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On the Theory of b-Functions

By

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Notation

X: a complex manifold

$\mathcal{O}_X$: the sheaf of holomorphic functions on X

$\mathcal{D}_X$: the sheaf of holomorphic linear differential operators of finite order on X

Θ_X: the sheaf of holomorphic vector fields on X

$\mathcal{D}_X[s]=\mathcal{D}_X\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}[s]$

$\boldsymbol{N}$: the set of natural numbers, $\boldsymbol{N}_0=\boldsymbol{N}\cup\{0\}$

$\boldsymbol{Z}$: the set of rational integers

g.c.d. = the greatest common divisor

l.c.m. = the least common multiple

Introduction

The purpose of this paper is to develop a general theory of b-functions with emphasis on the detailed study of examples. A b-function $b_f(s)$ associated with a local holomorphic function $f(x)$ is defined to be a generator of the ideal formed by polynomials in s satisfying

$$P(s, x, D)\, f(x)^{s+1}=b(s) f(x)^s, \tag{1}$$

for some linear differential operator $P(s, x, D)=\sum_{0\le j\le m} s^j P_j(x, D)$.

The following is a famous example of the equality of this type for a quadratic form $Q(x)=\sum_{i=1}^{n} x_i^2$,

$$\Delta Q^{s+1}=4(s+1)\left(s+\frac{n}{2}\right)Q^s, \tag{2}$$

where

$$\Delta=\sum_{i=1}^{n} D_i^2.$$

We note that the roots of $b_f(s)=0$ are strictly negative rational numbers.

It is well-known that b-function $b_f(s)$ plays important roles in analyzing hyperfunction f_+^s. In fact, define the gamma factor $\gamma(s)=\prod \Gamma(s+\alpha_i)$ when $b(s)=\prod(s+\alpha_i)$. Then (1) turns out to be

$$P(s, x, D)\frac{1}{\gamma(s+1)}f^{s+1}=\frac{1}{\gamma(s)}f^{s}.$$

In view of this formula, we can readily see that f_+^s depends meromorphically on s and its poles occur only at $-\alpha_i-\nu$, where ν runs over the non-negative integers.

For example, the factor $\left(s+\frac{n}{2}\right)$ in (2) has a relation with the following facts that (1) the poles of Q_+^s are located at $-\frac{n}{2}-\nu$, $\nu\in N_0$, (2) the local monodromy of $Q^{-1}(0)$ at 0 is $(-1)^n=\exp(2\pi i(-n/2))$, and (3) $I(h)=\int\exp((i/h)Q(x))\varphi(x)dx$ behaves asymptotically like $I(h)=0(h^{n/2})$, $(h\to 0)$.

At this point, it should be remarked that $b_f(s)$ is an invariant of the hypersurface $f^{-1}(0)$ finer than local monodromy (cf. §16 etc.).

We investigate $b_f(s)$ through the structure of Modules $\mathcal{N}=\mathcal{D}[s]f^s$ and $\mathcal{M}=\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$, where $\mathcal{D}[s]=\mathcal{D}\otimes C[s]$. Here s acts on them as $s: P(s)f^s\mapsto sP(s)f^s$. Then, one can state (1) as "$b_f(s)$ is the minimal polynomial of s in $\mathcal{M}$." $\mathcal{N}$ admits as pecial operation $t: P(s)f^s \mapsto P(s+1)f^{s+1}$. The commutation relation $ts-st=t$ plays an essential role. This standpoint was established by M. Kashiwara and M. Sato.

This paper is organized as follows. In Chapter I, we introduce the notion of $\mathcal{D}[t, s]$-Modules, thereby b-functions being generally introduced. General theory of these Modules are included in [32]. In Chapter II, the structure of $\mathcal{N}$ and $\mathcal{M}$ are studied. The author introduces a number $L(f)$, which measures non-quasi-homogeneity of f, and the notion of a functions of simplex type. He also gives a good presentation of a Module $(s+1)\mathcal{M}$ for f being isolated singularity with $L(f)=2, 3$, which is used in the next chapter. The methods to determine or to estimate b-functions are investigated in Chapter III. The author gives a join-formula for b-functions in § 16. Various examples are found in Chapter IV. In particular, some interesting explicit formulae are given for two-dimensional case. The determination of b-functions of all the canonical forms of isolated singularities with modality less than three is performed by the author and it is included in § 20 for corank $f=2$ and in [32] for corank $f=3$.

A part of the results of this work was announced in [27], [28], [29], [30], [34] and [37].

Historical remarks around the equation (1) is as follows.

I. M. Gelfand conjectured in Amsterdam Congress that the analytic properties of f_+^s could be well investigated by use of the desingularization theorem. In fact, I. N. Bernstein-S. I. Gelfand [39] and M. F. Atiyah [38] proved the meromorphic dependence of f_+^s in s and discribed its poles by the resolution theorem of H. Hironaka.

In 1961, M. Sato initiated a theory of b-functions for relative invariants on prehomogeneous vector spaces, in connection with the Fourier transforms and ζ-functions associated with these spaces [23], [26].

On the other hand, I. N. Bernstein independently took the equation (1) and proved an existence theorem of such b-functions which does not vanish identically when f is a polynomial [7]. J. E. Björk succeedingly generalized Bernstein's result for analytic functions [8].

Since then much effort has been forcused on the general theory of b-functions [12], [19], [27]. The author's contribution has been done since this stage. B. Malgrange pointed out a close connection between b-functions of f and the local monodromy of $f^{-1}(0)$ [16]. He proved that, when f has an isolated singularity, the eigenvalues of local monodromy are just $\exp(2\pi\sqrt{-1}\alpha)$ for roots α of b-function [17]. Afterwards, M. Kashiwara proved the rationality of the roots of b-function for general f in a completely different way [14].

As for the analytic property of f^α, we note also that an important result that f^α satisfies a holonomic system was proved by I. N. Bernstein [7] in a special case and by M. Kashiwara-T. Kawai [10] for any f. More generally, analytic property of $f^\alpha u$ for holonomic u is studied in [7] in a special case and in the author's subsequent paper [32] for general cases.

The b-functions associated with prehomogeneous vector spaces are well-investigated and they are determined by many people. The microlocal calculus finds its good application in the area of b-function theory, and that theme will be fully treated in M. Kashiwara-T. Kimura-M. Muro [41].

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Chapter I. Generarities

In this chapter, we study the basic features of general $\mathcal{D}[t, s]$-Modules and b-functions associated with them, which are indispensable to later chapters. The author develop the general theory of such b-functions and Modules in [32].

§ 1. $\mathcal{D}[t, s]$-Modules and b-Functions

Let $\boldsymbol{C}[t, s]$ be the associative algebra over $\boldsymbol{C}$ with generators s and t and defining relation

$$(1) \qquad ts - st = t.$$

Set $\mathcal{D}[t, s] = \mathcal{D} \underset{\boldsymbol{C}}{\otimes} \boldsymbol{C}[t, s]$. A $\mathcal{D}$-Module $\mathcal{M}$ is called a $\mathcal{D}[s]$-Module (respectively $\mathcal{D}[t, s]$-Module), if $\mathcal{M} \supset s\mathcal{M}$ (respectively $\mathcal{M} \supset s\mathcal{M}$, $\mathcal{M} \supset t\mathcal{M}$) holds. In this chapter, all Modules are $\mathcal{D}[t, s]$-Modules unless otherwise stated. Since $t^{\nu}s = (s+\nu)t^{\nu}$ in view of (1), $\operatorname{Ker} t^{\nu}$, $\operatorname{Coker} t^{\nu}$ and $\operatorname{Im} t^{\nu}$ are $\mathcal{D}[t, s]$-Modules along with a given $\mathcal{D}[t, s]$-Module.

Definition 1.1. *Let $\mathcal{L}$ be a $\mathcal{D}[s]$-Module. If $s \in \mathcal{E}nd_{\mathcal{D}}(\mathcal{L})$ has the non-zero minimal polynomial, we denote it by $d_{\mathcal{L}}(s)$, and say "$d_{\mathcal{L}}(s)$ exists." "b-functions" for a $\mathcal{D}[t, s]$-Module $\mathcal{N}$, are defined by $b_{\mathcal{N},\nu}(s) = d_{\mathcal{N}/t^{\nu}\mathcal{N}}(s)$, $\nu = 1, 2 \cdots$.*

Usually, $b_{\mathcal{N},1}$ is abbreviated as $b_{\mathcal{N}}$. As is easily seen, $b_{\mathcal{N},\nu}$ exist if and only if $b_{\mathcal{N}}$ exists.

It should be remarked that if $\mathcal{L}$ is a holonomic $\mathcal{D}[t, s]$-Module $d_{\mathcal{L}}(s)$ exists, since $\mathcal{E}nd_{\mathcal{D}}(\mathcal{L})_x$ $(x \in X)$ is finite dimensional and $\mathcal{E}nd_{\mathcal{D}}(\mathcal{L})$ is coherent [13].

Standard example of $\mathcal{D}[t, s]$-Module is constructed as follows. Let

f be a holomorphic function on $U \subset X$, let $\mathcal{L}$ be a coherent $\mathcal{D}$-Module and let u be its section over U. We denote the annihilator of u by $\mathcal{J}$, that is; $\mathcal{J}=\{Q\in\mathcal{D} \mid Qu=0\}$. Define the ideal $\mathcal{J}(s)\subset\mathcal{D}[s]$ by the condition that

$$P(s, x, D)\in\mathcal{J}(s) \quad \text{if and only if}$$

$$f^m P\Big(s, x, D+\frac{s}{f}\operatorname{grad} f\Big)\in \boldsymbol{C}[s]\otimes\mathcal{J}, \quad \text{for some } m.$$

We denote by $\mathcal{N}$ the Module $\mathcal{D}[s]/\mathcal{J}(s)$ and by $f^s u$ the class $(1 \bmod \mathcal{J}(s))$. $\mathcal{N}=\mathcal{D}[s]f^s u$ is a $\mathcal{D}[t, s]$-Module with actions of t and s given by,

$$t: P(s)\mapsto P(s+1)f, \quad s: P(s)\mapsto P(s)s.$$

The map t is *injective* in $\mathcal{N}$. In fact, if $P(s+1)f\in\mathcal{J}(s)$ then

$$f^m P\Big(s+1, x, D+\frac{s}{f}\operatorname{grad} f\Big)f=\sum Q_j s^j$$

for some m and $Q_j\in\mathcal{J}$. The left-hand side equals to

$$f^{m+1}P\Big(s+1, x, D+\frac{s+1}{f}\operatorname{grad} f\Big),$$

and the right-hand side can be rewritten in the form

$$\sum R_j(s+1)^j$$

for some $R_j\in\mathcal{J}$. Therefore,

$$f^{m+1}P\Big(s, x, D+\frac{s}{f}\operatorname{grad} f\Big)=\sum R_j s^j,$$

which implies $P(s)\in\mathcal{J}(s)$.

The $\mathcal{D}$-Module $\mathcal{D}f^s u$ is coherent, and if u is a holonomic section, $\mathcal{D}f^s u$ is subholonomic (see [32]).

Definition 1.2. *With a non-zero polynomial $p(s)$, we associate a number $w(p)\in\boldsymbol{N}_0$ in the following manner ($w(p)$ is called the width of p.)*

i) *If $p(s)\in\boldsymbol{C}^*$ then $w(p)=0$,*

ii) *If $p(s)=\prod_{i=0}^{k}(s+\alpha+i)^{\varepsilon_i}$, $\alpha\in\boldsymbol{C}$, $\varepsilon_0\varepsilon_k\neq 0$ then $w(p)=k+1$,*

iii) *If* $p(s)$ *has the form*

$$p(s) = \prod_{i=1}^{k} p_j(s), \text{ where each } p_j(s) \text{ is of the form in ii)},$$
$$p_j(s) = \prod (s+\alpha_j+i)^{\varepsilon_i(j)}, \text{ and } \alpha_j \not\equiv \alpha_{j'} \bmod \mathbf{Z} \ (j \neq j')\text{; then}$$
$$w(p) = \max_j w(p_j).$$

Theorem 1.3. *If* $d_{\mathcal{L}}(s)$ *exists, then* $t^{w(d_{\mathcal{L}})}\mathcal{L}=0$. *Furthermore if we assume that* t *is injective or surjective, then* $\mathcal{L}=0$.

Proof. we have

$$0 = d_{\mathcal{L}}(s)\mathcal{L} \supset d_{\mathcal{L}}(s)\, t^{w(d_{\mathcal{L}})}\mathcal{L},$$

and by virtue of (1),

$$0 = t^{w(d_{\mathcal{L}})} d_{\mathcal{L}}(s)\mathcal{L} = d_{\mathcal{L}}(s+w(d_{\mathcal{L}}))\, t^{w(d_{\mathcal{L}})}\mathcal{L}.$$

It follows from the definition of $w(d_{\mathcal{L}})$ that

$$\text{g.c.d.}\,(d_{\mathcal{L}}(s), d_{\mathcal{L}}(s+w(d_{\mathcal{L}}))) = 1.$$

Hence the assertion follows. When t is injective or surjective, it is obvious that $\mathcal{L}=0$. Q.E.D.

A coherent $\mathcal{D}$-Module $\mathcal{L}$ is called holonomic (resp. sub-holonomic) if $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{L}, \mathcal{D}) = 0$ for $i<n$ (resp. $i<n-1$). This condition is equivalent to $\operatorname{codim} \check{SS}(\mathcal{L}) \geq n$ (resp. $\operatorname{codim} \check{SS}(\mathcal{L}) \geq n-1$). $\mathcal{L}$ is called purely subholonomic if $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{L}, \mathcal{D}) = 0$ for $i \neq n-1$. It is known that for any coherent $\mathcal{D}$-Module, $\mathcal{E}xt^n_{\mathcal{D}}(\mathcal{L}, \mathcal{D})$ (resp. $\mathcal{E}xt^{n-1}_{\mathcal{D}}(\mathcal{L}, \mathcal{D})$) is holonomic (resp. sub-holonomic) and $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{L}, \mathcal{D}) = 0$, $i>n$. Let W be an irreducible component of $\check{SS}(\mathcal{L})$. Then the multiplicity of $\mathcal{L}$ at a generic point x_0 of an irreducible component of $\check{SS}(\mathcal{L})$ can be defined (which is denoted by $m_{x_0}(\mathcal{L})$), and has the additivity, that is, if

$$0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \mathcal{L}_3 \leftarrow 0,$$

is an exact sequence of coherent $\mathcal{D}$-Modules, $m_{x_0}(\mathcal{L}_2) = m_{x_0}(\mathcal{L}_1) + m_{x_0}(\mathcal{L}_3)$.

Corollary 1.4. *Let* $\mathcal{N}$ *be a sub-holonomic* $\mathcal{D}[t, s]$*-Module such that* $t\colon \mathcal{N} \to \mathcal{N}$ *is injective. Then,* $\mathcal{N}$ *is purely sub-holonomic.*

Proof. Consider the exact sequence

$$0 \leftarrow \mathcal{N}/t\mathcal{N} \leftarrow \mathcal{N} \xleftarrow{t} \mathcal{N} \leftarrow 0.$$

Set $\mathcal{L} = \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{N}, \mathcal{D})$. Then $\mathcal{L}$ is holonomic and the long exact sequence of $\mathcal{E}xt$ gives us the surjection $\mathcal{L} \xrightarrow{t} \mathcal{L} \to 0$. Therefore $\mathcal{L} = 0$ by virtue of Theorem 1. 3. Q.E.D.

Proposition 1. 5. *Upon the conditions in Corollary* 1. 4, $b_{\mathcal{N}}$ *exists.*

Proof. Consider an irreducible component W of $\check{SS}(\mathcal{N})$. Since t is injective, the multiplicity of $\mathcal{N}/t\mathcal{N}$ at a generic point of W vanishes. Therefore $\operatorname{codim} \check{SS}(\mathcal{N}/t\mathcal{N}) \geq n$ which implies that $\mathcal{N}/t\mathcal{N}$ is holonomic. Thus $b_{\mathcal{N}}$ exists (and so does $b_{\mathcal{N},\nu}$, by the argument after Definition 1. 1). Q.E.D.

The conditions in Corollary 1. 4 are satisfied for $\mathcal{N} = \mathcal{D}[s]f^s u$, if one of the following two conditions holds.

i) f is arbitrary holomorphic function, $u = 1$.

ii) f is quasi-homogeneous, $\mathcal{D}u$ is holonomic.

In the present paper, we restrict ourselves to case i). We investigate case ii) in [32], where the detailed structure of $b_{\mathcal{N},\nu}(s)$ and the relation between $\mathcal{N}_\alpha$ and $\mathcal{D}f^\alpha u$ $(\alpha \in \boldsymbol{C})$ are also discussed. The existence of $b_{\mathcal{N}}(s)$ for $\mathcal{N} = \mathcal{D}[s]f^s u$ with general f and $\mathcal{D}u$ being holonomic can be derived from that of case ii), following the technique in § 3 of [14]. (See [32])

§ 2. *b*-Functions of Holomorphic Functions

Let X be a complex manifold of dimension n, and let $f(x)$ be a holomorphic function. Hereafter we make use of the notations $f_i = \partial f/\partial x_i$, $\mathfrak{a} = \sum \mathcal{O} f_i$, for brevity.

The b-function of f, which we denote by $b_f(s)$, is defined by,

$$b_f(s) = b_{\mathcal{N}}(s), \quad \text{where} \quad \mathcal{N} = \mathcal{D}[s]f^s.$$

Here, $\mathcal{N}$ is a special case of $\mathcal{D}[s]f^s u$ for $u = 1$. We also define $b_{f,\nu}(s) = b_{\mathcal{N},\nu}(s)$. The existence of them will be assured later by Theorem

1.8.

It follows from the above definition that there are $P(s)$ and $P_\nu(s+\nu)\in\mathcal{D}[s]$, such that

(1) $$P(s)f^{s+1}=b_f(s)f^s,$$

(2) $$P_\nu(s+\nu)f^{s+\nu}=b_{f,\nu}(s)f^s,$$

and $b_f(s)$ and $b_{f,\nu}(s)$ are minimal among such polynomials in s.

When we emphasize the point $x\in X$ into consideration, we use the notation $b_{f,x}(s)$. Furthermore given a compact set $K\subset X$, we set $b_{f,K}(s)=\underset{x\in K}{\text{l.c.m.}}\, b_{f,x}(s)$.

If $f(x)\neq 0$, then $\frac{1}{f}f^{s+1}=f^s$. Hence $b_{f,x}(s)=1$.

If $f(x)=0$, setting $s=-1$ in (1), we know $(s+1)\,|\,b_{f,x}(s)$.

If $f(x)=0$, $\operatorname{grad} f(x)\neq 0$, then $b_{f,x}(s)=(s+1)$ by $\frac{1}{f_1}D_1f^{s+1}=(s+1)f^s$ (e.g. when $f_1(x)\neq 0$).

Therefore, our main concern is with $b_{f,x}(s)$ at a singular point of $f^{-1}(0)$.

If y is in a sufficiently small neighborhood of x, $b_{f,y}(s)\,|\,b_{f,x}(s)$ by (1). For $g(x)\in\mathcal{O}$, $g(x_0)\neq 0$, we have $b_{gf,x_0}(s)=b_{f,x_0}(s)$. Because, if $P(s,x,D)f^{s+1}=b_f(s)f^s$,

$$g^{-1}P(s,x,D-(s+1)\operatorname{grad}\log g)\,(gf)^{s+1}=b_f(s)\,(gf)^s,$$

and vice versa. Thus, $b_f(s)$ is an invariant of the hypersurface $\{f=0\}$ independent of the choice of its defining equation.

For later convenience we list up basic notations in b-function theory.

Definition 1.6.

i) $\mathcal{J}(s)=\{P(s)\in\mathcal{D}[s]\,|\,P(s)f^s=0\}$,

$\mathcal{J}_0=\mathcal{D}\cap\mathcal{J}(s)$, $\quad\mathcal{G}=\{X\in\Theta\,|\,Xf\in\mathcal{O}f\}$,

$\mathcal{M}=\mathcal{N}/t\mathcal{N}$, $\quad\widetilde{\mathcal{M}}=(s+1)\mathcal{M}$, $\quad\mathcal{N}_\alpha=\mathcal{N}/(s-\alpha)\mathcal{N}$.

ii) $W=\{(x,s\operatorname{grad}\log f)\,|\,s\in\mathbf{C},\ f(x)\neq 0\}^{\text{closure}}\subset T^*X$,

$W_0=\{(x,\xi)\in W\,|\,f(x)=0\}\cup\{(x,0)\,|\,x\in X\}$.

Proposition 1.7. $\mathcal{N}=\mathcal{D}[s]/\mathcal{J}(s)$,

$$\mathcal{M}=\mathcal{D}[s]/(\mathcal{J}(s)+\mathcal{D}[s]f).\quad \widetilde{\mathcal{M}}=\mathcal{D}[s]/(\mathcal{J}[s]+\mathcal{D}[s](\mathfrak{a}+\mathcal{O}f)),$$

$$\mathcal{N}_\alpha=\mathcal{D}[s]/(\mathcal{J}(s)+\mathcal{D}[s](s-\alpha))=\mathcal{D}/(\mathcal{J}(s)|_{s=\alpha}).$$

Proof. The isomorphisms of $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}_\alpha$ are easy to verify. That of $\widetilde{\mathcal{M}}$ is proved as follows. Let $P(s)$ be such that $P(s)(s+1)f^s=Q(s)f^{s+1}$. Setting $s=-1$, we have $Q(-1)=\sum q_i(x,D)D_i$. Hence,

$$P(s)(s+1)f^s=((s+1)R(s)+\sum q_i(x,D)D_i)f^{s+1}$$
$$=(s+1)(R(s)f+\sum q_i(x,D)f_i)f^s.$$
$$P(s)\in\mathcal{J}[s]+\mathcal{D}[s](\mathfrak{a}+\mathcal{O}f).\qquad\text{Q.E.D.}$$

If $\operatorname{grad} f(x)\neq 0$, $f(x)=0$ at $x\in X$, we can assume $f=x_1$. Then $\mathcal{D}[s]f^s=\mathcal{D}[s]/\mathcal{D}[s](s-x_1D_1)+\sum_{j=2}^n\mathcal{D}[s]D_j\simeq\mathcal{D}/\sum_{j=2}^n\mathcal{D}D_j$. Therefore, $\check{SS}(\mathcal{N})=\{(x,\xi)\,|\,\xi_2=\cdots=\xi_n=0\}=W$ in a neighborhood of x. Since $\check{SS}(\mathcal{N})$ is an analytic set, we have $\check{SS}(\mathcal{N})\supset W$.

We state the fundamental theorem of M. Kashiwara.

Theorem 1.8. i) *$\mathcal{N}$ is sub-holonomic and $\check{SS}(\mathcal{N})=W$.* ii) *$b_f(s)$ exists and all the roots of $b_f(s)=0$ are strictly negative rational.*

For the proof of this, we refer the reader to M. Kashiwara [14]. The existence of $b_f(s)$ can be derived from i) and Proposition 1.5. See also [32].

Corollary 1.9. *$\mathcal{M}$, $\widetilde{\mathcal{M}}$ and $\mathcal{N}_\alpha$ are holonomic. More precisely, $\check{SS}(\mathcal{M})\subset W\cap(f^{-1}(0))$, $\check{SS}(\widetilde{\mathcal{M}})\subset W\cap\operatorname{Sing}f^{-1}(0)$ and $\check{SS}(\mathcal{N}_\alpha)\subset W_0$.*

Proof. For, t gives an isomorphism on $W\backslash f^{-1}(0)$ in the exact sequence $0\to\mathcal{N}\xrightarrow{t}\mathcal{N}\to\mathcal{M}\to 0$, $\check{SS}(\mathcal{M})$ is contained in $f^{-1}(0)\cap W$ and hence a holonomic set.

Since $\quad fD_i-f_is\in\mathcal{J}(s)$, $\check{SS}(\mathcal{N}_\alpha)\subset W\cap(f^{-1}(0)\cup(\xi=0))$.

$$\check{SS}(\widetilde{\mathcal{M}})\subset\check{SS}(\mathcal{M})\cap(f_i=0,\ \forall i).\qquad\text{Q.E.D.}$$

When f is locally reduced, K. Saito proved the following:

Theorem 1.10. *$\mathcal{G}$ is a reflexive $\mathcal{O}_X$-Module. Let $X_i=\sum a_{ij}(x)D_j$ $i=1,\cdots,n$, be elements in $\mathcal{G}$. Then $X_1,\cdots,X_n$ is a locally free basis of $\mathcal{G}$, if and only if* $\det(a_{ij})=gf$, $g\in\mathcal{O}_X^*$.

Corollary 1.11. *Suppose* $\dim X=2$. *Then $\mathcal{G}$ has locally free basis X_1, X_2 $(X_i=\sum a_{ij}D_j)$ and $a_{11}a_{22}-a_{12}a_{21}=gf$, $g\in\mathcal{O}_X^*$. Conversely, if two vector fields X_i in $\mathcal{G}$ satisfy the above formula, they form a basis of $\mathcal{G}$.*

For the proof of these, we refer the reader to K. Saito [21]. When f is the square of the fundamental anti-invariant of a Coxeter group, considered as a function of fundamental invariants, $\mathcal{G}$ is a free module. This was pointed out by K. Saito [21]. For the determination of the structure of $\mathcal{G}$ and the microlocal structure of $\mathcal{D}[s]f^s$, we refer the reader to T. Yano [33] or T. Yano-J. Sekiguchi [35], [36]. They proved that the holonomic system $\mathcal{D}f^\alpha$ has multiplicity 1 on all the irreducible components of $\check{SS}(\mathcal{D}f^\alpha)$, and determined a basis of $\mathcal{G}$ concretely.

Corollary 1.11 was also noted by M. Sato and M. Kashiwara (not published).

Chapter II. Structure of the Ideal $\mathcal{J}(s)$

In this chapter, we shall restrict our attention to the structure of $\mathcal{J}(s)$. First of all, we introduce a number $L(f)$, which measures the non-quasi-homogeneity of f. We further define a class of functions called a convergent power series of simplex type, which plays an important role in later applications. In the case of such a function, corresponding $\mathcal{J}(s)$ contains a distinguished element (cf. Theorem 2.15). In §§ 6, 8, we shall determine the structure of $\mathcal{J}(s)$ upon the following two conditions that 1° $L(f)\leq 3$ and 2° the singularity is isolated. Section 8 is concerned with a delicate phenomenon about $\mathcal{J}(s)$, and given are counter examples against Sato-Kashiwara conjectures.

§ 3. Total Symbol

For the later purposes it is appropriate to modify the notion of order of an element of $\mathscr{D}[s]$ by regarding s as element of order 1. To be more precise, we define

Definition 2.1. *Given* $P(s)=\sum s^j P_j(x,D)\in\mathscr{D}[s]$, $\max_j(j+\operatorname{ord}P_j)$ *is called the total order of* P *and denoted by* $\operatorname{ord}^T(P(s))$. *Let* $l=\operatorname{ord}^T(P(s))$. *Then we call*

$$\sigma^T(P)(s,x,\xi)=\sum s^j\sigma_{l-j}(P_j),$$

the total symbol of P. *It follows that* $\sigma^T(P)$ *is a function on* $\boldsymbol{C}\times T^*X$ *having homogeneous degree* l *in* (s,ξ). *For an ideal* $\mathscr{I}(s)$ *in* $\mathscr{D}[s]$, *we define its total symbol ideal by*

$$\sigma^T(\mathscr{I}(s))=\{\sigma^T(P)\,|\,P\in\mathscr{I}(s)\}.$$

Let $\mathscr{I}$ be an ideal in $\mathscr{O}_{T^*X}[s]$ and S be a subset of $\boldsymbol{C}\times T^*X$. Then we denote by $V(\mathscr{I})$ and $\mathscr{I}(S)$ the null set of $\mathscr{I}$ and the ideal of functions that vanish on S, respectively.

Definition 2.2. i) *We define*

$$\check{SS}[s](\mathscr{L})=V(\sigma^T(\mathscr{I}(s))),$$

for a $\mathscr{D}[s]$*-Module* $\mathscr{L}=\mathscr{D}[s]/\mathscr{I}(s)$. *More generally we define*

$$\check{SS}[s](\mathscr{L})=\bigcup_{i=1}^{l}\check{SS}[s](\mathscr{D}[s]u_i),$$

for finitely generated $\mathscr{D}[s]$*-Module* $\mathscr{L}=\mathscr{D}[s]u_1+\cdots+\mathscr{D}[s]u_l$. ii) *Let* f *be a holomorphic function. The subset* $W[s]$ *in* $\boldsymbol{C}\times T^*X$ *is defined by*

$$W[s]=\{(s,x,s\operatorname{grad}\log f)\,|\,f\neq 0,\ s\in\boldsymbol{C}\}^{\text{closure}}.$$

Proposition 2.3.

$$\check{SS}[s](\mathscr{N})=W[s].$$

Proof. Let $P(s)\in\mathscr{D}[s]$, $\operatorname{ord}^T P=m$. Then

$$P(s)f^s=(s)_m\sigma^T(P)(f,x,df)f^{s-m}+(\text{lower order in } s),$$

Therefore, if $P(s) \ni \mathcal{J}(s)$, then $\sigma^T(P) \in \mathcal{I}(W[s])$. Let $p(s, x, \xi) \in \mathcal{I}(W[s])$. We shall prove that $\exists l$, $\exists P(s) \in \mathcal{J}(s)$ such that $\sigma^T(P(s)) = p(s, x, \xi)^l$. From this, 2.3 follows. Define the function on $\boldsymbol{C} \times X \ni (t, x)$ by $f'(t, x) = tf(x)$, and the function on $T^*(\boldsymbol{C} \times X)$ by $q(t, x, \tau, \xi) = p(t\tau, x, \xi)$. Since $p(s, x, sd\log f) = 0$, we have $q\left(t, x, \frac{s}{t}, sd\log f\right) = 0$. However, $\left(\frac{s}{t}, sd\log f\right) = sd_{(t,x)}\log f'$. Hence, q vanishes on $W' = \{(t, x, sd_{(t,x)}\log f') \mid tf \neq 0, s \in \boldsymbol{C}\}^{\text{closure}}$. By Theorem 1.8, we have $\check{SS}(\mathcal{N}') = W'$, where $\mathcal{N}' = \mathcal{D}_{\boldsymbol{C}\times X}[s]f'^s = \mathcal{D}_{\boldsymbol{C}\times X}f'^s$, whence there are $l \in \boldsymbol{N}$ and $Q \in \mathcal{J}_{f'}$ such that $\sigma(Q) = q^l$. We write Q_0 for the 0-th homogeneous part of Q with respect to t. Then obviously it follows that

$$Q_0(tD_t, x, D_x)f'^s = 0$$

and $\sigma(Q_0) = q^l$. Finally define $P(s, x, D_x) = Q_0(s, x, D_x)$. Then, we readily have $P(s, x, D_x)f^s = 0$ and $\sigma^T(P) = p^l$. Q.E.D.

The above Proposition 2.3 amounts to saying that $\sigma^T(\mathcal{J}(s)) \subset \mathcal{I}(W[s])$. There are examples of f for which this inclusion relation are a strict one (cf. § 8). A necessary condition for p to belong to $\sigma^T(\mathcal{J}(s))$ will be given in the following (1°, 2°, 3° below).

Let $p(s, x, \xi) \in \mathcal{O}_{T^*X}[s]$ be a homogeneous function of degree m in (s, ξ) : $p(s, x, \xi) = \sum_{|\alpha|=j} s^{m-j} a_{j,\alpha}\xi^\alpha$. In the sequel, we use the notation

$$R_k[p](s, x, \xi) = \sum_{|\alpha|=k+1} \frac{1}{\alpha!}(D_x^\alpha f)(x) \cdot (D_\xi^\alpha p)(s, x, \xi)$$

$$R_0[p](s, x, \xi) = p(s, x, \xi)$$

and

$$p(s, x, D) = \sum_j (s-j)_{m-j} \sum_{|\alpha|=j} a_{j,\alpha} D^\alpha.$$

Let $P(s)$ be an element in $\mathcal{D}[s]$ with $\operatorname{ord}^T(P(s)) = m$. Then,

$$P(s)f^s = a(s, x)f^{s-m},$$

where $a(s, x)$ is a polynomial of order not greater than m in s with coefficients in $\mathcal{O}$. We use the notation

(order less than $m+1$ in s)

for $a(s, x)f^{s-m}$. Let $a'(s, x)$ be a polynomial of order less than m in s with coefficients in $\mathcal{O}$. Then the formula of the form

$$a(s, x)f^{s-m} + a'(s, x)f^{s-m'}$$

is denoted by

$$a(s, x)f^{s-m} + (\text{lower order in } s).$$

First concider a homogeneous function $p(x, \xi) = \sum_{|\alpha|=k} a_\alpha \xi^\alpha$ not depending on s. Then we have

$$p(x, D)f^s = (s)_k p(x, df)f^{s-k} + \sum_{h=1}^{3} (s)_{k-h} R_h[p](x, df)f^{s-k+h}$$
$$+ (\text{lower order in } s).$$

From this, we have for $p(s, x, \xi) = \sum s^{m-k} p_k(x, \xi)$ with the degree of p_k in ξ is just k,

$$\begin{aligned}
p(s, x, D)f^s &= (s)_m \left(\sum f^{m-k} p_k(x, df)\right) f^{s-m} \\
&\quad + \sum_{h=1}^{3} \sum_k (s)_{k-h} (s-k)_{m-k} R_h[p_k] f^{s-k+h} \\
&\quad + (\text{order less than } m-3 \text{ in } s) \\
&= (s)_m p(f, x, df) f^{s-m} \\
&\quad + (s)_{m-1} R_1[p](f, x, df) f^{s-m+1} \\
&\quad - (s)_{m-2} R_1\left[\frac{\partial p}{\partial s}\right](f, x, df) f^{s-m+2} \\
&\quad + (s)_{m-3} R_1\left[\frac{\partial^2 p}{\partial s^2}\right](f, x, df) f^{s-m+3} \\
&\quad + (s)_{m-2} R_2[p](f, x, df) f^{s-m+2} \\
&\quad - 2(s)_{m-3} R_2\left[\frac{\partial p}{\partial s}\right](f, x, df) f^{s-m+3} \\
&\quad + (s)_{m-3} R_3[p](f, x, df) f^{s-m+3} \\
&\quad + (\text{order less than } m-3).
\end{aligned}$$

Now assume conditions:

1° $p(f, x, df) = 0$,

$2^\circ \quad R_1[p](f, x, df) \in (\mathfrak{a}+\mathcal{O}f)^{m-1}$.

With the aid of the condition 2°, there is a homogeneous polynomial p' of degree $m-1$ such that

$$p'(f, x, df) = R_1[p](f, x, df).$$

Set

$$P(s) = p(s, x, D) - p'(s, x, D).$$

Then

$$P(s)f^s = (s)_{m-2}\left(R_2[p] - R_1\left[\frac{\partial p}{\partial s}\right] - R_1[p']\right)(f, x, df)$$

$$+ (\text{order less than } m-2).$$

Thus we have the following theorem.

We define ideals $\mathfrak{c}_l$ $l=2, 3, \cdots$ by

$$\mathfrak{c}_l = \{R_1[q](f, x, df) \mid \text{hog. dem}_{(s,\xi)} q(s, x, \xi) = l, q(f, x, df) = 0\}.$$

Note that

$$\mathfrak{c}_l \subset (\mathfrak{a}+\mathcal{O}f)^{l-2}(\textstyle\sum \mathcal{O}f_{ij}).$$

Theorem 2.4. *Let $p(s, x, \xi) \in \mathcal{O}_{T^*X}[s]$ be a homogeneous polynomial of degree m in (s, ξ). Then, to impose the condition 1° and 2° is equivalent to ensure the existence of an operator $P(s)$ which has the following properties.*

$$\sigma^T(P(s)) = p,$$

$$P(s)f^s = (\textit{order less than } m-1 \textit{ in } s).$$

Assume 1° and 2°, and set

$$R(x) = \left(R_2[p] - R_1\left[\frac{\partial p}{\partial s}\right] - R_1[p']\right)(f, x, df),$$

with p' introduced just before this theorem. Then there exists $P'(s)$ such that

$$\sigma^T(P'(s)) = p,$$

$$P'(s)f^s = (\textit{order less than } m-2 \textit{ in } s),$$

if and only if the following condition 3° *holds.*

3° $R(x) \in (\mathfrak{a}+\mathcal{O}f)^{m-2}+\mathfrak{c}_{m-1}$.

Corollary 2.5. 1) *If* $P(s)$, $\mathrm{ord}^T P=m$, *satisfies*

1 $\sigma^T(P(s)) \in \sigma^T(\mathcal{J}(s))$,

2 $P(s)f^s=s^{m-1}Q(x)f^{s-m+1}+$ *(lower order in* s*),*

then $Q(x) \in (\mathfrak{a}+\mathcal{O}f)^{m-1}$.

2) *When* $p_2(s, x, \xi)=as^2+(\sum a_i\xi_i)s+\sum a_{ij}\xi_i\xi_j$ *satisfies* $p_2(f, x, df)=0$, *there exists* $P(s) \in \mathcal{J}(s)$ *such that* $\sigma^T(P)=p_2$ *if and only if*

$$\sum a_{ij}f_{ij} \in \mathfrak{a}+\mathcal{O}f.$$

§ 4. The Numbers $L(f)$ and $l(f)$

Let $f \in \mathcal{O}_X$ such that $V(\mathfrak{a}) \subset V(f)$. We denote by $l(f)$ the degree of integral dependence of f over $\mathfrak{a}$, whose existence is assured by the presence of

Theorem 2.6 (H. Hironaka). *f is integral over* $\mathfrak{a}$.*)

Corollary 2.7. *There exists* $P(s) \in \sum_{j=0}^{l} s^{l-j}P_j(x, D)$ *in* $\mathcal{J}(s)$ *such that* $\mathrm{ord}^T P=l$, $P_0(x, D)=1$.

Proof. The $p(s, x, \xi)$ in Theorem 2.6 belongs to $\mathcal{J}(W[s])$. It follows then from Proposition 2.3 that there exist k and $P(s) \in \mathcal{J}(s)$ satisfying

$$\sigma^T(P(s))=p^k.$$

Obviously, this $P(s)$ is an anounced one. Q.E.D.

We write $L(f)$ for the minimum of $\mathrm{ord}^T P$ where $P(s) \in \mathcal{J}(s)$ which is of the form specified in Corollary 2.7. We have $L(f)=1$ when and only when f is quasi-homogeneous.

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*) That is, there exists $p(s, x, \xi) \in \mathcal{O}_X[s, \xi]$, homogeneous in (s, ξ) and $p(f, x, df)=0$, $p(s, 0, 0)=s^{\deg p}$.

Proposition 2.9 (K. Saito [22]). *When f has an isolated singularity, the condition $L(f)\geq 2$ is equivalent to the condition*

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\in \mathfrak{a}+\mathcal{O}f.$$

§ 5. Functions of Simplex Type

Let $f(x)$ be a local holomorphic function at $0\in \boldsymbol{C}^n$, $f(0)=0$. We fix a local coördinate and expand f into convergent power series. The support of f in this coördinate is defined to be

$$\operatorname{supp}(f)=\{\alpha\in \boldsymbol{N}_0^n \,|\, a_\alpha\neq 0 \text{ in } f=\sum a_\alpha x^\alpha\}.$$

The set of subsets S of supp (f), satisfying $S+\boldsymbol{N}_0^n\supset \operatorname{supp}(f)$ has the minimal element, which is denoted by $\operatorname{inl}(f)$. This set can alternatively be described in the following way. Define the order relation $<$ on $\boldsymbol{N}_0^n$ by

$$\alpha<\alpha' \text{ if and only if } \alpha_i\leq \alpha'_i,\ \forall i.$$

Then $\operatorname{inl}(f)$ is characterized by

1° $\forall\beta\in \operatorname{supp}(f)$, $\exists\alpha\in \operatorname{inl}(f)$ such that $\alpha<\beta$,

and

2° $\forall\alpha, \alpha' (\alpha\neq\alpha')\in \operatorname{inl}(f)$, there is no relation $\alpha<\alpha'$.

Thus we can write $f(x)=\sum_{\alpha\in \operatorname{in}l(f)} a_\alpha(x)x^\alpha$, $a_\alpha(0)\neq 0$, and for any x^α, $x^{\alpha'}$, $\alpha, \alpha'\in \operatorname{inl}(f)$, x^α is not a divisor of $x^{\alpha'}$.

Incidentally, the following proposition holds.

Proposition 2.10. *If* $\operatorname{inl}(f)=\{\alpha^{(1)}, \cdots, \alpha^{(n)}\}$ *forms a set of vertices of* $(n-1)$*-simplex,* f *can be transformed to* $\sum y^{\alpha^{(i)}}$ *by an appropriate coördinate change* $(x)\mapsto(y)$.

Proof. Let A be the matrix $\begin{pmatrix}\alpha^{(1)}\\ \vdots\\ \alpha^{(n)}\end{pmatrix}$. Then it follows from the condition that A is invertible. Set $\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}=A^{-1}\begin{pmatrix}\log a_1\\ \vdots\\ \log a_n\end{pmatrix}$, where a_i's are

$f=\sum a_i(x)x^{\alpha^{(i)}}$, and set $y_i=\exp(b_i)x_i$. Then $f(x(y))=\sum_{i=1}^{n} y^{\alpha^{(i)}}$.

Q.E.D.

In this case, f can be considered to be weighted homogeneous with weight

$$(\gamma_1, \cdots, \gamma_n)=(1, \cdots, 1)\,{}^tA^{-1}.$$

Definition 2.11. *When* $\mathrm{inl}(f)=(\alpha^{(0)}, \alpha^{(1)}, \cdots, \alpha^{(n)})$ *forms a set of vertices of n-simplex, we call that f is of simplex type.*

In the sequel, we fix some coördinate system, and use the phraze, "of simplex type" to mean "of simplex type in that coördinate".

Proposition 2.12. *Let f be of simplex type. Then, by an appropriate change of coördinate* $(x)\to(y)$, *there is found a function* $g(y), g(0)\neq 0$, *such that*

$$f(x(y))=g(y)\sum_{i=0}^{n} y^{\alpha^{(i)}}.$$

Proof. First note that $A'=\begin{pmatrix}\alpha^{(0)} & 1\\ \vdots & \vdots\\ \alpha^{(n)} & 1\end{pmatrix}$ is invertible in this case. In fact, if it were not so, $\{\alpha^{(0)}, \cdots, \alpha^{(n)}\}$ should lie in some hyperplane. We set $\begin{pmatrix}b_1\\ \vdots\\ b_n\\ b\end{pmatrix}=A'^{-1}\begin{pmatrix}\log a_0\\ \vdots\\ \log a_{n-1}\\ \log a_n\end{pmatrix}$, $y_i=\exp(b_i)x_i$ and $g(y)=\exp(b)$. Then $f(x(y))=g(y)\sum_{i=0}^{i} y^{\alpha^{(i)}}$.

Q.E.D.

It is thus found that the b-function of f at 0 is equal to that of $\sum_{i=0}^{n} c_i x^{\alpha^{(i)}}$ for any $c_i\in \boldsymbol{C}^*$.

Theorem 2.13. *Let* $\alpha_0, \cdots, \alpha_n\in \boldsymbol{N}_0^n\backslash(0)$ *form a set of vertices of an n-simplex. Then, by an appropriate change of subscripts, if necessary, one can determine* I_j $j=1, 2, 3$ *uniquely which satisfy the following:*

$$I_1=\{\alpha_0, \cdots, \alpha_{k-1}\}, \quad I_2=\{\alpha_k, \cdots, \alpha_{\kappa-1}\}, \quad I_3=\{\alpha_\kappa, \cdots, \alpha_n\}.$$

There exist natural numbers $l_0, \cdots, l_\kappa$ *such that*

$$l_0\alpha^{(0)}+\cdots+l_{k-1}\alpha^{(k-1)}<l_k\alpha^{(k)}+\cdots+l_{\kappa-1}\alpha^{(\kappa-1)} \tag{1}$$

and

$$l_0+\cdots+l_{k-1}=l_k+\cdots+l_{\kappa-1}. \tag{2}$$

Proof. Since $\alpha^{(i)}$'s form a simplex in N_0^n, there exists a unique relation for $\beta_i\in \boldsymbol{Q}$

$$\beta_0\alpha^{(0)}+\cdots+\beta_n\alpha^{(n)}=0 \tag{3}$$

up to a constant multiple. Superscripts of α are so chosen that the following conditions for k and κ $(0\leq k<\kappa\leq n)$ are satisfied: 1° The coefficients $\beta_0, \cdots, \beta_{\kappa-1}$ are non-zero and $\beta_\kappa=\cdots=\beta_n=0$. 2° $\beta_0, \cdots, \beta_{k-1}$ have the same sign each other and $\beta_k, \cdots, \beta_{\kappa-1}$ have also the same sign which is opposite to that of $\beta_0, \cdots, \beta_{k-1}$, and 3° $|\beta_0+\cdots+\beta_{k-1}|>|\beta_k+\cdots+\beta_{\kappa-1}|$. (This is really an inequality because if this is an equality, $\{\alpha^{(0)}, \cdots, \alpha^{(\kappa-1)}\}$ must lie in some hyperplane.)

Let us tentatively assume $\beta_0, \cdots, \beta_{k-1}<0$. Then equation (3) becomes

$$(-\beta_0)\alpha^{(0)}+\cdots+(-\beta_{k-1})\alpha^{(k-1)}=\beta_k\alpha^{(k)}+\cdots+\beta_{\kappa-1}\alpha^{(\kappa-1)}.$$

We choose rational numbers $\gamma_0, \cdots, \gamma_{k-1}$ so that $0<\gamma_i\leq(-\beta_i)$ and

$$\gamma_0+\cdots+\gamma_{k-1}=\beta_k+\cdots+\beta_{\kappa-1}.$$

We can cancell denominators in this equation by multiplying some natural number N. We set $l_i=N\gamma_i$ for $(0\leq i\leq k-1)$ $l_i=N\beta_i$ for $(k\leq i\leq \kappa-1)$. Then, it is easy to see that condition (2) of 2.13 clearly holds and condition (1) thus follows from the very choice of γ_i's. Q.E.D.

Definition 2.14. *Let f be of simplex type. Then we divide* $\mathrm{inl}(f)$ *into three subsets according to Proposition* 3.13, $\mathrm{inl}(f)=I_1\cup I_2\cup I_3$, *and superscripts of α are also chosen as indicated there.*

We say that "a vector field X is associated with a hyperplane $h(x)=1$", if $X=\sum a^i x_i D_i$ and $h(x)=\sum a^i x_i$. If $\mathrm{inl}(f)$ is a simplex, there exists for each vertex $\alpha^{(i)}$ the unique face of that simplex which does not contain $\alpha^{(i)}$. Let $h_i(x)=1$ be its defining equation, and X_i be a vector field associated with it. When $h_j(x)$ is of the form $h_j(x) =\sum a_j^i x_i$, we put $c_j=\sum_j a_j^i \alpha_i^{(j)}-1$. Then it holds that

$$X_j x^{\alpha(j)}=(1+c_j)\, x^{\alpha(j)},$$

$$X_j x^{\alpha(i)}=x^{\alpha(i)}, \quad i\neq j.$$

Hence we have

$$X_j(\sum x^{\alpha(i)})=(\sum x^{\alpha(i)})+c_j x^{\alpha(j)}.$$

The following theorem says that $\mathcal{J}(s)$ contains a noteworthy element in it.

Theorem 2.15. *There exists an element $P(s)=P(s, x, \vartheta)$ in $\mathcal{J}(s)$ which has the form*

$$P(s)=P_k(s)\cdots P_{\kappa-1}(s)+Q(s, x, \vartheta),$$

with

$$P_j(s)=\prod_{\nu=0}^{l_j-1}(s-X_j+\nu c_j),$$

and

$$Q(s, 0, \vartheta)=0, \quad \mathrm{ord}^T Q\leq \mathrm{ord}^T P.$$

Thus

$$\mathrm{ord}^T(P)=l_0+\cdots+l_{\kappa-1}=l_k+\cdots+l_{\kappa-1}=l,$$

and

$$L(f)\leq l.$$

We used the notation

$$\vartheta=(\vartheta_1, \cdots, \vartheta_n)=(x_1D_1, \cdots, x_nD_n).$$

To prove 2.15, we here prepare two lemmata.

Lemma 2.16. *Let f be of simplex type. Then, there exist $L_j(s, x, \vartheta)$ of total order one which has the form*

$$L_j(s, x, \vartheta) = (s - X_j) + l_j(s, x, \vartheta),$$

with

$$l_j(s, 0, \vartheta) = 0,$$

and satisfies

$$L_j(s, x, \vartheta) f^s = (-a_j^0 c_j) x^{a^{(j)}} s f^{s-1}.$$

Here, we write $a_j^0 = a_j(0)$ $(\neq 0)$, when $f = \sum_{j=0}^{n} a_j(x) x^{a^{(j)}}$.

Proof. It follows from the definition of X_j that

$$(f - X_j f) = -a_j^0 c_j x^{a^{(j)}} + \sum b_k^j x^{a^{(k)}}, \ b_k^j \in \mathfrak{m}.$$

We regard these as equations in $x^{a^{(k)}}$ and get the solution

$$-a_j^0 c_j x^{a^{(j)}} = (f - X_j f) + \sum c_j^k (f - X_k f), \ c_j^k \in \mathfrak{m}.$$

Then, $L_j(s, x, \vartheta)$ is given by

$$L_j(s, x, \vartheta) = (s - X_j) + \sum c_j^k (s - X_k).$$

Q.E.D.

Lemma 2.17. *Let P and Q satisfy the equations*

$$P(s, x, \vartheta) f^s = p(s) x^\alpha \varphi(x) f^{s-l},$$

$$Q(s, x, \vartheta) f^s = q(s) x^\beta \psi(x) f^{s-m},$$

for some analytic functions φ and ψ, and polynomials in s, p and q. Then,

$$Q(s-l, x, \vartheta - \alpha) P(s, x, \vartheta) f^s$$

$$= p(s) q(s-l) x^{\alpha+\beta} \varphi\psi f^{s-l-m} + p(s) x^\alpha \sum \frac{D^\nu \varphi}{\nu!} (Q^{(\nu)}(s-l, x, \vartheta) f^{s-l}).$$

Here we have used the notation

$$R^{(\nu)} = \left[\left(\frac{\partial}{\partial \xi} \right)^\nu \left(\sum a_\alpha \xi^\alpha \right) \right]_{\xi \mapsto D}, \ \textit{when} \ R(s, x, D) = \sum a_\alpha D^\alpha.$$

The proof is straightforward.

Proof of Theorem 2.15. Let us set

$$Q_j(s)=L_j(s-(l_j-1), x, \vartheta-(l_j-1)\alpha^{(j)})\cdots$$
$$\cdots L_j(s-1, x, \vartheta-\alpha^{(j)})L_j(s, x, \vartheta).$$

It follows from Lemmata 2.16 and 2.17 that

$$Q_j(s)=P_j(s)+q_j(s, x, \vartheta)$$

with $q_j(s, 0, \vartheta)=0$, and

$$Q_j(s)f^s=(-a_j^0c_j)^{l_j}x^{l_j\alpha^{(j)}}(s)_{l_j}f^{s-l_j}.$$

Since $X_k x^{\alpha^{(j)}}=x^{\alpha^{(j)}}$ for $k\neq j$, we have

$$Q_0(s)Q_1(s)\cdots Q_{k-1}(s)f^s=\prod_0^{k-1}(-\alpha_jc_j)^{l_j}x^{\Sigma l_j\alpha^{(j)}}(s)_l f^{s-l},$$

$$Q_k(s)\cdots Q_{\kappa-1}(s)f^s=\prod_k^{\kappa-1}(-\alpha_jc_j)^{l_j}x^{\Sigma l_j\alpha^{(j)}}(s)_l f^{s-l}.$$

On the other hand, $x^{\sum_0^{k-1}l_j\alpha^{(j)}}$ divides $x^{\sum_k^{\kappa-1}l_j\alpha^{(j)}}$ by (2) of 2.11. Hence, if we put

$$P(s)=Q_k(s)\cdots Q_{\kappa-1}(s)-cx^{\sum_k^{\kappa-1}l_j\alpha^{(j)}-\sum_0^{k-1}l_j\alpha^{(j)}}Q_0(s)\cdots Q_{k-1}(s),$$

where

$$c=\prod_k^{k-1}(-\alpha_jc_j)^{l_j}/\prod_0^{k-1}(-\alpha_jc_j)^{l_j},$$

$P(s)$ turns out to belong to $\mathcal{J}(s)$ and has the required properties.

Q.E.D.

An important example of function of simplex type is the following.

Example 2.18.

$$f(x)=\sum_{i=1}^n x_i^{n_i}+x_1^{m_1}\cdots x_n^{m_n},$$

for $1\leq m_i<n_i$. Let us put $c=\sum\frac{m_i}{n_i}-1$. When $c=0$, f is weighted homogeneous. When $c\neq0$, there are two cases:

i) For $c<0$, $I_1=\{\alpha^{(0)}\}$, where $\alpha^{(0)}=(m_1, \cdots, m_n)$, $I_2=\{\alpha^{(1)}, \cdots, \alpha^{(n)}\}$, where $\alpha^{(i)}=(0, \cdots, n_i, \cdots, 0)$, $I_3=\emptyset$. In this case $X_0=\sum\frac{1}{n_i}\vartheta_i$, $X_i=X_0$

$-\dfrac{c}{m_i}\vartheta_i$ and $c_0=c$, $c_i=-(n_i/m_i)c$, $1\leq i\leq n$.

ii) For $c>0$, the convention of superscripts in 2.13 tells $I_1=\{\alpha^{(0)},\cdots,\alpha^{(n-1)}\}$ and $I_2=\{\alpha^{(n)}\}$. Tentatively, we change the superscripts in such a way that $I_1=\{\alpha^{(1)},\cdots,\alpha^{(n)}\}$, $I_2=\{\alpha^{(0)}\}$. Then all operators become of the same form as in i).

In both cases, we can take $l=\min_i(N_i)$ where $N_i=\dfrac{1}{n_i}\prod_{j=1}^{n}n_j$. The proof is omitted.

We finally note that in Example 2.18, the $P(s)$ in Theorem 2.14 is given by

$$\text{i)}\quad P(s)=\prod_{\nu=0}^{l-1}(s-X_0+\nu c)+Q(s),\qquad \text{if}\quad c>0$$

$$\text{ii)}\quad P(s)=\prod_{j=1}^{n}P_j(s)+Q(s),$$

$$P_j(s)=\prod_{\nu=0}^{l_j-1}(s-X_j+\nu c_j),\qquad \text{if}\quad c<0.$$

In particular we can take $l_j=1$, $l=n$ if $n_i\geq nm_i$ $(\forall i)$.

§ 6. Generators of $\mathcal{J}(s)$

In this section, we give a way of explicitly determining $\mathcal{J}(s)$. We always assume that f has an *isolated singularity*.

The ideal $\mathcal{J}_0=\mathcal{J}(s)\cap\mathcal{D}$ previously defined is determined by

Theorem 2.19.

$$\mathcal{J}_0=\sum_{i<j}\mathcal{D}X_{ij},\ X_{ij}=f_iD_j-f_jD_i.$$

To prove this, we begin by stating an algebraic lemma (cf. [12]).

Lemma 2.20. i) *The following conditions* 1 *through* 3 *on* g_i $(i=1,\cdots,n)$ *are equivalent.*

1. g_i *is not a zerodivisor of* $\mathcal{O}/\sum_{j=0}^{i-1}g_j\mathcal{O}$, $i=1,\cdots,n$, *where we understand* $g_0=0$.

2. $\bigoplus_{\nu\geq 0}\mathfrak{a}^{\nu}/\mathfrak{a}^{\nu+1}$ *is isomorphic to* $\mathcal{O}/\mathfrak{a}[\xi_1, \cdots, \xi_n]$ *under the homomorphism* $\xi_i \mapsto g_i$, *where* $\mathfrak{a}=\sum_{i=1}^{n} g_i\mathcal{O}$.

3. *Each homogeneous component of the kernel of* $\bigoplus_{\nu\geq 0}\mathfrak{a}^{\nu}\leftarrow\mathcal{O}[\xi_1, \cdots, \xi_n]$ *is generated by* $g_i\xi_j - g_j\xi_i$ *as an* $\mathcal{O}[\xi]$*-module.* ii) *If* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$ *of* $\mathcal{O}$ *is generated by n-elements* $g_1, \cdots, g_n$, $(g_1, \cdots, g_n)$ *satisfies the conditions in* i).

This lemma is the one known in the theory of local rings. When $(g_1, \cdots, g_n)$ satisfies one of the conditions in Lemma 3. 2 i), it is called an *$\mathcal{O}$-sequence.*

We remark that when f has an isolated singularity, $\mathfrak{a}=\sum \mathcal{O}f_i \supset \mathfrak{m}^N$ holds for large N (actually, we can take $N=\mu=\dim \mathcal{O}/\mathfrak{a}$). Thus $(f_1, \cdots, f_n)$ makes an $\mathcal{O}$-sequence.

Next, choose an element $P(x, D)$ from $\mathcal{J}_0$ with $\operatorname{ord} P=m$. Then the equation

$$0=P(x, D)f^s=\sigma_m(P)(x, df)(s)_m f^{s-m}+\text{(lower order in } s)$$

readily yields $\sigma_m(P)(x, df)=0$. Since $(f_1, \cdots, f_n)$ is an $\mathcal{O}$-sequence,

$$\sigma_m(P)(x, \xi)=\sum q_{ij}(x, \xi)(f_i\xi_j - f_j\xi_i)$$

by 3 of Lemma 2. 20. Thus, choosing $Q_{ij}\in\mathcal{D}$ such that $\sigma^T(Q_{ij})=q_{ij}$, we have

$$\operatorname{ord}(P-\sum Q_{ij}(x, D)X_{ij})<m.$$

Hence by induction on $\operatorname{ord}^T(P)$, we complete the proof of Theorem 2. 19.

Q.E.D.

When f is quasi-homogeneous, the relation $s-X_0\in\mathcal{J}(s)$ holds with a vector field X_0 such that $X_0 f=f$. Then for any $P(s)=\sum s^j P_j(x, D)$ $\in\mathcal{D}[s]$, $P(s)$ and $P_j(x, D)X_0^j$ are congruent modulo $\mathcal{J}(s)$. Hence

(4) $$\mathcal{N}\simeq\mathcal{D}/\mathcal{J}_0,$$

(5) $$\mathcal{M}\simeq\mathcal{D}/(\mathcal{J}_0+\mathcal{D}f).$$

Next, since $\mathcal{J}_0\subset\mathcal{D}\mathfrak{a}$ and $f\in\mathfrak{a}$, we have

(6) $$\tilde{\mathcal{M}}\simeq\mathcal{D}/\mathcal{D}\mathfrak{a}$$

by use of Proposition 2. 2 i).

When f is not quasi-homogeneous, there are $a_\nu(x, \xi)=\sum a_{\nu,i}(x)\xi_i$ such that

$$a_\nu(x)f+a_\nu(x, df)=0,$$

where $(a_1(x), \cdots, a_r(x))$ are the generators of $\mathfrak{a}:f$. Set

$$A_\nu(s, x, D)=a_\nu(x)s+a'_\nu(x, D),$$

where $a'_\nu(x, D)=\sum a_{\nu,i}(x)D_i$. Then we have the following:

Theorem 2. 21.

$$\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})=\sum_{\nu=1}^{r}\mathcal{D}A_\nu(s, x, D)+\mathcal{J}_0.$$

Proof. The proof will be carried out by induction on $l=\operatorname{ord}P(s)$ where $P(s)\in\mathcal{J}(s)$. The case $l=0$ is trivial. There are three cases when $P(s)=Qs+R$.

1. If $\sigma_l^T(P)=\sigma_{l-1}(Q)s$, then $Pf^s=0$ yields $\sigma_{l-1}(Q)(x, df)f=0$. Hence

$$\sigma_{l-1}(Q)(x, \xi)\in\sum\mathcal{O}[\xi](f_i\xi_j-f_j\xi_i),$$

and we can lower $\operatorname{ord}^T(P)$ by subtracting from P a suitable element in $\mathcal{J}_0\cdot s$.

2. If $\sigma_l^T(P)=\sigma_l(R)$, then, $\sigma_l(R)(x, df)=0$. The proof in this case reduces to that in 1, if we replace $\mathcal{J}_0\cdot s$ in 1 by $\mathcal{J}_0$.

3. If $\sigma_l^T(P)=\sigma_{l-1}(Q)s+\sigma_l(R)$, then

$$\sigma_{l-1}(Q)(x, df)f+\sigma_l(R)(x, df)=0. \tag{7}$$

Denote $\sigma_{l-1}(Q)(x, \xi)=\sum_{|\alpha|=l-1}a_{(\alpha)}\xi^\alpha$. Then the statement 2 in 2. 20 and (7) yield $a_{(\alpha)}f\in\mathfrak{a}$. Thus we can find $b_1^{(\alpha)}, \cdots, b_r^{(\alpha)}$ such that

$$a_{(\alpha)}=\sum_\nu b_\nu^{(\alpha)}(x)a_\nu(x).$$

Hence,

$$P'(s)=P(s)-\sum_\nu\left(\sum_\alpha b_\nu^{(\alpha)}(x)D^\alpha\right)A_\nu(s, x, D)$$

is either of total order less than l or reduces to the $P(s)$ in case 2.

Obviously, $\mathcal{J}_0\cdot s\subset\sum\mathcal{D}A_\nu+\mathcal{J}_0$ since $\mathfrak{a}:f\supset\mathfrak{a}$. Thus we have completed the proof of Theorem. Q.E.D.

We now proceed to the determination of the structure of $\mathcal{J}(s)$ when $L(s)=2$. Since $s^2+As+B\in\mathcal{J}(s)$ in this case, every element in $\mathcal{D}[s]$ is congruent to an element in $\mathcal{D}s+\mathcal{D}$ modulo $\mathcal{J}(s)$. Therefore we obtain the following:

Corollary 2.22. *When $L(f)=2$, $\mathcal{J}(s)$ is generated by $\mathcal{J}_0$ together with $A_\nu(s, x, D)$ and s^2+sA+B.*

Modules $\mathcal{N}$, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are generated by two elements $\bar{l}$ and $\bar{s}$, where the bar indicates the residue class of the element without bar. Their structure is characterized by the following theorem.

Theorem 2.23. *Modules $\mathcal{N}$, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have following presentations.*

i) *When $L(f)=1$,*

$$0\leftarrow\mathcal{N}\leftarrow\mathcal{D}\xleftarrow{(X_{ij})}\mathcal{D}^{\binom{n}{2}}, \tag{8}$$

$$0\leftarrow\mathcal{M}\leftarrow\mathcal{D}\xleftarrow{\begin{pmatrix}X_{ij}\\ f\end{pmatrix}}\mathcal{D}^{\binom{n}{2}+1}, \tag{9}$$

$$0\leftarrow\widetilde{\mathcal{M}}\leftarrow\mathcal{D}\xrightarrow{(f_i)}\mathcal{D}^{n}. \tag{10}$$

ii) *When $L(f)=2$,*

$$0\leftarrow\mathcal{N}\xleftarrow{\begin{pmatrix}1\\ s\end{pmatrix}}\mathcal{D}^2\xleftarrow{\begin{pmatrix}X_{ij} & 0\\ a'_\nu & a_\nu\end{pmatrix}}\mathcal{D}^{\binom{n}{2}+r}, \tag{11}$$

$$0\leftarrow\mathcal{M}\xleftarrow{\begin{pmatrix}1\\ s\end{pmatrix}}\mathcal{D}^2\xleftarrow{\begin{pmatrix}X_{ij} & 0\\ a'_\nu & a_\nu\\ f & 0\\ 0 & f\end{pmatrix}}\mathcal{D}^{\binom{n}{2}+r+2}, \tag{12}$$

$$0\leftarrow\widetilde{\mathcal{M}}\xleftarrow{\begin{pmatrix}1\\ s\end{pmatrix}}\mathcal{D}^2\xleftarrow{\begin{pmatrix}f_i & 0\\ f & 0\\ a'_\nu & a_\nu\end{pmatrix}}\mathcal{D}^{n+r+1}. \tag{13}$$

Remark: We can easily show that the sequence

$$0 \leftarrow \widetilde{\mathcal{M}} \leftarrow \mathcal{D}^2 \xleftarrow{\begin{pmatrix} f_i & 0 \\ f & 0 \\ 0 & f_i \\ 0 & f \\ a'_\nu & a_\nu \end{pmatrix}} \mathcal{D}^{2n+r+2}$$

is exact. It is the absence of f_i and f in the right column of the map in (13) that plays an important role in the following sections.

Proof of Theorem 2.23. We first note that sequences (8), (9) and (10) are direct consequences of (4), (5) and (7), Sequence (11) is derived from Theorem 2.20. Let us try to prove (13). Sequence (12) can be proved in a similar way.

First, suppose $P\bar{1}+Q\bar{s}=0$. Then there are $R_i(s)$ and $S(s)$ such that

$$(P+Qs)f^s=\sum R_i(s)f_if^s+S(s)f^{s+1},$$

where the first term of the right-hand side can further be written

$$\begin{aligned} R_i(s)f_if^s &= (R'_i(s)(s+1)+R_i(-1))f_if^s \\ &= R'_i(s)D_if^{s+1}+R_i(-1)f_if^s. \end{aligned}$$

Second, set $S'(s)=S(s)+\sum R'_i(s)D_i$. Then, by making use of

$$((s+1)^2+(s+1)A+B)f^{s+1}=0,$$

we find S_1 and $S_2 \in \mathcal{D}$ which satisfy

$$S'(s)f^{s+1}=(S_2+S_1s)f^{s+1}.$$

Hence we have

$$(P+Qs)f^s=\sum R_i(-1)f_if^s+(S_2+S_1s)f^{s+1}.$$

Next set $\sigma^T(s^2+sA+B)=s^2+(\sum a_i\xi_i)s+(\sum a_{ij}\xi_i\xi_j)$. Then, $\sum a_{ij}f_{ij} =\sum b_if_i+bf$ for some b_i and b by Corollary 2.5. Since $f^2+(\sum a_if_i)f +\sum a_{ij}f_if_j=0$, we have

$$fs+\sum a_ifD_i+\sum a_{ij}f_iD_j \in \mathcal{J}(s).$$

It follows that

$$sf^{s+1}=[\sum_i (a_i+b_i-\sum_j a_{ij}D_j)f_i-(\sum a_iD_i-b)f]f^s.$$

Consequently, there are R_i and $S\in\mathcal{D}$ such that

$$(P+Qs)f^s=(\sum R_i f_i)f^s+Sf^{s+1}.$$

This formula and Theorem 2. 20 implies (13) since $\mathcal{J}_0\subset\mathcal{D}\mathfrak{a}$. Q.E.D.

When $\dim X=2$, the statement in Theorem 2. 21 can be made simpler with the aid of Corollary 1. 11. First note that $f(x_1, x_2)$ is locally reduced since we assume $\mathfrak{a}\supset\mathfrak{m}^N$.

When $L(f)=1$, $\mathcal{J}(s)$ is generated by $s-X_0$ and X_{12}, (or $\mathcal{G}$ is generated by X_0 and X_{12}).

When $L(f)\geq 2$, $\mathfrak{a}:f$ is generated by two elements anounced in Theorem 1. 10 since so is $\mathcal{G}$. Let $\mathfrak{a}:f=\mathcal{O}a_1+\mathcal{O}a_2$ and $A_\nu(s, x, D)=a_\nu(x)s+(a_{\nu 1}(x)D_1+a_{\nu 2}(x)D_2)$. Then X_{12} must be represented by A_1 and A_2. Since (a_1, a_2) is also of an $\mathcal{O}$-sequence, the relations between a_1 and a_2 are generated by the following trivial relation:

$$(-a_2)a_1+a_1a_2=0.$$

Since $\mathcal{J}_0=\mathcal{D}X_{12}$, we obtain

$$a_1A_2-a_2A_1=\varphi X_{12}$$

where $\varphi(0)\neq 0$. Now we restate Theorem 2. 21 in the form:

Theorem 2. 24. *Suppose* $\dim X=2$. *Then,*

$$\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})=\sum_{\nu=1}^{2}\mathcal{D}A_\nu(s, x, D),$$

where $A_\nu(s, x, D)=a_\nu(x)s+(a_{\nu 1}(x)D_1+a_{\nu 2}(x)D_2)$ *with* $\mathfrak{a}:f=\mathcal{O}a_1+\mathcal{O}a_2$. *(When f is quasi-homogeneous, we understand $a_1=1$ and $a_2=0$.)*

Conversely, if there exist in $\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})$, A_1 *and* A_2 *with above form such that* $\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\varphi f$, $\varphi(0)\neq 0$, *then they generate* $\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})$.

§ 7. "Fundamental Conjecture" —Counter Examples—

Let $p(s, x, \xi)$ belong to $\mathcal{O}_X[s, \xi]$ and be homogeneous in (s, ξ) in such a way that

$$p(f, x, df)=0.$$

It follows from Proposition 2.3 that there is an integer l such that $p^l \in \sigma^T(\mathcal{J}(s))$.

In connexion with this fact, M. Sato and M. Kashiwara once conjectured in [12] that for such p, at least one of the following three statements should hold.

RS 1 $p \in \sigma^T(\mathcal{J}(s))$.

RS 2 There exists $m_0 \in \boldsymbol{N}$ for which the following holds. $p \in \sigma^T(\mathcal{J}(s))$, if $\deg p \geq m_0$.

RS 3 There exists $q(x, \xi)$ which is homogeneous in ξ such that $q(0, \xi) \neq 0$ and that $qp \in \sigma^T(\mathcal{J}(s))$.

However, they fail to hold in general, as is shown later.

It should be remarked that RS 1 is true for f being of isolated singularity with $L(f) \leq 2$. In fact, if $p(f, x, df)=0$, after dividing $p(s, x, \xi)$ by $s-\sum a_i x_i \xi_i$ or $s^2+\sigma_1(A)s+\sigma_2(B)$, one can use the argument of the proof of 2.18 or 2.20 to find the required operator $P(s)$.

Proposition 2.25. 1. *Let f satisfy RS1. Then, there exists an operator $P(s)$ such that*

$$P(s)f^{s+1}=b(s)f^{s*)} \text{ with } \operatorname{ord}^T P=\deg b.$$

2. *Let f satisfy RS2. Then, there are $P_\nu(s)$ and ν such that*

$$P_\nu(s+\nu)f^{s+\nu}=b_\nu(s)f^s \text{ with } \operatorname{ord}^T P_\nu=\deg b_\nu.$$

Corollary 2.26. *Assume that f has an isolated singularity at* 0 *and that $L(f) \leq 2$. Then one can find a "b-operator" $P(s)$ such that* $\operatorname{ord}^T(P)=\deg b$.

Proof of Proposition 2.25. Assume $\operatorname{ord}^T P > d=\deg b$. Then,

$$\sigma^T(P(s)f-b(s))=\sigma^T(P(s))f.$$

Thus $\sigma^T(P(s))(f, x, df)=0$. By RS 1, there exists an operator $P'(s)$

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*) Such an operator is called a "b-opepator" in the sequel.

in $\mathcal{J}(s)$ such that $\sigma^T(P'(s))=\sigma^T(P(s))$. Set $P''(s)=P(s)-P'(s+1)$. Then, $\mathrm{ord}^T P''<\mathrm{ord}^T P$ and $P''(s)f^{s+1}=b(s)f^s$. Proceeding in this way, we finally reach the stage that $\mathrm{ord}^T(P'''(s))=\deg b$.

The Proof of 2 is much the same. Q.E.D.

Proof of Corollary 2. 26. This can be directly proved by Proposition 2. 25. 1 and the argument preceding it.

We can also prove this Corollary by the aid of concrete process of determining operator $P(s)$ and by the simple fact that "If an ideal $\mathcal{J}$ in $\mathcal{D}$ is generated by elements $\{a_1, \cdots, a_k\}$ in $\mathcal{O}$, $(a_1, \cdots, a_k)$ forms an involutory basis of $\mathcal{J}$". See p. 161 and p. 163.

The following gives us a counter example against RS 1 and RS 2.

Example 2. 27. $f(x)=\dfrac{1}{n}(x^n+y^n+z^n)-\dfrac{1}{m}(xyz)^m$

$$n\geq 5m-2,\ m\geq 2.$$

This is of simplex type. We set $c=\dfrac{3m}{n}-1$ (<0),

$$X_0=\frac{1}{n}(xD_x+yD_y+zD_z),\ X_1=X_0-\frac{c}{m}D_x \text{ etc., and } \varphi=1-(xyz)^{n-3m}.$$

Let (i, j, k) be a permutation of $(1, 2, 3)$ and define the vector field X_{123}, for example, by

$$X_{123}=\frac{-1}{\varphi}(y^{m-1}z^{2m-1}D_x+x^{n-m-1}z^{m-1}D_y+x^{n-2m-1}y^{n-m-1}D_z).$$

Other operators are defined according to the permutation of variables. X_{ijk} satisfies

$$X_{ijk}f=x^{im-1}y^{jm-1}z^{km-1}.$$

We can verify the following:

1 $\mathfrak{a}\ni x^{n-1}-x^{m-1}(yz)^m=f_x$, $x^{m-1}y^{2m-1}z^{3m-1}$, $x^{2m-1}y^{n-m-1}$, $x^{n+2m-1}y^{m-1}$, x^{2n+m-1}, etc.

2 $\mathfrak{a}:f=(x^{m-1}y^{2m-1}, x^{n-m}-(yz)^m$, etc.)

3 $\dim \mathcal{O}/\mathfrak{a}=3n^2(m-1)+3n-1$,

$\dim \mathcal{O}/(\mathfrak{a}:f)=3n(m-1)^2-(m-1)(m^2-8m+1)+1,$

4 $\mathfrak{a}+\mathcal{O}f=(f_x,f_y,f_z,(xyz)^m),$

5 $\{(\mathfrak{a}^2+\mathfrak{a}f):f^2\}/(\mathfrak{a}:f)=(x^{2m-2},(xy)^{m-1},\cdots).$

Statements 1 through 4 hold even when $n\geq 3m+1$.

The structure of $\mathcal{J}(s)$ is as follows.

$$X_{ij},\ x^{m-1}y^{2m-1}(s-X_3)-\frac{c}{m}(x^m y^{2m-1}D_x+zX_{231}),$$

$$(x^{n-m}-(yz)^m)(s-X_1)-\frac{c}{m}x^{n-m}(xD_x)\ \text{etc.}$$

are generators of $\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})$. $L(f)\leq 3$ is guaranteed by the existence in $\mathcal{J}(s)$ of the operator

$$(s-X_1)(s-X_2)(s-X_3)-(xyz)^{n-3m}(s-X_0+2c)(s-X_0+c)(s-X_0).$$

The operator in $\mathcal{J}(s)$ with the leading term $(xy)^{m-1}s^2$ is

$$(xy)^{m-1}(s-X_3)(s-Y_3+c)+\left(\frac{c}{m}\right)^2 z^{n-m}D_xD_y+\left(\frac{c}{m}\right)^2 T$$

where

$$Y_3=X_0-\frac{c}{m}(xD_x+yD_y),$$

$$T=(xy)^{n-4m+1}x^{n-5m+2}X_{123}\cdot X_{321}+\frac{1}{\varphi}\{(xy)^{n-4m}z^{n-4m+1}$$
$$\times((3m-1)(yz)^m+(3m-2)x^{n-m})X_{321}$$
$$+(m-1)x^{2n-3m-1}y^{n-3m}z^{n-4m}D_y\}.$$

The situation for $x^{2m-2}\in(\mathfrak{a}^2+\mathfrak{a}f):f^2$ is delicate. We have

$$(s-X_2)(s-X_3)f^s=\left(\frac{c}{m}\right)^2(yz)^n s(s-1)f^{s-2},$$

and

$$Qf^s=x^{m-2}(yz)^{5m-2}s(s-1)f^{s-2}-\frac{m-1}{\varphi}x^{m-2}(yz)^{4m-2}sf^{s-1},$$

$$Q=X_{123}X_{132}+\frac{1}{\varphi}x^{n-3m-1}y^{2m-1}z^{m-1}$$
$$\times\{(3m-1)X_{312}+(2m-1)y^{n-3m}X_{231}\}.$$

Then setting

$$P(s)=x^{2m-2}(s-X_2)(s-X_3)-\left(\frac{c}{m}\right)^2(yz)^{n-5m+2}Q,$$

we obtain

$$P(s)f^s=\left(\frac{c}{m}\right)^2\frac{m-1}{\varphi}x^{m-2}(yz)^{n-m}sf^{s-1}. \tag{14}$$

On the other hand

$$x^{m-2}(yz)^{n-m}\notin(x^{n-1}-x^{m-1}(yz)^m,\cdots,(xyz)^m)=\mathfrak{a}+\mathcal{O}f.$$

For, if $x^{m-2}(yz)^{n-m}\in\mathfrak{a}+\mathcal{O}f$, we consider both sides mod x^{m-1} and have $(yz)^{n-m}\in(y^{n-1},z^{n-1})_{\mathcal{O}_{y,z}}$, which is impossible since $m\geq 2$. Therefore, (14) gives a counter example against RS 1 by Corollary 2.5, 1). We can find, however, following three operators.

$$xP(s)-\left(\frac{c}{m}\right)^2\frac{m-1}{\varphi}y^{n-3m+1}z^{n-4m+1}X_{123},$$

$$y^{m-1}\left\{x^{2m-2}\left(s-X_2-\frac{(m-1)c}{m\varphi}\right)(s-X_3)-\left(\frac{c}{m}\right)^2(yz)^{n-5m+2}Q\right\}$$
$$-\left(\frac{c}{m}\right)^2\frac{m-1}{\varphi}x^{m-2}z^{n-m}D_y,$$

and

$$z^{m-1}\left\{x^{2m-2}\left(s-X_3-\frac{(m-1)c}{m\varphi}\right)(s-X_2)-\left(\frac{c}{m}\right)^2(yz)^{n-5m+2}Q\right\}$$
$$-\left(\frac{c}{m}\right)^2\frac{m-1}{\varphi}x^{m-2}y^{n-m}D_z.$$

Their total symbols are $x\sigma^T(P)$, $y^{m-1}\sigma^T(P)$, and $z^{m-1}\sigma^T(P)$. Moreover, we can choose an element in $\mathcal{J}(s)$, whose total symbol is $\xi^{m-1}\sigma^T(P)$. Set

$$S_{m-2}=\sum_{j=1}^{m-1}(m-1)_jD_x^{m-1-j}\frac{1}{\varphi}x^{m-1-j},$$

$$R(s)=S_{m-2}(xyz)^{n-4m}$$
$$\times\left\{(s-X_0+c)(s-X_0)+\left(\frac{c}{m}\right)^2\frac{n-3m}{\varphi}y^{n-2m+1}z^{3m+1}X_{123}\right\}$$

or when $n \geq 5m-1$, set

$$R(s)=\left(\frac{c}{m}\right)^2 S_{m-2}x^{n-4m}(yz)^{n-5m+1}\left[\frac{n-3m}{\varphi}y^{n-m}z^{n-2m}X_{123}\right.$$

$$+\left\{xyzX_{123}\cdot X_{321}+\frac{x^{2m}}{\varphi}((3m-1)y^m X_{123}+x^{n-4m}\right.$$

$$\left.\left.\times((2m-1)(xz)^m+(m-1)y^{n-m})X_{321}\right\}\right].$$

Then,

$$\left\{(D_x^{m-1}x^{2m-2}+S_{m-2}x^{m-1})(s-X_2)(s-X_3)\right.$$

$$\left.-D_x^{m-1}\left(\frac{c}{m}\right)^2(yz)^{n-5m+2}Q\right\}-R(s)$$

is an element in $\mathcal{J}(s)$. Thus (14) cannot be a counter example against RS 3.

But there are no element in $\mathcal{J}(s)$ with its total symbol $\eta^{l_1}\zeta^{l_2}\sigma^T(P)$. In fact, if $D_y^{l_1}D_z^{l_2}P(s)+R'(s)\in\mathcal{J}(s)$, $\mathrm{ord}^T R'<l_1+l_2+2$, then

$$D_y^{l_1}D_z^{l_2}\varphi P(s)=\left(\frac{c}{m}\right)^2(m-1)f_y^{l_1}f_z^{l_2}x^{m-2}(yz)^{n-m}(s)_{l_1+l_2+1}f^{s-1-l_1-l_2}$$

$$+\text{(lower order in } s)$$

implies

$$x^{m-2}(yz)^{n-m}f_y^{l_1}f_z^{l_2}\in(\mathfrak{a}+\mathcal{O}f)^{l_1+l_2+1},$$

by Corollary 2. 5, 1). If this formula were true, there should be a homogeneous polynomial $F(t_1, \cdots, t_4)$ of degree l_1+l_2+1 with coefficients in $\mathcal{O}$, such that

$$x^{m-2}(yz)^{n-m}f_y^{l_1}f_z^{l_2}=F(f_x, f_y, f_z, (zyz)^m).$$

Considering both sides mod x^{m-1}, we conclude that

$$y^{(n-1)(l_1+1)-m+1}z^{(n-1)(l_2+1)-m+1}\in(\mathcal{O}_{yz}y^{n-1}+\mathcal{O}_{yz}z^{n-1})^{l_1+l_2+1}.$$

But this never occurs. Thus (14) also serves a counter example against RS 2. Set

$$P'(s)=\frac{1}{\varphi}(x^{2m-2}(s'-X_2)(s'-X_3)-\left(\frac{c}{m}\right)^2(yz)^{n-5m+2}X_{123}X_{132})\varphi$$

where

$$s' = s + c - 2 - \frac{3}{n}.$$

Then the following operator belongs to $\mathcal{J}(s)$.

$$P'(s)P(s) - \frac{m-1}{\varphi}\left(\frac{c}{m}\right)^2 (yz)^{n-5m+2}\Big[X_{123}\frac{1}{\varphi}\left(\frac{c}{m}\right)x^{m-3}y^{m-1}z^{n-2m-1}$$

$$\times\{(m-2)z^{2m}(s-X_2) + (xy)^{n-3m}(z^{n-m}+x^m y^m)(s-X_0)\}$$

$$+\frac{1}{\varphi}x^{2m-4}y^{2m-2}z^{m-2}\{(2m-3)y^m z^{2m} + (n+2m-1)x^{n-m}z^m$$

$$+(n+m-1)x^{n-2m}y^{n-m}\}(s-X_2)(s-X_3)\Big].$$

Thus $(\sigma^T(P))^2 \in \sigma^T(\mathcal{J}(s))$.

The following gives a counter example against RS 1, 2 and 3.

Example 2.28. $f = x^5(x+ty) - y^5$, t: a parameter. This is a μ-constant deformation of $x^6 - y^5$ $(\mu = 20)$. Set

$$X_0 = \frac{1}{6}xD_x + \frac{1}{5}yD_y,\quad X_1 = \frac{4}{25}xD_x + \frac{1}{5}yD_y.$$

For $t \neq 0$, $\mathcal{J}(s) \cap (\mathcal{D}s + \mathcal{D})$ is generated by

(a) $$(6x+5ty)\left(s - X_0 + \frac{t}{180}yD_x\right) - \frac{t^2y^2}{36}D_x,$$

and

(a′) $$y^2\left(s - X_0 + \frac{t}{180}yD_x\right) - \frac{t^2}{36}xX_{34},$$

where

$$X_{34} = \frac{1}{5}\left(1 - \frac{5^3t^5}{6^4}x\right)^{-1}\left(\frac{(6x)^4 - (5ty)^4}{6x+5ty}\cdot\frac{t}{6^4}D_x - x^3D_y\right).$$

We set

$$P(s) = \left(s - X_0 + \frac{t}{180}yD_x + \frac{1}{15}\right)\left(s - X_0 + \frac{t}{180}yD_x\right)$$

$$-\frac{5^3t^4}{(36)^2}x\left(s - X_1 - \frac{1}{15}\right)\left(t(s-X_1) - \frac{1}{25}xD_y\right).$$

Then

(15) $$P(s)f^s = ks(xy)^3 f^{s-1},\ k=\frac{4t^3}{5\cdot(36)^2}.$$

We can verify by direct calculation that

(16) $$(xy)^3 \notin \mathfrak{a}+\mathcal{O}f,$$

Formula (16) shows that (15) displays a counter example against RS 1. We can also show that this is a counter example against RS 2 by the same way as in 3. 24, considering $D_y^l P(s)$ and mod x^4.

Lastly, we show that this is a counter example against RS 3. Our argument relies on the following general proposition.

Proposition 2. 29. 1. *Let f have an isolated singularity at 0 with $l(f)=2$, and let $P(s)$ satisfy* $\operatorname{ord}^T(P)=m\geq 2$, *and*

(17) $$P(s)f^s = s^{m-1}a(x)f^{s-m+1} + (\text{lower order in } s).$$

If there exists $Q(x, D)\in\mathcal{D}$ such that $\sigma(Q)(0,\xi)\neq 0$, and

$$\sigma^T(Q(x,D)P(s))\in\sigma^T(\mathcal{J}(s)),$$

then $a(x)\in\mathfrak{a}+\mathcal{O}f$ when $m=2$, or $a(x)\in\mathfrak{a}$ when $m\geq 3$.

Especially when $m=2$, if (17) gives a counter example against RS 1, then it is also a counter example against RS 3.

2. *More generally, if $l(f)=\lambda\geq 2$ and* $\operatorname{ord}^T P\geq\lambda$, $a(x)\in\mathfrak{a}+\mathcal{O}f^{\lambda-1}$, *when* $\operatorname{ord}^T(P)=\lambda$, *or $a(x)\in\mathfrak{a}$ when* $\operatorname{ord}^T(P)>\lambda$.

Remark: This proposition is useful only when $m=\operatorname{ord}^T(P)=l(f)$ and $l(f)+1$. Since, regardress of the existence of Q, we see

$$a(x)\in(\mathfrak{a}+\mathcal{O}f+\textstyle\sum\mathcal{O}f_{ij})(\mathfrak{a}+\mathcal{O}f)^{m-2}.$$

Thus $a(x)\in\mathfrak{a}$ when $\operatorname{ord} P\geq l(f)+2$.

Proof. Since $f^2\in\mathfrak{a}^2+\mathfrak{a}f$, we can show by induction on k that

$$(\mathfrak{a}+\mathcal{O}f)^k=\mathfrak{a}^{k-1}(\mathfrak{a}+\mathcal{O}f).$$

Now, if $\sigma^T(QP)\in\sigma^T(\mathcal{J}(s))$,

(18) $$\sigma_l(Q)(x, df)\,a(x)\in(\mathfrak{a}+\mathcal{O}f)^{l+m-1}=\mathfrak{a}^{l+m-2}(\mathfrak{a}+\mathcal{O}f).$$

Therefore, since $m\geq 2$, there is a homogeneous polynomial $p_l(x, \xi)$ of degree l with coefficients in $\mathfrak{a}^{m-2}$ such that

$$\sigma_l(Q)(x, df)\,a(x)+p_l(x, df)f\in \mathfrak{a}^{l+m-1}.$$

By the aid of Theorem 2.19, 2 and $m\geq 2$, all the coefficients of $a(x)\sigma_l(Q)(x, \xi)+fp_l(x, \xi)$ are in $\mathfrak{a}$. Thus

$$a(x)\in \mathfrak{a}+\mathfrak{a}^{m-2}f,$$

owing to the condition $\sigma(Q)(0, \xi)\neq 0$.

The proof of 2 is almost the same. Q.E.D.

We apply 2.29 1 to (15). Then (16) implies that (15) is a counter example against RS 3.

There is $Q(s)\in \mathcal{J}(s)$ which assures $L(f)\leq 3$.

(c) $$Q(s)=\left(s-X_0+\frac{1}{10}\right)\left(s-X_0+\frac{1}{15}\right)\left(s-X_0+\frac{t}{180}yD_x\right)$$
$$-\frac{5^3t^5}{6^4}x\left(s-X_1-\frac{4}{15}\right)\left(s-X_1-\frac{1}{15}\right)\left(s-X_1-\frac{1}{25t}xD_y\right).$$

Therefore, our f has the property that $2=l(f)<L(f)=3$. (cf. § 8) $\mathcal{J}(s)$ is generated by (a), (a′), (c) and the following operator:

(b) $$yP(s)-kX_{34}.$$

Set

$$P'(s)=\left(s-X_0+\frac{t}{180}yD_x+\frac{1}{6}\right)\left(s-X_0+\frac{t}{180}yD_x+\frac{1}{10}\right)$$
$$-\frac{5^3t^4}{6^4}x\left(s-X_1+\frac{1}{25}\right)\left(t\left(s-X_1+\frac{2}{25}\right)-\frac{1}{25}xD_y\right).$$

Then

$$P'(s)P(s)-\frac{kt}{12}\left(s-X_0+\frac{t}{180}yD_x+\frac{1}{6}\right)x\left(t(s-X_1)-\frac{1}{25}xD_y\right)$$
$$-\frac{35kt^3}{6^3}y(s-X_1)(s-X_2)-\frac{25kt^2}{12}x\left(s-X_1+\frac{1}{25}\right)\left(s-X_0+\frac{t}{180}yD_x\right)$$

is an element in $\mathcal{J}(s)$. Thus $(\sigma^T(P))^2\in \sigma^T(\mathcal{J}(s))$.

We shall show later that several types of polynomials give us concrete counter examples against RS $1\sim 3$ (§18).

The next example satisfies $l(f)=3$. Corresponding operator of total order 3 fulfills conditions 1° and 2° in Theorem 2.1, but violates condition 3°.

Example 2.30. $f=\dfrac{1}{n_1}x^{n_1}+\dfrac{1}{n_2}y^{n_2}-tx^{m_1}y^{n_2-1}$.

$c=\dfrac{m_1}{n_1}-\dfrac{1}{n_2}$, $c'=\dfrac{n_1-m_1}{n_1(n_2-1)}-\dfrac{1}{n_1}$, t is a parameter. We assume $c>0$ and $(n_1-1)/5\leq m_1\leq (n_1-2)/4$. Consider the following operator:

$$\begin{aligned}P(s) &= (s-X_0+4c-ctx^{m_1}D_y)(s-X_0+2c-ctx^{m_1}D_y)\\ &\quad\times(s-X_0-ctx^{m_1}D_y)-m_1m_2{}^6t^7x^{5m_1-n_1+1}y^{n_2-7}\\ &\quad\times\left\{x^{m_1}(s-X_2+c'+c_2)-3\frac{(n_2-2)c}{m_2^2t}y\right\}\\ &\quad\times(s-X_2+c')\left(x^{m_1-1}(s-X_2)+\frac{c}{m_1m_2t}yD_x\right).\end{aligned}$$

It follows that

$$P(s)f^s=c^3t^3(n_2-1)(n_2-2)a(x)sf^{s-1} \tag{19}$$

where

$$a(x)=-x^{3m_1}y^{n_2-3}+(n_2-3)tx^{4m_1}y^{n_2-4}.$$

Thus conditions 1° and 2° of Theorem 2.4 hold for $\sigma^T(P)$.

We prove that (19) serves a counter example against RS 1 and RS 2 using condition 3°. Consider the operator $D_x^lP(s)$. If RS 2 holds for $\sigma^T(P)$, the following must hold by condition 3°, for $l\gg 0$.

$$f_x^la(x)\in(\mathfrak{a}+\mathcal{O}f)^{l+1}+\mathfrak{c}_{l+2}. \tag{20}$$

Claim: $\mathfrak{c}_{l+2}\subset\mathcal{O}x^{(n_1-1)(l+1)}+\mathcal{O}y^{n_2-3}$.

In fact, if $q(f,x,df)=0$, $\deg q=l+2$, then

$$q\left(\frac{x^{n_1}}{n_1},x,y;x^{n_1-1},0\right)\equiv 0 \mod \mathcal{O}y^{n_2-3}.$$

Hence

$$\frac{\partial^{l+2}q}{\partial\xi^{l+2}}(0,x,y;0,0)\in\mathcal{O}x+\mathcal{O}y^{n_2-3}.$$

Therefore the relation

$$R_1[q] \equiv \frac{n_1-1}{2} x^{n_1-2} \frac{\partial^2 q}{\partial \xi^2}\left(\frac{x^{n_1}}{n_1}, x, y; x^{n_1-1}, 0\right) \quad \bmod \mathcal{O} y^{n_2-3}$$

yields the Claim.

Considering both sides of (20) mod $\mathcal{O}y^{n_2-3}$, we obtain

$$x^{l(n_1-1)+4m_1} \in \mathcal{O}_x x^{(l+1)(n_1-1)}.$$

This is impossible since $4m_1 < n_1 - 1$. Therefore (19) is a counter example against RS 2 (and RS 1 when $l=0$).

Quite generally, if (n_1, n_2, m_1) satisfies

$$\frac{n_1-1}{k+1} \leq m_1 \leq \frac{n_1-2}{k}, \quad \frac{1}{n_2} < \frac{m_1}{n_1}, \quad n_2 \geq k+3,$$

$$\text{e.g.} \quad n_1 = k+2, \ n_2 = k+3, \ m_1 = 1,$$

we can prove

$$l(f) \leq \left[\frac{k}{2}\right] + 1, \quad L(f) \leq k.$$

Equalities holds for $k=1, 2, 3$. See § 19 types X_0, $X_0^{\flat}$ and $X_0^{\flat\flat}$.

§ 8. Generators of $\mathcal{J}(s)$ —continued—

In this section, we study the structure of $\mathcal{J}(s)$ when f has an isolated singularity and $L(f)=3$. Our goal is Theorem 2. 32. There exist an operator $Q(s)$ in $\mathcal{J}(s)$:

$$Q(s) = s^3 + Cs^2 + Ds + E, \tag{21}$$

with $\mathrm{ord}^T(Q) = 3$.

The case $L(f)=3$ will be divided into two cases: $2 = l(f) < L(f) = 3$ and $3 = l(f) = L(f)$. We call the latter "case (3, 3)". In the former case, there is an operator $P(s) = s^2 + As + B$, $\mathrm{ord}^T(P) = 2$ such that

$$P(s) f^s = a(x) s f^{s-1}, \tag{22}$$

where

$$a(x) \notin \mathfrak{a} + \mathcal{O}f.$$

We call this case "case (2, 3; a)". It is easy to see the following:

If $(\sigma^T(P))^{l'} \in \sigma^T(\mathcal{J}(s))$, then

$$(\mathcal{J}(W[s]))^{l'} \subset \sigma^T(\mathcal{J}(s)).$$

Recall that it was enough to choose $l'=2$ in Example 2. 28.

Let $(b_i(x))$ be a basis of an ideal $(\mathfrak{a}^2+\mathfrak{a}f) : f^2$. Then, there are operators

$$B_j(s) = b_j s^2 + b'_j s + b''_j, \ \mathrm{ord}^T(B_j) = 2, \tag{23}$$

with

$$B_j(s) f^s = b'''_j(x) s f^{s-1}.$$

Here, one may assume that $b'''_j \notin \mathfrak{a} + \mathcal{O}f$ for $j=1, \cdots, J$ and $b'''_j = 0$ for $j = J+1, \cdots, J+J'$. The following congruence relation can be proved in the same manner as in the proof of Theorem 2. 23.

$$b_j s f^{s+1} \equiv b'''_j f^s \quad \mathrm{mod}\ \mathcal{D}(\mathfrak{a} + \mathcal{O}f) f^s. \tag{24}$$

Let $\mathfrak{c}_2$ be the ideal used in Theorem 2. 4. Then

$$\mathfrak{c}_2 \subset \sum^J \mathcal{O} b'''_j \tag{25}$$

in view of the structure of $\mathcal{J}(s) \cap (\mathcal{D}s + \mathcal{D})$ and the definition of (b_j). Set

$$(\mathfrak{a} + \mathcal{O}f) : b'''_j = \sum_k \mathcal{O} b'''_{j,k} \quad \text{and} \quad b_{j,k} = b'''_{j,k} \cdot b_j.$$

(when $j > J$, $b_{j,k} = b_j$). Then, we can find operators $B_{j,k}(s)$ in $\mathcal{J}(s)$ such that

$$B_{j,k}(s) = b_{j,k} s^2 + b'_{j,k} s + b''_{j,k}, \mathrm{ord}^T(B_{j,k}) = 2. \tag{26}$$

In general, $\mathcal{J}(s) \cap (\mathcal{D}s^2 + \mathcal{D}s + \mathcal{D})$ cannot be generated only by $(\mathcal{D}s + \mathcal{D})(\mathcal{J}(s) \cap (\mathcal{D}s + \mathcal{D}))$ and $\sum \mathcal{D} B_{j,k}$, and we need operators of the following type (cf. Example 2. 27):

$$C_l(s) = c_l(x, D) s^2 + c'_l(x, D) s + c''_l(x, D), \tag{27}$$

with $\mathrm{ord}(c_l) \geq 1$.

As is easily seen, we can assume that

$$\operatorname{ord}^T(C_l) = \operatorname{ord}(c_l) + 2 = \operatorname{ord}(c_l'').$$

Moreover, it can be proved by the method employed in § 6 that coefficients of $\sigma(c_l)(x, \xi)$ belong to $\mathfrak{a} : f^2$. Set

$$q(s, x, \xi) = \sigma^T(Q) = s^3 + (\sum a_i\xi_i)s^2 + (\sum a_{ij}\xi_i\xi_j)s + \sum a_{ijk}\xi_i\xi_j\xi_k.$$

Theorem 2. 4 shows

$$R_1[q](f, x, df) = q'(f, x, df),$$

where

$$q'(s, x, \xi) = \sum b_{ij}\xi_i\xi_j + (\sum b_i\xi_i)s, \quad \text{for some } (b_{ij}, b_i).$$

Define the operator $\bar{Q}(s)$ by

$$\begin{aligned} \bar{Q}(s) = {} & fs(s-1) + (\sum a_i(D_if - f_i))(s-1) \\ & + \sum a_{ij}(D_iD_jf - D_if_j - D_jf_i) + \sum a_{ijk}D_jD_kf_i \\ & - \sum b_{ij}D_jf_i - \sum b_if_i. \end{aligned}$$

Then, $Q(s)f^s = 0$ entails

$$\bar{Q}(s)f^s = R(x)f^s, \tag{28}$$

where

$$R(x) = \left(R_2[q] - R_1\left[\frac{\partial q}{\partial s}\right] - R_1[q']\right)(f, x, df).$$

Making use of condition 3° of Theorem 2. 4 and (25), we have

$$R(x) \equiv 0 \mod \mathfrak{a} + \mathcal{O}f + \sum \mathcal{O}b_j''', \tag{29}$$

and hence by (28) and (29),

$$s^2f^{s+1} \equiv bf^s \mod (\mathcal{D}s + \mathcal{D})(\mathfrak{a} + \mathcal{O}f)f^s, \tag{30}$$

where

$$b \in \sum^J \mathcal{O}b_j'''.$$

Now we investigate "case (2, 3; a)." In this case, $b_1 = 1$, $J' = 0$, $b_1''' = a$, $J = 1$. We write b_k (b_k', b_k'' and B_k, respectively) for $b_{1,k}$ ($b_{1,k}'$, $b_{1,k}''$ and $B_{1,k}$, respectively), $k = 1, 2, \cdots, q$. Then we have the following

Proposition 2. 31. *In case "(2, 3; a)", it holds that*

$$\mathcal{J}(s) \cap (\mathcal{D}s^2 + \mathcal{D}s + \mathcal{D}) = \sum_{k=1}^{q} \mathcal{D}B_k(s) + \sum_{i=1}^{r} \mathcal{D}A_i(s, x, D) + \mathcal{J}_0.$$

Proof. Let $T(s) = Qs^2 + Rs + S$ be an element of $\mathcal{J}(s)$ with $\operatorname{ord}^T(T) = l$. The proof is carried out by induction on l. If $l \leq 1$, then the proof is straightforward. Assume $l \geq 2$.

Case i) $\operatorname{ord}(Q) < l-2$. In this case,

$$\sigma_{l-1}(R)f + \sigma_l(S) = 0.$$

Using the same argument as in Theorem 2.21, we can lower $\operatorname{ord}^T(T)$ by subtracting a suitable element in $(\mathcal{D}s + \mathcal{D}) \cap \mathcal{J}(s)$.

Case ii) $\operatorname{ord}(Q) = l-2$. In this case,

$$\{(R - QA)s + (S - QB)\}f^s = -Q(a(x)sf^{s-1}).$$

Now that the operator in the left-hand side is of order 1 in s, and the right-hand side is of order not greater than $l-1$ in s, there is an operator $R's + S' \in (\mathcal{D}s + \mathcal{D}) \cap \mathcal{J}(s)$ such that

$$\sigma_l^T(R's + S') = \sigma_l^T((R - QA)s + (S - QB)).$$

Therefore,

$$\sigma_{l-2}(Q)(x, df)\, a(x) \in (\mathfrak{a} + \mathcal{O}f)^{l-1}$$

by Corollary 2.5. Then, applying the argument of Proposition 2.20, we conclude that all the coefficients of $\sigma_{l-2}(Q)(x, \xi)$ belong to $(\mathfrak{a} + \mathcal{O}f) : a(x)$. Then, choosing appropriate elements $T_k \in \mathcal{D}$, we see that $T(s) - \sum T_k B_k$ is of total order less than l or reduces to an operator discussed in case i). Q.E.D.

Remark: The proof also shows that

$$(\mathfrak{a} + \mathcal{O}f) : a(x) \supset \mathfrak{a} : f, \tag{31}$$

if we consider $sA_i(s, x, D) \in \mathcal{J}(s)$.

The structure of $\tilde{\mathcal{M}}$ is given by the following theorem. Corresponding theorems for $\mathcal{N}$ and $\mathcal{M}$ are similarly given and we omit them.

Theorem 2.32. *When $L(f) = 3$, $\tilde{\mathcal{M}}$ has a following presentation.*

1) *case* (3, 3).

$$0 \longleftarrow \widetilde{\mathcal{M}} \xleftarrow{\begin{pmatrix}1\\ s\\ s^2\end{pmatrix}} \mathcal{D}^3 \xleftarrow{\begin{pmatrix} f_i & & \\ f & & \\ b'''_j & & \\ & f & \\ a'_\nu & a_\nu & \\ b''_{j,k} & b'_{j,k} & b_{j,k} \\ c''_l & c'_l & c_l \end{pmatrix}} \mathcal{D}^N \tag{32}$$

2) *case* (2, 3; a).

$$0 \longleftarrow \widetilde{\mathcal{M}} \xleftarrow{\begin{pmatrix}1\\ s\\ s^2\end{pmatrix}} \mathcal{D}^3 \xleftarrow{\begin{pmatrix} f_i & & \\ f & & \\ g & h & \\ a'_\nu & a_\nu & \\ b''_k & b'_k & b_k \end{pmatrix}} \mathcal{D}^{n+2+r+q} \tag{33}$$

where we can set the row $(g, h, 0)$ *either*

$$g=0,\, h=f \quad \text{or} \quad g=a,\, h=0.$$

In this case, the following inclusion relation holds.

$$(\mathfrak{a}+\mathcal{O}f) : a \supset \mathfrak{a} : f \supset \mathfrak{a}+\mathcal{O}f+\mathcal{O}a \tag{34}$$

Proof. 1) case (3, 3). Suppose

$$P\bar{1}+Q\bar{s}+R\bar{s}^2=0, \quad \text{in } \widetilde{\mathcal{M}}.$$

As in the first step of the proof of Theorem 3. 23, one can assume

$$(P+Qs+Rs^2)f^s=\sum R_i f_i f^s+(S_2+S_1 s+S_0 s^2)f^{s+1}, \tag{35}$$

for some R_i and $S_j \in \mathcal{D}$. Owing to (30), the right-hand side of (35) can be rewritten as follows:

$$\sum R'_i f_i f^s+(S'_2+S'_1 s)f^{s+1}+\sum T_i b'''_j f^s. \tag{36}$$

On the other hand, (24) shows

$$b'''_j \in \mathcal{J}(s)+(\mathcal{D}s+\mathcal{D})(\mathfrak{a}+\mathcal{O}f).$$

Thus (32) is proved.

2) case $(2, 3; a)$. The formula (24) yields

$$fs \equiv a \quad \text{mod } \mathcal{J}(s)+\mathcal{D}(\mathfrak{a}+\mathcal{O}f). \tag{37}$$

Thus Proposition 2. 31 and (32) proves (33).

Next we prove (34). The first inclusion is (31). Using the formula (22) under substitution $s\rightarrow s+1$, we obtain

$$(s^2+(A+2)s+(B+A+1))f^{s+1}=a(s+1)f^s. \tag{38}$$

We can eliminate s^2f^{s+1} and sf^{s+1} in the left-hand side of (38) by use of (30) and (37). Then we find

$$(as+H)f^s=0,$$

for some $H\in\mathscr{D}$ with ord $H\leq 2$. Therefore, Theorem 2. 21 (or its proof) proves

$$a\in\mathfrak{a}:f.$$

Q.E.D.

We remark that the similar argument of the last part of the proof shows

$$b_j'''\in\mathfrak{a}:f \tag{39}$$

for case (3, 3).

Chapter III. Determination of $b(s)$

In this chapter, we explain the method to determine or estimate b-functions, and give some explicit formulae in §§ 15~17. The relation with the local monodromy structure is also discussed.

When we exibit b-functions, we sometimes use $p(t)=\frac{1}{2\pi i}\oint\frac{d}{ds}\log b(s)\times t^{-s}ds$, where the path of integration encounters R_f counter clockwise. That is, if $b(s)=\prod_{i=1}^{l}(s+\alpha_i)^{c_i}$, $p(t)=\sum_{i=1}^{l}c_it^{\alpha_i}$. Similar notations are used for $\tilde{b}(s)$ and so on.

A. General Procedure

§ 9. Construction of Eigenvectors

One of the most effective way to seek factors of b-functions is the construction of eigenvectors. Let $\mathscr{L}$ be a $\mathscr{D}$-Module and u be its non-zero section satisfying

1. There exists $\alpha \in \boldsymbol{C}$ such that

$$Q(\alpha)u=0 \quad \text{for any} \quad Q(s) \in \mathcal{J}(s).$$

2. $fu=0$.

Then, $(s-\alpha)|b(s)$. In fact, $P(s)f^{s+1}=b(s)f^s$ yields $Q(s)=P(s)f-b(s) \in \mathcal{J}(s)$. Then $b(\alpha)u=P(\alpha)fu=0$. Hence, $b(\alpha)=0$.

Conditions 1 and 2 assure that the map $\mathcal{M} \to \mathcal{L}$ defined by $\bar{f}^s \mapsto u$ is a well-defined $\mathcal{D}$-homomorphism, and it is an eigenvector belonging to eigenvalue α of the action of s in $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{L})$. Since $\mathcal{M}$ is holonomic, if we take a holonomic $\mathcal{L}$, $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{L})$ has finite dimensional stalks [13], and the eigenvalues of s can be calculated. In this course, there follows an important result. The following Theorem 3.3 is due to Theorem 4.3 [12]. First note the following fact [13].

Proposition 3.1. *Let X be a complex manifold and let Y be its submanifold. Let $\mathcal{L}$ be a holonomic system on X. Then*

1. *There is a regular (in the sense of Whitney) stratification of X, $X=\bigcup X_\alpha$ such that $\check{SS}(\mathcal{L}) \subset \bigcup T^*_{X_\alpha}X$ and $\mathcal{H}om_{\mathcal{D}}(\mathcal{L}, \mathcal{B}_{X_\alpha|X})$ is locally constant sheaf of finite rank. Such stratification is called regular with respect to $\mathcal{L}$.*

2. *If $\check{SS}(\mathcal{L}) \subset T^*_Y X$ and* Supp $\mathcal{L} \subset Y$, *$\mathcal{L}$ is locally isomorphic to a finite direct sum of $\mathcal{B}_{Y|X}$.*

Definition 3.2. *Let $X=\bigcup X_\alpha$ be a regular stratification with respect to $\mathcal{M}$. We denote by $b^i(s)$ the minimal polynomal of s in $\bigoplus_{\operatorname{codim} X_\alpha=i} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B}_{X_\alpha|X})_{x_\alpha}$, $x_\alpha \in X_\alpha$. $\tilde{b}^i(s)$ is similarly defined for $\tilde{\mathcal{M}}$.*

Theorem 3.3.

$$\text{l.c.m.}\,(b^i) \mid b \mid \prod_{i=1}^{n} b^i, \tag{1}$$

$$\text{l.c.m.}\,(\tilde{b}^i) \mid \tilde{b} \mid \prod_{i=2}^{n} \tilde{b}^i. \tag{2}$$

Proof. It should be noted that $b^0=\tilde{b}^0=\tilde{b}^1=1$ and $b^1=s+1$. We set $\mathcal{M}^k=\prod_{i=0}^{k} b^i(s)\cdot\mathcal{M}$, and prove by induction that codim Supp $\mathcal{M}^k>k$. Since Supp $\mathcal{M}\subset\{f=0\}$, this is true for $k=0$.

Suppose codim Supp $\mathcal{M}^{k-1}=k$. Let Y be a k-codimensional irreducible component of Supp $\mathcal{M}^{k-1}$. Then, there is a non-singular manifold $Y'\subset Y$ such that $\operatorname{codim}(Y-Y')>k$ and $\check{S}S(\mathcal{M}^{k-1})\cap T_*X\underset{X}{\times}Y'\subset T^*_{Y'}X$. Therefore $\mathcal{M}^{k-1}\simeq(\mathcal{B}_{Y'|X})^N$ by 2 of Proposition 3.1. Since $b^k(s)\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B}_{Y'|X})_{y'}=0$, $b^k(s)\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{M}^{k-1})_{y'}=0$. Then the commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[d, "b^k(s)"'] \arrow[dr, "0"] & \\ \mathcal{M} \arrow[r, "\prod_{i=0}^{k-1} b^i(s)"'] & \mathcal{M}^{k-1} \end{tikzcd}$$

shows $\mathcal{M}^k_{y'}=0$. That is, codim Supp $\mathcal{M}^k>k$. Lastly, we obtain $\mathcal{M}^n=\prod_{i=0}^{n} b^i(s)\cdot\mathcal{M}=0$.

Therefore $b(s)\mid\prod_{i=0}^{n} b^i(s)$. l.c.m.$(b^i)\mid b$ was proved in the arguments at the beginning of this section. The proof of (2) is almost the same.

Q.E.D.

As a special case, if f has an *isolated singularity*, $b^n(s)$ turns out to be the minimal polynomial of s in $\widetilde{\mathcal{M}}$, and hence $b(s)=(s+1)b^n(s)$. Thus the determination of $b(s)$ is reduced to the study of $\widetilde{\mathcal{M}}$ in case of isolated singularity.

At this stage, we note a simple but useful proposition.

Proposition 3.4. *Assume f is a weighted homogeneous polynomial with $X_0=\sum a_ix_iD_i$ (isolated singularity or not) such that for a polynomial $p(x)\in\mathbf{C}[x]$, $(X_0+\sum a_i)p(x)=0$ yields $p(x)=0$. Then,*

$$\mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}},\mathcal{B}_{pt})\xrightarrow{\sim}\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B}_{pt}).$$

Proof. We have only to prove,

"If $\Delta\in\mathcal{B}_{pt}$ satisfies $X_{ij}\Delta=0$ and $f\Delta=0$, then $f_i\Delta=0$."

We set $\Delta=q(D)\delta(x)$. Since $\xi_jf_i(-D_\xi)q=\xi_if_j(-D_\xi)q$, there is an $r(\xi)$ such that $f_i(-D_\xi)q(\xi)=\xi_ir(\xi)$. Using the condition $f\Delta=0$,

$$\begin{aligned}0&=\sum a_i D_{\xi_i} f_i(-D_\xi) q(\xi)\\&=\sum a_i D_{\xi_i}\xi_i r(\xi)\\&=(\sum a_i\xi_i D_{\xi_i}+\sum a_i) r(\xi).\end{aligned}$$

By the condition on X_0, we can conclude that $r(\xi)=0$. Q.E.D.

M. Kashiwara conjectured that $b(s)=\prod b^i(s)$ ([12]). This, however, fails to hold in general, as seen in the following.

Example 3.5. $f=x^3+y^3z$ (cf. Example 4.20).

$\mathcal{J}(s)$ is generated by yD_y-3zD_z, $y^3D_x-3x^2D_z$, $y^2zD_x-x^2D_y$ and $s-\frac{1}{3}(xD_x+yD_y)$. The stratification is $X=\bigcup X_\alpha$, $X_0=(X\setminus\{f=0\})$, $X_1=\{f=0\}\setminus\{x=y=0\}$, $X_2=\{x=y=0\}\setminus\{0\}$, $X_3=\{0\}$. $\check{SS}(\mathcal{M})=T^*_{X_1}X\cup T^*_{X_2}X\cup T^*_{X_3}X$.

$$b^1(s)=s+1.\quad b^2(s)=\prod_{\nu=2}^{4}\left(s+\frac{\nu}{3}\right).\quad b^3(s)=\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right).$$

$b^2(s)$ can be calculated by the construction of eigenvectors $z^{-(k+1)/3}D_x^hD_y^k\delta(x,y)$ belonging to $-(h+k+2)/3$, $h,k=0,1$. $b^3(s)$ is known by eigenvectors $D_x^hD_y^2\delta(x,y,z)$, $h=0,1$, which belong to $-4/3$ and $-5/3$. On the other hand, explicit calculation of $P(s)$ (cf. § 11) shows

$$P(s)f^{s+1}=\prod_{\nu=2}^{5}\left(s+\frac{\nu}{3}\right)f^s,$$

where

$$P(s)=D_xD_y^2Q+\frac{1}{3}\left(s+\frac{5}{3}\right)\left(\frac{2}{3}D_x^3D_yy+D_y^3D_z\right)+\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)D_x^2,$$

$$Q=\frac{1}{3}\left(\frac{1}{3}D_x^2y^2+xD_yD_z\right).$$

Thus $b(s)=\prod_{\nu=2}^{5}\left(s+\frac{\nu}{3}\right)$. In this case, both $b^2(s)$ and $b^3(s)$ have a factor $\left(s+\frac{4}{3}\right)$, but $b(s)$ has $\left(s+\frac{4}{3}\right)$ as a simple factor.

§ 10. Construction of Operators

If we can find $P(s)\in\mathcal{D}[s]$ such that

(3) $$P(s)f^{s+1}=b'(s)f^{s},$$

$b(s)$ is a divisor of $b'(s)$. There is a systematic method to construct such $P(s)$ when f is a weighted homogeneous polynomial. This procedure was pointed out by M. Sato at the early stage of the theory of b-functions. For the simplicity of explanation, let us assume that $f(x, y)$ is a weighted homogeneous polynomial of 2-variables: $X_0f=f$, $X_0=\alpha xD_x+\beta yD_y$.

Suppose one can find operators $A(s)$ and $B(s)$ such that

$$A(s)f^{s+1}=a(s)x^{i+1}y^{j}f^{s},$$

$$B(s)f^{s+1}=a(s)x^{i}y^{j+1}f^{s}.$$

Then,

$$(\alpha D_xA+\beta D_yB)f^{s+1}=a(s)(\alpha D_xx+\beta D_yy)(x^iy^jf^s)$$
$$=a(s)(s+(i+1)\alpha+(j+1)\beta)x^iy^jf^s.$$

This process shows that if one has equalities

$$A_i(s)f^{s+1}=a_i(s)x^iy^{m-i}f^s \quad i=0, 1, \cdots, m,$$

then one can construct a $P(s)\in\mathcal{D}[s]$ such that (3) holds, and each roots of $b'(s)$ is that of l.c.m.$(a_i(s))$ or of the form $-(\alpha k+\beta l)$ $1\leq k, l$, $k+l\leq m+1$. More generally:

Proposion 3.6. i) *Let* $f(x_1, \cdots, x_k, y_1, \cdots, y_l)$ *be a polynomial satisfying* 1. $X_0f=f$ *for* $X_0=\sum_{i=1}^{k}a_ix_iD_i$ $a_i\in\boldsymbol{Q}$, 2. *For all multi-indices* α, $|\alpha|=m$, *there exist* $P_\alpha(s)$ *such that*

(4) $$P_\alpha(s)f^{s+1}=a_\alpha(s)x^\alpha f^s,\ a_\alpha(s)\in\boldsymbol{C}[s].$$

Then,

(5) $$b(s)\mid\prod_{h=k}^{m+k-1}\underset{\substack{\nu_i\geq1\\ \Sigma\nu_i=h}}{\text{l.c.m.}}(s+\sum a_i\nu_i)\cdot\text{l.c.m.}(a_\alpha(s)).$$

Assume further that $\text{ord}^TP_\alpha(s)=\deg a_\alpha(s)$ *in* (4). *Then, we can find* $P(s)$ *and* $b'(s)$ *in* (3) *such that* $\text{ord}^TP(s)=\deg b'(s)$.

ii) *When* $f(x)$ *is a weighted homogeneous polynomial with weight* $(a_1, \cdots, a_n)$ *and of isolated singularity,*

$$b_f(s) \mid (s+1) \prod_{h=n}^{\mu+n-1} \underset{\substack{\nu_i \geq 1 \\ \Sigma\nu_i = h}}{\text{l.c.m.}} (s + \sum a_i \nu_i) \tag{6}$$

where $\mu = \dim \mathcal{O}/\mathfrak{a}$.

Proof. Relation (5) can be proved analogously as preceding arguments. Last part of i) is obvious by the very construction of $P(s)$. When f has an isolated singularity, $\mathfrak{a} \supseteq \mathfrak{m}^\mu$, $\mu = \dim \mathcal{O}/\mathfrak{a}$. Therefore, (4) holds for any α, $|\alpha| = \mu$ with $a_\alpha(s) = s+1$. Q.E.D.

In the next section, we give the explicit formula for b_f in case ii). We present (6) here simply to show that our elementary procedure even proves the existence of b_f for some polynomials.

The estimate (5) is not the best possible one in general.

Example 3.7. $f = x^n + yz^m$.

$$\frac{1}{n} D_x f^{s+1} = (s+1) x^{n-1} f^s$$

$$D_y f^{s+1} = (s+1) z^m f^s.$$

Then,

$$\left(z^{m-1} \left(\frac{D_x}{n} \right)^2 + \frac{1}{m} x^{n-2} D_y D_z \right) f^{s+1} = (s+1) \left(s+1+\frac{n-1}{n} \right) x^{n-2} z^{m-1} f^{s+1}.$$

Thus, all calculations are carried out about the monomials $x^i y^j$ $0 \leq i \leq n-1$, $0 \leq j \leq m$. And we have

$$b(s) \mid (s+1) \underset{\substack{1 \leq k \leq n-1 \\ 1 \leq l \leq m}}{\text{l.c.m.}} \left(s + \frac{k}{n} + \frac{l}{m} \right).$$

Since $\delta(f)$, $y^{-1/m} D_x^{k-1} D_z^{l-1} \delta(x, z)$ and $D_x^{k-1} D_z^{m-1} \delta(x, y, z)$ belong to eigenvalues -1, $-\left(\frac{k}{n} + \frac{l}{m} \right)$ and $-\left(\frac{k}{n} + 1 \right)$, this is the equality.

Example of this type can be found in § 21.

B. Isolated Singularities

§ 11. Quasi-Homogeneous Isolated Singularities

Let f be a quasi-homogeneous analytic function with isolated singularity at $0\in \boldsymbol{C}^n$. In this case, a result of K. Saito [22] tells us that we can find a suitable coördinate transformation so that $X_0 f = f$ with $X_0 = \sum_{i=1}^{n} a_i x_i D_i$, $a_i \in \boldsymbol{Q}^+$. We shall show that b is determined by $(a_1, \cdots, a_n)$.

Applying the functors $\mathcal{H}om_{\mathcal{D}}(\cdot, \mathcal{B}_{pt})$ and $\Omega^n \otimes \cdot$ to the presentation II (10) of $\widetilde{\mathcal{M}}$, there are two exact sequences.

$$0 \to F \to \mathcal{B}_{pt} \to \mathcal{B}_{pt}^n,$$

$$0 \leftarrow F^* \leftarrow \Omega^n \leftarrow (\Omega^n)^n.$$

Here,

$$F = \mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}}, \mathcal{B}_{pt}) = \{\Delta(x) \in \mathcal{B}_{pt} | f_i(x)\Delta(x) = 0 \forall i\}$$

$$F^* = \Omega^n \underset{\mathcal{D}}{\otimes} \widetilde{\mathcal{M}} = \Omega^n / \mathfrak{a}\Omega^n \simeq \mathcal{O}/\mathfrak{a},$$

and they are dual to each other.

The action of s in F is X_0, and that in $\mathcal{O}/\mathfrak{a}$ is $X_0^* = -X_0 - \sum a_i$.

Since we can take monomials as a basis of $\mathcal{O}/\mathfrak{a}$, s is diagonalizable. The following theorem was proved by Kashiwara-Sato-Miwa [19]. Here, we give a simple proof of it.

As we have shown in § 9, $b(s) = (s+1)\tilde{b}(s)$, where $\tilde{b}(s)$ is a minimal polynomial of s in $\mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}}, \mathcal{B}_{pt})$. We denote by $\widetilde{B}(s)$ the characteristic polynomial of s, and by $\widetilde{P}(t)$, the associated trace-type formula.

Theorem 3. 8. $\widetilde{P}(t) = \prod \dfrac{t^{a_i} - t}{1 - t^{a_i}}$.

Proof. Let $\{m_i\}$ $1 \leq i \leq \mu$ be monomials such that $\{m_i \bmod \mathfrak{a}\}$ generates $\mathcal{O}/\mathfrak{a}$.

Claim:

$$\boldsymbol{C}[x_1, \cdots, x_n] \simeq \boldsymbol{C}[f_1, \cdots, f_n] \otimes \sum_{i=1}^{\mu} \boldsymbol{C} m_i. \tag{7}$$

In fact, since $(f_1, \cdots, f_n)$ is algebraically independent, $\boldsymbol{C}[f_1, \cdots, f_n]$ is isomorphic to a polynomial ring. One can define the map from the right to the left-hand side naturally. The inverse can be constructed as follows. Take $\varphi(x)\in \boldsymbol{C}[x_1, \cdots, x_n]$. Then, there are unique $a_i\in \boldsymbol{C}$ such that $\varphi(x)=\sum a_i m_i+\sum b_i f_i$, since $\mathcal{O}/\mathfrak{a}$ is a vector space. We can rewrite further

$$\sum_i b_i f_i=\sum_i (\sum_j a_{ij} m_j+b_{ij} f_j) f_i,$$

and a_{ij} are uniquely determined since $\mathfrak{a}/\mathfrak{a}^2\simeq(\mathcal{O}/\mathfrak{a}[T])^{(1)}$ by Lemma 2. 20. Here, $(\mathcal{O}/\mathfrak{a}[T])^{(\nu)}$ denotes the ν-th homogeneous part of $\mathcal{O}/\mathfrak{a}[T]$. Using $\mathfrak{a}^\nu/\mathfrak{a}^{\nu+1}\simeq(\mathcal{O}/\mathfrak{a}[T])^{(\nu)}$, we can proceed further and find the unique $(a_i, a_{ij}, a_{ijk}, \cdots)$ such that

$$\varphi(x)=\sum a_i m_i+\sum a_{ij} m_i f_j+\sum a_{ijk} m_i f_j f_k+\cdots.$$

When the weight of φ is d, this series terminates at most $[2d]+1$ terms.

Consider the Poincaré polynomials of both sides of (7). Then,

$$\prod (1-t^{a_i})^{-1}=\prod (1-t^{1-a_i})^{-1}\times q(t).$$

$q(t)$ is nothing but $\mathrm{tr}(t^{X_0}: \mathcal{O}/\mathfrak{a})$. Therefore,

$$\begin{aligned}\mathrm{tr}(t^{-s}: \mathcal{O}/\mathfrak{a}) &= \mathrm{tr}(t^{X_0+\Sigma a_i}: \mathcal{O}/\mathfrak{a})\\ &=\prod t^{a_i}\prod\frac{1-t^{1-a_i}}{1-t^{a_i}}\\ &=\prod\frac{t^{a_i}-t}{1-t^{a_i}}.\end{aligned}$$

Q.E.D.

Corollary 3.9. i) $\dim F=\dim F^*=\mu=\prod\frac{1-a_i}{a_i}$.

ii) *Let* $\tilde{P}(t)=\sum_{\alpha\in \boldsymbol{Q}_+} q_\alpha t^\alpha$ *be the expansion into the polynomial with fractional power. Then,*

$$b(s)=(s+1)\prod_{q_\alpha\neq 0}(s+\alpha).$$

iii) $\tilde{b}(-n-s)=(-)^{\tilde{d}}\tilde{b}(s), \quad \tilde{d}=\deg \tilde{b}$.

The proof is obvious.

We remark that the eigenvector belonging to $-\sum a_i$ is 1 in $\mathcal{O}/\mathfrak{a}$ and $\delta(x)$ in F, and that to $-(n-\sum a_i)$ is $\mathrm{Hess}(f)$ in $\mathcal{O}/\mathfrak{a}$.

Since $\widetilde{\mathcal{M}}\simeq\mathcal{D}/\mathcal{D}\mathfrak{a}$, one can rewrite

$$\tilde{b}(X_0)=\sum p_i(x,D)f_i\,.$$

Then, defining $P(x,D)=\sum p_i(x,D)D_i$, we obtain

$$\begin{aligned}P(x,D)f^{s+1}&=(s+1)\sum p_i(x,D)f_if^s,\\&=(s+1)\tilde{b}(X_0)f^s,\\&=b(s)f^s.\end{aligned}$$

Put $\tilde{d}=\deg\tilde{b}$. If $d=\max(\operatorname{ord}p_i)>\tilde{d}$, then $\sum\sigma_d(p_i)f_i=0$. Therefore, by setting $p_i(x,D)=\sum D^\alpha a_{i,\alpha}(x)$ for $\sigma_d(p_i)=\sum a_{i,\alpha}(x)\xi^\alpha$, we obtain $\sum p_i'(x,D)f_i=0$. Then

$$\tilde{b}(X_0)=\sum(p_i-p_i')f_i\,,\ \operatorname{ord}(p_i-p_i')<\operatorname{ord}p_i\,.$$

Therefore, we can choose p_i's such that $\operatorname{ord}p_i\leq\deg\tilde{b}$ and at least one of them is an equality. Then

$$\operatorname{ord}P=\max(\operatorname{ord}p_i+1)=\deg b\,.$$

Thus we again find Cor. 2. 26. For example, when $f=\sum x_i^2$, $\tilde{b}(s)=s+\frac{n}{2}$.

$$\tilde{b}(X_0)=\frac{1}{2}\sum x_iD_i+\frac{n}{2}=\frac{1}{2}\sum D_ix_i=\frac{1}{4}\sum D_if_i\,.$$

Therefore $P(x,D)=\frac{1}{4}\sum D_i^2=\frac{1}{4}\Delta$.

§ 12. $L(f)=2$

We use the presentation II (13). Applying the functor $\mathcal{H}om_{\mathcal{D}}(\cdot,\mathcal{B}_{pt})$ and $\Omega^n\underset{\mathcal{D}}{\otimes}\cdot$, we have,

$$0\to F\to\mathcal{B}^2_{pt}\to\mathcal{B}^{n+r+1}_{pt}\,,\tag{9}$$

$$0\leftarrow F^*\leftarrow(\Omega^n)^2\leftarrow(\Omega^n)^{n+r+1}.\tag{10}$$

Here

$$\begin{aligned}F&=\mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}},\mathcal{B}_{pt})\\&=\left\{\begin{pmatrix}u\\v\end{pmatrix}\in\mathcal{B}^2_{pt}\,\middle|\,u\in F_1,\ a_\nu(x)v+a_\nu'(x,D)u=0,\ 1\leq\nu\leq r\right\},\end{aligned}$$

$$F_1 = \{u \in \mathcal{B}_{pt} | (\mathfrak{a} + \mathcal{O}f) u = 0\}.$$

The sequence (10) will be analysed later.

Set $F_2 = \{v \in \mathcal{B}_{pt} | (\mathfrak{a} : f) v = 0\}$. Since $L(f) = 2$, we have $f^2 \in \mathfrak{a}f + \mathfrak{a}^2$, and hence $F_2 \subset F_1$. Set $\mu_1 = \dim F_1 = \dim \mathcal{O}/(\mathfrak{a} + \mathcal{O}f)$, $\mu_2 = \dim F_2 = \dim \mathcal{O}/(\mathfrak{a} : f)$. Then, $\mu_1 + \mu_2 = \mu = \dim \mathcal{O}/\mathfrak{a}$, since

$$\mathrm{Coker}\,(\mathcal{O} \xrightarrow{f} \mathcal{O}/\mathfrak{a}) = \mathcal{O}/(\mathfrak{a} + \mathcal{O}f)$$

and

$$\mathrm{Coim}\,(\mathcal{O} \xrightarrow{f} \mathcal{O}/\mathfrak{a}) = \mathcal{O}/(\mathfrak{a} : f).$$

We choose a basis of F_2 and F_1 such that $(u_1, \cdots, u_{\mu_2})$ is a basis of F_2 and $(u_1, \cdots, u_{\mu_2}, u_{\mu_2+1}, \cdots, u_{\mu_1})$ is one of F_1. If $\sum e_\nu(x, D) a_\nu(x) = 0$,

$$\begin{aligned}&(\textstyle\sum e_\nu(x, D) a'_\nu(x, D)) f^s \\ &\quad = -s(\textstyle\sum e_\nu(x, D) a_\nu(x)) f^s = 0 .\end{aligned}$$

Therefore $\sum e_\nu(x, D) a'_\nu(x, D) \in \mathcal{J}_0 \subset \mathcal{D}\mathfrak{a}$. Then we can solve the system of equations for v_i, $1 \leq i \leq \mu_1$,

$$a_\nu(x) v_i = -a'_\nu(x, D) u_i \quad 1 \leq \nu \leq r ,$$

and v_i's are determined mod F_2. Thus, $\begin{pmatrix} 0 \\ u_i \end{pmatrix}$, $1 \leq i \leq \mu_2$ and $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$, $1 \leq i \leq \mu_1$ forms a basis of F.

This can be summarized in

Theorem 3.9. *Let f be a holomorphic function having isolated singularity and of $L(f) = 2$. Then $\mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}}, \mathcal{B}_{pt})$ is μ-dimensional and its basis is given by the form*

$$\left\{ \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ u_{\mu_2} \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \cdots, \begin{pmatrix} u_{\mu_1} \\ v_{\mu_1} \end{pmatrix} \right\}$$

where $(u_1, \cdots, u_{\mu_2})$ forms a basis of $\{u \in \mathcal{B}_{pt} | (\mathfrak{a} : f) u = 0\}$, $(u_1, \cdots, u_{\mu_2}, u_{\mu_2+1}, \cdots, u_{\mu_1})$ forms one of $\{u \in \mathcal{B}_{pt} | (\mathfrak{a} + \mathcal{O}f) u = 0\}$ and v_i satisfy equations

$$a_\nu(x) v_i = -a'_\nu(x, D) u_i \quad 1 \leq \nu \leq r .$$

$\tilde{b}(s)$ can be calculated as a minimal polynomial of

$$s:\begin{pmatrix}u\\v\end{pmatrix}\mapsto\begin{pmatrix}0 & 1\\-B & -A\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\quad\text{in } F\,.$$

Let b_{ij} be the components of a matrix

$$\tilde{b}\left(\begin{pmatrix}0 & 1\\-B & -A\end{pmatrix}\right)=\begin{pmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{pmatrix}.$$

We have $(b_{11}, b_{12})\begin{pmatrix}f^s\\sf^s\end{pmatrix}=\tilde{b}(s)f^s$, and

$$(b_{11}, b_{12})=\sum(c^i, 0)f_i+(c, 0)f+\sum d_\nu(a'_\nu(x, D), a_\nu(x)),$$

by the presentation of $\widetilde{\mathcal{M}}$ and the definition of $\tilde{b}(s)$. Set

$$P(s, x, D)=\sum c^i D_i+(s+1)\cdot c\,.$$

Then, $P(s, x, D)f^{s+1}=b(s)f^s$. This construction of P is applicable even if $L(f)\geq 3$, when $\mathcal{J}(s)$ is determined and $\tilde{b}(s)$ is known.

We note that we can take $P(s)$ such that $\operatorname{ord}^T P=\deg b$ (Corollary 2.26). This can be seen directly as follows. By the definition of b_{ij}, $\operatorname{ord} b_{11}\leq\tilde{d}$, and $\operatorname{ord} b_{12}\leq\tilde{d}-1$, where $\tilde{d}=\deg\tilde{b}$. Owing to the relation $b_{12}=\sum d_\nu a_\nu(x)$, we can choose $\operatorname{ord} d_\nu\leq\tilde{d}-1$, by the same reasoning as in p. 161. Then, $\operatorname{ord}(b_{11}-\sum d_\nu a_\nu(x, D))\leq\tilde{d}$ and hence we can take $\max(\operatorname{ord} c^i, \operatorname{ord} c)\leq\tilde{d}$ owing to the relation

$$b_{11}-\sum d_\nu a_\nu(x, D)=\sum c^i f_i+cf\,.$$

Therefore, $\operatorname{ord}^T P(s)\leq\tilde{d}+1=\deg b$, and since the converse inequality is obvious, this must be an equality.

Let $u=\begin{pmatrix}\Delta\\\Delta'\end{pmatrix}\in\mathcal{B}^2_{pt}$.

Then u satisfies $\begin{pmatrix}0 & 1\\-B & -A\end{pmatrix}u=\alpha u$, and $u\in F$ if and only if Δ satisfies

$$\Delta'=\alpha\Delta\,,\quad f\Delta=f_i\Delta=0\,,$$

(11) $$A_\nu(\alpha, x, D)\Delta=0\,,$$

(12) $$(\alpha^2+A\alpha+B)\Delta=0\,.$$

Therefore, we sometimes call Δ itself, instead of $u=\begin{pmatrix}\Delta\\\alpha\Delta\end{pmatrix}$ an eigen-vector.

Next, let $u'=\begin{pmatrix}\varDelta'\\ \varDelta''\end{pmatrix}$ and $u=\begin{pmatrix}\varDelta\\ \alpha\varDelta\end{pmatrix}$ be root vectors belonging to an eigenvalue α:

$$s\cdot(u, u')=(u, u')\begin{pmatrix}\alpha & 1\\ & \alpha\end{pmatrix}.$$

Then

$$\varDelta''=\alpha\varDelta'+\varDelta\ ,\quad f\varDelta'=f_i\varDelta'=0\ ,$$

$$A_\nu(\alpha, x, D)\,\varDelta'=-a_\nu(x)\,\varDelta\ , \tag{13}$$

$$(\alpha^2+A\alpha+B)\,\varDelta'=-(2\alpha+A)\,\varDelta\ . \tag{14}$$

Conversely, if $\varDelta$ is an eigenfunction and $\varDelta'$, $\varDelta''$ satisfy above formulae, u and u' are root vectors: Especially when

$$(\mathfrak{a}:f)\,\varDelta=0 \tag{15}$$

and

$$(2\alpha+A)\,\varDelta=0\ , \tag{16}$$

$u=\begin{pmatrix}\varDelta\\ \alpha\varDelta\end{pmatrix}$ and $u'=\begin{pmatrix}0\\ \varDelta\end{pmatrix}$ form those belonging to an eigenvalue α.

We can also use II(32). In this case, the isomorphism holds

$$F^*\simeq(\mathcal{O}/(\mathfrak{a}+\mathcal{O}f))\begin{pmatrix}1\\0\end{pmatrix}\oplus(\mathcal{O}/\mathfrak{a}:f)\begin{pmatrix}0\\1\end{pmatrix}, \tag{19}$$

as a vector space. The action of s is, under the above isomorphism,

$$s:\begin{pmatrix}q\\r\end{pmatrix}\mapsto\begin{pmatrix}0 & -B^*\\ 1 & -A^*\end{pmatrix}\begin{pmatrix}q\\r\end{pmatrix}.$$

§ 13. $L(f)=3$

First, we consider case $(2, 3; \mathrm{a})$. Define the spaces of δ-functions F_i as follows:

$$F_1=\{u\in\mathcal{B}_{pt}\,|\,(\mathfrak{a}+\mathcal{O}f+\mathcal{O}a)\,u=0\}\ ,$$

$$F_2=\{u\in\mathcal{B}_{pt}\,|\,(\mathfrak{a}:f)\,u=0\}\ ,$$

$$F_3=\{u\in\mathcal{B}_{pt}\,|\,((\mathfrak{a}+\mathcal{O}f):a)\,u=0\}\ .$$

The relation II (34) shows there are canonical inclusions:

$$F_3 \subset F_2 \subset F_1 . \tag{20}$$

Since $\dim F_1 + \dim F_3 = \dim \mathcal{O}/(\mathfrak{a}+\mathcal{O}f)$, we have

$$\sum_{i=1}^{3} \dim F_1 = \dim \mathcal{O}/\mathfrak{a} = \mu .$$

Using the presentation II (33) with $g=a$ and $h=0$, we obtain the following theorem. The method of proof is similar to that of Theorem 3. 9 and we omit it.

Theorem 3. 10. *Let f be a holomorphic function having isolated singularity and $2=l(f)<L(f)=3$. Then $\mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}}, \mathcal{B}_{pt})$ is μ-dimensional and its basis is given by the form*

$$\left\{\begin{pmatrix}0\\0\\u_1\end{pmatrix}\cdots\begin{pmatrix}0\\0\\u_{\mu_3}\end{pmatrix}, \begin{pmatrix}0\\u_1\\v_1'\end{pmatrix}, \cdots, \begin{pmatrix}0\\u_{\mu_2}\\v_{\mu_2}'\end{pmatrix}, \cdots, \begin{pmatrix}u_1\\v_1\\w_1\end{pmatrix}, \cdots, \begin{pmatrix}u_{\mu_1}\\v_{\mu_1}\\w_{\mu_1}\end{pmatrix}\right\} .$$

where $(u_1, \cdots, u_{\mu_i})$ is a basis of F_i for $i=1, 2, 3$ and $(u, v, v'w)$ satisfy the following equations:

$$b_k v_i' + b_k' u_i = 0 ,$$

$$a_\nu v_i + a_\nu' u_i = 0 ,$$

$$b_k w_i + b_k' v_i + b_k'' u_i = 0 .$$

The action of s in $F=\mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}}, \mathcal{B}_{pt})$ is given by

$$s: \begin{pmatrix}u\\v\\w\end{pmatrix} \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ -E & -D & -C\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix} .$$

Let c_{ij} be the components of a matrix

$$\tilde{b}\left(\begin{pmatrix} & 1 & \\ & & 1 \\ -E & -D & -C\end{pmatrix}\right) = \begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33}\end{pmatrix} .$$

The presentation II (33) with $g=0$, $h=f$ gives

$$(c_{11}, c_{12}, c_{13}) = \sum (c^i, 0, 0) f_i + (c_1, c_2, 0) f + \sum d_\nu (a_\nu', a_\nu, 0)$$
$$+ \sum e_k (b_k'', b_k', b_k) .$$

Define

$$P(s) = \sum c^i D_i + (s+1) c_1 + s(s+1) c_2 .$$

Then

$$P(s) f^{s+1} = b(s) f^s.$$

We remark that this b-operator $P(s)$ can be so chosen that

$$\text{(21)} \qquad \operatorname{ord}^T(P) = \deg b .$$

It should be noted that we cannot apply Proposition 2. 25 since RS 3 (and hence RS 2 and RS 1 also) does not hold for f. The argument to prove the existence of $P(s)$ with (21) is similar to that for case $L(f) = 2$ (cf. p. 163) and we omit the proof.

The situation becomes complicated for case (3, 3). Example 2. 27 shows that $f \notin \mathfrak{a} : f$, and there exist $C_l(s)$'s in general. The detailed discussion will be found in a subsequent paper of the author.

§ 14. Examples of Calculation

In order to demonstrate how to use the procedures given in the last section, we calculate some examples. The following two examples are $L(f) = 2$,, and double root occurs.

Example 3. 11. $f = \frac{1}{4}(x^4 + y^4 + z^4) - xyz$ $\mu = 11$, $\mathfrak{a} \supset \mathfrak{m}^5$, $\mathfrak{a} : f = \mathfrak{m}$.
We use the following notations. $X_0 = (xD_x + yD_y + zD_z)/4$, $X_1 = X_0 + xD_x/4$, $X_2 = X_0 + yD_y/4$, $X_3 = X_0 + zD_z/4$ and $\varphi = 1 - xyz$.

$\mathcal{J}(s)$ is generated by X_{ij} and following four operators.

$$x(s - X_3) - \frac{z^2}{4\varphi}(y^2 D_x + zD_y + x^2 yD_z),$$

$$y(s - X_1) - \frac{x^2}{4\varphi}(y^2 zD_x + z^2 D_y + xD_z),$$

$$z(s - X_2) - \frac{y^2}{4\varphi}(yD_x + xz^2 D_y + x^2 D_z),$$

$$(s - X_1)(s - X_2) - c^2 xy P_1 P_2 - \frac{cxy}{\varphi} z(3s - X_2 + 2X_0),$$

where

$$P_1 = \frac{-1}{\varphi}(y^2 z D_x + z^2 D_y + x D_z),$$

and

$$P_2 = \frac{-1}{\varphi}(z^2 D_x + x^2 z D_y + y D_z).$$

Then, calculating eigenvectors, we have

$$e_1 = \begin{pmatrix} 0 \\ \delta \end{pmatrix},\quad e_2 = \begin{pmatrix} \delta \\ -\delta \end{pmatrix},\quad e_3 = \begin{pmatrix} D_x\delta \\ \frac{-5}{4} D_x\delta \end{pmatrix},\quad e_4 = \begin{pmatrix} D_y\delta \\ \frac{-5}{4} D_y\delta \end{pmatrix},$$

$$e_5 = \begin{pmatrix} D_z\delta \\ \frac{-5}{4} D_z\delta \end{pmatrix},\quad e_6 = \begin{pmatrix} D_x^2\delta \\ \frac{-3}{2} D_x^2\delta \end{pmatrix},\quad \cdots,\quad e_9 = \begin{pmatrix} (D_x^3 - 6D_yD_z)\delta \\ \frac{-7}{4}(D_x^3 - 6D_yD_z)\delta \end{pmatrix} \cdots.$$

By the aid of this basis of $\mathcal{H}om_{\mathcal{D}}(\tilde{\mathcal{M}}, \mathcal{B}_{pt})$, the action of s can be written

$$s(e_1, e_2, e_3, \cdots, e_{11}) = (e_1, e_2, \cdots, e_{11}) \begin{bmatrix} -1 & & & & & & \\ 1 & -1 & & & & & \\ & & -5/4 & & & & \\ & & & \ddots & & & \\ & & & & -3/2 & & \\ & & & & & \ddots & \\ & & & & & & -7/4 \\ & & & & & & & \ddots \end{bmatrix}.$$

Therefore,

$$\tilde{b}(s) = (s+1)^2\left(s+\frac{5}{4}\right)\left(s+\frac{6}{4}\right)\left(s+\frac{7}{4}\right).$$

Example 3.12. $f = xy(x+y^2)(x^2+y)$

$$\mu = 13,\ \mathfrak{a}: f = \mathfrak{m}.\ X_1 = \frac{1}{3}xD_x + \frac{1}{6}yD_y,\ Y_1 = \frac{1}{6}xD_x + \frac{1}{3}yD_y.$$

$\mathcal{J}(s)$ is generated by three operators:

$$x(s - Y_1) - \frac{y}{30(4 - 13xy + 9x^2y^2)}\{(7x^2 - 40y - 9x^3y)xD_x$$
$$+ 2(14x^2 + 5y + 18x^3y)yD_y\},$$

and similar operator, exchanging x and y in the above, and

$$s^2-sA-B,\quad A=\frac{1-\frac{1}{10}xy}{2\left(1-\frac{5}{4}xy\right)}(xD_x+yD_y),\quad B=-\frac{1-xy}{1-\frac{5}{4}xy}X_1Y_1.$$

The operators can be chosen in another way (cf. § 19). $\begin{pmatrix}0\\ \delta\end{pmatrix}$ and $\begin{pmatrix}\delta\\ -\frac{1}{2}\delta\end{pmatrix}$ form a root subspace belonging to an eigenvalue $-1/2$. $\begin{pmatrix}D_xD_y\delta\\ -D_xD_y\delta\end{pmatrix}$ belongs to -1. Others are $D_x\delta, \cdots\leftrightarrow -2/3$, $D_x^2\delta, \cdots\leftrightarrow -5/6$, $D_x^3\delta, \cdots \leftrightarrow -1$, $\left(D_x^2D_y+\frac{1}{6}D_x^4\right)\delta, \cdots\leftrightarrow -7/6$, $\left(D_x^3D_y-D_y^4+\frac{1}{20}D_x^5\right)\delta, \cdots\leftrightarrow -4/3$.

Thus,

$$\tilde{b}(s)=\left(s+\frac{1}{2}\right)^2\left(s+\frac{2}{3}\right)\left(s+\frac{5}{6}\right)(s+1)\left(s+\frac{7}{6}\right)\left(s+\frac{4}{3}\right).$$

C. Local Monodromy

§ 15. Relation with Local Monodromy

In the preceding sections, the equality $\dim F=\dim \mathcal{O}/\mathfrak{a}$ holds. This is based on the deep connexion between the local monodromy and the theory of b-function.

The local monodromy of $f^{-1}(0)$ around 0 is a linear operator in $H^i(f^{-1}(\varepsilon)\cap U; \boldsymbol{C})$, where $U=\{x\in \boldsymbol{C}^n \mid |x|<\varepsilon, |f(x)|<\delta, 0<\varepsilon\ll\delta<1\}$. When f has an isolated singularity at 0, only $H^0\simeq \boldsymbol{C}$ and $H^{n-1}\simeq \boldsymbol{C}^{\mu}$ do not vanish.

In general, the local monodromy and b-function relate each other through the hypercohomologies of relative differential form. Since we have not yet completed the argument in general, we discuss here the case of isolated singularity. In this case, the linear map in H^{n-1} is usually called the local monodromy.

Professor B. Malgrange proved in [17],

Theorem 3.13. *Let f have an isolated singularity at* 0. *Then,* $\exp(2\pi is)|_F$ *is equivalent to the local monodromy of $f^{-1}(0)$ around* 0.

Therefore, if we determine the action of s in F, we can also determine the local monodromy of $f^{-1}(0)$. At this stage, it should be emphasized that, as an invariant of $f^{-1}(0)$, b-function is stronger than the local monodromy. For example, let f_t be a μ-constant family of isolated singularities. Then, the local monodromy of f_0 and that of f_t are known to be equivalent. b-function, however, varies. This situation is extensively analyzed in §§ 18, 19. Here we give an easy example.

Example 3.14. $f = x^5 + y^5 + tx^3y^3$. $\mu = 16$.
For $t = 0$, this is weighted homogeneous and

$$\tilde{b}(s) = \prod_{\nu=2}^{8}\left(s + \frac{\nu}{5}\right)$$

$$\tilde{B}(s) = \left(s+\frac{2}{5}\right)\left(s+\frac{3}{5}\right)^2\left(s+\frac{4}{5}\right)^3\left(s+\frac{5}{5}\right)^4\left(s+\frac{6}{5}\right)^3\left(s+\frac{7}{5}\right)^2\left(s+\frac{8}{5}\right).$$

For $t \neq 0$, the factor $\left(s+\frac{8}{5}\right)$ changes into $\left(s+\frac{3}{5}\right)$ and then

$$\tilde{b}(s) = \prod_{\nu=2}^{7}\left(s + \frac{\nu}{5}\right),$$

$$\tilde{B}(s) = \left(s+\frac{2}{5}\right)\left(s+\frac{3}{5}\right)^3\left(s+\frac{4}{5}\right)^3\left(s+\frac{5}{5}\right)^4\left(s+\frac{6}{5}\right)^3\left(s+\frac{7}{5}\right)^2.$$

When $t = 0$, $D_x^3 D_y^3 \delta(x, y)$ is an eigenvector belonging to eigenvalue $-8/5$, and $\delta(x, y)$ is one belonging to $-2/5$. Whereas, when $t \neq 0$, the former cannot be an eigenvector since $f \cdot D_x^3 D_y^3 \delta(x, y) \neq 0$, and the later belongs to two eigenvalues $-2/5$ and $-3/5$. Local monodromy does not change since $-3/5 \equiv -8/5 \mod \boldsymbol{Z}$.

§ 16. Join Formula for b-Functions

Let $f(x)$ and $g(y)$ be holomorphic functions with different variables. Then, we can know the b-function of $f(x) + g(y)$ in terms of those of f and g. Put $n = \dim X$, $m = \dim Y$.

Theorem 3.15. *Let $f(x) \in \mathcal{O}_X$, $g(y) \in \mathcal{O}_Y$, $f(0) = g(0) = 0$, and assume $g(y)$ is quasi-homogeneous and of isolated singularity. Then,*

$$\tilde{b}^{n+m}_{f+g}(s)\,|\,\text{l.c.m.}(\tilde{b}^{n}_{f}(s-\alpha)\,|\,\alpha\in\tilde{R}_g).$$

Proof. We set $h(x,y)=f(x)+g(y)$, $\mathcal{D}=\mathcal{D}_{X\times Y}$, $\mathcal{O}=\mathcal{O}_{X\times Y}$ and Y_0 be the vector field of quasi-homogeneity of g; $Y_0g=g$. Then, the following inclusion holds.

Claim:

$$\mathcal{J}_h(s)\supset\mathcal{J}_f(s-Y_0). \tag{22}$$

The proof of this Claim is given at the last of this section.

Next, since $\mathfrak{a}_h+\mathcal{O}h=\mathcal{O}\cdot\mathfrak{a}_f+\mathcal{O}f+\mathcal{O}\cdot\mathfrak{a}_g$, if we set

$$\mathcal{M}'=\mathcal{D}[s]/(\mathcal{D}[s]\mathcal{J}_f(s-Y_0)+\mathcal{D}[s]f+\mathcal{D}[s]\mathfrak{a}_f+\mathcal{D}[s]\mathfrak{a}_g),$$

there is a canonical surjection

$$\mathcal{M}'\to\tilde{\mathcal{M}}_h\to 0. \tag{23}$$

As we know, $\tilde{\mathcal{M}}_g=\mathcal{D}_Y/\mathcal{D}_Y\mathfrak{a}_g\simeq\bigoplus_{i=1}^{\mu}\mathcal{F}_i$, where $\mathcal{F}_i\simeq\mathcal{B}_{pt}$, $\mu=\dim\mathcal{O}_Y/\mathfrak{a}_g$ and s acts on each component separately,

$$s: u\mapsto\alpha_i u,\quad u\in\mathcal{F}_i,$$

and $\tilde{B}_g(s)=\prod(s-\alpha_j)$. Since the action of s is that of Y_0 in $\tilde{\mathcal{M}}_g$, we have,

$$\mathcal{M}'\simeq\mathcal{D}_{X\times Y}\underset{\mathcal{D}_X}{\otimes}[\bigoplus_{i=1}^{\mu}\{\mathcal{D}_X[s]/(\mathcal{D}_X[s]\mathcal{J}_f(s-\alpha_i)+\mathcal{D}_X[s](\mathfrak{a}_f+\mathcal{O}_Xf))\}].$$

This proves $\tilde{b}_h(s)\,|\,\underset{j}{\text{l.c.m.}}(\tilde{b}_f(s-\alpha_j))$. Q.E.D.

Corollary 3.16. *Upon the conditions of Theorem* 3.15, *we further assume $f(x)$ is of isolated singularity. Then,*

$$\tilde{b}_{f+g}(s)=\underset{j}{\text{l.c.m.}}(\tilde{b}_f(s-\alpha)\,|\,\alpha\in\tilde{R}_g),$$

$$\tilde{B}_{f+g}(s)=\prod_{i=1}^{\mu}B_f(s-\alpha_i),\ \textit{where}\ \prod(s-\alpha_i)=\tilde{B}_g(s).$$

Proof. In this case, $h=f+g$ has an isolated singularity at $0\in X\times Y$. $\mathcal{M}'$ defined in the proof of Theorem 3.15 satisfies $\check{SS}(\mathcal{M}')\subset T^*_{\{0\}}(X\times Y)$. We apply the functor $\mathcal{H}om_{\mathcal{D}}(\cdot,\mathcal{B}_{pt|X\times Y})$ to (23) and have

$$0\to F_h\to F', \tag{24}$$

where $F_h = \mathcal{H}om_{\mathcal{D}}(\widetilde{\mathcal{M}}_h, \mathcal{B}_{pt|X\times Y})$, $F' = \bigoplus_{i=1}^{\mu} \mathcal{H}om_{\mathcal{D}_X}(\widetilde{\mathcal{M}}_f|_{s\to s-\alpha_i}, \mathcal{B}_{pt})$. Owing to the Theorem 3. 13, the equalities

$$\dim F_h = \dim \mathcal{O}/\mathfrak{a}_h = \dim \mathcal{O}_X/\mathfrak{a}_f \times \dim \mathcal{O}_Y/\mathfrak{a}_g = \dim F'$$

holds. Therefore, (24) is an isomorphism. Q.E.D.

Corollary 3. 17. *When $f(x)$ has an isolated singularity at $0\in X$, the b-function of $h(x, y) = f(x) + \sum_{i=1}^{k} y_i^2$ at $0 \in X\times Y$, is*

$$\tilde{b}_h(s) = \tilde{b}_f\left(s+\frac{k}{2}\right), \quad \widetilde{B}_h(s) = \widetilde{B}_f\left(s+\frac{k}{2}\right).$$

We note that there is an isomorphism, in Corollary 3. 16,

$$\mathcal{M}' \xrightarrow{\sim} \widetilde{\mathcal{M}}_h,$$

by applying the functor $\mathcal{H}om_{\boldsymbol{C}}(\cdot, \mathcal{B}_{pt|X\times Y})$ to (24). This isomorphism can be proved (and hence also 3·16) directly, i.e. without using Theorem 3. 13, when $L(f)\leq 2$, or, case (2, 3; a).

In order to prove Claim in the proof of Theorem 3. 15, we prepare

Lemma 3. 18. *There are natural numbers $c_j^{(l)}$ $j\geq 0$ such that*

$$(s)_h = \sum_{j=0}^{h} (-)^j c_j^{(l)} (h)_j (s+l)_{h-j}, \quad 1\leq l.$$

Proof. We use the induction on l. When $l=1$, one can prove by induction on h that

$$(s)_h = \sum_{j=1}^{h} (-)^j (h)_j (s+1)_{h-j}. \tag{25}$$

Then, it follows from the hypothesis of induction and (25) that

$$\begin{aligned}(s)_h &= \sum_j (-)^j c_j^{(l)} (h)_j (s+l)_{h-j} \\ &= \sum_j (-)^j c_j^{(l)} (h)_j \sum_{k=0}^{h-j} (-)^k (h-j)_k (s+l+1)_{h-j-k} \\ &= \sum_j (-)^{j+k} c_j^{(l)} (h)_{j+k} (s+l+1)_{h-j-k}.\end{aligned}$$ Q.E.D.

$c_j^{(l)}$ are determined by

$$c_j^{(0)}=0\,,\quad j\neq 0\,,\quad c_0^{(l)}=1\,,\quad l=0,1,\cdots$$
$$c_{j+1}^{(l)}=c_j^{(l)}+c_{j+1}^{(l-1)}\,.$$

Choose an operator of the form

$$P(s,x,D)=\sum_{k=0}^{m}(s-k)_{m-k}p_k(x,D)\quad\text{with}\quad p_k(x,D)=\sum_{|\alpha|=k}a_{k,\alpha}D^{\alpha}\,.$$

We apply this Lemma 3.18 for $(s-Y_0-k)_{m-k}$ in the following formula.

$$\begin{aligned}(26)\quad & P(s-Y_0,x,D)(f+g)^s\\ &=\sum_{l\geq k\geq 0}(s)_{k-l}(s-Y_0-k)_{m-k}R_l[p_k](x,df)(f+g)^{s-k+l}\\ &=\sum(s)_{k-l}\Big(\sum_j(-)^j c_j^{(l)}(m-k)_j(s-Y_0-k+l)_{m-k-j}\Big)\\ &\qquad\times R_l[p_k](f+g)^{s-k+l}\\ &=\sum(s)_{k-l}\sum(-)^j c_j^{(l)}(m-k)_j(s-k+l)_{m-k-j}f^{m-k-j}\\ &\qquad\times R_l[p_k](f+g)^{s-m+l+j}\\ &=\sum(s)_{m-l-j}(-)^j c_j^{(l)}\sum_k(m-k)_j f^{m-k-j}R_l[p_k](f+g)^{s-m+l+j}\\ &=\sum(s)_{m-l-j}(-)^j c_j^{(l)}R_l\Big[\frac{\partial^j}{\partial s^j}\sigma^T(p)\Big](f,x,df)(f+g)^{s-m+l+j}.\end{aligned}$$

Similarly,

$$\begin{aligned}(27)\quad & P(s,x,D)f^s\\ &=\sum(s)_{m-l-j}(-)^j c_j^{(l)}R_l\Big[\frac{\partial^j}{\partial s^j}\sigma^T(p)\Big](f,x,df)f^{s-m+l+j}.\end{aligned}$$

Any $P(s)\in\mathcal{D}_X[s]$ can be uniquely written as

$$P(s,x,D)=\sum_{\mu=0}^{m}P_{\mu}(s,x,D),$$

where

$$P_{\mu}(s,x,D)=\sum_{k=0}^{\mu}(s-j)_{\mu-k}p_{\mu,k}(x,D),$$

$$p_{\mu,k}(x,D)=\sum_{|\alpha|=k}a_{\mu,k,\alpha}D^{\alpha}.$$

Then, the preceding equalities (26) and (27) say that the coefficient of $(s)_d(f+g)^{s-d}$ in $P(s-Y_0,x,D)(f+g)^s$ and that of $(s)_d f^{s-d}$ in

$P(s, x, D)f^s$ are the same. Therefore, $P(s-Y_0, x, D)\in\mathcal{J}_{f+g}(s)$ when $P(s, x, D)\in\mathcal{J}_f(s)$. Q.E.D. of Claim.

Chapter IV. Results of $b(s)$

In this chapter, we investigate several examples of b-functions. Sections 18~20 are devoted to the study of non-quasi-homogeneous isolated singularities in $\dim X=2, 3$. We add some remarks in § 21, about the b-functions of isolated singularities with modality not greater than 2. Its detailed arguments in case corank $(f)=3$ will be found in [32]. Examples of non-isolated singularities are given in § 22.

§ 17. Two-Dimensional Case

When the space dimension is 2, we can apply Theorem 2. 24. As is shown below, we find "explicit formulae" under some assumptions on f. Let us explain the situation.

First, we assume that f is a locally reduced non-quasi-homogeneous function at $0\in\boldsymbol{C}^2$ such that

(a) $\mathfrak{a}: f=(x^a, y^b)$.

Next, we assume that generators of $\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})$ are given by

$$A_1(s, x, D)=x^a(s-X_2)+A_1'(x, D),$$

and

$$A_2(s, x, D)=y^b(s-X_1)+A_2'(x, D),$$

where $X_k=a_{k1}xD_x+a_{k2}yD_y$, $k=1, 2$, $a_{ij}\in\boldsymbol{Q}^+$, and they satisfy the condition

(b) the weight of $A_1'(x, D)$ ($A_2'(x, D)$, respectively) is greater than that of $a_{21}a$ in X_2 ($a_{12}b$ in X_1, respectively).

Set $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$. There are two cases.

1° rank $A=1$.

Assume $(a_{11}, a_{12})=c\cdot(a_{21}, a_{22})$ $c\in\boldsymbol{Q}$. We write f in the following form: $f=f_0+g$, where f_0 is the sum of monomials in f which have minimal weight, say w, with respect to (a_{21}, a_{22}). Then,

$$x^a(f_0+g)=x^a(wf_0+X_2g)-A_1'(x, D)(f_0+g)$$

$$y^b(f_0+g)=y^b(cwf_0+cX_2g)-A_2'(x, D)(f_0+g).$$

Comparing the terms with minimal weight in these formulae, we have $w=1$ and $c=1$. Thus, $X_1=X_2=X_0=\alpha xD_x+\beta yD_y$, $f=f_0+g$, $X_0f_0=f_0$ and g has the weight greater than 1 with respect to X_0. This shows that, when rank $A=1$, f can be considered as a higher order deformation of weighted homogeneous polynomial. Since $y^bA_1-x^aA_2=\varphi(f_xD_y-f_yD_x)$, $\varphi(0)\neq 0$, we have

$$1>(a+1)\alpha+(b+1)\beta. \tag{2}$$

$2°$ rank $A=2$.

In this case, inequalities $a_{11}\neq a_{21}$, $a_{12}\neq a_{22}$ holds in general. Then the relation $x^aA_2-y^bA_1=\varphi(f_xD_y-f_yD_x)$ again shows

$$\begin{aligned}1&=(a+1)a_{11}+(b+1)a_{12}\\&=(a+1)a_{21}+(b+1)a_{22}.\end{aligned}$$

That is, A can be written in the form

$$A=(1-(a+1)a_{21}-(b+1)a_{12})\begin{bmatrix}\frac{1}{a+1} & \\ & \frac{1}{b+1}\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}(a_{21}, a_{12}).$$

Taking the determinant of coefficients of $\begin{bmatrix}A_1\\A_2\end{bmatrix}$, we have

$$f=c\cdot x^{a+1}y^{b+1}+g,\quad c\neq 0,$$

where g is the sum of monomials whose weight are strictly greater than that of $x^{a+1}y^{b+1}$ with weight of X_1 or X_2.

Moreover, we impose the condition

(c) $L(f)=2$.

Upon these conditions, we conjecture that the action of s is determined by a, b and A. The explicit formulae for b-functions are given as follows.

Conjecture 4.0 (EEF).

$1°$ rank $A=1$. $A=\begin{bmatrix}1\\1\end{bmatrix}(\alpha,\beta)$. *Then,* $\mu=\dfrac{1-\alpha}{\alpha}\cdot\dfrac{1-\beta}{\beta}$,

$$\tilde{P}(t)=\frac{(t^{\alpha}-t)(t^{\beta}-t)}{(1-t^{\alpha})(1-t^{\beta})}+t^{1-a\alpha-b\beta}(1-t)\frac{(1-t^{a\alpha})(1-t^{b\beta})}{(1-t^{\alpha})(1-t^{\beta})}.$$

Moreover, s is semisimple.

$2°$ rank $A=2$.

$$A=\begin{bmatrix}\beta' & \beta\\ \alpha & \alpha'\end{bmatrix}=(1-(a+1)\alpha-(b+1)\beta)\begin{bmatrix}\frac{1}{a+1} & \\ & \frac{1}{b+1}\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}(\alpha,\beta).$$

Then, $\mu=1+\dfrac{b}{\alpha}+\dfrac{a}{\beta}$, *and*

$$\tilde{P}(t)=t+\frac{t^{\alpha+\alpha'}(1-t)(1-t^{b\alpha'})}{(1-t^{\alpha})(1-t^{\alpha'})}+\frac{t^{\beta+\beta'}(1-t)(1-t^{a\beta'})}{(1-t^{\beta})(1-t^{\beta'})}.$$

Set $d=\text{g.c.d.}\ (a+1,b+1)$. *Then,* $-\dfrac{\nu}{d}$, $\nu=1,\cdots,d-1$ *are non semisimple eigenvalue of s of height two.*

We call the formula and proviso about semisimplicity of s in $1°$ (respectively in $2°$) as "EEF" type $1°$ (resp. $2°$).

The common case where formulae type $1°$ and $2°$ could cover formally is the following.

$$A=\begin{bmatrix}1\\1\end{bmatrix}(\alpha,\beta)\quad\text{and}\quad 1=(a+1)\alpha+(b+1)\beta.$$

In this case, even though these two formulae seem very different, they give the same result as directly seen. Of course, this case can never occur according to the restriction (2).

We also conjecture that a second order operator showing that $L(f)=2$ can be chosen in the following way

Type $1°$

$$(s-X_0+c')(s-X_0)+sA'+B',\tag{3}$$

$$c'=1-(a+1)\alpha-(b+1)\beta,$$

where each term in $sA'+B'$ has strictly positive weight with respect to X_0. Note that c' is positive by the inequality (2).

Type 2°

(4) $$(s-X_1)(s-X_2)+sA'+B',$$

where $sA'+B'$ has strictly positive weight with respect to both X_1 and X_2.

Especially when $a=b=1$ in type 1°, we can also take

$$(s+1-\alpha-\beta)(s-X_0)+sA'+B'+(\alpha D_x A_1'+\beta D_y A_2').$$

We add some remarks to "EEF" type 1°.

According to the analysis in case 1°,

$$f=f_0+\text{(higher weight)},\quad X_0 f_0=f_0.$$

The first term of $\tilde{P}_f(t)$ of type 1° is the same with $\tilde{P}_{f_0}(t)$. Since there is a factor $(1-t)$ in the second term, $\tilde{P}_f(t)$ and $\tilde{P}_{f_0}(t)$ can be expanded into the fractional polynomial of the form:

$$\tilde{P}_f(t)=\sum_{\gamma\in C_0} t^\gamma+\sum_{\gamma\in C} t^\gamma,$$

and

$$\tilde{P}_{f_0}(t)=\sum_{\gamma\in C_0} t^\gamma+\sum_{\gamma\in C} t^{\gamma+1}.$$

Note that

(5) $$\min C_0<\min C$$

owing to the inequality (2), because $\min C_0=\alpha+\beta$ and $\min C=1-a\alpha-b\beta$.

There is a natural generalization of "EEF" type 1° to n-dimensional case. That is, if

$(\mathrm{a})_n$ $$\mathfrak{a}: f=(x_1^{a_1},\cdots,x_n^{a_n}),$$

and first order operators associated to it are

$(\mathrm{b})_n$ $$x_i^{a_i}(s-X_0)+\text{(higher weight)},$$

with $X_0=\sum\alpha_i x_i D_i$, and

$(\mathrm{c})_n$ $$L(f)=2,$$

then

$$\tilde{P}(t)=\prod\frac{t^{\alpha_i}-t}{1-t^{\alpha_i}}+t^{(n-1)-\Sigma a_i\alpha_i}(1-t)\prod\frac{1-t^{a_i\alpha_i}}{1-t^{\alpha_i}}. \tag{6}$$

Moreover, s is semisimple.

We refer this formula with proviso as "EEF" type $1_{(n)}$. There are several cases where these conjectures can be verified, as we will discuss later on.

Conditions (a), (b) and (c) are essential. In fact, types $W^{\#}_{1,2q}$ and $W^{\#}_{1,2q-1}$ in § 21 satisfy (a) (with $a=1$, $b=2$) and (c) but violates (b). $\tilde{P}(t)$ is given in [32] and is different from both type 1° and 2°.

The next example satisfies (a) and (b) but violates (c).

Example 4. 1. $f=\frac{1}{n_1}x^{n_1}+\frac{1}{n_2}(y-tx^{m_1})(y+(n_2-1)tx^{m_1})^{n_2-1}$.

t is a non-zero parameter. We impose conditions

$$(n_1-1)/4\leq m_1\leq(n_1-2)/3,\quad 1/n_2<m_1/n_1.$$

Then $\mathfrak{a}: f=(x^{n_1-2m_1-1}, y)$, and first order operators associated with it are the following: $(m_2=n_2-1)$

$$y(s-X_0+(m_1m_2-c)tx^{m_1}D_y)-cm_2t^2x^{2m_1}D_y,$$

$$x^{n_1-2m_1-1}(s-X_0+(m_1m_2-c)tx^{m_1}D_y)-cm_2t^2\frac{(y+m_2tx^{m_1})^{n_2-5}}{\varphi}Q,$$

where

$$Q=(y+m_2tx^{m_1})^3(D_x-m_1m_2tx^{m_1-1}D_y)$$
$$+\frac{m_1tx^{m_1-1}}{y}\{(y^4+m_2tx^{m_1})^4-(m_2tx^{m_1})^4\}D_y$$

$$\varphi=1-m_1m_2^4t^5x^{5m_1-n_1}(y+m_2tx^{m_1})^{n_2-5}.$$

Owing to the inequality $1/n_2<m_1/n_1$, we can check the condition (b). However, condition (c) does not hold. In fact, $2=l(f)<L(f)=3$ in this case. $\tilde{P}(t)$ is given by the following and does not coinside with formula type 1° or 2°.

$$\tilde{P}(t)=\frac{(t^{1/n_1}-t)(t^{1/n_2}-t)}{(1-t^{1/n_1})(1-t^{1/n_2})}+t^{1-(n_1-2m_1-1)/n_1-1/n_2}(1-t)\frac{1-t^{(n_1-2m_1)/n_1}}{1-t^{1/n_1}}$$

$$+t^{1-(n_1-3m_1-1)/n_1-2/n_2}(1-t)\frac{1-t^{(n_1-3m_1)/n_1}}{1-t^{1/n_1}}.$$

See type $X_0^{\flat\flat}$ in § 18 and [32].

§ 18. $x^{n_1}+y^{n_2}+x^{m_1}y^{m_2}$

In this section, we study the typical example

$$f(x, y)=\frac{1}{n_1}x^{n_1}+\frac{1}{n_2}y^{n_2}-tx^{m_1}y^{m_2},$$

where t is a parameter.

We can assume $1\leq m_i\leq n_i-1$ owing to Proposition 2. 10. In the following, c always denotes $\sum\frac{m_i}{n_i}-1$.

When $c=0$, f is weighted homogeneous polynomial with weight $\left(\frac{1}{n_1}, \frac{1}{n_2}\right)$, and hence by Theorem 3. 6, we have

$$\tilde{P}(t)=\frac{(t^{1/n_1}-t)(t^{1/n_2}-t)}{(1-t^{1/n_1})(1-t^{1/n_2})}. \tag{7}$$

When $c\neq 0$, f is of simplex type, and when $c>0$, f is a μ-constant deformation of

$$\frac{1}{n_1}x^{n_1}+\frac{1}{n_2}y^{n_2}. \tag{8}$$

Therefore, the local monodromy of f is the same as that of (8). But $\tilde{P}(t)$ is not given by (7) as is shown below. When $c<0$, $b_f(s)$ may have double roots. Then so do the local monodromy. In the sequel, we use the following notation.

$$X_0=\frac{1}{n_1}xD_x+\frac{1}{n_2}yD_y,\quad X_1=\frac{(n_2-m_2)}{m_1n_2}xD_x+\frac{1}{n_2}yD_y,$$

$$X_2=\frac{1}{n_1}xD_x+\frac{(n_1-m_1)}{m_2n_1}yD_y.$$

First of all, we determine $\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})$.

Proposition 4. 2. $\mathfrak{a}: f$ *are* $\mathcal{J}(s)\cap(\mathcal{D}s+\mathcal{D})$ *are given as follows.*

1. $1\leq m_1\leq n_1/2,\ 1\leq m_2\leq n_2/2$: (x^{m_1-1}, y^{m_2-1})

$$x^{m_1-1}(s-X_2)+\frac{c}{m_1m_2t}y^{n_2-m_2}D_x-\frac{1}{m_1m_2t^2}x^{n_1-m_1-1}y^{n_2-2m_2}(s-X_0),$$

$$y^{m_2-1}(s-X_1)+\frac{c}{m_1m_2t}x^{n_1-m_1}D_y-\frac{1}{m_1m_2t^2}x^{n_1-2m_1}y^{n_2-m_2-1}(s-X_0).$$

2. $n_1/2\leq m_1\leq n_1-1,\ n_2/2\leq m_2\leq n_2-1$: $(x^{n_1-m_1-1}, y^{n_2-m_2-1})$

$$x^{n_1-m_1-1}(s-X_0)-cty^{m_2}D_x-m_1m_2t^2x^{m_1-1}y^{2m_2-n_2}(s-X_2),$$

$$y^{n_2-m_2-1}(s-X_0)-ctx^{m_1}D_y-m_1m_2t^2x^{2m_1-n_1}y^{m_2-1}(s-X_1).$$

3. $n_1+1\leq 2m_1,\ n_2-1\geq 2m_2$:

$$((x^{n_1-m_1}-m_1ty^{m_2}),\ (x^{n_1-m_1-1}y^{n_2-2m_2-1}-m_1^2m_2t^3x^{2m_1-n_1-1}y^{m_2-1})).$$

$$x^{n_1-m_1}(s-X_0)-m_1ty^{m_2}(s-X_1),$$

$$y^{n_2-2m_2-1}\{x^{n_1-m_1-1}(s-X_0)-cty^{m_2}D_x\}$$
$$-m_1^2m_2t^3x^{2m_1-n_1-1}\left\{y^{m_2-1}(s-X_1)+\frac{c}{m_1m_2t}x^{n_1-m_1}D_y\right\}.$$

4. $n_2+1\leq 2m_2,\ n_1-1\geq 2m_1$: $\mathfrak{a}$: *f and first order operators are the same with those of case* 3, *if we exchange x and y, and subscripts* 1 *and* 2.

Proof. One can prove by direct calculation that the operators listed above belong to $\mathcal{J}(s)$. The proof that they actually form a basis is based on Theorem 1.10. We rewrite them in the form $a_\nu(x)s+a_{\nu 1}(x)D_x+a_{\nu_2}(x)D_y$, and calculate $d=\det\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$. Then, we have

$$d=\begin{cases} ct(1-m_1m_2t^2x^{2m_1-n_1}y^{2m_2-n_2})f & \cdots 2\\ -\dfrac{c}{m_1m_2t}\left(1-\dfrac{1}{m_1m_2t^2}x^{n_1-2m_1}y^{n_2-2m_2}\right)f & \cdots 1\\ m_1ct^2f & \cdots 3\\ m_2ct^2f & \cdots 4\end{cases}.$$

Thus they form a basis. Q.E.D.

In general, $L(f)\leq\min(n_1, n_2)$ (cf. Example 3.18).

There are special types of functions for which $L(f)\leq 2$ holds. They are listed in the following table. We denote $a\to b$ under m_ν, to indicate $a\leq m_\nu\leq b$.

Table 4.3

		m_1	m_2
i) $c<0$			
1. X_1		$1\to(n_1+1)/2$	1
2. $X_1^{\#}$		$(n_1+2)/2\to(2n_1+1)/3$	1
3. X_2		1	$1\to(n_2+1)/2$
4. $X_2^{\#}$		1	$(n_2+2)/2\to(2n_2+1)/3$
5. S		$2\to n_1/2$	$2\to n_2/2$
6. $S^{\#}$		$2\to n_1/3$	$(n_2+1)/2$
	and	$(n_1+1)/2$	$2\to n_2/3$
ii) $c>0$			
7. X_0		$(n_1-1)/2\to n_1-1$	n_2-1
	and	n_1-1	$(n_2-1)/2\to n_2-1$
8. $X_0^{\flat}$		$(n_1-1)/3\to(n_1-2)/2$	n_2-1
	and	n_1-1	$(n_2-1)/3\to(n_2-2)/2$
9. Y		$n_1/2\to n_1-2$	$n_2/2\to n_2-2$
10. $Y^{\flat}$		$(2/3)n_1\to n_1-2$	$(n_2-1)/2$
	and	$(n_1-1)/2$	$(2/3)n_2\to n_2-2$
iii) Special cases of i) and ii)			
11. $X_{1,\min}^{\#}$		$(n_1+2)/2$	1
12. $X_{2,\min}^{\#}$		1	$(n_2+2)/2$
13. $S_{\min}(S_{\min}^{\#})$		2	2
14. $X_{0,\max}^{\flat}$		$(n_1-2)/2$	n_2-1
	and	n_1-1	$(n_2-1)/2$
15. $Y_{\max}(Y_{\max}^{\flat})$		n_1-2	n_2-2
$S_{\min}^{\#}$ and $Y_{\max}^{\flat}$ appear only when $\min(n_1, n_2)=3$.			

Using the notations above we can state the following

Theorem 4.4. 1) *The function f enjoys the property* 1*A* (2*A*, 3*A*, *respectively*) *if and only if it has the property* 1*B* (2*B*, 3*B*, *respectively*).

	A	B
1	*Quasi-homogeneous*	X_0, X_1, X_2, *or* $c=0$.
2	$\mathfrak{a}: f=(x^a, y^b)$ $a\geq 1$, $b\geq 1$.	$X_1^{\#}, X_2^{\#}, S, S^{\#}, X_0^{\flat}, Y, Y^{\flat}$.
3	$\mathfrak{a}: f=(x, y)$	$X_{1,\min}^{\#}, X_{2,\min}^{\#}, S_{\min}, S_{\min}^{\#}$ $X_{0,\max}^{\flat}, Y_{\max}, Y_{\max}^{\flat}$.

2) *When f has the property that $\mathfrak{a}: f=(x^a, y^b)$, $a\geq 1$, $b\geq 1$, in* 1) *the equality $L(f)=2$ holds.*

Proof. 1): 1A. If f is of type X_0 and $m_1=n_1-1$, $m_2\geq n_2/2$, then use 2 of Proposition 5.2. If $m_1=n_1-1$, $m_2=(n_2-1)/2$, then use 3. If f

is of type X_1 and $m_2=1$, $m_1\leq n_1/2$, then use 1. If $m_2=1$, $m_1=(n_1+1)/2$, then use 3. The proof for the case X_2 can be done in the same manner.

2A and 3A. These can be derived from Proposition 5. 2 by direct calculation.

To prove 2), we list up second order operators, which certificate that $L(f)=2$.

$$S:\quad (s-X_1)(s-X_2)-\frac{1}{m_1m_2t^2}x^{n_1-2m_1}y^{n_2-2m_2}(s-X_0+c)(s-X_0).$$

$$Y:\quad (s-X_0+c)(s-X_0)-m_1m_2t^2x^{2m_1-n_1}y^{2m_2-n_2}(s-X_1)(s-X_2).$$

$$S^{\#}:\quad (s-X_1)(s-X_2)-\frac{1}{m_1m_2^2t^3}x^{n_1-3m_1}\left(s-X_0+c+\frac{m_1}{n_1}-\frac{1}{n_2}\right)$$
$$\times\{y^{n_2-m_2-1}(s-X_0)-ctx^{m_1}D_y\}\qquad (m_2=(n_2+1)/2).$$

$$X_1^{\#}:\ \left(s-X_1-\frac{n_1}{m_1}c\right)\left\{(s-X_1)+\frac{c}{tm_1}x^{n_1-m_1}D_y\right\}-Q(s),$$

$$Q(s)=\begin{cases}\dfrac{1}{m_1^2t^3}x^{2n_1-3m_1}y^{n_2-3}(s-x_0+c)(s-X_0),\\[2ex] \dfrac{1}{m_1^3t^4}x^{2n_1-3m_1+1}y^{n_2-4}\left(s-X_0+c-\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)\\ \qquad\qquad\times\{x^{n_1-m_1-1}(s-X_0)-ctyD_x\}.\end{cases}$$

$$Y^{\flat}:\ (s-X_0+c)(s-X_0)-m_1^2m_2t^3x^{3m_1-2n_1}$$
$$\times\left(s-X_1-\frac{n_1}{m_1}c+\frac{m_2}{n_2}-\frac{1}{n_2}\right)\left\{y^{m_2-1}(s-X_1)+\frac{c}{m_1m_2t}x^{n_1-m_1}D_y\right\}$$
$$(m_2=(n_2-1)/2).$$

$$X_0^{\flat}:\ \left(s-X_0+c+\frac{m_1}{n_1}-\frac{1}{n_2}\right)\left(s-X_0-ctx\frac{m_1}{n_1}D_y\right)-Q(s),$$

$$Q(s)=\begin{cases}m_1m_2^2t^3x^{3m_1-n_1}y^{n_2-3}(s-X_1)(s-X_2),\\[2ex] m_2^3t^3x^{3m_1-n_1+1}y^{n_2-4}\left(s-X_2-\dfrac{c}{m_2}-\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)\\[2ex] \qquad\times\left\{m_1tx^{m_1-1}(s-X_2)+\dfrac{c}{m_2}yD_x\right\}\quad (m_2=n_2-1).\end{cases}$$

The operator for $X_2^{\#}$ is similar to that of $X_1^{\#}$. In view of these operators, we can conclude that $L(f)=2$. Q.E.D.

Note that second order operators in case 1), 3 can be taken in the form (cf. 175)

$$(s+1-\alpha-\beta)(s-X_0)+\cdots.$$

For instance, in type $Y_{\max}$, $m_1=n_1-2$, $m_2=n_2-2$,

$$\left(s+1-\frac{1}{n_1}-\frac{1}{n_2}\right)(s-X_0)+\frac{m_1m_2t^2}{c}x^{n_1-4}y^{n_2-4}(s-X_1)(s-X_2)$$

$$+\frac{ct^2}{\varphi}x^{n_1-5}y^{n_2-5}\left\{\frac{m_1}{n_1}x^2P_1+\frac{m_2}{n_2}y^2P_2+\left(\frac{m_1}{n_1}+\frac{m_2}{n_2}\right)\frac{m_1m_2}{c}xy(s-X_0)\right\}$$

$$-ct\left\{\frac{y^{n_2-4}}{n_1}P_1^2+\frac{x^{n_1-4}}{n_2}P_2^2\right\},$$

$$P_1=\frac{-1}{\varphi}(yD_x+m_1tx^{n_1-3}D_y),\quad P_2=\frac{-1}{\varphi}(m_2ty^{n_2-3}D_x+xD_y),$$

$$\varphi=1-m_1m_2t^2x^{n_1-4}y^{n_2-4}.$$

We exibit some special cases.

1. $n_1=3$. Then f is quasi-homogeneous or $L(f)=2$: types $X_2^\sharp$, $Y^\flat$, $X_0^\flat$, $S^\sharp$.
2. $n_1=4$. Then f is quasi-homogeneous or $L(f)=2$, except next three cases.

① $m_1=3$; $(n_2-1)/3>m_2\geq 2$, $m_2\neq n_2/4$.

They are case (2, 3; a).

② $m_1=1$; $m_2=n_2-2$ or $2(n_2+1)/3$. They are also (2, 3; a).

③ $m_1=1$; $n_2-3\geq m_2>2(n_2+1)/3$, $m_2\neq 3n_2/4$. $L(f)\leq 3$.

In general, following four types listed in Table 4. 5 are case (2, 3; a). $(\mathfrak{a}+\mathcal{O}f)$: $a=(x^{a'}, y)$

Table 4.5 i)

	m_1	m_2	condition	a'
$S^{\sharp\sharp}$	$2\to n_1/4$	$(n_2+2)/2$	$\frac{m_1}{n_1}-\frac{1}{n_2}<\frac{1}{2}$	m_1-1
$X_0^{\flat\natural}$	$(n_1-1)/3\to(n_1-2)/2$	n_2-2	$\frac{m_1}{n_1}>\frac{2}{n_2}$	n_1-2m_1-1
$X_0^{\flat\flat}$	$(n_1-1)/4\to(n_1-2)/3$	n_2-1	$\frac{m_1}{n_1}>\frac{1}{n_2}$	n_1-3m_1-1
$X_1^{\sharp\cdot}$	$2(n_1+1)/3$	1	$\frac{2}{n_1}+\frac{3}{n_2}<1$	1

Same notations are used when we exchange x and y and subscripts 1 and 2.

The ideal $\mathfrak{a}: f$ and $a(x) = x^{k_1} y^{k_2}$ are listed up in the following.

Table 4.5 ii)

	$\mathfrak{a}:f$	k_1	k_2
$S^{\sharp\sharp}$	$x^{m_1-1}y,\ y^{(n_2-2)/2}-m_2tx^{m_1}$	n_1-m_1	$(n_2-2)/2$
$X_0^{\flat\natural}$	$x^{n_1-2m_1-1}y,\ y^2-m_2tx^{m_1}$	$2m_1$	n_2-4
$X_0^{\flat\flat}$	$x^{n_1-2m_1-1},\ y-m_2tx^{m_1}$	$3m_1$	n_2-3
$X_1^{\sharp\cdot}$	$x^{(n_1+1)/3},\ x^{(n_1-2)/3}-m_1ty$	m_1-2	n_2-2

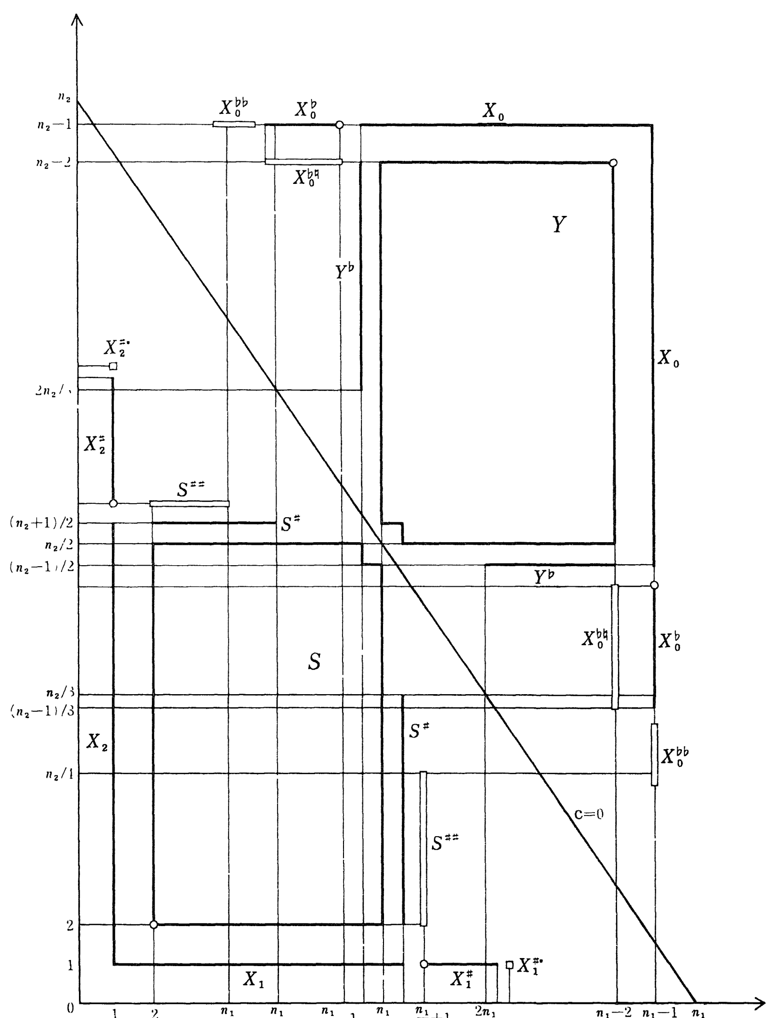

Fig. 4.6

The structure of $\mathcal{J}(s)$ in these four types and the determination of $b(s)$ is found in [32].

Note that Example 2. 29 is of type $X_0^{\flat\flat}$, and Example 4. 1 is also of type $X_0^{\flat\flat}$ if we performe the coördinate transform $X=x$, $Y=y-(n_2-1)tx^{m_1}$.

The followings are proved to be $L(f)\leq 3$. We conjecture that equality holds.

$$(n_1+2)/2\leq m_1\leq 2n_1/3\,,\quad 2\leq m_2\leq n_2/3\,.\quad \text{e.g. } S^{\#\#}.$$

$$2n_1/3\leq m_1\leq n_1-2\,,\quad n_2/3\leq m_2\leq (n_1-2)/2\,.\quad \text{e.g. } X_0^{\flat\natural}.$$

$\mathfrak{a}: f$ is given by $(x^{2m_1-n_1-1}y^{m_2-1}, x^{n_1-m_1}-m_1ty^{m_2})$ and $(x^{n_1-m_1-1}y^{n_2-2m_2-1}, x^{n_1-m_1}-m_1ty^{m_2})$ respectively.

Theorem 4. 7. *When $\mathfrak{a}: f=(x^a, y^b)$, "EEF" holds for f. Parameters are listed in the following table. Moreover, we can choose second order operators as indicated in* (3) *and* (4).

Type 1°.

	α	β	a	b
$Y, Y^{\flat}$	$1/n_1$	$1/n_2$	n_1-m_1-1	n_2-m_2-1
$X_0^{\flat}$	$1/n_1$	$1/n_2$	$\left(\begin{array}{l}1\\ n_1-2m_1-1\end{array}\right.$	$\begin{array}{l}n_2-2m_2-1\\ 1\end{array}$
$X_1^{\#}$	$(n_2-m_2)/m_1n_2$	$1/n_2$	$2m_1-n_1-1$	1
$X_2^{\#}$	$1/n_1$	$(n_1-m_1)/m_2n_1$	1	$2m_2-n_2-1$

Type 2°.

$S, S^{\#}$ $\qquad a=m_1-1\,,\quad b=m_2-1\,.$

$$A=\begin{pmatrix}\dfrac{n_2-m_2}{m_1n_2} & \dfrac{1}{n_2}\\[2ex] \dfrac{1}{n_1} & \dfrac{n_1-m_1}{m_2n_1}\end{pmatrix}.$$

Proof. Generators of $\mathcal{J}(s)$ were already discussed. In order to prove that $\tilde{b}(s)$ is given by "EEF", we determine eigenvectors and root vectors of s in $\mathcal{H}om_{\mathcal{D}}(\tilde{\mathcal{M}}, \mathcal{B}_{pt})$ explicitly. Here, we performe this procedure taking as examples type Y and S. Calculation for other types can be given similarly.

We use the following notation:

$(a)_b = a(a-1)\cdots(a-b+1)$,

$[a]_b = a(a+1)\cdots(a+b-1)$.

$(p(s))_{\mathrm{red}} = \prod(s+\alpha_i)$, for a polynomial $p(s) = \prod(s+\alpha_i)^{\varepsilon_i}$, $\varepsilon_i \neq 0$, $\alpha_i \neq \alpha_j$, $i \neq j$.

For the δ-functions, we use the notation

$$\delta[i, j] = D_x^i D_y^j \delta(x, y).$$

Since indices i and j are complicated, we do not adopt the usual notation $\delta^{(i,j)}$.

Y: The generators of $\mathcal{J}(s)$ are

$$x^{n_1-m_1-1}\{(s-X_0) - m_1 m_2 t^2 x^{k_1} y^{k_2}(s-X_2)\} - cty^{m_2}D_x,$$

$$y^{n_2-m_2-1}\{(s-X_0) - m_1 m_2 t^2 x^{k_1} y^{k_2}(s-X_1)\} - ctx^{m_1}D_y,$$

$$(s-X_0+c)(s-X_0) - m_1 m_2 t^2 x^{k_1} y^{k_2}(s-X_1)(s-X_2),$$

$$k_1 = 2m_1 - n_1, \quad k_2 = 2m_2 - n_2.$$

We set $s_{ij} = \dfrac{i+1}{n_1} + \dfrac{j+1}{n_2}$. Define delta functions Δ_{ij}, Δ'_{ij} and u_{ij} as follows.

$$\Delta_{ij} = \delta[i, j] + \sum_{l \geq 1} c^l_{i,j}\delta[i-lk_1, j-lk_2]$$

$$c^l_{i,j} = \frac{(-)^{l(k_1+k_2)}}{(2l)!}(n_1 n_2 t^2)^l \left[\frac{i+1}{n_1}\right]_l \left[\frac{j+1}{n_2}\right]_l (i)_{lk_1}(j)_{lk_2}.$$

$$\Delta'_{ij} = \delta[i, j] + \sum_{l \geq 1} c'^l_{i,j}\delta[i-lk_1, j-lk_2],$$

$$c'^l_{i,j} = \frac{(-)^{l(k_1+k_2)}}{(2l+1)!}(n_1 n_2 t^2)^l \left[\frac{i+m_1+1}{n_1}\right]_l \left[\frac{j+m_2+1}{n_2}\right]_l (i)_{lk_1}(j)_{lk_2}.$$

$$u_{ij} = \Delta_{ij} + (-)^{n_1+m_1+m_2}\frac{(j)_{m_2}t}{(i+n_1-m_1)_{n_1-m_1-1}}\Delta'_{i+n_1-m_1, j-m_2}$$

$$+ (-)^{n_2+m_1+m_2}\frac{(i)_{m_1}t}{(j+n_2-m_2)_{n_2-m_2-1}}\Delta'_{i-m_1, j+n_2-m_2}.$$

Then, u_{ij} where $(0 \leq i \leq n_1-2,\ 0 \leq j \leq m_2-1)$ or $(0 \leq i \leq m_1-1,\ 0 \leq j \leq n_2-2)$ are eigenvectors belonging to eigenvalue $-s_{ij}$.

When $(0 \leq i \leq n_1-m_1-2,\ 0 \leq j \leq n_2-m_2-2)$, Δ'_{ij} are also eigenvectors belonging to $-s_{ij}-c$.

Therefore,

$$\tilde{P}(t)=\frac{(t^{1/n_1}-t)\,(t^{1/n_2}-t)}{(1-t^{1/n_1})\,(1-t^{1/n_2})}+t^{1-(n_1-m_1)/n_1-(n_2-m_2)/n_2}$$

$$\times(1-t)\frac{(t^{1/n_1}-t^{(n_1-m_1)/n_1})\,(t^{1/n_2}-t^{(n_2-m_2)/n_2})}{(1-t^{1/n_1})\,(1-t^{1/n_2})}.$$

$$\tilde{b}(s)=\Big(\prod_{\substack{\binom{0\le i\le n_1-2}{0\le j\le m_2-1}\\ \binom{0\le i\le m_1-1}{0\le j\le n_2-2}}}(s+s_{ij})\prod_{\substack{0\le i\le n_1-m_1-2\\ 0\le j\le n_2-m_2-2}}(s+s_{ij}+c)\Big)_{\mathrm{red}}.$$

S: The generators of $\mathcal{J}(s)$ are

$$x^{m_1-1}\left\{(s-X_2)-\frac{1}{m_1m_2t^2}x^{h_1}y^{h_2}(s-X_0)\right\}+\frac{c}{m_1m_2t}y^{n_2-m_2}D_x,$$

$$y^{m_2-1}\left\{(s-X_1)-\frac{1}{m_1m_2t^2}x^{h_1}y^{h_2}(s-X_0)\right\}+\frac{c}{m_1m_2t}x^{n_1-m_1}D_y,$$

$$(s-X_1)\,(s-X_2)-\frac{1}{m_1m_2t^2}x^{h_1}y^{h_2}(s-X_0+c)\,(s-X_0),$$

$$h_1=n_1-2m_1,\quad h_2=n_2-2m_2.$$

Set

$$s_{ij}^1=\frac{(n_2-m_2)\,(i+1)+m_1(j+1)}{m_1n_2},$$

$$s_{ij}^2=\frac{m_2(i+1)+(n_1-m_1)\,(j+1)}{m_2n_1}.$$

$d=\text{g.c.d.}\,(m_1,m_2)$, $m_1=dm_1'$, $m_2=dm_2'$. s_{ij}^1 and s_{ij}^2 coincide for $i=rm_1'-1$, $j=rm_2'-1$, $1\le r\le d$. We denote this value r/d as s_r.

We make use of the following delta functions, $0\le i\le n_1-2$, $0\le j\le n_2-2$.

$$\Delta_{ij}^1=\delta[i,j]+\sum_{l\ge1}c_{i,j}^{l,1}\delta[i-lh_1,j-lh_2]$$

$$c_{i,j}^{l,1}=\frac{(i)_{lh_1}(j)_{lh_2}(-)^{l(h_1+h_2)}\left[\dfrac{i+1}{m_1}\right]_{2l}}{(n_1n_2t^2)^l\,l!\,[d_{ij}^1+1]_l}$$

$$d_{ij}^1=\frac{m_2}{n_2}\left(\frac{i+1}{m_1}-\frac{j+1}{m_2}\right).$$

Similarly, we define Δ^2_{ij} by exchanging i and j and subscripts 1 and 2 in the above formulae. Since $d^1_{i,j}+1=\frac{m_2}{n_2}\left(\frac{i+1}{m_1}+\frac{n_2-j-1}{m_2}\right)$ and $d^2_{i,j}+1=\frac{m_1}{n_1}\left(\frac{n_1-i-1}{m_1}+\frac{j+1}{m_2}\right)$ are positive when $0\leq i\leq n_1-2$, $0\leq j\leq n_2-2$, they are well-defined.

Then,

$$\Delta_r=\Delta^1_{rm_1'-1,rm_2'-1}=\Delta^2_{rm_2'-1,rm_2'-1},\quad 1\leq r\leq d$$

are eigenfunction belonging to eigenvalue $-s_r$.

For other values of (i, j), we set

$$u^1_{ij}=\Delta^1_{ij}+\frac{(j)_{n_2-m_2}(-)^{n_2+m_1+m_2}}{m_1n_2td^1_{ij}(i+m_1)_{m_1-1}}\Delta^1_{i+m_1,j-n_2+m_2},$$

for $0\leq i\leq n_1-2$, $0\leq j\leq n_2-2$, and also set u^2_{ij} for $0\leq i\leq n_1-2$, $0\leq j\leq m_2-2$, in a usual manner.

Then, u^1_{ij} and u^2_{ij} are eigenvectors belonging to $-s^1_{ij}$ and $-s^2_{ij}$.

When $d\geq 2$, non-semisimple roots appear. We define the series e^l_r by the following recursion formula.

$$\begin{cases} e^0_r=0 \\ e^l_r=\dfrac{1}{n_1n_2l^2t^2}(i-(l-1)h_1)_{h_1}(j-(l-1)h_2)_{h_2}(-)^{h_1+h_2} \\ \qquad\times\left(\dfrac{r}{d}+2l-2\right)\left(\dfrac{r}{d}+2l-1\right)e^{l-1}_r \\ \qquad-\dfrac{1}{c}\left\{\dfrac{c+1}{l}-\left(\dfrac{1}{\frac{r}{d}+2l-2}+\dfrac{1}{\frac{r}{d}+2l-1}\right)\right\}c^l_r, \end{cases}$$

where $c^l_r=c^{l,1}_{rm_1'-1,rm_2'-1}=c^{l,2}_{rm_1'-1,rm_2'-1}$.

Set

$$\Delta_r'=\sum_{l\geq 1}e^l_r\delta[rm_1'-1-lh_1,\ rm_2'-1-lh_2],$$

$$\Delta_r''=\sum_{l\geq 0}\left(c^l_r-\frac{r}{d}e^l_r\right)\delta[\ {}''\ ,\ {}''\ ].$$

Then, $u_r'=\begin{bmatrix}\Delta_r'\\ \Delta_r''\end{bmatrix}$ and $u_r=\begin{bmatrix}\Delta_r\\ -\frac{r}{d}\Delta_r\end{bmatrix}$ form a root subspace belonging to eigenvalue $-\frac{r}{d}$ for $1\leq r\leq d-1$.

$$s(u_r', u_r) = (u_r', u_r)\begin{bmatrix} -\frac{r}{d} & \\ 1 & -\frac{r}{d} \end{bmatrix}.$$

Thus, b-functions are determined.

$$\tilde{b}(s) = \prod_{r=1}^{d-1}\left(s+\frac{r}{d}\right)\Big(\prod_{\substack{0\le i\le m_1-2\\ 0\le j\le n_2-2}}(s+s_{ij}^1)\prod_{\substack{0\le i\le n_1-2\\ 0\le j\le m_2-2}}(s+s_{ij}^2)\,(s+1)\Big)_{\mathrm{red}},$$

$$\tilde{P}(t) = t + \frac{t^{a_{11}+a_{12}}(1-t)\,(1-t^{(m_1-1)a_{11}})}{(1-t^{a_{11}})\,(1-t^{a_{12}})} + \frac{t^{a_{21}+a_{22}}(1-t)\,(1-t^{(m_2-1)a_{22}})}{(1-t^{a_{21}})\,(1-t^{a_{22}})},$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{n_2-m_2}{m_1 n_2} & \dfrac{1}{n_2} \\ \dfrac{1}{n_1} & \dfrac{n_1-m_1}{m_2 n_1} \end{bmatrix}.$$

$$\chi(\tau) = \frac{(\tau^{m_2 n_1}-1)\,(\tau^{m_1 n_2}-1)}{(\tau^{n_1}-1)\,(\tau^{n_2}-1)}(\tau-1)$$

is the characteristic polynomial of the local monodromy of $f^{-1}(0)$, when g.c.d.$((n_1-m_1), m_2)$ = g.c.d.$(m_1, (n_2-m_2)) = 1$.

§ 19. Other Examples

In this section, we show four examples in 2-dimensional case and two examples in 3-dimensional case. Interesting examples in 3-dimension can be found in [32].

Reduced quasi-homogeneous polynomial in 2-variables is essentially one of the following three types.

$$x^{n_1}+y^{n_2}, \quad x(x^{n_1}+y^{n_2}), \quad xy(x^{n_1}+y^{n_2}).$$

In § 18, we investigated non-quasi-homogeneous functions derived from the first type. Examples 4. 8～11 are dealt with those derived from the second and the third type. In this section, we restrict ourselves to the most typical classes, that is, those similar to Y and S in § 18. It should be noted that one can take

$$(x^{n_1-m_1}+y^{m_2})\,(x^{m_1}+y^{n_2-m_2})$$

instead of type S owing to Proposition 2. 12. We adopt this form in

Examples xS and xyS.

We exhibit generators of $\mathcal{J}(s)$, and values of parameters in "EEF" in Table 4. 12. The determination of the action of s for these examples is similar to that for type Y and S and we omit the details.

Example 4. 8. xY

$$f=x\left(\frac{1}{n_1}x^{n_1}+\frac{1}{n_2}y^{n_2}-x^{m_1}y^{m_2}\right).$$

$$n_1/2\leq m_1\leq n_1-2,\quad n_2/2\leq m_2\leq n_2-1.$$

In this case,

$$\mu=(n_1+1)(n_2-1)+1.$$

Set

$$c=(m_1m_2-(n_1-m_1)(n_2-m_2))/(n_1+1)n_2,$$

$$k=((m_1+1)n_2-m_2)/n_2,$$

$$k'=\{(m_1+1)^2n_2+m_2^2-(m_1+1)m_2n_2\}/n_2m_2^2,$$

$$\varphi=1-kx^{2m_1-n_1}y^{2m_2-n_2},$$

and

$$X_0=(n_2xD_x+n_2yD_y)/(n_1+1)n_2.$$

$\mathcal{J}(s)$ is generated by

$$x^{n_1-m_1}(s-X_0)-\frac{c}{\varphi}\{m_2x^{2m_1-n_1+1}y^{m_2-1}D_x+(x^{m_1}+kxy^{2n_2-m_2-1})D_y\}$$

$$y^{n_2-m_2-1}(s-X_0)-\frac{c}{\varphi}y^{m_2}\{xy^{n_2-m_2-1}D_x-(kx^{m_1-1}+k'y^{n_2-m_2})D_y\}$$

and

$$(s-X_0+c)(s-X_0)-\frac{k}{\varphi}x^{2m_1-n_1}y^{2m_2-n_2}$$
$$\times\{(2X_0-X_1-X_2-c)s+X_1X_2-X_0^2+X_0c\}.$$

Example 4. 9. xyY

$$f=xy(x^{n_1}+y^{n_2}-x^{m_1}y^{m_2})$$

$$n_1/2 \leq m_1 \leq n_1 - 1, \quad n_2/2 \leq m_2 \leq n_2 - 1, \quad \mu = (n_1+1)(n_2+1).$$

Set

$$c = \{m_1 m_2 - (n_1 - m_1)(n_2 - m_2)\} / \{(n_1+1)(n_2+1) - 1\}$$

and

$$X_0 = (n_2 x D_x + n_1 y D_y) / \{(n_1+1)(n_2+1) - 1\}.$$

$\mathcal{J}(s)$ is generated by

$$x^{n_1 - m_1}(s - X_0) + \frac{c}{c_1} y^{m_2}(s - X_1),$$

$$y^{n_2 - m_2}(s - X_0) + \frac{c}{c_2} x^{m_1}(s - X_2),$$

and

$$(s - X_0 + c)(s - X_0) - \frac{c^2}{c_1 c_2} x^{2m_1 - n_1} y^{2m_2 - n_2}(s - X_1)(s - X_2).$$

Example 4. 10. xS

$$f = x(x^m + y^n)(x^\mu + y^\nu), \quad \mu \geq m, \ n \geq \nu.$$

Milnor $\sharp = (m+1)(n+\nu) + (\nu - 1)(m + \mu)$.

Set

$$c = n\mu - \nu m, \quad m' = m + \mu + 1, \quad k = \frac{n(\mu(n+\nu) + \nu)}{\nu(m(n+\nu) + n)}.$$

$$\Theta = \nu x D_x + \mu y D_y, \quad \Psi = n x D_x + m y D_y,$$

$$X_1 = \Psi / \{m(n+\nu) + n\}, \quad X_2 = \Theta / \{\mu(n+\nu) + \nu\}.$$

$$Y_1 = \Theta / \nu m', \quad Y_2 = \Psi / n m'.$$

$\mathcal{J}(s)$ is generated by

$$x^m(s - Y_1) - \frac{c y^n}{m m' \nu}(s - x D_x) - \frac{n\mu}{m\nu} x^\mu y^{n-\nu}(s - X_1)$$

$$y^{\nu-1}(s - X_1) - k x^{\mu - m} y^{n-\nu}(s - Y_1) + \frac{c x^\mu}{\nu(m(n+\nu) + n)} D_y,$$

and

$$(s - X_1)(s - Y_1) - k x^{\mu - m} y^{n-\nu}(s - X_2)(s - Y_2).$$

Example 4.11. xyS

$$f=xy(x^m+y^n)(x^\mu+y^\nu), \quad \mu\geq m, \quad n\geq\nu.$$

Milnor $\sharp=(n+1)(n+\nu)+(\nu+1)(m+\mu)+1.$

Notations c, Θ and Ψ are same as in 4.10. Set

$$c'=(m+\mu+1)(n+\nu+1)-1,$$

$$k_{X_1}=m(n+\nu)+m+n, \quad k_{X_2}=\mu(n+\nu)+\mu+\nu.$$

$$k_{Y_1}=\nu(\mu+m)+\mu+\nu, \quad k_{Y_2}=n(m+\mu)+m+n,$$

$$X_1=\Psi/k_{X_1}, \quad X_2=\Theta/k_{X_2}, \quad Y_1=\Theta/k_{Y_1}, \quad Y_2=\Psi/k_{Y_2}.$$

$\mathcal{J}(s)$ is generated by

$$x^m(s-Y_1)+\frac{k_{X_2}}{k_{Y_1}}y^n(s-X_2),$$

$$y^\nu(s-X_1)+\frac{k_{Y_2}}{k_{X_1}}x^\mu(s-Y_2),$$

and

$$(s-X_1)(s-Y_1)-\frac{k_{X_2}k_{Y_2}}{k_{X_1}k_{Y_1}}x^{\mu-m}y^{n-\nu}(s-X_2)(s-Y_2).$$

One can apply "EEF" for preceding four examples by setting parameters in the following Table

Table 4.12

Type 1°	α	β	a	b
xY	$1/(n_1+1)$	$n_1/\{(n_1+1)n_2\}$	n_1-m_1	n_2-m_2-1
xyY	$n_2/\{(n_1+1)(n_2+1)-1\}$	$n_1/\{(n_1+1)(n_2+1)-1\}$	n_1-m_1	n_2-m_2

Type 2°	A	a	b
xS	$\begin{pmatrix} \frac{n}{m(\nu+n)+n} & \frac{m}{m(\nu+n)+n} \\ \frac{1}{m+\mu+1} & \frac{\mu}{\nu(m+\mu)+\nu} \end{pmatrix}$	m	$\nu-1$
xyS	$\begin{pmatrix} \frac{n}{m(n+\nu)+m+n} & \frac{m}{m(n+\nu)+m+n} \\ \frac{\nu}{\nu(\mu+m)+\mu+\nu} & \frac{\mu}{\nu(\mu+m)+\mu+\nu} \end{pmatrix}$	m	ν

Example 4.13.

$$T_{p,q,r}:\ f=\frac{1}{p}x^p+\frac{1}{q}y^q+\frac{1}{r}z^r-axyz$$

$$\mu=p+q+r-1.\quad c=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1,\quad a:\ \text{parameter}$$

$$X_0=\frac{1}{p}xD_x+\frac{1}{q}yD_y+\frac{1}{r}zD_z,\quad X=\frac{1}{3}(xD_x+yD_y+zD_z)$$

$$X_1=X_0-cxD_x,\quad X_2=X_0-cyD_y,\quad X_3=X_0-czD_z.$$

$$X_{210}=\frac{-1}{\varphi}(y^{q-2}z^{r-3}D_x+az^{r-2}D_y+a^2xD_z),$$

and X_{120} etc. are defined by permutation of variables and $\{p, q, r\}$.

$$Q=\left(\frac{1}{3}-\frac{1}{p}\right)x^{p-3}X_{201}\cdot X_{210}+\left(\frac{1}{3}-\frac{1}{q}\right)y^{q-3}X_{120}\cdot X_{021}$$
$$+\left(\frac{1}{3}-\frac{1}{r}\right)z^{r-3}X_{012}\cdot X_{102}$$

$$\varphi=a^3-x^{p-3}y^{q-3}z^{r-3}.$$

The generators of $\mathcal{J}(s)$ are X_{ij}'s and

$$y(s-X_1)-cx^{p-2}X_{210},\ z(s-X_2)-cy^{q-2}X_{021},\ x(s-X_3)-cz^{r-2}X_{102},$$

$$(s-X_0+c)(s-X)-\frac{2c}{\varphi}x^{p-3}y^{q-3}z^{r-3}\left(s-\frac{1}{2}(X_0+X)\right)+acQ,$$

or

$$(s-X_1)(s-X_2)-c^2x^{p-3}y^{q-3}X_{210}\cdot X_{120}$$
$$-\frac{cx^{p-3}y^{q-3}z^{r-3}}{\varphi}(3s-X_2+2X_0),\ \text{etc.}$$

$\begin{pmatrix}\delta\\-\delta\end{pmatrix}$ and $\begin{pmatrix}0\\\delta\end{pmatrix}$ forms a root subspace belonging to eigenvalue -1. Other eigenvectors are

$$\Delta_\nu^{(1)}=D_x^\nu\delta,\quad 1\le\nu\le p-2\leftrightarrow -1-\frac{\nu}{p}$$

$$\Delta_{p-1}^{(1)}=\left(D_x^{p-1}+\frac{(p-1)!}{a}(-)^{p-1}D_yD_z\right)\delta\leftrightarrow -1-\frac{p-1}{p},$$

and similar delta functions by exchanging x, y, z.

$$\tilde{b}(s) = (s+1)^2\Bigl(\prod_{\substack{1\leq\nu\leq p-1\\ 1\leq\nu'\leq q-1\\ 1\leq\nu''\leq r-1}}\Bigl(s+1+\frac{\nu}{p}\Bigr)\Bigl(s+1+\frac{\nu'}{q}\Bigr)\Bigl(s+1+\frac{\nu''}{r}\Bigr)\Bigr)_{\mathrm{red}}.$$

It is remarkable that

$$\tilde{P}(t) = -t+t(1-t)\Bigl(\frac{1}{1-t^{1/p}}+\frac{1}{1-t^{1/q}}+\frac{1}{1-t^{1/r}}\Bigr)$$

and exponents $(1/p, 1/q, 1/r)$ can be found through the coefficients of X_1, X_2, X_3 in the following manner.

$$\begin{pmatrix} 1-\frac{1}{q}-\frac{1}{r} & \frac{1}{q} & \frac{1}{r} \\ \frac{1}{p} & 1-\frac{1}{p}-\frac{1}{r} & \frac{1}{r} \\ \frac{1}{p} & \frac{1}{q} & 1-\frac{1}{p}-\frac{1}{q} \end{pmatrix} = \Bigl(1-\frac{1}{p}-\frac{1}{q}-\frac{1}{r}\Bigr)I_3$$

$$+\begin{pmatrix}1\\1\\1\end{pmatrix}(1/p, 1/q, 1/r).$$

This is similar to "EEF" type 2°.

Example 4.14. $f=\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{1}{4}z^4-axyz^2$. a: parameter.

$\mu=12$, $c=1/6$. Set

$$X_0=\frac{1}{3}xD_x+\frac{1}{3}yD_y+\frac{1}{4}zD_z,\quad \varphi=1-2a^3z^2.$$

$$Q_1=\frac{a}{\varphi}\Bigl(\frac{z}{a}D_x+2axzD_y+yD_z\Bigr),\quad Q_2=\frac{a}{\varphi}\Bigl(2ayzD_x+\frac{z}{a}D_y+xD_z\Bigr).$$

$$Q=Q_1Q_2+Q_2Q_1.$$

$\mathcal{J}(s)$ is generated by X_{ij} and

$$x(s-X_0)-\frac{a}{6\varphi}z^2(yD_x+az^2D_y+a^2xzD_z)$$

$$y(s-X_0)-\frac{a}{6\varphi}z^2(az^2D_x+xD_y+a^2yzD_z)$$

$$z(s-X_0)-\frac{a}{6\varphi}\{(axz^2+y^2)azD_x+(ayz^2+x^2)azD_y+xyD_z\}$$

$$(s-X_0+c)(s-X_0)+\frac{a^2z^2}{6^2\varphi}\left(5s+\frac{11}{6}(xD_x+yD_y)+\frac{3}{2}zD_z\right)-\frac{a^2z^2}{12}Q.$$

The correspondence of eigenvectors and eigenvalues are as follows. (We omit the minus sign).

$$\delta\leftrightarrow\frac{11}{12},\frac{13}{12},\quad D_x\delta,\ D_y\delta\leftrightarrow\frac{5}{4},\quad D_xD_y\delta\leftrightarrow\frac{19}{12},\quad \begin{matrix}(D_yD_z^2-aD_x^2)\delta\\(D_xD_z^2-aD_y^2)\delta\end{matrix}\leftrightarrow\frac{7}{4},$$

$$D_z\delta\leftrightarrow\frac{7}{6},\quad \begin{matrix}D_yD_z\delta\\D_xD_z\delta\end{matrix}\leftrightarrow\frac{3}{2},\quad D_z^2\delta\leftrightarrow\frac{17}{12},\quad \left(D_xD_yD_z+\frac{a}{3}D_z^3\right)\delta\leftrightarrow\frac{13}{12},$$

Thus b-function is determined as follows.

$$\tilde{b}(s)=\left(s+\frac{3}{2}\right)\left(s+\frac{5}{4}\right)\left(s+\frac{7}{4}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{11}{6}\right)$$

$$\times\left(s+\frac{11}{12}\right)\left(s+\frac{13}{12}\right)\left(s+\frac{17}{12}\right)\left(s+\frac{19}{12}\right)$$

$$\tilde{P}(t)=\frac{(t^{1/3}-t)(t^{1/3}-t)(t^{1/4}-t)}{(1-t^{1/3})(1-t^{1/3})(1-t^{1/4})}+t^{2-1/3-1/3-1/4}(1-t).$$

In view of this, $\tilde{P}(t)$ is given by "EEF" type $1_{(3)}$.

§ 20. Remarks on the Canonical Forms of Isolated Singularities

According to V. I. Arnold, isolated singularities with modality not greater than two are completely classified, up to stable equivalence, by the following lists [2], [5].

1. 0-modal case

   A_k, D_k, E_6, E_7, E_8.

2. 1-modal case (with parameter a)

   ① P_8, X_9, J_{10}.

   ② P_{p+5}, $R_{p,q}$, $T_{p,q,r}$, X_{p+5}, $Y_{p,q}$, J_{p+4}

   ③ 14 exceptional families.

3. 2-modal case (with parameters b and c or $\boldsymbol{a}=a_0+a_1y$)

   ① $J_{3,0}$, $Z_{1,0}$, $W_{1,0}$, $Q_{2,0}$, $S_{1,0}$, $U_{1,0}$

② $J_{3,p}$, $Z_{1,p}$, $W_{1,p}$, $W^{\#}_{1,2q-1}$, $W^{\#}_{1,2q}$, $Q_{2,p}$, $S_{1,p}$, $S^{\#}_{1,2q-1}$, $S^{\#}_{1,2q}$, $U_{1,2q-1}$, $U_{1,2q}$.

③ 14 exceptional families.*

The case 1 is weighted homogeneous. 2-① and 3-① ($c=0$ or $a_1=0$) are weighted homogeneous 1-parameter family. As for 2-③ and 3-③, they are weighted homogeneous when $a=0$ and $a_0=a_1=0$ respectively, and forms μ-constant family of deformation. They are non-quasi-homogeneous and of simplex type, and hence if $a_0\neq 0$ in 3-③, we can assume $a_1=0$ by Proposition 3. 12.

2-② and 3-② except $W^{\#}_{1,p}$ are also of simplex type, and we can assume $a_1=0$ in 3-②, except $W^{\#}_{1,p}$.

Theorem 4. 15. *In all the canonical forms with modality less than three, $L(f)=2$ holds if f is non-quasi-homogeneous.* $\mathfrak{a}:f$ *is given as follows.*

(x, y) *or* (x, y, z)	2-②, 2-③ *with* $a\neq 0$, 3-① *with* $c\neq 0$ *or* $a_1\neq 0$, *and* 3-③ *with* $a_0=0$, $a_1\neq 0$.
(x, y^2) *or* (x, y^2, z)	3-②, *and* 3-③ *with* $a_0\neq 0$.

We can determine the action of s in $\mathcal{H}om_{\mathcal{D}}(\tilde{\mathcal{M}}, \mathcal{B}_{pt})$ and know $\tilde{b}(s)$ and local monodromy. Especially,

1. s is semisimple in cases 2-③, 3-② and 3-③.
2. In cases P_{p+5}, $R_{p,q}$, $T_{p,q,r}$, $\tilde{b}(s)$ has a double factor $(s+1)^2$, and in X_{p+5}, $Y_{p,q}$, J_{p+4}, $\left(s+\frac{1}{2}\right)^2$.
3. In the cases of two variables except $W^{\#}_{1,p}$, and three variables in 2-③ and 3-③, *"EEF" holds.* Although $W^{\#}_{1,p}$ satisfies conditions (a) and (c) in § 17, it does not satisfy (b).

As for the proof of this theorem, the author restrict himself to cases corank $(f)=2$ except $W^{\#}_{1,p}$, $J_{3,0}$, $Z_{1,0}$ and $W_{1,0}$. The proof for cases referred to above and f being corank $(f)=3$ is included in [32].

* In V. I. Arnold's papers [4], [5] and [40], y^8 should be read y^{11} in E_{19} and E_{20}.

Proof) In the next table, we give the correspondence between V. I. Arnold's classification and the author's. Most of those types have been already discussed. Example 4. 19 below gives special types corresponding to Arnold's class Z_* and $Z_{1,p}$.

Table 4. 17 is concerned with classes in 3-①, which are not of simplex type, together with more general classes $J_{k,0}$ and $Z_{i,0}$ and parameters appearing in "EEF". The detailed structure of them are included in [32], with the stuucture of $W^\#_{1,p}$. Q.E.D.

Table 4. 16

2-②		α	β	3-②		α	β
J_{p+4}	$S^\#_{\min}$	1/3	$1/p$	$J_{3,p}$	$S^\#$	1/3	$1/(9+p)$
X_{p+5}	$S_{\min}$	1/4	$1/p$	$W_{1,p}$	S	1/4	$1/(6+p)$
$Y_{p,q}$	$S_{\min}$	$1/p$	$1/q$	$Z_{1,p}$	$yS^\#$	$\left((6+p)\big/3(7+p)\right)$	$1/(7+p)$
2-③		α	β	3-③		α	β
E_{12}	$Y^\flat_{\max}$	1/3	1/7	E_{18}	$Y^\flat$	1/3	1/10
E_{13}	$X^\#_{2,\min}$	1/3	2/15	E_{19}	$X^\#_2$	1/3	2/21
E_{14}	$Y^\flat_{\max}$	1/3	1/8	E_{20}	$Y^\flat$	1/3	1/11
Z_{11}	$yY^\flat_{\max}$	4/15	1/5	Z_{17}	$yY^\flat$	7/24	1/8
Z_{12}	$yX^\#_{2,\min}$	3/11	2/11	Z_{18}	$yX^\#_2$	5/17	2/17
Z_{13}	$yY^\flat_{\max}$	5/18	1/6	Z_{19}	$yY^\flat$	8/27	1/9
W_{12}	$Y_{\max}$	1/4	1/5	W_{17}	$X^\#_2$	1/4	3/20
W_{13}	$X^\#_{2,\min}$	1/4	3/16	W_{18}	Y	1/4	1/7

Table 4. 17

3-①	similar type	α	β	b
$J_{3,0}$	$Y^\flat_{\max}$	1/3	1/9	1
$Z_{1,0}$	$yY^\flat_{\max}$	2/7	1/7	1
$W_{1,0}$	$Y_{\max}$	1/4	1/6	1
$J_{k,0}$	$Y^\flat$	1/3	$1/3k$	k-2-d
$Z_{i,0}$	$yY^\flat$	$(i+1)/(3i+4)$	$1/(3i+4)$	i-d

The number d is determined by

$$c \equiv 0 \mod y^d, \quad \not\equiv 0 \mod y^{d+1}.$$

Among the Arnold's classification, following eight types in Table 4. 18 are also of simplex type. $J_{k,i}$ and E_* ($Z_{i,p}$ and Z_*, respectively) includes parameter $\boldsymbol{a} = a_0 + \cdots + a_{k-2}y^{k-2}$ ($\boldsymbol{b} = b_0 + \cdots + b_i y^i$, respectively), and $a_0 \neq 0$ ($b_0 \neq 0$, respectively) in $J_{k,i}$ ($Z_{i,p}$, respectively). $\mathfrak{a}: f = (x, y^b)$. The number d in types E_* and Z_* is determined by

$$a \equiv 0 \bmod y^d, \quad \not\equiv 0 \bmod y^{d+1}, \quad \text{for } E_* .$$

$$b \equiv 0 \bmod y^d, \quad \not\equiv 0 \bmod y^{d+1}, \quad \text{for } Z_* .$$

Table 4.18

		α	β	b
$J_{k,i}$	$S^\sharp$	$1/3$	$1/(3k+i)$	$k-1$
$Z_{i,p}$	$yS^\sharp$	$(3i+p+3)/3(3i+p+4)$	$1/(3i+p+4)$	$i+1$
E_{6k}	$Y^\flat$	$1/3$	$1/(3k+1)$	$k-1-d$
E_{6k+1}	$X_2^\sharp$	$1/3$	$2/3(2k+1)$	$k-1-d$
E_{6k+2}	$Y^\flat$	$1/3$	$1/(3k+2)$	$k-1-d$
Z_{6i+11}	$yY^\flat$	$(3i+4)/3(3i+5)$	$1/(3i+5)$	$i+1-d$
Z_{6i+12}	$yX_2^\sharp$	$(2i+3)/(6i+11)$	$2/(6i+11)$	$i+1-d$
Z_{6i+13}	$yY^\flat$	$(3i+5)/9(i+2)$	$1/(3i+6)$	$i+1-d$

Operators and ideals $\mathfrak{a}: f$ for $Z_{i,p}$ and Z_* are included in the following example Z.

Example 4.19.

$$Z: f = \frac{1}{3}x^3y + \frac{1}{n}y^n - tx^{m_1}y^m$$

$$1 \quad 2 \quad 0$$

i) $m_1 = 1,\ 2m \geq n+2,\ n > m.\quad c = \frac{1}{3}\left(1 - \frac{1}{n}\right) + \frac{m}{n} - 1\ (\neq 0)$

$$x(s - X_2) - \frac{3}{(3m-1)t}y^{n-m}(s - X_0)$$

$$y^{2m-n-1}\left(s - X_2 + \frac{3c}{(3m-1)t}y^{n-m}D_x\right) - \frac{3x}{(3m-1)t}(s - X_0)$$

a) $2n+1 > 3m$ $\mathfrak{a}: f = (x, y^{2m-n-1})$ type $yX_2^\sharp$, Z_{6i+12}: $\begin{matrix} n = 3i+6+d \\ m = 2i+4 \end{matrix}$

$$\left(s - X_2 - \frac{3nc}{3m-1}\right)\left(s - X_2 + \frac{3c}{(3m-1)t}y^{n-m}D_x\right)$$

$$-\frac{1}{t}\left(\frac{3}{3m-1}\right)^2 y^{2n-3m+1}(s - X_0 + c)(s - X_0)$$

b) $2n+1<3m$ $\mathfrak{a}: f=(x, y^{n-m})$ type $yY^{\flat}$, $Z_{6i+11}: \begin{array}{l} n=3i+5 \\ m=2i+4+d \end{array}$ $Z_{6i+13}: \begin{array}{l} n=3i+6 \\ m=2i+5+d \end{array}$

$$(s-X_0+c)(s-X_0)-\left(m-\frac{1}{3}\right)^2 ty^{3m-2n-1}$$

$$\times\left(s-X_2-\frac{3nc}{3m-1}\right)\left(s-X_2+\frac{3c}{(3m-1)t}y^{n-m}D_x\right)$$

ii) $m_1=2,\ n\geq 2m,\ m\geq 2.\ c=\frac{2}{3}\left(1-\frac{1}{n}\right)+\frac{m}{n}-1\ (\neq 0).$

$$x(s-X_0)-2ty^{m-1}(s-X_1),$$

$$y^{n-2m+1}(s-X_0-cty^{m-1}D_x)-\frac{2(3m-2)}{3}t^2x(s-X_2).$$

a) $n+2>3m$ $\mathfrak{a}: f=(x, y^{m-1})$ type $yS^{\#}$ $Z_{i,p}: \begin{array}{l} n=3i+p+4 \\ m=i+2 \end{array}$

$$(s-X_1)(s-X_2)-\frac{3y^{n-3m+2}}{4(3m-2)t^3}(s-X_0+2c)(s-X_0-cty^{m-1}D_x)$$

b) $n+2<3m$ $\mathfrak{a}: f=(x, y^{n-2m+1})$ type $yX_0^{\flat}$

$$(s-X_0+2c)(s-X_0-cty^{m-1}D_x)-4\left(m-\frac{2}{3}\right)t^3y^{3m-n-2}(s-X_1)(s-X_2).$$

§ 21. Non-isolated Singularities

We give some examples of b-functions of non-isolated singularities.

Example 4.20. $f=x^n+y^lz^m.$ $d=\text{g.c.d.}(l, m).$

1. $d=1.$

$$b(s)\ (s+1)\left(\prod_{i=0}^{n-2}\left(s+\frac{i+1}{n}+1\right)\prod_{\substack{0\leq i\leq n-2\\0\leq j\leq l-2}}\left(s+\frac{i+1}{n}+\frac{j+1}{l}\right)\right.$$

$$\left.\times\prod_{\substack{0\leq i\leq n-2\\0\leq k\leq m-2}}\left(s+\frac{i+1}{n}+\frac{k+1}{m}\right)\right)_{\text{red}}.$$

2. $d\geq 2.$ Put $l'=l/d,\ m'=m/d.$

$$b(s)\mid(s+1)\prod_{\substack{1\le t\le d-1\\0\le i\le n-2}}\left(s+\frac{i+1}{n}+\frac{t}{d}\right)\left(\prod_{i=0}^{n-2}\left(s+\frac{i+1}{n}+1\right)\right.$$

$$\left.\times\prod_{\substack{0\le i\le n-2\\0\le j\le l-2\\j\ne tl'-1}}\left(s+\frac{i+1}{n}+\frac{j+1}{l}\right)\prod_{\substack{0\le i\le n-2\\0\le k\le m-2\\m\ne tm'-1}}\left(s+\frac{i+1}{n}+\frac{k+1}{m}\right)\right)_{\mathrm{red}}.$$

We can prove these formulae by explicit construction of differential operator $P(s)$ such that $P(s)f^{s+1}=b'(s)f^s$, and the estimate in Theorem 3.3.

Recall that the integral local monodromy of f is

$$H_0=\mathbb{Z} \qquad\qquad id$$

$$H_1=(n-1)(d-1)\mathbb{Z}\quad \underbrace{\begin{bmatrix} & & -1\\ 1 & & \vdots\\ & \ddots & \vdots\\ & 1 & \vdots\\ & & 1-1\end{bmatrix}}_{n-1}\otimes\underbrace{\begin{bmatrix} & & -1\\ 1 & & \vdots\\ & \ddots & \vdots\\ & 1 & \vdots\\ & & 1-1\end{bmatrix}}_{d-1}$$

$$H_2=(n-1)d\mathbb{Z}\quad \underbrace{\begin{bmatrix} & & & -1\\ 1 & & & \vdots\\ & 1 & & \vdots\\ & & \ddots & \vdots\\ & & 1 & \vdots\\ & & & 1-1\end{bmatrix}}_{n-1}\otimes\underbrace{\begin{bmatrix} & & & 1\\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & \\ & & & 1\end{bmatrix}}_{d}.$$

Example 4.21. $f=\sum_{i=1}^{k}x_{2i-1}x_{2i}^{p_i}$

$$b(s)=(s+1)\left(\prod_{0\le i_j\le p_j-1}\left(s+\frac{i_1+1}{p_1}+\cdots+\frac{i_k+1}{p_k}\right)\right)_{\mathrm{red}}.$$

Example 4.22. $f=\sum_{i=1}^{k}(x_{2i-1}x_{2i})^2$

$$b(s)\mid(s+1)\prod_{i=0}^{k}\left(s+\frac{k}{2}+\frac{i}{2}\right)^{k+1-i}.$$

Note the equality

$$\frac{1}{4}\left(\sum x_{2i}^2D_{2i-1}^2\right)f^{s+1}=(s+1)\left(s+\frac{k}{2}\right)\left(\prod x_{2i}^2\right)f^s.$$

Example 4.23. Cubic cones in $\mathbb{C}^3$

Cubic cones in $\boldsymbol{C}^3$ are classified in nine types.

1. $x^3+y^3+z^3-3\lambda xyz$ $\quad (s+1)^2\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{6}{3}\right)$

   $\lambda^3\neq 1$

2. x^3+y^3-3xyz $\quad (s+1)^3\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)$

3. x^2z-y^3 $\quad (s+1)\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$

4. x^3+xyz $\quad (s+1)^3\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)$

5. x^2z+yz^2 $\quad (s+1)^2\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right)$

6. xyz $\quad (s+1)^3$

7. x^3+y^3 $\quad (s+1)^2\left(s+\frac{2}{3}\right)\left(s+\frac{4}{3}\right)$

8. x^2y $\quad (s+1)^2\left(s+\frac{1}{2}\right)$

9. x^3 $\quad (s+1)\left(s+\frac{1}{3}\right)\left(s+\frac{2}{3}\right).$

The polynomials in s written in the righthand side are the b-functions of the lefthand side except 2 and 4. In cases 2 and 4, the factor $(s+1)^3$ might be $(s+1)^2$ for b-function. The diagram of specialization is as follows.

$$\begin{array}{ccccccccc} 1 & & & & & & & & \\ \downarrow & & & & & & & & \\ 2 & \to & 4 & \to & 6 & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ 3 & \to & 5 & \to & 7 & \to & 8 & \to & 9\,. \end{array}$$

It should be noted that the maximal root of $b(s)=0$ increases along the arrows.

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