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On the Fixed Point Algebra of a UHF Algebra under a Periodic Automorphism of Product Type

By

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Abstract

We study the fixed point algebra $\mathfrak{A}^\alpha$ of a UHF algebra $\mathfrak{A}$ under a periodic automorphism α of product type. We show an example of $\mathfrak{A}^\alpha$ which is simple and has more than two tracial states and we characterize the case where $\mathfrak{A}^\alpha$ has only one tracial state. Next we show that $\mathfrak{A}^\alpha$ is a UHF algebra if and only if $\mathfrak{A}$ is generated by an infinite family of mutually commuting α-invariant type I_p subfactors whose fixed point algebras are abelian and by a UHF subalgebra of $\mathfrak{A}^\alpha$ which commutes with the former (where p denotes the period of α).

§ 1. Introduction

E. Størmer [7] showed that the even CAR algebra is isomorphic to the CAR algebra itself. The CAR algebra is the UHF algebra of type (2^n) and the even CAR algebra is the fixed point algebra of the CAR algebra under a specific periodic automorphism with period 2.

In this note we study the fixed point algebra $\mathfrak{A}^\alpha$ of a UHF algebra $\mathfrak{A}$ under a periodic automorphism α of product type with period p, where α is of product type if $\mathfrak{A}$ is the C^*-tensor product of finite type I factors $\mathfrak{A}_n$ and α is the product of $\alpha_n \in \mathrm{Aut}\, \mathfrak{A}_n$. The case studied by Størmer corresponds to $p=2$ and $\mathfrak{A}_n$ of type I_2. In general $\mathfrak{A}^\alpha$ is not necessarily a UHF algebra. In Theorem 4. 4, we give several equivalent conditions that $\mathfrak{A}^\alpha$ is a UHF algebra. In particular, this is the case if and only if $(\mathfrak{A}, \alpha)$ is isomorphic to $(\mathfrak{A}_0 \otimes \mathfrak{A}_p, \iota \otimes \alpha_p)$ where $\mathfrak{A}_0$ is a UHF algebra, ι is the identity map and $(\mathfrak{A}_p, \alpha_p)$ is the following specific example:

Let M be the full $p \times p$ matrix algebra and $e_{ij}\,(\mathrm{i}, \mathrm{j}=1, \cdots, p)$ its matrix units. Let α be the periodic automorphism of M with period p implemented by the unitary $\exp(2\pi i p^{-1} \Sigma k e_{kk})$. We let $\mathfrak{A}_p$ be the C^*-

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tensor product of countably infinite copies of M and α_p the corresponding product automorphism of α.

The other main result in this note is the characterization of the case where $\mathfrak{A}^\alpha$ has a unique tracial state, given in Theorem 3. 10. One of the characterizations is that $\mathfrak{A}^\alpha$ contains sufficiently large UHF subalgebras in the following sense: For any $\varepsilon>0$ there exist a projection e of $\mathfrak{A}^\alpha$ with $\tau(e)>1-\varepsilon$, a UHF subalgebra $\mathfrak{B}$ with e as identity and a sequence $\{e_n\}$ of projections of $\mathfrak{B}$ with $\tau(e_n)\to\tau(e)$ as $n\to\infty$ such that any x of $\mathfrak{A}^\alpha$ has a sequence $\{x_n\}\subset\mathfrak{B}$ satisfying $\|e_nxe_n-x_n\|\to 0$ as $n\to\infty$, where τ is the unique tracial state of $\mathfrak{A}$.

It has been shown in [6] that $\mathfrak{A}^\alpha$ is simple if and only if the invariant $\Gamma(\alpha)$ is equal to $Z_p\equiv Z/pZ$.

The three situations for $\mathfrak{A}^\alpha$ mentioned above have the following mutual relations: If $\mathfrak{A}^\alpha$ has a unique trace, then $\mathfrak{A}^\alpha$ is simple (c.f. [6, Th. 2]) but the converse does not hold as is shown in Remark 3. 12. If $\mathfrak{A}^\alpha$ is a UHF algebra, $\mathfrak{A}^\alpha$ has the unique trace, as is well known, but the converse does not hold (see Remark 4. 5).

§ 2. Invariant $\Gamma(\alpha)$

Let G be a compact abelian group and let $(\mathfrak{A}_n, G, \alpha^{(n)})$ be a sequence of C^*-dynamical systems, i.e. $\mathfrak{A}_n$ is a C^*-algebra with 1 and $\alpha^{(n)}$ is a continuous homomorphism of G into Aut $\mathfrak{A}_n$. Let $\mathfrak{A}$ be the infinite C^*-tensor product of $\mathfrak{A}_n$, $n=1, 2, \cdots$, and let α_g be the automorphism $\otimes\alpha_g{}^{(n)}$ of $\mathfrak{A}$ for each $g\in G$. Then $(\mathfrak{A}, G, \alpha)$ is a C^*-dynamical system. The $\Gamma(\alpha)$ is defined to be the intersection of $\mathrm{Sp}(\alpha|\mathfrak{B})$ where $\mathfrak{B}$ runs over all non-zero α-invariant hereditary C^*-subalgebras of $\mathfrak{A}$ [5].

For each $t\in\widehat{G}$ let N_t be the set consisting of n such that $\mathrm{Sp}\alpha^{(n)}\ni t$. Let H be the set of t such that the cardinality of N_t is infinite.

Lemma 2. 1. *$\Gamma(\alpha)$ contains the subgroup generated by H.*

Proof. Let x be a positive element of $\mathfrak{A}^\alpha$ with $\|x\|=1$. Then there are a positive integer n and a positive element x_0 of $(\otimes_1{}^n\mathfrak{A}_m)^\alpha$ with $\|x_0\|=1$ and $\|x-x_0\|<2^{-1}$. For any non-zero $y\in\otimes_{n+1}^{\infty}\mathfrak{A}_m$, xyx does not vanish since $\|xyx\| \geqq \|x_0yx_0\| - \|x_0yx_0 - xyx\| = \|y\| - \|x_0yx_0 - xyx\| \geqq \|y\| - 2\|x - x_0\| \|y\| > 0$. Thus $\mathrm{Sp}(\alpha | \overline{x\mathfrak{A}x})$ contains $\mathrm{Sp}(\otimes_{n+1}^{\infty}\alpha^{(m)})$, in particular the subgroup generated by H. Now it is easy to complete the proof (c.f. Lemma 4. 1 in [4]).

In the following sections we take as $\mathfrak{A}_n$ a finite type I factor. The existence of minimal projections of $\otimes_1{}^n\mathfrak{A}_m$ in $(\otimes_1{}^n\mathfrak{A}_m)^{\alpha}$ for any $n < \infty$ obviously implies

Proposition 2. 2. *$\Gamma(\alpha)$ is the subgroup generated by H when $\mathfrak{A}_n$ are finite type I factors.*

In addition we remark that $\Gamma(\alpha)$ is a closed subgroup of $\widehat{G}$ in general [5].

§ 3. Fixed Point Algebra

Let $\mathfrak{A}_n$ be a finite type I_{d_n} factor ($d_n \geqq 2$) and let α_n be a periodic automorphism of $\mathfrak{A}_n$ satisfying $\alpha_n{}^p = \iota$ where ι is the trivial automorphism and p is a fixed positive integer. Then there exist matrix units e_{ij} of $\mathfrak{A}_n$ and a function φ_n on $\mathfrak{X}_n = \{1, 2, \cdots, d_n\}$ into $Z_p = Z/pZ$ such that α_n is implemented by the unitary $\exp[\Sigma_j i2\pi p^{-1}\varphi_n(j) e_{jj}^{(n)}]$ where $\varphi_n(j)$ is any representative in Z of class $\varphi_n(j) \in Z_p$.

Let $\mathfrak{A}$ be the infinite C^*-tensor product of $\mathfrak{A}_n$, $n = 1, 2, \cdots$, and α be the corresponding product automorphism of α_n. Then α is, of course, a periodic automorphism of the UHF algebra $\mathfrak{A}$ such that $\alpha^p = \iota$. Now we assume that α has period p and we want to describe the fixed point algebra which is an AF algebra [1, Lemma 5. 3].

Let $\mathfrak{A}(n) = \otimes_1^n \mathfrak{A}_m$ and let $\mathfrak{A}(0) = \mathbf{C}\cdot 1$. Then $\mathfrak{A}(n)^{\alpha}$ is the direct sum of at most p finite type I factors. We construct each factor by the following procedure [1, Lemma 5. 2]: Fix $t \in Z_p$ and set

$$S_t(n) = \{(i, j) \in \Pi_1{}^n\mathfrak{X}_m \times \Pi_1{}^n\mathfrak{X}_m ; \Sigma_1{}^n\varphi_m(i_m) = \Sigma_1{}^n\varphi_m(j_m) = t\}.$$

For each $(i, j) \in S_t(n)$ let $e(i, j) = e_{i_1j_1}^{(1)} e_{i_2j_2}^{(2)} \cdots e_{i_nj_n}^{(n)}$. Then $\{e(i, j)\}$ forms matrix units of a finite type I subfactor of $\mathfrak{A}(n)^{\alpha}$ which we denote by $M_{n,t}$. Then

$$\mathfrak{A}(n)^{\alpha} = \bigoplus_{t \in Z_p} M_{n,t} .$$

The embedding of $\mathfrak{A}(n)^{\alpha}$ into $M_{n+1,t}$ is as follows:

$$\bigoplus_{j=1}^{d_{n+1}} M_{n,t-\varphi_{n+1}(j)} = \bigoplus_{s \in Z_p} n_s \cdot M_{n,t-s}$$

where n_s denotes the multiplicity of $M_{n,t-s}$ in $M_{n+1,t}$: the number of $\{j : \varphi_{n+1}(j) = s\}$. We know that $\mathfrak{A}^{\alpha}$ is generated by the increasing sequence $\mathfrak{A}(n)^{\alpha}$.

Let $x_n (n=0, 1, 2, \cdots)$ be a random variable with values in Z_p such that $x_n = t$ occurs with probability n_t/d_{n+1}, i.e. $P(x_n = t) = n_t/d_{n+1}$. Suppose that the family $\{x_n\}$ are mutually independent. For $m \leqq n$, let

$$S(m, n) = \sum_{j=m}^{n} x_j .$$

Denoting by $\Gamma(\alpha)$ the invariant Γ of the action of Z_p on $\mathfrak{A}$ by $t \in Z_p \to \alpha^t$, we can easily show on the basis of Proposition 2. 2, the following:

Proposition 3. 1. *$\Gamma(\alpha) = Z_p$ holds if and only if for any positive integer m and any $t \in Z_p$ there exists an $n \geqq m$ such that $P(S(m, n) = t) > 0$.*

Now we consider a stronger condition on $\{x_n\}$:

Condition 3. 2. *For each positive integer m, $S(m, n)$ converges in distribution, as $n \to \infty$, to a random variable which takes each value with equal probability, i.e.* $\lim_n P(S(m, n) = t) = p^{-1}$ for any $t \in Z_p$.

In other words the condition is satisfied if and only if for any nonzero $t \in Z_p$ and any positive integer m,

$$\lim_n \langle \exp i2\pi p^{-1} t S(m, n) \rangle = \lim \prod_{m}^{n} \langle \exp i2\pi p^{-1} t x_j \rangle = 0$$

where $\langle \cdot \rangle$ denotes the mean.

Proposition 3. 3. If $\{t \in Z_p ; \Sigma_{n \in N_t} d_n^{-1} = \infty\}$ generates Z_p, then Condition 3. 2 is satisfied (where N_t is defined in section 2). In particular if $\Gamma(\alpha)=Z_p$ and $\{d_n\}$ is bounded, then Condition 3. 2 is satisfied.

Proof. Let $t\in Z_p$ be non-zero. Then there is an $s\in Z_p$ with $\exp i2\pi p^{-1}ts\neq 1$ such that $\Sigma_{n\in N_s}d_n^{-1}=\infty$. Then

$$|\langle\exp i2\pi p^{-1}tS(m,n)\rangle|\leqq\prod\left\{1-2d_j^{-1}(1-d_j^{-1})\left(1-\cos\frac{2\pi}{p}\right)\right\}^{1/2}$$

$$\leqq\prod\left\{1-(2d_j)^{-1}\left(1-\cos\frac{2\pi}{p}\right)\right\}$$

where the products are taken over $\{j\in N_s: m\leqq j\leqq n\}$. The right hand side converges to zero as $n\to\infty$ if and only if $\Sigma_{i\in N_s}d_j^{-1}=\infty$. Q.E.D.

Before going into discussions of our main result in this section we first show that Condition 3. 2 does not depend on the choice of $(\mathfrak{A}_n, \alpha_n)$, $n=1, 2, \cdots$. Let $\mathfrak{B}(n)$ be an increasing sequence of α-invariant finite type I subfactors of $\mathfrak{A}$ such that $\mathfrak{A}=\overline{\cup\mathfrak{B}(n)}$. Then we have

Proposition 3. 4. *Let* $\mathfrak{A}=\overline{\cup\mathfrak{A}(n)}=\overline{\cup\mathfrak{B}(n)}$ *be as above. Then there is an automorphism* θ *of* $\mathfrak{A}$ *with* $\theta\circ\alpha=\alpha\circ\theta$ *such that for every positive integer* n *there exists a positive integer* m *such that* $\theta(\mathfrak{B}(n))\subset\mathfrak{A}(m)$ *and* $\mathfrak{A}(n)\subset\theta(\mathfrak{B}(m))$.

The proposition implies that $\{x_n\}$ defined through $\mathfrak{A}(n)$ satisfies Condition 3. 2 if and only if $\{x_n\}$ defined through $\mathfrak{B}(n)$ satisfies Condition 3. 2. Thus we have our assertion.

The proof of Proposition 3. 4 is the same as that of Lemma 2. 6 in [1] if we show that the unitaries u_i and v_i there can be chosen in $\mathfrak{A}^\alpha$. This will be easily shown if we prove the following lemma corresponding to Lemma 2. 3 in [1].

Lemma 3. 5. *Let* $\mathfrak{B}$ *be an* α*-invariant finite-dimensional subalgebra of* $\mathfrak{A}$ *such that* $\alpha|\mathfrak{B}$ *is inner. Then for all* $\varepsilon>0$ *there exist a unitary operator* $u\in\mathfrak{A}^\alpha$ *and a positive integer* n *such that* $\|u-1\|<\varepsilon$ *and* $u\mathfrak{B}u^*\subset\mathfrak{A}(n)$.

Proof. We may assume $1\in\mathfrak{B}$. By applying Lemma 2. 3 of [1] to $\mathfrak{A}^{\alpha}=\overline{\bigcup\mathfrak{A}(n)^{\alpha}}$ and $\mathfrak{B}^{\alpha}$ we may assume that $\mathfrak{B}^{\alpha}\subset\mathfrak{A}(n_0)^{\alpha}$ for some n_0. Let $\{f_{i,j}^{(k)}\}_{k=1}^{m}$ be matrix units for $\mathfrak{B}$ such that $(f_{ij}^{(k)}f_{pr}^{(l)}=\delta_{kl}\delta_{jp}f_{ir}^{(k)}, f_{ij}^{(k)}=f_{ji}^{(k)*}$ and) $\alpha(f_{ij}^{(k)})=\exp\{i(\psi_k(i)-\psi_k(j))\}f_{ij}^{(k)}$ with suitable functions $\psi_k\,(k=1,\cdots,m)$. Then for any $\delta>0$ we can find an integer $n\geqq n_0$ and a family $\{g_{ij}^{(k)}\}$ of matrix units in $\mathfrak{A}(n)$ such that $\|f_{ij}^{(k)}-g_{ij}^{(k)}\|<\delta$ and $f_{ii}^{(k)}=g_{ii}^{(k)}$ (c.f. Lemma 1. 10 of [2]). Let

$$g_{ij}^{\prime(k)}=p^{-1}\sum_{l=0}^{b-1}\exp\{il(\psi_k(j)-\psi_k(i)\}\alpha^l(g_{ij}^{(k)}).$$

Then $g_{ij}^{\prime(k)}\in\mathfrak{A}(n)$, $\alpha(g_{ij}^{\prime(k)})=\exp\{i(\psi_k(i)-\psi_k(j))\}g_{ij}^{\prime(k)}$, $f_{ii}^{(k)}g_{ij}^{\prime(k)}=g_{ij}^{\prime(k)}f_{jj}^{(k)}=g_{ij}^{\prime(k)}$ and $\|f_{ij}^{(k)}-g_{ij}^{\prime(k)}\|<\delta$. If δ is sufficiently small, the partial isometry $e_{ij}^{(k)}$ obtained from the polar decomposition of $g_{ij}^{\prime(k)}$, which is an element of $\mathfrak{A}(n)$, satisfies $\alpha(e_{ij}^{(k)})=\exp\{i(\psi_k(i)-\psi_k(j))\}e_{ij}^{(k)}$ and $\|f_{ij}^{(k)}-e_{ij}^{(k)}\|<\varepsilon$. Let $u=\Sigma_k\Sigma_j e_{j1}^{(k)}f_{1j}^{(k)}$. Then u satisfies the above conditions. Q.E.D.

Let τ be the unique tracial state of $\mathfrak{A}$ and let $(\pi_\tau,\mathfrak{H}_\tau,\Omega_\tau)$ be the GNS representation of $\mathfrak{A}$ associated with τ. Let $\bar{\alpha}$ be the automorphism of the factor $M=\pi_\tau(\mathfrak{A})''$ such that $\bar{\alpha}\circ\pi_\tau=\pi_\tau\circ\alpha$. Then it is shown by Connes [2, Th. 2. 4. 1] that $M^{\bar{\alpha}}$ is a factor if and only if $\Gamma(\bar{\alpha})=\mathrm{Sp}\,\bar{\alpha}(=Z_p)$. Since Ω_τ is separating, $M^{\bar{\alpha}}$ is isomorphic to $M^{\bar{\alpha}}|[M^{\bar{\alpha}}\Omega_\tau]$. Thus, as $M^{\bar{\alpha}}=\pi_\tau(\mathfrak{A}^{\alpha})''$, we have:

Lemma 3. 6. *Let $(M=\pi_\tau(\mathfrak{A})'',\bar{\alpha})$ be as above. Then τ is a factor state of $\mathfrak{A}^{\alpha}$ if and only if $\Gamma(\bar{\alpha})=Z_p$.*

Since π_τ is faithful, we have that $\Gamma(\bar{\alpha}|\pi_\tau(\mathfrak{A}))=\Gamma(\alpha)$. Let $\mathfrak{B}$ be a non-zero $\bar{\alpha}$-invariant hereditary C^*-subalgebra of $\pi_\tau(\mathfrak{A})$. Then there is a projection e of $M^{\bar{\alpha}}$ such that eMe is the weak closure $\bar{\mathfrak{B}}$ of $\mathfrak{B}$ (c.f. [5]). Since $\mathrm{Sp}(\bar{\alpha}|\bar{\mathfrak{B}})=\mathrm{Sp}(\bar{\alpha}|\mathfrak{B})$ the definitions of $\Gamma(\bar{\alpha})$ and $\Gamma(\alpha)$ imply:

Lemma 3. 7. $\Gamma(\bar{\alpha})\subset\Gamma(\alpha)$.

Let $C(Z_p)$ be the space of real valued functions on Z_p. Let T_n and $T_n'\,(n=1,2,\cdots)$ be the linear transformations on $C(Z_p)$ defined by

$$(T_n f)(t) = d_n^{-1}\Sigma_{j=1}^{d_n} f(t+\varphi_n(j)) = \langle f(t+x_{n-1})\rangle;$$

$$(T_n' g)(t) = d_n^{-1}\Sigma_{j=1}^{d_n} g(t-\varphi_n(j)) = \langle g(t-x_{n-1})\rangle.$$

Then we have

$$\Sigma_{t\in Z_p}(T_n f)(t) g(t) = \Sigma_{t\in Z_p} f(t)(T_n' g)(t).$$

Lemma 3.8. *There exists a one-to-one correspondence between the set of all tracial positive linear functionals* τ' *of* $\mathfrak{A}^\alpha$ *and the set of all sequences* $\{f_n\}_{n=0}^{\infty}$ *of positive functions of* $C(Z_p)$ *satisfying* $T_n f_n = f_{n-1}$ $(n=1, 2, \cdots)$, where the correspondence is given by

$$f_n(t) = \prod_{1}^{n} d_m \cdot \tau'(f_t^{(n)}) \tag{3.1}$$

for $M_{n,t} \neq (0)$ *with* $f_t^{(n)}$ *being any minimal projection of* $M_{n,t}$. *Furthermore* $\tau'(1) = f_0(0)$ *holds for any pair* τ' *and* $\{f_n\}$ *which satisfy* (3.1) *and there exists a constant* M *such that* $\|f_n\|_\infty \leqq M$ *for any* n *and for any* $\{f_n\}$ *satisfying the above condition and* $f_0(0) = 1$.

Proof. Since $\tau'(f_t^{(n)})$ does not depend on the choice of $f_t^{(n)}$ by the property of the trace on $M_{n,t}$, the mapping $\tau' \mapsto \{f_n(t);\ M_{n,t} \neq (0)\}$ defined by (3·1) is well-defined. The component of a projection $f_t^{(n-1)} \neq 0$ in $M_{n,t+s}$ is the sum of n_s orthogonal minimal projections of $M_{n,t+s}$, which implies that $T_n f_n(t) \equiv d_n^{-1}\sum_j t_n(t+\varphi_n(\tau)) = f_{n-1}(t)$ for $\{f_n(t)\}$ defined by τ' through (3.1). The equality (3.1) defines f_n for sufficiently large n and so the relations $T_n f_n = f_{n-1}$ consistently define a unique sequence $\{f_n\}$ through (3.1).

Conversely let $\{f_n\} \subset C(Z_p)_+$ be such that $T_n f_n = f_{n-1}$. Let τ_n' be the unique tracial positive linear functional on $\mathfrak{A}(n)^\alpha$ satisfying (3.1). Then $T_n f_n = f_{n-1}$ implies that $\tau_n' | \mathfrak{A}(n-1)^\alpha = \tau_{n-1}'$. Hence $\{\tau_n'\}$ defines the unique tracial positive linear functional τ' on $\mathfrak{A}^\alpha$ such that $\tau' | \mathfrak{A}(n)^\alpha = \tau_n'$.

$\tau'(1) = f_0(0)$ follows from the definition.

To prove the last assertion, let g_n be a function on Z_p for each $n=0, 1, 2, \cdots$ such that $g_n(t) = \tau(e_t^{(n)})$ with $e_t^{(n)}$ being the identity of $M_{n,t}$. In particular $g_0(0) = 1$ and $g_0(t) = 0$ for $t \neq 0$. Then $\{g_n\}$ satisfies that $T_n' g_{n-1} = g_n$ $(n=1, 2, \cdots)$.

If n_0 is a positive integer such that $\mathrm{Sp}(\alpha|\mathfrak{A}(n_0))=Z_p$, then there is $\delta>0$ such that $g_{n_0}\geqq\delta$. If $g_n\geqq\delta$, then $g_{n+1}(t)=\langle g_n(t-x_n)\rangle\geqq\delta$. Thus we know that $g_n\geqq\delta$ for all $n\geqq n_0$

Let $\{f_n\}$ be a sequence satisfying the condition and $f_0(0)=1$. Then

$$\Sigma_t f_n(t)g_n(t)=\Sigma f_n(t)(T_n'g_{n-1})(t)$$
$$=\Sigma(T_nf_n)(t)g_{n-1}(t)=\Sigma f_{n-1}(t)g_{n-1}(t).$$

Thus we know that $\Sigma f_n(t)g_n(t)=f_0(0)=1$. Hence $f_n(t)\leqq\delta^{-1}$ holds for all $t\in Z_p$ and all $n\geqq n_0$. This completes the proof.

The trivial solution of $T_nf_n=f_{n-1}$ with $f_0(0)=1$ is $\{f_n\equiv1\}$, which corresponds to the restriction of τ to $\mathfrak{A}^\alpha$.

Lemma 3.9. *Let K be a subset of $t\in Z_p$ such that*

$$\lim_n|\langle\exp 2\pi ip^{-1}tS(m,n)\rangle|>0$$

for sufficiently large m. Then K forms a subgroup of Z_p and the order of K is the number of extremal tracial states of $\mathfrak{A}^\alpha$. Furthermore the central decomposition of the restriction of τ to $\mathfrak{A}^\alpha$ gives all extremal tracial states of $\mathfrak{A}^\alpha$.

Proof. For $t\in K$, let $\{n_j\}$ be a subsequence of positive integers such that

$$\lim_j\langle\exp 2\pi ip^{-1}tS(n_1,n_j)\rangle\neq0\,.$$

Then $\{\exp 2\pi ip^{-1}tS(n_1,n_j)\}_j$ forms a fundamental sequence in the mean of order 2 and hence of order 1. This implies that K is a group.

Let t_0 be a non-zero minimal element of $K\subset\{0,1,\cdots,p-1\}$ (which devides p) and let $\{n_j\}$ be as above for $t_0\in K$. Let λ be the limit of $\exp 2\pi ip^{-1}t_0S(n_1,n_j)$ in the mean of order 2. Then $\lambda^q\equiv1$ with $q=pt_0^{-1}$, the order of K. Hence we have a random variable S_{n_1} taking values in Z_q identified with $\{0,1,\cdots,q-1\}$ such that $\exp 2\pi iq^{-1}S_{n_1}=\lambda$. Let ρ be the quotient map from Z_p onto Z_q. Then $\rho(S(n_1,n_j))$ converges to S_{n_1}. For any non-negative function $f\not\equiv0$ of $C(Z_q)$ let $f_n\in C(Z_p)$ be such that

$$
\begin{aligned}
(3.2)\quad f_n(t) &= \langle f(\rho(t)+\rho(S(n, n_1-1))+S_{n_1})\rangle && \text{if } n \leqq n_1-1\,, \\
&= \langle f(\rho(t)+S_{n_1})\rangle && \text{if } n=n_1\,, \\
&= \langle f(\rho(t)-\rho(S(n_1, n-1)+S_{n_1})\rangle && \text{if } n \geqq n_1+1\,.
\end{aligned}
$$

Then we can show that $T_n f_n = f_{n-1}$. For example when $n \geqq n_1+1$, $(T_n f_n)(t) = \langle f(\rho(t)+\rho(x_{n-1})-\rho(S(n_1, n-1))+S_{n_1})\rangle$ due to the independence of $\rho(x_{n-1})$ with $S_{n_1}-\rho(S(n_1, n-1))$. As $f \not\equiv 0$, $f_n \not\equiv 0$. Thus by Lemma 3.8 we obtain a tracial positive linear functional τ_f corresponding to f.

The transformation S on $C(Z_q)$ defined by $(Sf)(t) = \langle f(t+S_{n_1})\rangle$ is not degenerate since $\langle \exp 2\pi i q^{-1} t S n_1\rangle \neq 0$ for any $t \in Z_q$. Thus, since $f \in C(Z_q)_+ \to \tau_f$ is affine, we have an injective linear mapping from $C(Z_q)$ into the space of all continuous self-adjoint tracial functionals of $\mathfrak{A}^\alpha$, which is order-preserving.

Let τ' be a tracial state of $\mathfrak{A}^\alpha$ and let $\{f_n\}$ be the corresponding sequence in $C(Z_p)$ as in Lemma 3.8. Since $\{f_n\}$ is uniformly bounded, we have a subsequence $\{m_j\}$ of $\{n_j\}$ such that $f_{m_j}(t)$ converges, say to $f'(t)$, for each $t \in Z_p$. Let $m(k)$ be a subsequence of $\{m_j\}$ such that $S(m, m(k))$ converges in distribution, say to S_m'. Then $f_m(t) = \langle f'(t+S_m')\rangle$ follows from the property $T_n f_n = f_{n-1}$ and the independence of $\{x_n\}$. Since $S_m'+q$ and S_m' have the same distribution due to the fact that $\langle \exp 2\pi i p^{-1} t S_m'\rangle = 0$ for $t \notin K$, we know that $f_m = f_m \circ \rho$. Then we can show that $\{f_n\}$ is obtained as in (3.2) with $f = f'|K$. This implies that the space of all continuous self-adjoint tracial functionals of $\mathfrak{A}^\alpha$ is order-isomorphic to $C(Z_q)$.

Let δ_s be a function on Z_q such that $\delta_s(t)=0$ for $t \neq s$ and $\delta_s(s)=1$ and let $f_s = \delta_s/(\delta_s)_0(0)$. Let τ_s be the tracial state corresponding to f_s. Then τ_s are extremal tracial states of $\mathfrak{A}^\alpha$ and the following equality holds:

$$
(3.3)\qquad \tau = \sum_{s \in Z_q} (\delta_s)_0(0)\,\tau_s
$$

since $\Sigma(\delta_s)_0(0)(f_s)_n(t) = 1$ for any $t \in Z_q$ and n. The decomposition (3.3) of τ is the central decomposition of τ. Q.E.D.

Now we state our main result in this section:

Theorem 3.10. *Let* $(\mathfrak{A}=\otimes\mathfrak{A}_n, \alpha=\otimes\alpha_n)$ *and* $(M=\pi_\tau(\mathfrak{A})'', \bar{\alpha})$ *be as above. Then the following statements are equivalent:*

(i) $\mathfrak{A}^\alpha$ *has a unique tracial state;*

(ii) τ *is a factor state of* $\mathfrak{A}^\alpha$;

(iii) $\Gamma(\bar{\alpha})=Z_p$

(iv) *Condition* 3.2 *is satisfied;*

(v) *For any* $\varepsilon>0$ *there exist a projection* e *of* $\mathfrak{A}^\alpha$ *with* $\tau(e)>1-\varepsilon$, *a UHF subalgebra* $\mathfrak{B}$ *with* e *as identity and a sequence* $\{e_n\}$ *of projections of* $\mathfrak{B}$ *with* $\tau(e_n)\to\tau(e)$ *such that any* $x\in\mathfrak{A}^\alpha$ *has a sequence* $x_n\in\mathfrak{B}$ *satisfying* $\|e_nxe_n-x_n\|\to 0$ *as* $n\to\infty$;

(vi) *In* (v) $\{e_n\}$ *can be chosen so that* $\|[e_n, x]\|\to 0$ *as* $n\to\infty$ *for any* $x\in\mathfrak{A}^\alpha$.

If $\Gamma(\alpha)\neq Z_p(=\mathrm{Sp}\,\alpha)$, all the statements do not hold and hence are equivalent since in this case the center of $\mathfrak{A}^\alpha$ is not trivial [6, Th. 2]. Hence in the following we assume $\Gamma(\alpha)=Z_p$.

The equivalence of (ii) with (iii) is proved in Lemma 3.6 and the equivalences of (i), (ii) and (iv) are proved in Lemma 3.9. The implication (vi)$\Rightarrow$(v) is trivial.

Proof. (v)$\Rightarrow$(i) Suppose that (v) holds for some $\varepsilon<1$. Let τ' be a tracial state of $\mathfrak{A}^\alpha$. For $x\in e\mathfrak{A}^\alpha e$ we have

$$\tau'(x)=\tau'(e_nx)+\tau'((e-e_n)x).$$

The first term of the right hand side tends to $\tau'(e)\tau(e)^{-1}\tau(x)$ as $n\to\infty$ since $\tau'(x)=\tau'(e)\tau(e)^{-1}\tau(x)$ for $x\in\mathfrak{B}$ by the uniqueness of a tracial state of the UHF subalgebra $\mathfrak{B}$. The second term is smaller than $\|x\|\times\tau'(e-e_n)=\|x\|\tau'(e)\tau(e)^{-1}\tau(e-e_n)$. Thus we have $\tau'(x)=\tau'(e)\tau(e)^{-1}\times\tau(x)$ for $x\in e\mathfrak{A}^\alpha e$. For any x and y of $\mathfrak{A}^\alpha$ we have

$$\begin{aligned}\tau'(xey)&=\tau'(eyxe)=\tau'(e)\tau(e)^{-1}\tau(eyxe)\\&=\tau'(e)\tau(e)^{-1}\tau(xey).\end{aligned}$$

This implies that $\tau'=\tau$ since $\mathfrak{A}^\alpha e\mathfrak{A}^\alpha$ is dense in $\mathfrak{A}^\alpha$ by simplicity of $\mathfrak{A}^\alpha$.

Q.E.D.

Proof. (iv)$\Rightarrow$(vi) If $g_n(t)=\tau(e_t^{(n)})$ as in the proof of Lemma 3.8, we have

$$g_n(t)=(T_n'\cdot T_{n-1}'\cdots T_1'g_0)(t)=P(S(0,n-1)=t).$$

Then Condition 3.2 implies that $g_n(t)\to p^{-1}$ as $n\to\infty$. Hence for any $\varepsilon>0$ there are n and a projection e of $\mathfrak{A}(n)$ such that $\tau(e)>1-\varepsilon$ and $\tau(ee_t^{(n)})=p^{-1}\tau(e)$ for all $t\in Z_p$. Set $g_n'(t)=\tau(ee_t^{(m)})$ for $m\geqq n$. Then $T'_{m+1}g_m'=g'_{m+1}$ and hence $g_m'(\cdot)$ is constant for all $m\geqq n$. Thus the AF algebra $e\mathfrak{A}^\alpha e$ is defined by the increasing sequence

$$e\mathfrak{A}(n)^\alpha e\subset e\mathfrak{A}(n+1)^\alpha e\subset\cdots$$

of the finite dimensional algebras where the direct summands of each $e\mathfrak{A}(m)^\alpha e$ are of the same type with each other.

Now we complete the proof by applying the following lemma to the system $(e\mathfrak{A}e,\alpha|e\mathfrak{A}e)$.

Lemma 3.11. *Let* $(\mathfrak{A},\alpha)$ *be as above and suppose that* $M_{1,t}$ $(t\in Z_p)$ *are isomorphic with each other and that Condition* 3.2 *is satisfied. Then the statement* (vi) *in Theorem* 3.10 *holds with* $e=1$.

Proof. As we have remarked above the lemma, the direct summands of $\mathfrak{A}(n)^\alpha$ are of the same type with each other, say of type $I_{q(n)}$. Now we shall construct a subsequence n_k of positive integers with $n_1=1$, an increasing sequence of subfactors $\mathfrak{B}_k$ of type $I_{q(n_k)}$ of $\mathfrak{A}(n_k)^\alpha$ and a sequence of projections $e_k(k\geqq2)$ of $\mathfrak{B}_k$ such that $\tau(e_k)>1-k^{-1}$ and $e_kx=xe_k\in\mathfrak{B}_k$ for any $x\in\mathfrak{A}(n_{k-1})^\alpha$. If this is done, the UHF subalgebra $\mathfrak{B}=\overline{\bigcup\mathfrak{B}_k}$ and the projections $\{e_k\}$ satisfy the condition in (vi) of the theorem.

Let $\mathfrak{B}_1$ be any full $q(n_1)\times q(n_1)$ matrix subalgebra in $\mathfrak{A}(n_1)^\alpha$. Suppose that we have $\{n_k\}$, $\{\mathfrak{B}_k\}$ and $\{e_k\}$ satisfying the above conditions for $k\leqq m$. Let $h_l^{(t)}(s)=\tau(e_t^{(n_m)}e_s^{(l)})$ for $l\geqq n_m$. Then $T'_{l+1}h_l^{(t)}=h_{l+1}^{(t)}$ and so

$$h_l^{(t)}(s)=\langle h_{n_m}^{(t)}(s-S(n_m,l-1)\rangle.$$

Hence there is an $l\equiv n_{m+1}$ such that for all s_1 and s_2,

$$|h_l^{(t)}(s_1)-h_l^{(t)}(s_2)|<p^{-2}(m+1)^{-1}.$$

Since $e_t^{(n_m)}$ and $e_s^{(l)}$ all belong to $\mathfrak{A}(l)^\alpha\cap\mathfrak{B}_m'$ whose direct summands are of the same type with each other, we have a projection $e_{l,t}$ in $\mathfrak{A}(l)^\alpha\cap\mathfrak{B}_m'$ such that $e_{l,t}\leqq e_t^{(n_m)}$, $\tau(e_t^{(n_m)}-e_{l,t})<p^{-1}(m+1)^{-1}$ and $\tau(e_{l,t}e_s^{(l)})$ are independent of s. Let $e_{m+1}=\Sigma_t e_{l,t}$ and let $\mathfrak{B}_{n+1}$ be a type $I_{q(l)}$ subfactor (with 1) of $\mathfrak{A}(l)^\alpha$ containing $e_{l,t}(t\in Z_p)$ and $\mathfrak{B}_m$. Since $\mathfrak{A}(n_m)^\alpha$ is generated by $\mathfrak{B}_m$ and $\{e_t^{(n_m)}\}$ we have $e_{m+1}\mathfrak{A}(n_m)^\alpha e_{m+1}\subset\mathfrak{B}_{m+1}$ and by definition we have $e_{m+1}\in(\mathfrak{A}(n_m)^\alpha)'$. Thus we have constructed n_{m+1}, $\mathfrak{B}_{m+1}$ and e_{m+1} satisfying the conditions. This completes the proof by induction.

Remark 3.12 Let q be a positive integer such that q devides p. Then there is an example $(\mathfrak{A},\alpha)$ where $\mathfrak{A}^\alpha$ is simple and has q extremal tracial states. Let $\mathfrak{A}_n$ be a type I_{n^2} factor and let α_n be the automorphism of $\mathfrak{A}_n$ implemented by $\exp\{2\pi ip^{-1}e_n\}$ where e_n is a one-dimensional projection of $\mathfrak{A}_n$ if n is odd and e_n is a q times $n^2/2$-dimensional projection of $\mathfrak{A}_n$ if n is even. We consider the system $(\mathfrak{A}=\otimes\mathfrak{A}_n,\alpha=\otimes\alpha_n)$. Since $P(S(m,m+2p)=t)>0$ for all $t\in Z_p$ and m, we have $\Gamma(\alpha)=Z_p$ and hence $\mathfrak{A}^\alpha$ is simple [6]. For any $t\in Z_p$,

$$|\langle\exp 2\pi ip^{-1}tS(m,n)\rangle|=\prod_{m\leqq 2k+1\leqq n}|1+(2k+1)^{-2}(\exp 2\pi ip^{-1}t-1)|$$

$$\times\prod_{m\leqq 2k\leqq n}2^{-1}|1+\exp 2\pi ip^{-1}qt|.$$

This implies that $\lim_n|\langle\exp 2\pi ip^{-1}tS(m,n)\rangle|\neq 0$ if and only if $t\in(p/q)Z_p$. Thus by Lemma 3.9 we have the assertion.

§ 4. The Condition for $\mathfrak{A}^\alpha$ to Be UHF

Keep the definitions and notations in section 3. For $t\in Z_p$ let $\mathfrak{A}^\alpha(\{t\})$ be the set of $x\in\mathfrak{A}$ with $\alpha(x)=\exp\{2\pi ip^{-1}t\}x$ and let $\mathcal{U}$ be the unitary group of $\mathfrak{A}$.

Lemma 4.1. *The following statements are equivalent*:

(i) $\mathfrak{A}^\alpha(\{1\})\cap\mathcal{U}\neq\emptyset$,

(ii) $\mathfrak{A}^\alpha(\{1\})\cap\mathfrak{A}(n)\cap\mathcal{U}\neq\emptyset$ *for sufficiently large* n,

(iii) $\mathfrak{A}$ *contains an* α*-invariant type* I_p *factor* M *such that* M^α *is* *abelian,*

(iv) *For sufficiently large n,* $\mathfrak{A}(n)$ *contains an* α*-invariant type* I_p *factor* M *such that* $\mathfrak{A}(n)\cap M'\subset\mathfrak{A}^\alpha$ *and* M^α *is abelian,*

(v) $P(S(0,n)=t)=p^{-1}$ *for any* $t\in Z_p$ *for sufficiently large n.*

Proof. (iv)$\Leftrightarrow$(v) and (iv)$\Rightarrow$(iii)$\Rightarrow$(i) are obvious. (i)$\Rightarrow$(ii) follows from the fact that $\bigcup_n(\mathfrak{A}^\alpha(\{1\})\cap\mathfrak{A}(n)\cap\mathcal{U})$ is dense in $\mathfrak{A}^\alpha(\{1\})\cap\mathcal{U}$. Suppose that (ii) holds. Let u be a unitary in $\mathfrak{A}^\alpha(\{1\})\cap\mathfrak{A}(n)$, e a minimal projection of the center of $\mathfrak{A}(n)^\alpha$ and M the algebra generated by $e_{k,l}=u^keu^{*l}(k,l=1,\cdots,p)$. Since $e_{k,l}$ forms matrix units for M and M contains the center of $\mathfrak{A}(n)^\alpha$, it is easy to see that M satisfies the condition in (iv). Q.E.D.

Proposition 4.2. *If one of the conditions in Lemma* 4.1 *is satisfied, then* $e=1$ *is possible in the statements (v) and (vi) in Theorem* 3.10.

Proof. This is easily seen from the proof (iv)$\Rightarrow$(vi) of Theorem 3.10 and from Lemma 4.1(v).

Lemma 4.3. *The following statements are equivalent:*

(i) $\mathfrak{A}^\alpha(\{1\})\cap\mathcal{U}$ *contains a central sequence ;*

(ii) *There exists a subsequence* n_k *of positive integers such that* $\mathfrak{A}^\alpha(\{1\})\cap\mathfrak{A}(n_{k+1})\cap\mathfrak{A}(n_k)'\cap\mathcal{U}\neq\emptyset$;

(iii) $\mathfrak{A}$ *contains a central sequence* M_k *of* α*-invariant type* I_p *factors such that* ${M_k}^\alpha$ *are abelian ;*

(iv) *There exists a subsequence* n_k *of positive integers such that* $\mathfrak{A}(n_{k+1})\cap\mathfrak{A}(n_k)'$ *contains an* α*-invariant type* I_p *factor* M *with abelian* M^α *satisfying* $\mathfrak{A}(n_{k+1})\cap\mathfrak{A}(n_k)'\cap M'\subset\mathfrak{A}^\alpha$.

(v) *There exists a subsequence* n_k *of positive integers such that* $P(S(n_k,n_{k+1}-1)=t)=p^{-1}$ *for any* $t\in Z_p$ *and* $k=1,2,\cdots$.

Proof. (iv)$\Leftrightarrow$(v) and (iv)$\Rightarrow$(iii)$\Rightarrow$(i) are obvious (where $\{M_k\}$ is called a central sequence if $\|[x_k,y]\|$ converges to zero as $k\to\infty$ for any bounded sequence $x_k\in M_k$ and any $y\in\mathfrak{A}$). Suppose (i) and let u_k

be a central sequence of unitaries of $\mathfrak{A}^{\alpha}(\{1\})$. Then for any $\varepsilon>0$ and n there is a k such that $\|u_k-x_1\|<\varepsilon/2$ holds for some $x_1\in\mathfrak{A}\cap\mathfrak{A}(n)'$. Further there is an $m>n$ such that $\|u_k-x_2\|<\varepsilon$ for some $x_2\in\mathfrak{A}(m)\cap\mathfrak{A}(n)'$. This implies that there is an $x_3\in\mathfrak{A}(m)\cap\mathfrak{A}(n)'\cap\mathfrak{A}^{\alpha}(\{1\})$ with $\|u_k-x_3\|<\varepsilon$. If ε is sufficiently small, the partial isometry obtained from the polar decomposition of x_3 is a unitary in $\mathfrak{A}^{\alpha}(\{1\})$. Thus we have (i)$\Rightarrow$(ii). The proof (ii)$\Rightarrow$(iv) is the same as that in Lemma 4.1. Q.E.D.

Now we recall $(\mathfrak{A}_p, \alpha_p)$ defined in section 1.

Theorem 4.4. *Let $(\mathfrak{A}, Z_p, \alpha)$ be as above. Then the following statements are equivalent:*

(i) *$\mathfrak{A}^{\alpha}$ is isomorphic to $\mathfrak{A}$;*

(ii) *$\mathfrak{A}^{\alpha}$ is a UHF algebra;*

(iii) *$(\mathfrak{A}, \alpha)$ is isomorphic to $(\mathfrak{A}_0\otimes\mathfrak{A}_p, \iota\otimes\alpha_p)$ where ι is the trivial automorphism of a UHF algebra $\mathfrak{A}_0$;*

(iv) *One of the conditions in Lemma 4.3 is satisfied.*

Proof. (i)$\Rightarrow$(ii) is obvious. Suppose (ii). Then by Lemma 2.6 of [1] there are an increasing sequence $\mathfrak{B}(n)$ of type I subfactors of $\mathfrak{A}^{\alpha}$ and a subsequence m_n of positive integers such that $\mathfrak{A}^{\alpha}=\overline{\bigcup\mathfrak{B}(n)}$ and $\mathfrak{A}(n)^{\alpha}\subset\mathfrak{B}(n)\subset\mathfrak{A}(m_n)^{\alpha}$, $n=1, 2, \cdots$. Hence the proportionality $P(S(n, m_n-1)=s-t)$ of the multiplicity of $M_{n,t}$ embedded in $M_{m_n,s}$ as s varies, is independent of $t\in Z_p$. This implies that $P(S(n, m_n-1)=t)=p^{-1}$ for any $t\in Z_p$. Thus we have (ii)$\Rightarrow$(iv). If (iv) holds, we have (iii) by using Lemma 4.3 (iv). Thus we have only to show that $\mathfrak{A}_p^{\alpha_p}$ is isomorphic to $\mathfrak{A}_p$.

For the system $(\mathfrak{A}_p, \alpha_p)$ we construct an increasing sequence $\mathfrak{B}(n)$ of type I_{p^n} subfactors such that $\mathfrak{A}(n)^{\alpha}\subset\mathfrak{B}(n)\subset\mathfrak{A}(n+1)^{\alpha}$. Let $\mathfrak{B}$ be a subfactor of type $I_{p^{n-1}}$ of $\mathfrak{A}(n)$ and let $e_t^{(n)}$, $t\in Z_p$, be a set of distinct minimal projections of the center of $\mathfrak{A}(n^{\alpha})$. Then $\mathfrak{A}(n)^{\alpha}$ is generated by $\mathfrak{B}$ and $\{e_t^{(n)}, t\in Z_p\}$ and $e_t^{(n)}e_s^{(n+1)}$ is a minimal projection of $\mathfrak{A}(n+1)^{\alpha}\cap\mathfrak{B}'$ for any t and s in Z_p. Hence there exists a subfactor $\mathfrak{B}_1$ (of type I_p) of $\mathfrak{A}(n+1)\cap\mathfrak{B}'$ such that $\mathfrak{B}_1\ni e_t^{(n)}$, $t\in Z_p$. Let $\mathfrak{B}(n)$ be the algebra generated by $\mathfrak{B}$ and $\mathfrak{B}_1$. Then $\mathfrak{A}(n)^{\alpha}\subset\mathfrak{B}(n)\subset\mathfrak{A}(n+1)^{\alpha}$ and $\mathfrak{B}(n)$ is a type I_{p^n} factor. Thus $\overline{\bigcup\mathfrak{B}(n)}=\overline{\bigcup\mathfrak{A}(n)^{\alpha}}=\mathfrak{A}^{\alpha}$ which completes the proof.

Remark 4.5. There is an example $(\mathfrak{A}, \alpha)$ where $\mathfrak{A}^{\alpha}$ is not a UHF algebra but has a unique tracial state.

Let $\mathfrak{A}_n$ be a type I_{p+1} factor and let α_n be the automorphism of $\mathfrak{A}_n$ implemented by $\exp\{2\pi ip^{-1}\Sigma_1^{p+1}ke_k\}$ where $\{e_k\}_1^{p+1}$ is a family of orthogonal projections of $\mathfrak{A}_n$. We consider the system $(\mathfrak{A}=\otimes\mathfrak{A}_n, \alpha=\otimes\alpha_n)$. Then $(\mathfrak{A}, \alpha)$ satisfies Condition 3.2 but $\mathfrak{A}$ does not contain a UHF subalgebra of type (p^n). By Theorems 3.10 and 4.4 this proves our assertion.

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