In* $H^*(BX_6; Z_2)$ *the following relations hold* mod *decomposables.*

$$Sq^2y_4 = y_6, \quad Sq^1y_6 = y_7, \quad Sq^4y_6 = y_{10}, \quad Sq^8y_{10} = y_{18},$$

$$Sq^{16}y_{18} = y_{34}, \quad Sq^{16}y_{32} = y_{48}.$$

(ii) $H^*(BX_6; Z_2)$ *is generated by* y_4 *and* y_{32} *over* $\mathcal{A}_2$.

Theorem C. *For any* $X_7 \in \{E_7; 2\}$,

$$E_0H^*(BX_7; Z_2) \cong Z_2[y_4, y_6, y_7, y_{10}, y_{11}, y_{18}, y_{19}, y_{34}, y_{35}, y_{64}, y_{66}, y_{67}, y_{96}, y_{112}]/R,$$

as an algebra, where R *is the ideal generated by* (3.9) *and* (3.10).

Theorem D. (i) *In* $H^*(BX_7; Z_2)$ *the following relations hold* mod *decomposables.*

(4) $Sq^2y_4=y_6$, $Sq^1y_6=y_7$, $Sq^4y_6=y_{10}$, $Sq^1y_{10}=y_{11}$,

$Sq^8y_{10}=y_{18}$, $Sq^1y_{18}=y_{19}$, $Sq^{16}y_{18}=y_{34}$, $Sq^1y_{34}=y_{35}$,

$Sq^{32}y_{34}=y_{66}$, $Sq^1y_{66}=y_{67}$, $Sq^{32}y_{64}=y_{96}$, $Sq^{16}y_{96}=y_{112}$.

(ii) *$H^*(BX_7; Z_2)$ is generated by y_4 and y_{64} over $\mathcal{A}_2$.*

Needless to say, Theorems A, B, C, D give the module structure of $H^*(BE_j; Z_2)$ $(j=6, 7)$ over $\mathcal{A}_2$. These are simpler proof than those of [6] and [7].

Remark. Let G_2 and F_4 be the compact, 1-connected, simple exceptional Lie groups of rank 2 and 4 respectively. Let $X_2\in\{G_2; 2\}$ and $X_4\in\{F_4; 2\}$. The structure of $H^*(BX_i; Z_2)$ $(i=2, 4)$ over $\mathcal{A}_2$ is obtained more easily by our argument. We leave them to the reader.

The paper is organized as follows. In § 1 we recollect the Singer's results on the two kinds of squaring operations in the Eilenberg-Moore spectral sequence. In § 2 we review that these operations coincide with those defined algebraically on $\mathrm{Cotor}_A(Z_2, Z_2)$ through the isomorphism $E_2\cong\mathrm{Cotor}_A(Z_2, Z_2)$. In § 3 we calculate squaring operations on $\mathrm{Cotor}_A(Z_2, Z_2)$ for $A=H^*(X_6; Z_2)$ and $H^*(X_7; Z_2)$. § 4 and § 5 show that the Eilenberg-Moore spectral sequences for X_6 and X_7 collapse and this leads us to our results. The final section, § 6, will be used to prove a lemma which is used in § 5.

§ 1. Squaring Operations in the Eilenberg-Moore Spectral Sequence

Let $S_*(T)$ denote the normalized singular Z_2-chain complex of a space T with all vertices at the base point. Put $S^*(T)=\mathrm{Hom}(S_*(T), Z_2)$.

Let X be a connected, associative H-space and BX the classifying space of X [8]. A special case of the dual statement to Théorème 3. 1 of Moore [9] states that there is an isomorphism

(1. 1) $H^*(BX; Z_2)\cong\mathrm{Cotor}_{S^*(X)}(Z_2, Z_2)$ (or $\mathrm{Ext}_{S_*(X)}(Z_2, Z_2)$).

Let K denote the coalgebra $S^*(X)$. Let $\overline{C}(K)$ denote the cobar construction of K, in which $\overline{C}^s(K)=\overline{K}\otimes\cdots\otimes\overline{K}$ (s-times) with $\overline{K}=\sum_{t>0} K^t$. Then $\overline{C}(K)$ is a double complex with the external differential induced from the coalgebra structure of K and the internal differential induced from the differential in K. Let $\mathrm{Tot}\,\overline{C}(K)$ denote the total complex of $\overline{C}(K)$. Then $\mathrm{Cotor}_K(Z_2, Z_2)$ is, by definition, the cohomology of $\mathrm{Tot}\,\overline{C}(K)$. The total complex $\mathrm{Tot}\,\overline{C}(K)$ has a filtration such that

$$F^r\,\mathrm{Tot}^n\,\overline{C}(K)=\sum_{\substack{p+q=n\\ p\geq r}}\overline{C}^{p,q}(K),$$

where the first index p is the external degree and the second one q is the internal degree. This gives rise to a spectral sequence $\{E_r\}$ such that

$$E_2\cong\mathrm{Cotor}_{H^*(X;Z_2)}(Z_2, Z_2)\Rightarrow H^*(BX; Z_2). \tag{1.2}$$

We call the spectral sequence (1.2) the Eilenberg-Moore spectral sequence for X.

Remark. This is dual to the spectral sequence

$$E^2\cong\mathrm{Tor}^{H_*(X;Z_2)}(Z_2, Z_2)\Rightarrow H_*(BX; Z_2),$$

which is constructed in [9].

Now we recollect the Singer's results [12] for our purpose. Singer shows that products and squaring operations are defined in $\mathrm{Cotor}_{S^*(X)}(Z_2, Z_2)$ as well as in $H^*(BX; Z_2)$ and the isomorphism (1.1) preserves them (Proposition 1.1 of [12 I], Proposition 7.1 of [12 II]). This enables us to introduce products and squaring operations in the Eilenberg-Moore spectral sequence.

Proposition 1.1 (Propositions 1.2, 1.3, 1.5 of [12 I]). *In the Eilenberg-Moore spectral sequence $\{E_r\}$ for an associative H-space X the following properties hold*:

(1) *Each $E_r (r\geq 2)$ is a differential algebra and products on E_2 determine those on E_r $(r\geq 2)$.*

(2) *There are squaring operations*

$$Sq^k: E_r^{p,q}\to E_r^{p,q+k}\quad(0\leq k\leq q),$$

$$Sq^k: E_r^{p,q}\to E_r^{p+k-q,2q}\quad(k\geq q),$$

and the squaring operations on E_2 *determine those on* E_r $(r\geq 2)$.

(3) *Let* $\rho: F^pH^{p+q}(BX; Z_2)\to E_\infty^{p,q}$ *be the natural projection.*

For $u\in F^pH^{p+q}(BX; Z_2)$ *and* $v\in F^rH^*(BX; Z_p)$, *we have*

i) $uv\in F^{p+r}H^*(BX; Z_2)$ *and* $\rho(uv)=\rho(u)\rho(v)$,

ii) *if* $0\leq k\leq q$, *then* $Sq^ku\in F^pH^*(BX; Z_2)$ *and* $\rho Sq^ku=Sq^k\rho u$,

iii) *if* $q\leq k$, *then* $Sq^ku\in F^{p+k-q}(BX; Z_2)$ *and* $\rho Sq^ku=Sq^k\rho u$.

The operation $Sq^k: E_r^{q,p}\to E_r^{p,q+k}$ will be called *a vertical squaring operation* and $Sq^k: E_r^{p,q}\to E_r^{p+k-q,2q}$ *a diagonal squaring operation.*

Proposition 1.2 (Proposition 1.4 of [12 I]). *Let* $u\in E_r^{p,q}$ $(r\geq 2)$.

i) *If* $k\leq q-r+1$, *then* $d_rSq^ku=Sq^kd_ru$ *in* E_r.

ii) *If* $q-r+1\leq k\leq q$, *then* Sq^ku *survives to* $E_t^{p,q+k}$, *where* $t=2r+k-q-1$, Sq^kd_ru *survives to* $E_t^{p+t,2q-2r+2}$ *and* $d_t[Sq^ku]=[Sq^kd_ru]$.

iii) *If* $q\leq k$, *then* Sq^ku *survives to* $E_t^{p+k-q,2q}$, *where* $t=2r-1$, Sq^kd_ru *survives to* $E_t^{p+t+k-q,2q-2r+2}$ *and* $d_t[Sq^ku]=[Sq^kd_ru]$.

Remark. We sometimes regard the vertical operation Sq^k: $E_r^{p,q}\to E_r^{p,q+k}$ is zero if $k>q$. In this sense the differentials commute with vertical operations, i.e., $d_rSq^ku=Sq^kd_ru$ in E_r for every $k\geq 0$ and $r\geq 2$.

§ 2. Squaring Operations on the E_2-Term

Let X be an associative H-space. Put $A=H^*(X; Z_2)$.

Proposition 2.1 (Theorem 2.2 of [10]).

$$E_2\cong \mathrm{Cotor}_A(Z_2, Z_2) \text{ as algebras}.$$

We recall the two kinds of squaring operations on $\mathrm{Cotor}_A(Z_2, Z_2)$.

Let $\overline{C}(A)$ be the cobar construction of A. Let $\alpha=[x_1|\cdots|x_p]\in\overline{C}^{p,q}(A)$. Define an operation $Sq^k{}_V$: $\overline{C}^{p,q}(A)\to\overline{C}^{p,q+k}(A)$ by

$$(2.1)\qquad Sq^k{}_V\alpha=\sum[Sq^{k_1}x_1|\cdots|Sq^{k_p}x_p],\ k_1+\cdots+k_p=k.$$

Then $Sq^k{}_V$ commutes with the coboundary in $\overline{C}(A)$, since A is the coalgebra over the Steenrod algebra. Hence this induces

$$Sq^k{}_V: \mathrm{Cotor}_A^{p,q}\to\mathrm{Cotor}_A^{p,q+k}.$$

Let $\overline{B}(A)$ be the bar construction of A, i.e.,

$$\overline{B}^s(A)=\overline{H}_*(X;Z_2)\otimes\cdots\otimes\overline{H}_*(X;Z_2)\quad (s\text{-times}).$$

There is a map with external degree $i\geq 0$,

$$\Delta_i\colon \overline{B}(A)\to\overline{B}(A)\otimes\overline{B}(A),$$

satisfying $d\Delta_i+\Delta_i d=\Delta_{i-1}+T\Delta_{i-1}$ $(\Delta_{-1}=0)$. The cup-i-product

$$\underset{i}{\cup}\colon \overline{C}^p(A)\otimes\overline{C}^q(A)\to\overline{C}^{p+q-i}(A)$$

is defined by

$$(\alpha\underset{i}{\cup}\beta)(c)=(\alpha\otimes\beta)\Delta_i(c)\quad\text{for } \alpha\in\overline{C}^p(A),\ \beta\in\overline{C}^q(A),$$

$$c\in\overline{B}^{p+q-i}(A)$$

and satisfies

$$\delta(\alpha\underset{i}{\cup}\beta)=\delta\alpha\underset{i}{\cup}\beta+\alpha\underset{i}{\cup}\delta\beta+\alpha\underset{i-1}{\cup}\beta+\beta\underset{i-1}{\cup}\alpha.$$

Then an operation $Sq^k{}_D\colon \overline{C}^{p,q}(A)\to\overline{C}^{p+k,2q}(A)$ is defined by

(2.2) $$Sq^k{}_D\alpha=\alpha\underset{p-k}{\cup}\alpha+\delta\alpha\underset{p-k+1}{\cup}\alpha\quad\text{for}\quad \alpha\in\overline{C}^{p,q}(A).$$

This commutes with the coboundary and induces

$$Sq^k{}_D\colon \mathrm{Cotor}_A{}^{p,q}\to\mathrm{Cotor}_A{}^{p+k,2q}.$$

The construction of $Sq^k{}_D$ is essentially due to [1]. The explicit formula for the cup-i-product may be found in [14]. Especially, we recall the formulae:

$$[x_1|\cdots|x_s]\underset{0}{\cup}[x_{s+1}|\cdots|x_{s+r}]=[x_1|\cdots|x_{s+r}],$$

$$[x_1|\cdots|x_s]\underset{1}{\cup}[x_{s+1}|\cdots|x_{s+r}]=\sum_{i=1}^{s}[x_1|\cdots|x_{i-1}|x_i{}^{(1)}x_{s+1}|$$
$$\cdots|x_i{}^{(r)}x_{s+r}|x_{i+1}|\cdots|x_s],$$

$$[x_1|\cdots|x_s]\underset{s}{\cup}[x_1|\cdots|x_s]=[x_1{}^2|\cdots|x_s{}^2],$$

where $\psi^{(r-1)}(x)=\sum x^{(1)}\otimes\cdots\otimes x^{(r)}$, $\psi^{(r-1)}\colon A\to A\otimes\cdots\otimes A$ (r-times), is the $(r-1)$-iterated diagonal map.

Proposition 2.2 (Propositions 7.2, 7.3 of [12 II]). *Through the* *isomorphism* $E_2 \cong \mathrm{Cotor}_A(Z_2, Z_2)$,

i) *if* $0 \leq k \leq q$, *then the vertical squaring operation* Sq^k *on* E_2 *coincides with* $Sq^k{}_V$ *on* $\mathrm{Cotor}_A(Z_2, Z_2)$,

ii) *if* $q \leq k$, *then the diagonal squaring operation* Sq^k *on* E_2 *coincides with* $Sq^{k-q}{}_D$ *on* $\mathrm{Cotor}_A(Z_2, Z_2)$.

Corollary 2. 3. *Let* $\sum[x_1|\cdots|x_p] \in \bar{C}^{p,q}(A)$ *and* $\sum[x_{p+1}|\cdots|x_{p+r}] \in \bar{C}^{r,s}(A)$ *represent* $u \in E_2^{p,q}$ *and* $v \in E_2^{r,s}$ *respectively. Then*

i) $\sum[x_1|\cdots|x_p|x_{p+1}|\cdots|x_{p+r}] \in \bar{C}^{p+r,q+s}(A)$ *represents* $uv \in E_2^{p+r,q+s}$,

ii) *if* $0 \leq k \leq q$, *then*

$$\sum_{k_1+\cdots+k_p=k} \sum[Sq^{k_1}x_1|\cdots|Sq^{k_p}x_p] \in \bar{C}^{p,q+k}(A)$$

represents $Sq^k u \in E_2^{p,q+k}$.

iii) *if* $q \leq k$, then

$$\sum[x_1|\cdots|x_p] \underset{p-k+p}{\cup} [x_1|\cdots|x_p] \in \bar{C}^{p+k-q,2q}$$

represents $Sq^k u \in E_2^{p+k-q,2q}$.

Proof. Immediate from Propositions 2. 1, 2. 2 and (2. 1), (2. 2). q.e.d.

Here we remark, for later use:

Proposition 2. 4. *As for the vertical squaring operation, the Cartan formula holds on* E_2, *i.e.*,

$$Sq^k(uv) = \sum_{i+j=k} Sq^i u Sq^j v \text{ for } u, v \in E_2$$

and E_r $(r \geq 2)$ *inherits this formula.*

Proof. We confirm this by Corollary 2. 3, i), ii), and Proposition 1. 1, (1), (2), though this may be proved by the standard argument. q.e.d.

Let ϕ be the diagonal map of $A = H^*(X; Z_2)$. Let L be a quotient coalgebra of $\bar{A}$ over the Steenrod algebra $\mathcal{A}_2$ with projection θ: $A \to L$.

$\bar{\phi}'$ denotes the diagonal map of L. Note that L is not equipped with unit. Construct the tensor algebra $T(sL)$ with product ψ, where s is the suspension, that is, the operation to make a copy with external degree added by one. Let I be the two-sided ideal generated by $\psi\circ(s\theta\otimes s\theta)\circ\phi(Ker\,\theta)$. Let $\bar{X}=T(sL)/I$. The differential $\bar{d}$ on $\bar{X}$ is induced so that $\bar{d}=\psi\circ\bar{\phi}'\circ s^{-1}: sL\to T(sL)$ is derivative. Then $\bar{d}(I)\subset I$ and $\bar{d}\circ\bar{d}=0$, and this is well-defined. $\bar{X}$ is the quotient of $\bar{C}(A)$ as differential algebra with projection $p: \bar{C}(A)\to\bar{X}$ such that $p[x_1|\cdots|x_n]=s\theta x_1\cdots s\theta x_n$ (see [11]). The (vertical) squaring operation on $\bar{X}$ is defined by

$$Sq^k{}_V x=\sum_{k_1+\cdots+k_n=k} sSq^{k_1}x_1\cdots sSq^{k_n}x_n\,,\ x=sx_1\cdots sx_n\in T(sL)$$

for $k\geq 0$.

Proposition 2.5. *The projection $p: \bar{C}(A)\to\bar{X}$ preserves the operation $Sq^k{}_V$.*

Proof. Immediate from Propositions 2.1, 2.2 and Corollary 2.3.

q.e.d.

Corollary 2.6. *Assume that $p: \bar{C}(A)\to\bar{X}$ induces an isomorphism on cohomology. Let $\sum sx_1\cdots sx_p\in\bar{X}^{p,q}$ represent $u\in E_2^{p,q}$. Then if $0\leq k\leq q$, the element*

$$\sum_{k_1+\cdots+k_p=k}\sum sSq^{k_1}x_1\cdots sSq^{k_p}x_p\in\bar{X}^{p,q+k}$$

represents $Sq^k u\in E_2^{p,q+k}$.

Proof. Immediate from Corollary 2.3 and Proposition 2.5.

q.e.d.

§ 3. Squaring Operations on Cotor_A for $A=H^*(X_6; Z_2)$ and $H^*(X_7; Z_2)$

Let $X_6\in\{E_6; 2\}$ and $X_7\in\{E_7, 2\}$.

By definition and [2], we have

$$H^*(X_6; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda(x_5, x_9, x_{15}, x_{17}, x_{23}),$$

$$H^*(X_7; Z_2) = Z_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27}).$$

The reduced diagonal map is given by Theorem 3.1 of [6] and Theorem 1.8 of [7], namely,

(3.1) *for* X_6, X_7, $\bar{\phi}(x_i) = 0 \quad (i = 3, 5, 9, 17)$,

(3.2) *for* X_6, $\bar{\phi}(x_{15}) = x_9 \otimes x_3^2$,

$\bar{\phi}(x_{23}) = x_{17} \otimes x_3^2$,

(3.3) *for* X_7, $\bar{\phi}(x_{15}) = x_5 \otimes x_5^2 + x_9 \otimes x_3^2$,

$\bar{\phi}(x_{23}) = x_5 \otimes x_9^2 + x_{17} \otimes x_3^2$,

$\bar{\phi}(x_{27}) = x_9 \otimes x_9^2 + x_{17} \otimes x_5^2$.

The squaring operations on the elements are given by [2] and [13], namely,

(3.4) *for* X_6, X_7, $x_5 = Sq^2 x_3$, $x_9 = Sq^4 x_5$, $x_{17} = Sq^8 x_9$, $x_{23} = Sq^8 x_{15}$,

(3.5) *for* X_7, $x_{27} = Sq^4 x_{23}$,

(3.6) *for* X_6, X_7, $x_{17} = Sq^2 x_{15}$.

Proposition 3.1. (i) *Let* $A = H^*(X_6; Z_2)$. *Then as an algebra,*

$$E_2 \cong \mathrm{Cotor}_A(Z_2, Z_2) \cong Z_2[y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{34}, y_{48}]/R,$$

where the gradings of generators are given by

$$y_4 \in (1, 3),\ y_6 \in (1, 5),\ y_7 \in (1, 6),\ y_{10} \in (1, 9),$$

$$y_{18} \in (1, 17),\ y_{32} \in (2, 30),\ y_{34} \in (2, 32),\ y_{48} \in (2, 46),$$

($y \in (i, j)$ *means* $y \in E_2^{i,j}$) *and* R *is the ideal generated by*

(3.7) $y_7 y_{10}$, $y_7 y_{18}$, $y_7 y_{34}$, $y_{34}^2 + y_{10}^2 y_{48} + y_{18}^2 y_{32}$.

(ii) *The following relations hold in* E_2:

$$Sq^2 y_4 = y_6,\ Sq^1 y_6 = y_7,\ Sq^4 y_6 = y_{10},\ Sq^8 y_{10} = y_{18},\ Sq^{16} y_{32} = y_{48}.$$

Proof. The calculation of $\mathrm{Cotor}_A(Z_2, Z_2)$ is purely algebraic, and hence (i) follows from Theorem 2.3 of [6]. To determine the squaring operations, recall the outline of their calculation. Let $L = \{x_3, x_5, x_3^2, x_9, x_{17}, x_{15}, x_{23}\}$ and denote the corresponding elements by $sL = \{a_4, a_6, a_7, a_{10},$

$a_{18}, b_{16}, b_{24}\}$. Then by (3.4) we have

$$Sq^2a_4=a_6,\ Sq^1a_6=a_7,\ Sq^4a_6=a_{10},\ Sq^8a_{10}=a_{18},\ Sq^8b_{16}=b_{24}. \tag{3.8}$$

Form a differential algebra $(\bar{X}, \bar{d})$ as in § 2. Explicitly $\bar{X}$ is isomorphic to the polynomial algebra $Z_2[a_4, a_6, a_7, a_{10}, a_{18}, b_{16}, b_{24}]$. Then the projection $p: \bar{C}(A) \to \bar{X}$ induces an isomorphism on cohomology, i.e., $\mathrm{Cotor}_A(Z_2, Z_2) \cong H(\bar{X}, \bar{d})$. Each y_i is represented in $\bar{X}$ as follows:

$$y_i\colon a_i \quad (i=4, 6, 7, 10, 18),$$

$$y_{32}\colon b_{16}^2, \qquad y_{48}\colon b_{24}^2, \qquad y_{34}\colon a_{10}b_{24}+a_{18}b_{16}.$$

Note that the squaring operations on y_i follow immediate from Corollary 2.6 and (3.8). q.e.d.

We next turn to X_7.

Proposition 3.2. (i) *Let $A=H^*(X_7; Z_2)$. Then as an algebra,*

$$E_2 \cong \mathrm{Cotor}_A(Z_2, Z_2) \cong Z_2[y_4, y_6, y_7, y_{10}, y_{11}, y_{18}, y_{19}, y_{34}, y_{35},$$
$$y_{66}, y_{67}, y_{64}, y_{96}, y_{112}]/R,$$

where the gradings of generators are given by

$$y_4 \in (1, 3),\ y_6 \in (1, 5),\ y_7 \in (1, 6),\ y_{10} \in (1, 9),$$
$$y_{11} \in (1, 10),\ y_{18} \in (1, 17),\ y_{19} \in (1, 18),\ y_{34} \in (2, 32),$$
$$y_{35} \in (2, 33),\ y_{66} \in (3, 63),\ y_{67} \in (3, 64),\ y_{64} \in (4, 60),$$
$$y_{96} \in (4, 92),\ y_{112} \in (4, 108),$$

and R is the ideal generated by

$$y_6y_{11}+y_{10}y_7,\ y_6y_{19}+y_{18}y_7,\ y_{10}y_{19}+y_{18}y_{11}, \tag{3.9}$$
$$y_{11}^3+y_{19}y_7^2,\ y_{11}y_{19}^2,\ y_{19}^3,\ y_7y_{34}+y_6y_{35}+y_{19}y_{11}^2,$$
$$y_{11}y_{34}+y_{10}y_{35}+y_7y_{19}^2,\ y_{19}y_{34}+y_{18}y_{35},\ y_{11}y_{35}^2+y_7^2y_{67},$$
$$y_{19}y_{35}^2+y_{11}^2y_{67},\ y_{19}^2y_{67},\ y_{34}^4+y_{18}^4y_{64}+y_{10}^4y_{96}+y_6^4y_{112},$$
$$y_{35}^4+y_{19}^4y_{64}+y_{11}^4y_{96}+y_7^4y_{112},\ y_{66}^2+y_{10}^2y_{112}+y_{18}^2y_{96},$$
$$y_{67}^2+y_{11}^2y_{112}+y_{19}^2y_{96},$$

(3. 10) $y_7y_{66}+y_6y_{67}+y_{19}^2y_{35},\ y_{11}y_{66}+y_{10}y_{67},\ y_{19}y_{66}+y_{18}y_{67}$,

$y_{35}^2y_{67}+y_7^2y_{11}y_{112}+y_{11}^2y_{19}y_{96}$,

$y_{34}y_{67}+y_{35}y_{66},\ y_{66}y_{67}+y_{10}y_{11}y_{112}+y_{18}y_{19}y_{96}$,

$y_{34}^2y_{35}^2+y_{18}^2y_{19}^2y_{64}+y_{10}^2y_{11}^2y_{96}+y_6^2y_7^2y_{112}$.

(ii) *The vertical squaring operations in* E_2 *are given by*

$$Sq^2y_4=y_6,\ Sq^1y_6=y_7,\ Sq^4y_6=y_{10},\ Sq^1y_{10}=y_{11},$$
$$Sq^8y_{10}=y_{18},\ Sq^1y_{18}=y_{19},\ Sq^1y_{34}=y_{35},\ Sq^1y_{66}=y_{67},$$
$$Sq^{32}y_{64}=y_{96},\ Sq^{16}y_{96}=y_{112}.$$

Proof. The calculation of $\mathrm{Cotor}_A(Z_2, Z_2)$ is the same as that given by [7], although the relations (3. 10) are dropped there. Recall the outline of their calculation. Let $L=\{x_3, x_5, x_3^2, x_9, x_5^2, x_{17}, x_9^2, x_{15}, x_{23}, x_{27}\}$ and denote the corresponding elements by $sL=\{a_4, a_6, a_7, a_{10}, a_{11}, a_{18}, a_{19}, b_{16}, b_{24}, b_{28}\}$. Then by (3. 4) and (3. 5) we have

(3. 11) $Sq^2a_4=a_6,\ Sq^1a_6=a_7,\ Sq^4a_6=a_{10},\ Sq^1a_{10}=a_{11}$,

$Sq^8a_{10}=a_{18},\ Sq^1a_{18}=a_{19},\ Sq^8b_{16}=b_{24},\ Sq^4b_{24}=b_{28}$.

Form a differential algebra $(\bar{X}, \bar{d})$ as in § 2. Explicitly

$$\bar{X}\cong Z_2\{a_i, b_j\}/I,\ i=4, 6, 7, 10, 11, 18, 19,\ j=16, 24, 28,$$

where I is the ideal generated by all possible $[a_m, a_n]$ and $[b_p, b_q]$ and by $[a_i, b_j]$ except $(i, j)=(6, 16)$, $(10, 16)$, $(6, 24)$, $(10, 28)$ and $[a_6, b_{16}]+a_{11}^2$, $[a_{10}, b_{16}]+a_{19}a_7$, $[a_6, b_{24}]+a_{11}a_{19}$, $[a_{10}, b_{28}]+a_{19}^2$. Then the projection $p: \bar{C}(A)\to\bar{X}$ induces an isomorphism on cohomology, i.e., $\mathrm{Cotor}_A(Z_2, Z_2)\cong H(\bar{X}, \bar{d})$. Each y_i is represented in $\bar{X}$ as follows.

(3. 12) y_i: a_i $(i=4, 6, 7, 10, 11, 18, 19)$

y_{34}: $a_{18}b_{16}+a_{10}b_{24}+a_6b_{28}$, y_{35}: $a_{19}b_{16}+a_{11}b_{24}+a_7b_{28}$,

y_{66}: $a_{10}b_{28}^2+a_{18}b_{24}^2+a_{19}^2b_{28}$, y_{67}: $a_{11}b_{28}^2+a_{19}b_{24}^2$,

y_{4j}: b_j^4 $(j=16, 24, 28)$.

Remark that the representative of y_{66} in [7] is incorrect. Now the squaring operations on y_i follow immediate from Corollary 2. 6 and (3. 11).

q.e.d.

Proposition 3. 3. *$Sq^1{}_D y_{34} = y_{67}$ in* $\mathrm{Cotor}_A(Z_2, Z_2)$, *and hence $Sq^{33}y_{34} = y_{67}$ in E_2.*

Proof. Let C be a representative of y_{34} in the cobar construction $\overline{C}(A)$. The explicit form of C is given by

$$C = [x_{17}|x_{15}] + [x_9|x_{23}] + [x_5|x_{27}] + [x_5x_{17}|x_5^2] + [x_9x_{17}|x_3^2] + [x_5x_9|x_9^2].$$

Then $Sq^1{}_D y_{34}$ is represented by $C \underset{1}{\cup} C$. By using the explicit formula for the cup-1-product, we have

$$C \underset{1}{\cup} C = [x_9^2|x_{23}|x_{23}] + [x_5^2|x_{27}|x_{27}] + r,$$

where $r \in Ker(p\colon \overline{C}(A) \to \overline{X})$. Hence

$$p(C \underset{1}{\cup} C) = a_{11}b_{28}^2 + a_{19}b_{24}^2.$$

Therefore $C \underset{1}{\cup} C$ represents y_{67}, and we have $Sq^1{}_D y_{34} = y_{67}$ in $\mathrm{Cotor}_A(Z_2, Z_2)$. The latter relation $Sq^{33}y_{34} = y_{67}$ in E_2 follows from Proposition 2. 2.

q.e.d.

For later use we note

Lemma 3. 4. *$Sq^1y_{64} = Sq^2y_{64} = Sq^4y_{64} = 0$ and $Sq^8y_{64} = y_{18}^4$ in E_2.*

Proof. Since $Sq^1b_{16} = Sq^4b_{16} = 0$ for dimensional reasons and $Sq^2b_{16} = a_{18}$ in $\overline{X}$ by (3. 6), and since y_{64} is represented by b_{16}^4, the lemma follows from Corollary 2. 6. q.e.d.

§ 4. Collapsing of the Spectral Sequence for X_6

Let $X_6 \in \{E_6; 2\}$ and put $A = H^*(X_6; Z_2)$. Consider the Eilenberg-Moore spectral sequence for X_6:

(4. 1) $$E_2 \cong \mathrm{Cotor}_A(Z_2, Z_2) \Rightarrow H^*(BX_6; Z_2),$$

where the E_2-term is given by Proposition 3. 1.

Theorem 4. 1. *The Eilenberg-Moore spectral sequence* (4. 1) *for* X_6 *collapses.*

This will follow from the following lemmas.

Lemma 4. 2. *The element* y_4 *survives, and hence so do* y_6, y_7, y_{10}, y_{18}.

Proof. For dimensional reasons y_4 survives and so do the other elements by Propositions 1. 2 and 3. 1. q.e.d.

We need the following facts.

(4. 2) i) $y_{18}^2 \neq 0$ *in* $H^*(BX_6; Z_2)$,

ii) $y_7{}^5 \neq 0$ *in* E_3,

iii) $y_4{}^2 y_6{}^2 y_7{}^2 \neq 0$, $y_4 y_6{}^2 y_7{}^3 \neq 0$ *in* E_4,

iv) $y_4{}^5 y_7{}^2 \neq 0$ *in* E_5.

Proof is clear for dimensional reasons.

Lemma 4. 3. *The element* y_{34} *survives.*

Proof. Denote $F^p = F^p H^*(BX_6; Z_2)$. First note that $Sq^{18} y_{18} = y_{18}^2 \neq 0$ in $H^*(BX_6; Z_2)$ by (4. 2). Remark that $y_{18}^2 \in F^2$. By Adem relation

$$y_{18}^2 = Sq^{18} y_{18} = Sq^2 Sq^{16} y_{18} + Sq^{17} Sq^1 y_{18}\ .$$

For dimensional reasons $Sq^1 y_{18} \in F^3$, and hence $Sq^{17} Sq^1 y_{18} \in F^3$ by Proposition 1. 1. Now assume that y_{34} does not survive, then $Sq^{16} y_{18} \in F^3$, and hence $Sq^2 Sq^{16} y_{18} \in F^3$. This is a contradiction to $y_{18}^2 \in F^2$. Thus y_{34} survives and furthermore we must have

$$Sq^{16} y_{18} \equiv y_{34} \bmod F^3.$$

This completes the proof. q.e.d.

In the above proof we have shown

Proposition 4. 4. $Sq^{16}y_{18} \equiv y_{34}$ mod *decomposables in* $H^*(BX_6; Z_2)$.

Lemma 4. 5. $Sq^1y_{32} = Sq^2y_{32} = 0$ *in* E_2.

Proof. Recall that y_{32} is represented by b_{16}^2 in $\bar{X}$ (See the proof of Proposition 3. 1) and $Sq^1b_{16} = 0$ for dimensional reasons. Hence Sq^2y_{32} is represented by

$$Sq^2b_{16}^2 = Sq^1b_{16}Sq^1b_{16} = 0 \qquad \text{by Corollary 2. 6}.$$

Therefore we have $Sq^2y_{32} = 0$ in E_2. It is easier to see $Sq^1y_{32} = 0$.

q.e.d.

Lemma 4. 6. *The element* y_{32} *survives, and so does* y_{48}.

Proof. We first show that $y_{32} \in E_2^{2,30}$ is a permanent cocycle. Consider $d_r: E_r^{2,30} \to E_r^{2+r,31-r}$ $(r \geq 2)$. For dimensional reasons the possible elements to be killed by y_{32} are

$$E_2^{4,29} \ni y_4^2y_7y_{18} = 0, \ y_6y_7y_{10}^2 = 0,$$

$$E_3^{5,28} \ni y_4y_6^2y_7y_{10} = 0, \ y_6^2y_7^3,$$

$$E_4^{6,27} \ni y_4^4y_7y_{10} = 0, \ y_4^2y_6^3y_7, \ y_4^3y_7^3,$$

$$E_5^{7,26} \ni y_4^5y_6y_7.$$

Put $d_3(y_{32}) = ay_6^2y_7^3$ with $a \in Z_2$. Applying Sq^2, we have

$$0 = d_3(Sq^2y_{32}) = Sq^2d_3(y_{32}) = ay_7^5$$

by Propositions 1. 2, 3. 1 and Lemma 4. 5. Then since $y_7^5 \neq 0$ by (4. 2) we have $a=0$. Next put $d_4(y_{32}) = ay_4^2y_6^3y_7 + by_4^3y_7^3$ with $a, b \in Z_2$. Applying Sq^1, we have $0 = d_4(Sq^1y_{32}) = Sq^1d_4(y_{32}) = ay_4^2y_6^2y_7^2$. Since $y_4^2y_6^2y_7^2 \neq 0$ by (4. 2), we have $a=0$. Then applying Sq^2 to $d_4(y_{32}) = by_4^3y_7^2$, we have $0 = d_4(Sq^2y_{32}) = Sq^2d_4(y_{32}) = by_4y_6^2y_7^3$. Since $y_4y_6^2y_7^3 \neq 0$ by (4. 2), we have $b=0$. Thus $d_4(y_{32}) = 0$. Finally put $d_5(y_{32}) = ay_4^5y_6y_7$ with $a \in Z_2$. Applying Sq^1,

$$0 = d_5(Sq^1(y_{32})) = Sq^1d_5y_{32} = ay_4^5y_7^2.$$

Since $y_4^5y_7^2 \neq 0$ by (4. 2), we have $a=0$ and $d_5(y_{32}) = 0$. Thus we have shown that y_{32} is a permanent cocycle. Since y_{32} is not killed for dimensional reasons, we conclude that y_{32} survives, and hence $y_{48}=Sq^{16}y_{32}$ survives by Proposition 1. 2. q.e.d.

Now Theorem 4. 1 follows from Lemmas 4. 2, 4. 3 and 4. 6.

Theorems A and B follow immediately from Propositions 3. 1 and 4. 4 and Theorem 4. 1.

§ 5. Collapsing of the Spectral Sequence for X_7

Let $X_7\in\{E_7, 2\}$ and put $A=H^*(X_7; Z_2)$. Consider the Eilenberg-Moore spectral sequence for X_7:

(5. 1) $$E_2\cong \mathrm{Cotor}_A(Z_2, Z_2)\Rightarrow H^*(BX_7; Z_2),$$

where the E_2-term is given by Proposition 3. 2.

Theorem 5. 1. *The Eilenberg-Moore spectral sequence for X_7 collapses.*

This will follow from the following lemmas.

Lemma 5. 2. *The element y_4 survives, and so do y_6, y_7, y_{10}, y_{11}, y_{18}, y_{19}.*

Proof. The element y_4 survives for dimensional reasons, and so do the other elements by Propositions 1. 2 and 3. 2. q.e.d.

Lemma 5. 3. *The element y_{34} survives and so does y_{35}.*

Proof is quite similar to that of Lemma 4. 3, though the existence of the element of degree 19 may make a proof a little bit complicated.

As an analogous result to Proposition 4. 4 we can show

Proposition 5. 4. $Sq^{16}y_{18}\equiv y_{34}$ mod *decomposables in* $H^*(BX_7; Z_2)$.

Lemma 5. 5. *The element* y_{66} *survives and so does* y_{67}.

Proof. Denote $F^p = F^p H^*(BX_7; Z_2)$. By Proposition 3. 3 the relation $Sq^{33}y_{34} = y_{67}$ holds in E_2. Hence the element y_{67} survives to E_∞ by Proposition 1. 2, and we obtain

$$y_{67} = Sq^{33}y_{34} \equiv Sq^1 Sq^{32} y_{34} \mod F^4$$

in $H^*(BX_7; Z_2)$. Assume that y_{66} does not survive. Then $Sq^{32}y_{34} \in F^4$ for dimensional reasons. So $y_{67} \equiv 0 \mod F^4$, which is a contradiction to $y_{67} \in F^3$. Therefore y_{66} survives and furthermore we must have

$$Sq^{32}y_{34} \equiv y_{66} \mod F^4 . \qquad \text{q.e.d.}$$

In the proof we have obtained

Proposition 5. 6. $Sq^{32}y_{34} \equiv y_{66}$ mod *decomposables in* $H^*(BX_7; Z_2)$.

Lemma 5. 7. $d_r(Sq^i y_{64}) = 0$ *for* $i = 1, 2, 4, 8$ *and for all* $r \geq 2$.

Proof. Immediate from Lemma 3. 4. q.e.d.

Lemma 5. 8. *The element* y_{64} *survives and hence so do* y_{96} *and* y_{112}.

(The proof will be given in § 6.)

Now Theorem 5. 1 follows from Lemmas 5. 2, 5. 3, 5. 5 and 5. 8.

Theorems C and D follow from Propositions 3. 2, 5. 4 and 5. 6 and Theorem 5. 1.

§ 6. Proof of Lemma 5. 8

The proof of Lemma 5. 8 given here is quite analogous to that of Lemma 4. 3, although it is much more complicated. To prove the lemma, it suffices to show that the element y_{64} survives, since $y_{96} = Sq^{32}y_{64}$ and $y_{112} = Sq^{16}y_{96}$ by Proposition 3. 2. For dimensional reasons y_{64} is not killed, and hence we need only to check that $y_{64} \in E_2^{4,60}$ is a permanent cocycle.

Let $S(n)$ be the set of monomials in $E_2^{*,*}$

(6. 1) $$y_4^a y_6^b y_7^c y_{10}^d y_{11}^e y_{18}^f y_{19}^g y_{34}^h y_{35}^i$$

with

(6. 2. n) $$4a+6b+7c+10d+11e+18f+19g+34h+35i=n\,.$$

Note that the Z_2-module generated by $S(n)$ is closed under the vertical squaring operations. The set $S(n)$ is ordered lexicographically from the right, for example, $y_4 y_6^2 y_{10}^3 y_{19} > y_6^2 y_7^2 y_{10} y_{19}$ in $S(65)$. Since there are relations

(6. 3) $$y_6 y_{11} + y_{10} y_7 = 0,\ y_6 y_{19} + y_{18} y_7 = 0,\ y_{10} y_{19} + y_{18} y_{11} = 0,$$
$$y_{11}^3 + y_{19} y_7^2 = 0,\ y_{11} y_{19}^2 = 0,\ y_{19}^3 = 0,\ y_7 y_{34} + y_6 y_{35} + y_{19} y_{11}^2 = 0,$$
$$y_{11} y_{34} + y_{10} y_{35} + y_7 y_{19}^2 = 0,\ y_{19} y_{34} + y_{18} y_{35} = 0,$$

the monomials of $S(n)$ satisfying one of the following

(6. 4) i) $c \geq 1$, $d \geq 1$, ii) $c \geq 1$, $f \geq 1$, iii) $e \geq 1$, $f \geq 1$,

iv) $e \geq 3$, v) $e \geq 1$, $g \geq 2$, vi) $g \geq 3$, vii) $e \geq 2$, $g \geq 1$,

viii) $c \geq 1$, $g \geq 2$, ix) $g \geq 1$, $h \geq 1$,

can be reduced either to a trivial one or to a linear combination of the other monomials of higher order. A monomial is irreducible unless it satisfies one of the relations (6. 3). Thus the set of the irreducible monomials of degree n forms a Z_2-basis of $S(n)$.

Remark that the first (possibly) non-trivial differential is

$$d_r\colon\ E_r^{4,60} \to E_r^{4+r,61-r},$$

since the elements y_i ($i=4, 6, 7, 10, 11, 18, 19, 34, 35$) are cocycles. So the following lemmas are clear for dimensional reasons.

Lemma 6. 1. *The irreducible monomials* $y_4^a y_6^b y_7^c y_{10}^d y_{11}^e y_{18}^f y_{19}^g y_{34}^h y_{35}^i$ *are non-trivial in* $E_r^{p,q}$ *for the following cases*:

(1) $p+q \leq 68$ *and* $p+q \neq 65$, *when* $a>0$,

(2) $p+q=69$ *and* 73, *when* $a=0$.

Lemma 6. 2. *The non-negative integer solutions of the equation* (6. 2. 65) *and*

(6. 5) $$a+b+c+d+e+f+g+2h+2i=4+r$$

except the cases (6. 4) *gives a basis* $\{m_{r,i}\}$ *of* $E_r^{4+r,61-r}$.

Using this basis, each element of $E_r^{4+r,61-r}$ is expressed as $\sum_i k_i m_{r,i}$ with $k_i \in Z_2$. Explicitly we have

(6. 6) i) $E_2^{6,59}$: $k_1 y_4^3 y_{18} y_{35} + k_2 y_4^2 y_{11}^2 y_{35} + k_3 y_6^2 y_7 y_{11} y_{35} + k_4 y_4 y_6 y_{10}^2 y_{35}$
$+ k_5 y_4 y_6 y_{10} y_{11} y_{34} + k_6 y_6 y_7^2 y_{11} y_{34} + k_7 y_4^2 y_{10}^2 y_{18} y_{19}$
$+ k_8 y_6^3 y_{10} y_{18} y_{19} + k_9 y_6 y_{10}^4 y_{19}$,

ii) $E_3^{7,58}$: $k_1 y_4^3 y_7 y_{11} y_{35} + k_2 y_4^2 y_6^2 y_{10} y_{35} + k_3 y_4 y_6^2 y_7^2 y_{35} + k_4 y_6^5 y_{35}$
$+ k_5 y_4^2 y_6^2 y_{11} y_{34} + k_6 y_4 y_6 y_7^3 y_{34} + k_7 y_4^3 y_6 y_{10} y_{18} y_{19}$
$+ k_8 y_4 y_6^4 y_{18} y_{19} + k_9 y_7^5 y_{11} y_{19} + k_{10} y_4 y_6^2 y_{10}^3 y_{19}$
$+ k_{11} y_4 y_{10}^5 y_{11}$,

iii) $E_4^{8,57}$: $k_1 y_4^5 y_{10} y_{35} + k_2 y_4^4 y_7^2 y_{35} + k_3 y_4^3 y_6^3 y_{35} + k_4 y_4^5 y_{11} y_{34}$
$+ k_5 y_4^3 y_6^2 y_7 y_{34} + k_6 y_4^4 y_6^2 y_{18} y_{19} + k_7 y_4^2 y_6 y_7^3 y_{11} y_{19}$
$+ k_8 y_4 y_6^4 y_7 y_{11} y_{19} + k_9 y_4^4 y_{10}^3 y_{19} + k_{10} y_4^2 y_6^3 y_{10}^2 y_{19}$
$+ k_{11} y_6^6 y_{10} y_{19} + k_{12} y_4 y_7^6 y_{19} + k_{13} y_6^3 y_7^4 y_{19}$
$+ k_{14} y_4^2 y_6 y_{10}^4 y_{11} + k_{15} y_6^4 y_{10}^3 y_{11}$,

iv) $E_5^{9,56}$: $k_1 y_4^6 y_6 y_{35} + k_2 y_4^6 y_7 y_{34} + k_3 y_4^4 y_6^2 y_7 y_{11} y_{19} + k_4 y_4^5 y_6 y_{10}^2 y_{19}$
$+ k_5 y_4^3 y_6^4 y_{10} y_{19} + k_6 y_4^3 y_6 y_7^4 y_{19} + k_7 y_4^2 y_6^4 y_7^2 y_{19}$
$+ k_8 y_4 y_6^7 y_{19} + k_9 y_4^2 y_7^5 y_{11}^2 + k_{10} y_4 y_6^3 y_7^3 y_{11}^2$
$+ k_{11} y_6^6 y_7 y_{11}^2 + k_{12} y_4^3 y_6^2 y_{10}^3 y_{11} + k_{13} y_4 y_6^5 y_{10}^2 y_{11}$
$+ k_{14} y_6^2 y_7^6 y_{11}$,

v) $E_6^{10,55}$: $k_1 y_4^7 y_7 y_{11} y_{19} + k_2 y_4^6 y_6^2 y_{10} y_{19} + k_3 y_4^5 y_6^2 y_7^2 y_{19}$
$+ k_4 y_4^4 y_6^5 y_{19} + k_5 y_4^4 y_6 y_7^3 y_{11}^2 + k_6 y_4^3 y_6^4 y_7 y_{11}^2$
$+ k_7 y_4^6 y_{10}^3 y_{11} + k_8 y_4^4 y_6^3 y_{10}^2 y_{11} + k_9 y_4^2 y_6^6 y_{10} y_{11}$

$$+k_{10}y_4^3y_7^6y_{11}+k_{11}y_4^2y_6^3y_7^4y_{11}+k_{12}y_4y_6^6y_7^2y_{11}$$

$$+k_{13}y_6^9y_{11}+k_{14}y_4y_6^2y_7^7+k_{15}y_6^5y_7^5,$$

vi) $E_7^{11,54}$: $k_1y_4^9y_{10}y_{19}+k_2y_4^8y_7^2y_{19}+k_3y_4^7y_6^3y_{19}+k_4y_4^6y_6^2y_7y_{11}^2$

$$+k_5y_4^7y_6y_{10}^2y_{11}+k_6y_4^5y_6^4y_{10}y_{11}+k_7y_4^5y_6y_7^4y_{11}$$

$$+k_8y_4^4y_6^4y_7^2y_{11}+k_9y_4^3y_6^7y_{11}+k_{10}y_4^4y_7^7+k_{11}y_4^3y_6^3y_7^5$$

$$+k_{12}y_4^2y_6^6y_7^3+k_{13}y_4y_6^9y_7,$$

vii) $E_8^{12,53}$: $k_1y_4^{10}y_6y_{19}+k_2y_4^9y_7y_{11}^2+k_3y_4^8y_6^2y_{10}y_{11}$

$$+k_4y_4^7y_6^2y_7^2y_{11}+k_5y_4^6y_6^5y_{11}+k_6y_4^6y_6y_7^5$$

$$+k_7y_4^5y_6^4y_7^3+k_8y_4^4y_6^7y_7,$$

viii) $E_9^{13,52}$: $k_1y_4^{11}y_{10}y_{11}+k_2y_4^{10}y_7^2y_{11}+k_3y_4^9y_6^3y_{11}$

$$+k_4y_4^8y_6^2y_7^3+k_5y_4^7y_6^5y_7,$$

ix) $E_{10}^{14,51}$: $k_1y_4^{12}y_6y_{11}+k_2y_4^{11}y_7^3+k_3y_4^{10}y_6^3y_7,$

x) $E_{11}^{15,50}$: $k_1y_4^{13}y_6y_7.$

The above elements are the candidates to be killed off by y_{64}. That is,

$$d_r(y_{64})=\sum_i k_im_{r,i} \quad\text{with}\quad k_i\in Z_2 \tag{6.7}$$

for $d_r: E_r^{4,60}\to E_r^{4+r,61-r}$ $(r\geq 2)$. We will show that all the coefficients k_i are zero in the following way.

First we apply Sq^1 on both sides of (6.7). Since $Sq^1d_ry_{64}=d_rSq^1y_{64}=0$ by Lemma 5.7, we have

$$\sum_i k_iSq^1m_{r,i}=0,$$

where $Sq^1m_{r,i}$ is calculated by Proposition 3.2 and by the Cartan formula. Then the linear independency of $\{Sq^1m_{r,i}\}$ by Lemma 6.1 implies that $k_i=0$. By this argument we get

Lemma 6.3. *k_i is trivial for*

$i=1, 6, 7, 8, 9$ *in* (6.6.i),

$i=4, 6, 7, 8, 10, 11$ *in* (6.6.ii),

$i=6, 7, 9, 10, 11, 13, 14, 15$ *in* (6. 6. iii),

$i=4, 5, 6, 8, 10, 12, 13$ *in* (6. 6. iv),

$i=2, 4, 5, 7, 8, 9, 11, 13, 15$ *in* (6. 6. v),

$i=1, 3, 5, 6, 7, 9, 11, 13$ *in* (6. 6. vi),

$i=1, 3, 5, 6, 8$ *in* (6. 6. vii),

$i=1, 3, 5$ *in* (6. 6. viii),

$i=1, 3$ *in* (6. 6. ix),

$i=1$ *in* (6. 6. x).

Then by applying Sq^2 on both sides of (6. 7), we get by Lemma 5. 7

$$\sum k_i Sq^2 m_{r,i}=0,$$

where the summation runs over i not listed in Lemma 6. 3.

The linear independency of $\{Sq^2 m_{r,i}\}$ by Lemma 6. 1 implies

Lemma 6. 4. *k_i is trivial for*

$i=3, 4, 5$ *in* (6. 6. i),

$i=3, 5$ *in* (6. 6. ii),

$i=1, 3, 4, 5, 8, 12$ *in* (6. 6. iii),

$i=3, 11, 14$ *in* (6. 6. iv),

$i=1, 3, 6, 10, 12, 14$ *in* (6. 6. v),

$i=4, 12$ *in* (6. 6. vi),

$i=2, 4, 7$ *in* (6. 6. vii),

$i=4$ *in* (6. 6. viii),

$i=2$ *in* (6. 6. ix).

Corollary 6. 5. (1) *$y_4y_6^2y_7y_{11}y_{35}$ and $y_4^3y_{11}^2y_{35}$ are not trivial in $E_r^{7,62}$.*

(2) *$y_4^4y_{11}^2y_{35}$ is not trivial in $E_r^{8,63}$.*

(3) *$y_4^8y_6^2y_7^2y_{11}$ is not trivial in $E_r^{13,56}$.*

(4) *$y_4^4y_6^3y_{35}$, $y_4^6y_{10}y_{35}$, $y_4^4y_6^2y_7y_{34}$, $y_4^6y_{11}y_{34}$, $y_4^2y_7^6y_{19}$ are not trivial in $E_r^{9,60}$.*

Proof. (1) and (2): The elements $y_6^2y_7y_{11}y_{35}$ and $y_4^2y_{11}^2y_{35}$ are not d_r-images of y_{64}, since $k_2=k_3=0$ in (6. 6. i). So $y_4y_6^2y_7y_{11}y_{35}$ and $y_4^a y_{11}^2 y_{35}$ for $a=3$, 4 are not trivial, since $d_r=0$ in these degrees.
(3) follows from that $k_4=0$ in (6. 6. vii).
(4) follows from that $k_i=0$ for $i=1$, 3, 4, 5, 12 in (6. 6. iii).
q.e.d.

Then by applying Sq^4 on the both sides of (6. 7) we get the following lemma by virtue of Lemma 6. 1 and Corollary 6. 5.

Lemma 6. 6. *k_i is trivial for*

$i=2$	*in* (6. 6. i),
$i=1, 2, 9$	*in* (6. 6. ii),
$i=1, 2, 7, 9$	*in* (6. 6. iv),
$i=2$	*in* (6. 6. viii).

Corollary 6. 7. *$y_4^8y_7y_{34}$, $y_4^8y_6y_{35}$ and $y_4^4y_6^4y_7^2y_{19}$ are not trivial in $E_r^{11,62}$.*

Proof. This follows from that $k_1=k_2=k_7=0$ in (6. 6. iv).
q.e.d.

Now we apply Sq^8 on the both sides of (6. 7) and by Lemma 6. 1 and Corollaries 6. 5 and 6. 7 we get

Lemma 6. 8. *k_i is trivial for*

$i=2$	*in* (6. 6. iii),
$i=2, 8, 10$	*in* (6. 6. vi).

Thus we have shown that all k_i are trivial. This completes the proof of Lemma 5. 8.

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