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G-Homotopy Types of G-Complexes and Representations of G-Cohomology Theories

By

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Dedicated to Professor Ryoji Shizuma on his 60^{th} birthday

Let G be a *finite* group throughout the present work. Bredon [2] discussed CW-complexes on which G acts nicely, and called them G-complexes. In the present work we discuss first, about G-spaces having G-homotopy types of G-complexes, parallel properties to Milnor [8] in § 2, where main results are Theorem 2. 3 and Corollary 2. 6. Then, in § 3 we apply Corollary 2. 6 to representations of G-equivariant cohomology theories, defined by Segal [9], by Ω-G-spectra (Theorems 3. 3 and 3. 4).

§ 1. G-Complexes

By a *G-complex* X we mean a CW-complex X on which G acts as a group of automorphisms of its cell-structure such that X^g, the fixed-point set of g, is a subcomplex for each $g \in G$, Bredon [2]. We refer the basic properties of G-complexes to [2]. By *G-maps* and *G-homotopies* we mean equivariant maps between G-spaces and equivariant homotopies between G-maps, respectively, for simplicity.

First we quote two basic properties of G-complexes.

G-Homotopy Extension Property ([2], Chap. I, § 1). *Let (X, A) be a pair of G-complex X and its G-subcomplex A, and Y a G-space. Given a G-map $f: X \to Y$ and a G-homotopy $F: A \times I \to Y$ such that $F|A \times \{0\} = f|A$, then there exists a G-homotopy $\tilde{F}: X \times I \to Y$ such that $\tilde{F}|X \times \{0\} = f$ and $\tilde{F}|A \times I = F$.*

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G-Cellular Approximation Theorem ([2], Chap. II, Proposition (5.6)). *Let (X, A) be a pair of G-complexes and Y another G-complex. Given a G-map $f: X \to Y$ such that $f|A$ is cellular, then there exists a G-homotopy $F: X \times I \to Y$ such that $F|A \times \{t\} = f|A$, $0 \leqq t \leqq 1$, $F|X \times \{0\} = f$ and $F|X \times \{1\}$ is cellular.*

Because of these two properties we can make constructions such as mapping-cylinders, mapping-cones, equalizers, telescopes, G-cofibration sequences (Puppe sequences) etc., in the category of (pointed) G-complexes.

Secondly we quote

Theorem of J.H.C. Whitehead for G-Complexes ([2], Chap. II, Corollary (5.5)). *Let $f: X \to Y$ be a G-map between two G-complexes. f is a G-homotopy equivalence iff $f^H = f|X^H$: $X^H \to Y^H$ is a weak homotopy equivalence for every subgroup H of G.*

As to the denomination of the above theorem we refer to Matumoto [5].

Now, in our case X^H is a CW-complex for each subgroup H of G by definition. Thus the above theorem can be restated in the following form:

Proposition 1.1. *Let $f: X \to Y$ be a G-map between two G-complexes. f is a G-homotopy equivalence iff $f^H: X^H \to Y^H$ is a homotopy equivalence for every subgroup H of G.*

The above proposition holds also for pairs of G-complexes. By "$f \simeq_G g$" we denote that two G-maps f and g are G-homotopic.

Proposition 1.2. *Let $f: (X, A) \to (Y, B)$ be a G-map between two pairs of G-complexes. f is a G-homotopy equivalence iff $f^H: (X^H, A^H) \to (Y^H, B^H)$ is a homotopy equivalence for every subgroup H of G.*

Proof.[1] The "only if" part is clear. To prove the "if" part we

[1] We would like to appreciate Professor Peter S. Landweber who kindly communicated us the error of our original proof.

need the following

Lemma. *Let (K, L) be a pair of G-complexes and $f: X \to Y$ be a G-homotopy equivalence of G-spaces. Let $g: L \to X$ and $h: K \to Y$ be G-maps such that $h|L = f \circ g$. Then there exists a G-map $\tilde{g}: K \to X$ such that $\tilde{g}|L = g$ and $f \circ \tilde{g} \simeq_G h$ relative to L.*

This lemma follows from [2], Chap. II, Lemma (5. 2), if we replace f by an inclusion map making use of the mapping cylinder of f.

Now suppose that $f^H: (X^H, A^H) \to (Y^H, B^H)$ is a homotopy equivalence for every subgroup H of G. Then $f^H: X^H \to Y^H$ and $(f|A)^H: A^H \to B^H$ are homotopy equivalences for all subgroups H of G. Thus, by Proposition 1. 1 we see that $f: X \to Y$ and $f|A: A \to B$ are G-homotopy equivalences.

Let $g_B: B \to A$ be a G-homotopy inverse of $f|A$, and $H_B: (f|A) \circ g_B \simeq_G 1_B$ a G-homotopy. By G-homotopy extension property we have a G-map H_1: $Y \times I \to Y$ such that $H_1|B \times I = H_B$ and $H_1|Y \times 1 = 1_Y$. Put $h = H_1|Y \times 0$, then $h|B = (f|A) \circ g_B$. Apply the above lemma to the pair (h, g_B) of G-maps, and get a G-map $g: Y \to X$ such that $g|B = g_B$ and $f \circ g \simeq_G h$ relative to B. Let $H_2: f \circ g \simeq_G h$ be this G-homotopy relative to B. The sum $H_2 + H_1: f \circ g \simeq_G 1_{(Y,B)}$ is a G-homotopy of G-maps $(Y, B) \to (Y, B)$ of pairs by construction. As is easily seen, $H_2 + H_1|B \times I$ can be equivariantly deformed to H_B relative to $B \times 0 \cup B \times 1$. Then, by G-homotopy extension property we can deform $H_2 + H_1$ to a G-homotopy $H: f \circ g \simeq_G 1_{(Y,B)}$ such that $H|B \times I = H_B$.

Take a G-homotopy $H_A: (g|B) \circ (f|A) \simeq_G 1_A$ and apply the same argument as above to $(f|A, g)$, then we get a G-map $\tilde{f}: (X, A) \to (Y, B)$ and a G-homotopy $H': g \circ \tilde{f} \simeq_G 1_{(X,A)}$ of G-maps of pairs such that $\tilde{f}|A = f|A$ and $H'|A \times I = H_A$. Then

$$f \simeq_G f \circ g \circ \tilde{f} \simeq_G \tilde{f}$$

as G-maps of pairs. Thus

$$g \circ f \simeq_G g \circ \tilde{f} \simeq_G 1_{(X,A)}$$

as G-maps of pairs. q.e.d.

More generally we obtain

Proposition 1.3. *Let* $f:(A; A_1, \cdots, A_{n-1}) \to (B; B_1, \cdots, B_{n-1})$ *be a G-map between two n-ads of G-complexes. f is a G-homotopy equivalence iff* $f^H:(A^H; A_1^H, \cdots, A_{n-1}^H) \to (B^H; B_1^H, \cdots, B_{n-1}^H)$ *is a homotopy equivalence for every subgroup H of G.*

Proof. Again the "only if" part is clear.

Suppose that $f^H:(A^H; A_1^H, \cdots, A_{n-1}^H) \to (B^H; B_1^H, \cdots, B_{n-1}^H)$ is a homotopy equivalence of n-ads for every subgroup H of G. Then $f|A_{i_1} \cap \cdots \cap A_{i_s}$: $A_{i_1} \cap \cdots \cap A_{i_s} \to B_{i_1} \cap \cdots \cap B_{i_s}$ is a G-homotopy equivalence for every subset $\{i_1, \cdots, i_s\} \subset \{1, \cdots, n-1\}$. Using the same argument as in the above proof we can construct a right G-homotopy inverse $g:(B; B_1, \cdots, B_{n-1}) \to (A; A_1, \cdots, A_{n-1})$ of f and G-homotopy $H: f\circ g \simeq_G 1$ of G-maps of n-ads by stepwise construction of $g|A_{i_1} \cap \cdots \cap A_{i_s}$ and $H|A_{i_1} \cap \cdots \cap A_{i_s} \times I$ so that they extends G-maps and G-homotopies already constructed, starting from $g|A_1 \cap \cdots \cap A_{n-1}$ and $H|A_1 \cap \cdots \cap A_{n-1} \times I$. Then we construct a right G-homotopy inverse $\tilde{f}$ of g as G-maps of n-ads in the same way. Finally we see that $f \simeq_G f\circ g\circ \tilde{f} \simeq_G \tilde{f}$ as G-maps of n-ads and that g is a left G-homotopy inverse of f. q.e.d.

A *simplicial G-complex* K is a simplicial complex K endowed with a group G of automorphisms of its simplicial structure. Then, its geometric realization K_w (or K_s) in the weak (or strong) topology is a G-space, but not always a G-complex in our sense. As is easily seen, K_w is a G-complex when, for each $g \in G$ and each simplex σ of K,

$g\sigma = \sigma$ iff g fixes all vertices of σ.

In particular we have

Proposition 1.4. *Let K be a simplicial G-complex. The barycentric subdivision* $Sd\,K_w$ *of* K_w *is a G-complex.*

In virtue of this proposition we regard K_w as a G-complex.

A *simplicial G-set* K is a simplicial set K together with a group G of automorphisms of its simplicial structure. For each $g \in G$ its action on K commutes with all structure maps of K; hence its fixed-point set K^g is a simplicial subset of K. Let $|K|$ be the geometric realization of

K in the weak topology (Milnor [7]). G-actions on K induce G-actions on $|K|$. As is easily seen we have

Proposition 1.5. *Let K be a simplicial G-set. Then $|K|$ is a G-complex and $|K|^H = |K^H|$ for any subgroup H of G.*

Let X be a G-space. Its singular complex $S(X)$ is a simplicial G-set with induced G-actions. Then $S(X^g) = S(X)^g$ for any $g \in G$ as is easily seen. By a routine argument we obtain the following

Proposition 1.6. *The assignment*

$$X \mapsto |S(X)|$$

is functorial on the category of G-spaces, and the map

$$\alpha : |S(X)| \to X,$$

defined by $\alpha(\sigma, y) = \sigma(y)$ for $(\sigma, y) \in |S(X)|$, is a natural G-map.

Let $\mathbb{X} = (X; X_1, \cdots, X_{n-1})$ be an n-ad of G-spaces. Then $S(\mathbb{X}) = (S(X); S(X_1), \cdots, S(X_{n-1}))$ is an n-ad of simplicial G-sets, and $|S(\mathbb{X})| = (|S(X)|; |S(X_1)|, \cdots, |S(X_{n-1})|)$ is an n-ad of G-complexes.

Proposition 1.7. *Let $\mathbb{X} = (X; X_1, \cdots, X_{n-1})$ be an n-ad of G-complexes. Then*

$$\alpha : |S(\mathbb{X})| \to \mathbb{X}$$

is a G-homotopy equivalence of n-ads.

Proof. Since $|S(\mathbb{X})|^H = (|S(X^H)|; |S(X_1^H)|, \cdots, |S(X_{n-1}^H)|)$ for each subgroup H of G, we see that

$$\alpha | |S(\mathbb{X})|^H : |S(\mathbb{X})|^H \to \mathbb{X}^H = (X^H; X_1^H, \cdots, X_{n-1}^H)$$

is a homotopy equivalence for each H by Milnor [7], Theorem 4, and [8], Lemma 1. Thus Proposition 1.3 completes the proof.

§ 2. *G*-Homotopy Types of *G*-Complexes

In this section we discuss G-spaces having G-homotopy types of G-complexes in a parallel way to Milnor [8]. Roughly to say, since G is finite, averaging procedures over G allow us parallel arguments to [8].

Let X be a G-space. A covering $\mathcal{U}=\{U_\lambda\}_{\lambda\in\Lambda}$ of X is called a *G-covering* when $gU_\lambda\in\mathcal{U}$ for each $g\in G$ and $\lambda\in\Lambda$. Then, putting

$$gU_\lambda=U_{g\lambda},$$

G acts on the indexing set Λ.

Let $\mathcal{U}=\{U_\lambda\}_{\lambda\in\Lambda}$ be an open G-covering of a G-space X. A partition of unity $\{p_\lambda\}_{\lambda\in\Lambda}$ subordinate to $\mathcal{U}$ is called a *G-partition of unity* (subordinate to $\mathcal{U}$) when

$$p_\lambda(g^{-1}x)=p_{g\lambda}(x)$$

for $g\in G$ and $x\in X$.

First we prove an analogue of Milnor [8], Theorem 2. We denote by $\mathcal{W}^G$ the category of G-spaces having G-homotopy types of G-complexes and by $\mathcal{W}_n^G$ the category of n-ads of G-spaces having G-homotopy types of n-ads of G-complexes.

Theorem 2. 1. *The following restrictions on an n-ad* $\boldsymbol{A}=(A;A_1,\cdots,A_{n-1})$ *are equivalent:*

(a) $\boldsymbol{A}$ *belongs to the category* $\mathcal{W}_n^G$,

(b) $\boldsymbol{A}$ *is G-dominated by an n-ad of G-complexes,*

(c) $\boldsymbol{A}$ *has the G-homotopy type of an n-ad of simplicial G-complexes in the weak topology.*

(d) $\boldsymbol{A}$ *has the G-homotopy type of an n-ad of simplicial G-complexes in the strong topology,*

Proof. The implications (c)$\Rightarrow$(a)$\Rightarrow$(b) are clear (by Proposition 1. 4). Remark that, for an n-ad $\boldsymbol{A}$ of G-spaces, the barycentric subdivision of $|S(\boldsymbol{A})|$ is an n-ad of simplicial G-complexes in the weak topology. Because of Proposition 1. 7 we get a proof that (b)$\Rightarrow$(c) by the same argument as [8], p.275, (using the same diagram).

Proof that (c)$\Leftrightarrow$(d). Let $\boldsymbol{K}=(K;K_1,\cdots,K_{n-1})$ be an n-ad of simplicial G-complexes, and $\boldsymbol{K}_w$ and $\boldsymbol{K}_s$ denote the n-ads of geometric realizations of $\boldsymbol{K}$ in the weak and strong topology respectively. Recall that the topology of K_s is given by the standard metric d defined by barycentric coordinates which is G-invariant.

Let $\{\beta\}$ be the set of vertices of K and $\mathcal{U}=\{U_\beta\}$ be the locally finite open covering defined as in [8], p.276. $\mathcal{U}$ is a G-covering as is easily seen. Let $p_\beta: K_s \to R$ be defined by

$$p_\beta(x)=d(x, K_s-U_\beta)/\sum d(x, K_s-U_\gamma),$$

$x \in K_s$, for each vertex β of K, where the summation runs over all vertices γ of K. Then $\{p_\beta\}$ is a G-partition of unity subordinate to $\mathcal{U}$. Define $p: \boldsymbol{K}_s \to \boldsymbol{K}_w$ by letting $p(x)$ be the point in K_w with barycentric coordinates $p_\beta(x)$. Now it is clear that p is a continuous G-map, and maps each $(K_j)_s$ into $(K_j)_w$.

Let

$$i: \boldsymbol{K}_w \to \boldsymbol{K}_s$$

be the canonical map which is obviously equivariant. The composition $i \circ p: \boldsymbol{K}_s \to \boldsymbol{K}_s$ maps each simplex into itself equivariantly, hence a linear homotopy gives a G-homotopy of $i \circ p$ to the identity. Similarly $p \circ i: \boldsymbol{K}_w \to \boldsymbol{K}_w$ is G-homotopic to the identity. q.e.d.

Let X be a G-space. $X \times X$ is a G-space by diagonal actions. X is called to be *G-ELCX* (*G-equi-locally convex*) if there exists a G-invariant neighborhood U of the diagonal in $X \times X$ and a G-map

$$\lambda: U \times I \to X$$

(which will be called the *structure map*) satisfying Milnor's conditions (1), (2) and (3) of [8], p.277. Even though we do not assume the open covering $\mathcal{V}=\{V_\beta\}$ of X by *convex* set (which we call the *convex covering* of X) to be a G-covering, we can actually choose $\mathcal{V}$ so as to be a G-covering by adding all gV_β to $\mathcal{V}$, $g \in G$ and $V_\beta \in \mathcal{V}$, because of equivariancy of the structure map λ. This will be called the *convex G-covering* of X.

An n-ad $\boldsymbol{X}=(X; X_1, \cdots, X_{n-1})$ is called a *G-ELCX n-ad when* X is *G-ELCX*, X_i is a closed G-subspace for each i, $1 \leqq i \leqq n-1$, and $\boldsymbol{X}$ is an

ELCX n-ad in the sense of Milnor [8].

Here we remark the following. Let X be a paracompact G-space and $\mathcal{U}$ an open G-covering of X, then we can choose a locally finite G-covering $\mathcal{W}$ of X which refines $\mathcal{U}$. (Choose any locally finite refinement $\mathcal{W}'$ of $\mathcal{U}$, and add all g-transforms of elements of $\mathcal{W}'$ to $\mathcal{W}'$; the resulting G-covering $\mathcal{W}$ is still locally finite since G is finite.) Next, let $\mathcal{W}$ be a locally finite open G-covering of a paracompact X. We can choose a G-partition of unity subordinate to $\mathcal{W}$ (by averaging over G an arbitrary chosen partition of unity subordinate to $\mathcal{W}$).

Lemma 2. 2. *Every n-ad of simplicial G-complexes in the strong topology is G-ELCX.*

Proof. Let $\boldsymbol{K}=(K; K_1, \cdots, K_{n-1})$ be an n-ad of simplicial G-complexes in the strong topology. Use the same constructions and notations as [8], p.278, Proof of Lemma 2. It is easy to check that U is G-invariant and the maps

$$\mu: U \to K \quad \text{and} \quad \lambda: U \times I \to K$$

are G-maps. q.e.d.

Theorem 2. 3. *The following restrictions on an n-ad* $\boldsymbol{A}=(A; A_1, \cdots, A_{n-1})$ *are equivalent:*

(i) $\boldsymbol{A}$ *belongs to* $\mathcal{W}_n^G$,

(ii) $\boldsymbol{A}$ *has a G-homotopy type of a metrizable G-ELCX n-ad,*

(iii) $\boldsymbol{A}$ *has a G-homotopy type of a paracompact G-ELCX n-ad.*

Proof. Since simplicial complexes in the strong topology are metrizable, Theorem 2. 1 and Lemma 2. 2 imply that (i) $\Rightarrow$ (ii). Since metrizable spaces are paracompact, it is obvious that (ii) $\Rightarrow$ (iii).

Proof that (iii) $\Rightarrow$ (i). This part corresponds to Lemma 4 of [8]. Let $\boldsymbol{A}=(A; A_1, \cdots, A_{n-1})$ be a paracompact G-*ELCX* n-ad. Because of Theorem 2. 1, it is sufficient to prove that $\boldsymbol{A}$ is G-dominated by an n-ad of G-complexes.

Let $\mathcal{V}=\{V_\beta\}$ be the convex G-covering of A. Since A is fully normal, we can find an open covering $\mathcal{W}'=\{W_\gamma'\}$ of A which is sufficiently fine so that the star of any point a of A with respect to $\mathcal{W}'$ is contained in some convex set V_β (as in [8], p.279).

Since G is finite and A_i's are closed, we can choose at every point a of A a sufficiently small open neighborhood W_a of a such that i) W_a is G_a-invariant and $W_a\cap gW_a=\phi$ for $g\in G-G_a$, where G_a is the isotropy subgroup of G at a, ii) W_a is contained in some W_γ', iii) $gW_a=W_{ga}$ for any $g\in G$, and iv) if $W_a\cap A_i\neq\phi$ then $a\in A_i$. We call each open set W_a an *admissible* open set *centered* at a. The totality $\mathcal{W}=\{W_a; a\in A\}$ of these admissible centered open sets forms an open G-covering which refines $\mathcal{W}'$; hence $\mathcal{W}$ is also sufficiently fine so that the star of any point a of A with respect to $\mathcal{W}$ is contained in some convex set V_β.

Practically we need only a G-subsystem of $\mathcal{W}$ which covers A. So, choosing one representative among G-orbits in $\mathcal{W}$ which coincide mutually as families of subsets of A, we may assume that $W_a\neq W_b$ if $a\neq b$.

Let $\mathcal{U}=\{U_\delta\}$ be a locally finite open G-covering of A which refines $\mathcal{W}$. Let N denote the nerve of $\mathcal{U}$, considered as a geometric simplicial complex in the weak topology. Define subcomplexes N_i such that the vertices $\delta_0, \cdots, \delta_k$ span a simplex of N_i iff $U_{\delta_0}\cap\cdots\cap U_{\delta_k}$ intersects A_i. Then we obtain an n-ad $\mathbb{N}=(N; N_1, \cdots, N_{n-1})$ of simplicial G-complexes in the weak topology. Choose a G-partition of unity $\{p_\delta\}$ subordinate to $\mathcal{U}$. Define $p:A\to N$ by letting $p(a)$ be the point in N with barycentric coordinates $p_\delta(a)$. p is clearly continuous and determines a G-map

$$p: \mathbb{A}\to\mathbb{N}$$

of n-ads of G-spaces.

Next we define a G-map

$$q: \mathbb{N}\to\mathbb{A}$$

of n-ads. Let $Sd\,N$ be the barycentric subdivision which is a G-complex. Vertices of $Sd\,N$ corresponds to simplices of N which are mutually identified by an abuse of notations. Order vertices of $Sd\,N$ so that $\sigma<\sigma'$ iff $\sigma\supset\sigma'$ in N. Then G-actions on $Sd\,N$ preserve this ordering. Set

$$Sd\mathcal{U}=\{U_\sigma'=U_{\delta_0}\cap\cdots\cap U_{\delta_k}\},$$

where $\sigma=\langle\delta_0, \cdots, \delta_k\rangle$ runs over all simplices of N. For each $U_\sigma'\in Sd\mathcal{U}$

we choose from $\mathcal{W}$ an admissible open set W_σ centered at a_σ so that $U'_\sigma \subset W_\sigma$ and $gW_\sigma = W_{g\sigma}$ for any $g \in G$.

Now we define the wanted G-map q as follows, by induction on the skeletons of $Sd\, N$. For each vertex σ of $Sd\, N$ set $q(\sigma) = a_\sigma$. Consider any k-simplex ξ in $Sd\, N$ with vertices $\sigma_0 < \cdots < \sigma_k$. Each point x of ξ can be written uniquely in the form $x = (1-t)\sigma_0 + ty$, $0 \leqq t \leqq 1$, where y lies in the $(k-1)$-face opposite to the leading vertex σ_0. Put

$$q(x) = \lambda(a_{\sigma_0}, q(y), t),$$

assuming q is defined and a G-map on the $(k-1)$-skeleton inductively by the above formula. q is well defined and continuous on the k-skeleton. As G-actions preserve ordering of $Sd\, N$, it is easy to see that q is a G-map on the k-skeleton. Suppose q maps the $(k-1)$-skeleton of $Sd\, N_i$ to A_i for each i, $1 \leqq i \leqq n-1$. If ξ is a k-simplex of $Sd\, N_i$ with vertices $\sigma_0 < \cdots < \sigma_k$, then U_{σ_0} intersects A_i and $a_{\sigma_0} \in A_i$ by our choices; hence $q(x) \in A_i$ for any point x of ξ by definition of *G-ELCX* n-ad. Thus q maps the k-skeleton of $Sd\, N_i$ to A_i for each i, $1 \leqq i \leqq n-1$, completing the induction.

For each point $a \in A$, let V_β be a convex set which contains the star of a with respect to $\mathcal{W}$. Then $q \circ p(a)$ is a convex combination of points in V_β, whence $(a, q \circ p(a)) \in V_\beta \times V_\beta \subset U$. Therefore the formula

$$(a, t) \mapsto \lambda(a, q \circ p(a), t)$$

defines a G-homotopy between $q \circ p$ and the identity of A. q.e.d.

Corresponding to Proposition 3 of [8] we obtain the following

Proposition 2.4. *If $\boldsymbol{A}$ belongs to $\mathcal{W}_n^G$ and $\boldsymbol{B}$ belongs to $\mathcal{W}_m^G$ then $\boldsymbol{A} \times \boldsymbol{B}$ belongs to $\mathcal{W}_{n+m-1}^G$.*

Proof. The product $\boldsymbol{A} \times \boldsymbol{B}$ is an $(n+m-1)$-*ad* as defined in [8], p.277. Because of Theorem 2.3 we may suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ to be metrizable and *G-ELCX*. Then $\boldsymbol{A} \times \boldsymbol{B}$ is metrizable by product-metric. Using products of convex sets as convex sets, and the product of the structure maps as the structure map, it is routine to check that $\boldsymbol{A} \times \boldsymbol{B}$ is *G-ELCX*. q.e.d.

If X and Y are G-spaces then the function space $F(X;Y)$ from X to Y, endowed with compact-open topology, is a G-space by the formula

$$(g\varphi)(x)=g\varphi(g^{-1}x)$$

for $\varphi\in F(X;Y)$, $x\in X$ and $g\in G$.

The following theorem corresponds to Theorem 3 of [8].

Theorem 2.5. *If $\mathcal{A}=(A;A_1,\cdots,A_{n-1})$ belongs to $\mathcal{W}_n^G$ and if $\mathcal{C}=(C;C_1,\cdots,C_{n-1})$ is an n-ad of compact G-spaces, then the n-ad*

$$(F(C;A);F(C,C_1;A,A_1),\cdots,F(C,C_{n-1};A,A_{n-1}))$$

belongs to $\mathcal{W}_n^G$.

Proof. By Theorem 2.3 we may assume that $\mathcal{A}$ is metrizable and G-$ELCX$. Since A is metrizable and C is compact, $F(C;A)$ is metrizable; and $F(C,C_i;A,A_i)$ is its closed G-subspaces for each i, $1\leqq i\leqq n-1$.

Define the neighborhood U' of diagonals in $F(C;A)\times F(C;A)$, the structure map λ' and convex sets of $F(C;A)$ as in [8], Proof of Lemma 3. It is easy to check that U' is G-invariant and λ' is a G-map. Thus $F(C;A)$ is G-$ELCX$, and the n-ad mentioned in the theorem is also G-$ELCX$. q.e.d.

Let V be a finite-dimensional G-module, and Σ^V denote the one-point compactification of V. Let X be a pointed G-space with base point x_0. We put

$$\Omega^V X=F(\Sigma^V,*;X,x_0),$$

which we call the (dim V)*-fold loop space* of X with G-actions of type V in parameters. $\Omega^V X$ is a pointed G-space with the constant map e as base point.

Corollary 2.6. *If a pair (X,x_0) belongs to $\mathcal{W}_2^G$, then the pair $(\Omega^V X,e)$ also belongs to $\mathcal{W}_2^G$.*

This corollary corresponds to Corollary 3 of [8], and will be used in the next section.

§ 3. Representations of G-Cohomology Theories

Segal [9] proposed to discuss generalized G-equivariant cohomology theories with degrees in the real representation ring $RO(G)$ of G. These are called *G-cohomology theories* for the sake of simplicity. Here we discuss to represent G-cohomology theories by Ω-G-spectra (defined below) in virtue of the method of Brown [3, 4].

A reduced G-cohomology theory will be defined as follows. Let $\mathcal{W}_0^G$ and $\mathcal{F}_0^G$ be the categories of pointed G-spaces and G-maps whose objects have G-homotopy types of G-complexes and of finite G-complexes, respectively; and let $\mathcal{C}\mathcal{W}_0^G$ and $\mathcal{C}\mathcal{F}_0^G$ denote the full subcategories of them with pointed G-complexes and finite G-complexes as objects, respectively. When we are given with an abelian-group-valued contravariant functor $\tilde{h}^\alpha$ for each $\alpha \in RO(G)$ simultaneously on the category $\mathcal{W}_0^G$ or $\mathcal{F}_0^G$, satisfying the following two axioms A1) and A2), then we call the system

$$\tilde{h}_G^* = \{\tilde{h}^\alpha; \alpha \in RO(G)\}$$

a *reduced G-cohomology theory* on $\mathcal{W}_0^G$ or on $\mathcal{F}_0^G$.

A1) *Each $\tilde{h}^\alpha$ is a G-homotopy functor satisfying wedge axiom and Mayer-Vietoris axiom on $\mathcal{C}\mathcal{W}_0^G$ or on $\mathcal{C}\mathcal{F}_0^G$.* (Cf., *Adams* [1] *and Brown* [4].)

A2) *For each finite-dimensional G-module V, the natural suspension isomorphism*

$$\sigma^V: \tilde{h}^\alpha(X) \approx \tilde{h}^{\alpha+V}(\Sigma^V X)$$

is defined for every $\alpha \in RO(G)$ (where $\Sigma^V X = \Sigma^V \wedge X$).

Take an infinite-dimensional G-module W which contains a discrete countable G-subset S such that every finite subset of S is linearly independent and, for every subgroup H of G, there exists an infinite number of points of S at which the isotropy groups of G are H. Let L be the simplicial complex consisting of all simplices spanned by finite subsets of S. L is a simplicial G-complex and every finite simplicial G-complex is isomorphic to a G-subcomplex of L. Let $\mathcal{C}_0^G$ be the full subcategory of $\mathscr{C}\mathscr{F}_0^G$ having all finite G-subcomplexes of L as objects. $\mathscr{C}_0^G$ is a small category and contains countably-infinite many objects. Now the pairs $(\mathscr{C}\mathscr{W}_0^G, \mathscr{C}_0^G)$ and $(\mathscr{C}\mathscr{F}_0^G, \mathscr{C}_0^G)$ are homotopy categories and all functors $\tilde{h}^\alpha$, restricted to $\mathscr{C}\mathscr{W}_0^G$ or $\mathscr{C}\mathscr{F}_0^G$, are homotopy functors in the sense of Brown [4]; and we can apply Brown's theory to our functor $\tilde{h}^\alpha$.

Here we remark the following. Every finite G-complex is G-homotopy equivalent to a finite simplicial G-complex (by simplicial approximations of attaching maps of cells); hence the set of G-homotopy types of finite G-complexes is countable, and we can choose a representative system $\mathscr{K}=\{K, K', \cdots\}$ such that all elements of K belongs to $\mathscr{C}_0^G$. Next, for any two complexes K and K' in $\mathscr{K}$, the set $[K, K']^G$ is countable (where $[\ \ ,\ \ ]^G$ stands for the set of G-homotopy classes of pointed G-maps), because any G-map $f: K\to K'$ can be G-approximated by a simplicial G-map of some subdivisions of K and K' (i.e., take barycentric subdivisions $Sd\,K$ and $Sd\,K'$ first to make them G-complexes in our sense, secondly subdivide $Sd\,K$ sufficiently fine so that we can apply the usual simplicial approximation to f, then, taking care of G-equivariancy, we can apply the usual argument of simplicial approximation to get simplicial G-approximation of f). These remarks will be used later to apply the device of Adams [1], § 3, to our case.

Let $\mathscr{C}$ be a full subcategory of $\mathscr{C}\mathscr{W}_0^G$ and h a Brown's homotopy functor on $\mathscr{C}$ (in the sense of G-homotopy). Let Y be an object of $\mathscr{C}$ and $u\in h(Y)$. The map

$$T_u: [X, Y]^G \to h(X),$$

defined by $T_u[f]=f^*u$, is a natural transformation of functors on $\mathscr{C}$, and the correspondence

$$u\mapsto T_u$$

gives a bijection

$$h(Y)\approx \text{Nat Trans}([\ \ , Y]^G, h),$$

[3], Lemma 3.1. When T_u is an isomorphism for each object X of $\mathscr{C}$, Y is called a *representing* complex of h as usual.

Let $\mathscr{C}\mathscr{W}_*^G$ and $\mathscr{C}\mathscr{F}_*^G$ be the full subcategories of $\mathscr{C}\mathscr{W}_0^G$ and

$\mathscr{C}\mathscr{F}_0^G$, respectively, in which objects are G-complexes X such that X^H are arcwise connected for all subgroups H of G.

As is easily seen

$$[(G/H)^+\wedge S^n, Y]^G \approx \pi_n(Y^H)$$

for all $n\geqq 0$ and all subgroups H of G (where G acts trivially on S^n and Y is a pointed G-complex). Hence, if $f:Y\to Y'$ is a map in $\mathscr{C}\mathscr{W}_*^G$ such that

$$f_*:[X, Y]^G \approx [X, Y']^G$$

for all G-complexes X in $\mathscr{C}\mathscr{F}_*^G$, then f is a G-homotopy equivalence by J.H.C. Whitehead's theorem for G-complexes. Thus we can apply [4], Theorem 2. 8, to a Brown's homotopy functor on $\mathscr{C}\mathscr{W}_*^G$ and we obtain

Proposition 3. 1. *Let h be a Brown's homotopy functor defined on $\mathscr{C}\mathscr{W}_*^G$. There exists a representing couple (Y, u) of h, where Y lies in $\mathscr{C}\mathscr{W}_*^G$ and $u\in h(Y)$,* i.e.,

$$T_u:[X, Y]^G \approx h(X),$$

a natural isomorphism of sets for X in $\mathscr{C}\mathscr{W}_*^G$. Y is unique up to G-homotopy equivalence.

(Let $\mathscr{K}_*$ be the subset of $\mathscr{K}$ consisting of all elements which belong to $\mathscr{C}\mathscr{F}_*^G$. Remark that we can use only elements of $\mathscr{K}_*$ as attaching data in the constructions in the proof of Theorem 2. 8 of [4], which supplements the proof of the above proposition.)

Before discussing representations of Brown's homotopy functor on $\mathscr{C}\mathscr{F}_*^G$, we remark the following

Lemma. *Every G-complex X in $\mathscr{C}\mathscr{W}_*^G$ can be expressed as a union of finite G-subcomplexes which belong to $\mathscr{C}\mathscr{F}_*^G$.*

Proof. It is clear that X can be expressed as a union of finite G-subcomplexes. Hence it is sufficient to show that, for arbitrary finite G-subcomplex K' of X, we can find a finite G-subcomplex K of X such that $K \supset K'$ and K belongs to $\mathscr{C}\mathscr{F}^G_*$.

Let H be a subgroup of G. We want to find a finite G-subcomplex K_1 of X satisfying that $K_1 \supset K'$ and, for every vertex v of K_1 such that G_v is contained in H, v can be joined to the base point by a path in $K_1^{G_v}$. Suppose we obtained a finite G-subcomplex K_2 of X satisfying that $K_2 \supset K'$ and, for every vertex w of K_2 such that G_w is a *proper* subgroup of H, w can be joined to the base point by a path in $K_2^{G_w}$. Now, for each vertex v of K_2 such that $G_v = H$, we can find a path L_v which is a subcomplex of X^H and joins v to the base point. Set

$$K_1 = K_2 \cup (\cup_v GL_v)$$

where v runs over all vertices of K_2 such that $G_v = H$. K_1 is the wanted G-complex.

Now, inductively on inclusions of subgroups H of G, after a finite times of the above construction we obtain a finite G-subcomplex K of X such that $K \supset K'$ and every vertex v of K can be joined to the base point by a path in K^{G_v}, which is equivalent to saying that K belongs to $\mathscr{C}\mathscr{F}^G_*$.

q.e.d.

Let h be a group-valued Brown's homotopy functor on $\mathscr{C}\mathscr{F}^G_*$. Put

$$\hat{h}(X) = \varprojlim_{r} h(X_r)$$

for each G-complex X in $\mathscr{C}\mathscr{W}^G_*$, where X_r runs over all finite G-subcomplexes of X which belong to $\mathscr{C}\mathscr{F}^G_*$. $\hat{h}$ is a *weak* G-homotopy functor on $\mathscr{C}\mathscr{W}^G_*$ in the parallel sense to "weak homotopy" in [1]. For each object Y in $\mathscr{C}\mathscr{W}^G_*$ and $u \in \hat{h}(Y)$, the maps

$$T_u : [X, Y]^G \to h(X), \quad X \in \mathscr{C}\mathscr{F}^G_*,$$

and

$$\hat{T}_u : [X', Y]^G_w \to \hat{h}(X') \quad X' \in \mathscr{C}\mathscr{W}^G_*,$$

defined by $T_u[f] = f^*u$ and $\hat{T}_u[g] = g^*u$, respectively, are natural transformations of functors and the correspondences

$$u \mapsto T_u \quad \text{and} \quad u \mapsto \hat{T}_u$$

give rise to bijections of sets

$$\hat{h}(Y) \approx \text{Nat Trans}([\quad, Y]^G, h)$$
$$\approx \text{Nat Trans}([\quad, Y]^G_w, \hat{h}),$$

where $[\quad,\quad]^G_w$ stands for the set of weak G-homotopy classes of G-maps, [3], Lemma 3. 3, and [1], Lemma 4. 1.

By the earlier remarks and the above lemma we can apply the arguments of [1], § 3, to the present case. In particular, the functor $\hat{h}$ on $\mathscr{C}\mathscr{W}^G_*$ satisfies the Wedge axiom, the isomorphism with inverse limits and the Mayer-Vietoris axiom in the weak sense, [1], Lemma 3. 3, Lemma 3. 4 and Proposition 3. 5, without any countability assumption on h.

Now we can do the same arguments and constructions as [1], Lemma 4. 2 and Proposition 4. 4, by utilizing only elements of $\mathscr{K}_*$ as attaching data, and we obtain representations of h, that is,

Proposition 3. 2. *Let h be a group-valued Brown's homotopy functor defined on $\mathscr{C}\mathscr{F}^G_*$. There exists a representing couple (Y, u) of h, where Y lies in $\mathscr{C}\mathscr{W}^G_*$ and $u \in \hat{h}(Y)$*, i.e.,

$$T_u : [X, Y]^G \approx h(X),$$

a natural isomorphism of sets for X in $\mathscr{C}\mathscr{F}^G_*$. Y is unique up to G-homotopy equivalence.

We can also prove an analogue of [1], Theorem 1. 9, and introduce a certain Hopf-space-structure to Y to make T_u an isomorphism of groups. But we don't need it to represent G-cohomology theories.

Now we shall discuss representations of G-cohomology theories. Let $\tilde{h}^*_G = \{\tilde{h}^\alpha ; \alpha \in RO(G)\}$ be a reduced G-cohomology theory defined on $\mathscr{W}^G_0$ or $\mathscr{F}^G_0$. Since discussions of both cases are quite parallel and since the first case is a bit simpler, we shall discuss only the second case, i.e., we suppose $\tilde{h}^*_G$ is defined on $\mathscr{F}^G_0$.

By Proposition 3. 2 we have a representing complex Y' of $\tilde{h}^\alpha | \mathscr{C}\mathscr{F}^G_*$ for each $\alpha \in RO(G)$, i.e., we have a natural isomorphism

$$[X, Y'_\alpha]^G \approx \tilde{h}^\alpha(X), \quad X \in \mathscr{C}\mathscr{F}^G_*,$$

for each $\alpha \in RO(G)$.

By ΣY and ΩY we denote the suspension and the loop space of a pointed G-space Y with trivial G-actions on parameters. Put

$$Y_\alpha = \Omega Y'_{\alpha+1},$$

where 1 denotes the real 1-dimensional trivial G-module. Y_α is a Hopf-space (H-space) with the multiplication defined by usual loop compositions. Moreover, this multiplication in Y_α commutes with every g-action, $g \in G$. In this sense we call Y_α a *Hopf-G-space*. By Corollary 2. 6 Y_α belongs to $\mathcal{W}_0^G$; hence we may assume that Y_α is a *Hopf-G-complex* (replacing by a G-homotopy equivalent one if necessary). Then Y_α^H is a Hopf-subcomplex of Y_α for any subgroup H of G.

ΣX belongs to $\mathscr{C}\mathcal{W}_*^G$ for any G-complex X. Thus we have isomorphisms

$$\tilde{h}^\alpha(X) \overset{\sigma}{\approx} \tilde{h}^{\alpha+1}(\Sigma X) \approx [\Sigma X, Y'_{\alpha+1}]^G \approx [X, Y_\alpha]^G$$

for each X in $\mathscr{C}\mathscr{F}_0^G$ and $\alpha \in RO(G)$, where σ is the suspension isomorphism. Moreover, the above isomorphisms are group isomorphisms by a usual argument, endowing $[X, Y_\alpha]^G$ a group structure induced by the Hopf-G-structure of Y_α. Thus Y_α represents $\tilde{h}^\alpha$ on $\mathscr{C}\mathscr{F}_0^G$ as a group-valued functor.

Let $\hat{h}^\alpha$ be the associated functor to $\tilde{h}^\alpha$, i.e.,

$$\hat{h}^\alpha(X) = \varprojlim_\gamma \tilde{h}^\alpha(X_\gamma)$$

for X in $\mathscr{C}\mathcal{W}_0^G$, where X_γ runs over all finite G-subcomplexes of X. Since $[X, Y_\alpha]_w^G = \varprojlim_\gamma [X_\gamma, Y_\alpha]^G$ as is easily seen, we have a natural isomorphism

$$[X, Y_\alpha]_w^G \approx \hat{h}^\alpha(X)$$

of groups for each X in $\mathscr{C}\mathcal{W}_0^G$ and $\alpha \in RO(G)$, i.e., Y_α represents $\hat{h}^\alpha$.

Let V be a finite-dimensional G-module. Passing to the inverse limit of suspension isomorphisms $\sigma^V\colon \tilde{h}^\alpha(X_\gamma) \approx \tilde{h}^{\alpha+V}(\Sigma^V X_\gamma)$, we obtain a natural isomorphism

$$\hat{\sigma}^V\colon \hat{h}^\alpha(X) \approx \hat{h}^{\alpha+V}(\Sigma^V X), \quad X \in \mathscr{C}\mathcal{W}_0^G.$$

Again, passing to the inverse limit of the canonical natural isomorphism $[\Sigma^V X_\gamma, Y_{\alpha-V}]^G \approx [X_\gamma, \Omega^V Y_{\alpha+V}]^G$, we have a natural isomorphism

$$[\Sigma^V X, Y_{\alpha+V}]^G_w \approx [X, \Omega^V Y_{\alpha+V}]^G_w, \quad X \in \mathscr{C}\mathscr{W}^G_0 .$$

Combining the above three natural isomorphisms, we obtain a natural isomorphism

$$[X, Y_\alpha]^G_w \approx [X, \Omega^V Y_{\alpha+V}]^G_w, \quad X \in \mathscr{C}\mathscr{W}^G_0 ,$$

of groups, where $\Omega^V Y_{\alpha+V}$ is a Hopf-G-space with structures induced from those of $Y_{\alpha+V}$, and the group structure of the right hand side of the above isomorphism is induced from Hopf-G-structures of $Y_{\alpha+V}$.

By Corollary 2. 6 $\Omega^V Y_{\alpha+V}$ belongs to $\mathscr{W}^G_0$. And we may suppose that $\Omega^V Y_{\alpha+V}$ itself is a Hopf-G-complex. Putting $X = Y_\alpha$ in the above isomorphism, we obtain a G-map

$$f_{\alpha,V}: Y_\alpha \to \Omega^V Y_{\alpha+V}$$

such that $[f_{\alpha,V}]$ corresponds to the class of the identity map of Y_α. Next, putting $X = \Omega^V Y_{\alpha+V}$ in the same isomorphism, we obtain a G-map

$$g_{\alpha,V}: \Omega^V Y_{\alpha+V} \to Y_\alpha$$

which corresponds to the class of the identity map of $\Omega^V Y_{\alpha+V}$. By the above choices we see easily that $(f_{\alpha,V})_* = (g_{\alpha,V})^{-1}_*$ which is the same as the above natural isomorphism.

This shows, on one hand, that $g_{\alpha,V} \circ f_{\alpha,V}$ and $f_{\alpha,V} \circ g_{\alpha,V}$ are weakly G-homotopic to the identity maps; and, on the other hand, the fact that $f_{\alpha,V}$ and $g_{\alpha,V}$ induce group isomorphisms implies that $f_{\alpha,V}$ and $g_{\alpha,V}$ are weak morphisms of Hopf-G-complexes (i.e., they commute with Hopf-structure maps up to weak G-homotopy).

Then, for each subgroup H of G, we see easily that $(f_{\alpha,V})^H$ is a weak morphism of Hopf-complexes, and

$$(g_{\alpha,V})^H \circ (f_{\alpha,V})^H \simeq_w 1 \quad \text{and} \quad (f_{\alpha,V})^H \circ (g_{\alpha,V})^H \simeq_w 1 ,$$

where "$\simeq_w$" denotes "weak homotopy", which implies isomorphisms

$$(f_{\alpha,V})^H_*: \pi_n(Y^H_\alpha) \approx \pi_n((\Omega^V Y_{\alpha+V})^H)$$

for all $n \geqq 0$. Hence, $(f_{\alpha,V})^H$ is a weak morphism of Hopf-complexes, induces one-one correspondence of path-components, and gives a weak homotopy equivalence of e-components. Thus $(f_{\alpha,V})^H$ is a weak homotopy equivalence by a classically well-used argument. Finally, J.H.C. Whitehead's theorem for G-complexes concludes that $f_{\alpha,V}$ is a G-homotopy equivalence.

Summarizing the above arguments we obtain

Theorem 3.3. *Let* $\tilde{h}_G^* = \{\tilde{h}^\alpha; \alpha \in RO(G)\}$ *be a reduced G-cohomology theory defined on* $\mathcal{W}_0^G$ *or* $\mathcal{F}_0^G$. *There exists, for each* $\alpha \in RO(G)$, *a G-complex* Y_α *in* $\mathcal{C}\mathcal{W}_0^G$ *which is a Hopf-G-complex and represents* $\tilde{h}^\alpha$ *as a group-valued functor. Furthermore, for each finite-dimensional G-module V, there exists a G-homotopy equivalence*

$$f_{\alpha, V}: Y_\alpha \simeq_G \Omega^V Y_{\alpha+V},$$

which is a morphism or weak morphism of Hopf-G-spaces (depending on the categories) and induces the suspension isomorphism σ^V *for each* $\alpha \in RO(G)$.

Let ω be a G-module containing exactly one copy of each irreducible G-module (including a trivial one) as a direct summand. A *G-spectrum* $\boldsymbol{E}$[2] consists of a G-space E_n in $\mathcal{W}_0^G$ and a G-map $\varepsilon_n: \Sigma^\omega E_n \to E_{n+1}$ for each $n \in \boldsymbol{Z}$. Let $\varepsilon_n': E_n \to \Omega^\omega E_{n+1}$ be the adjoint G-map of ε_n for each $n \in \boldsymbol{Z}$. $\boldsymbol{E}$ is called an *Ω-G-spectrum* if ε_n' is a G-homotopy equivalence for every $n \in \boldsymbol{Z}$. Since ω contains a 1-dimensional trivial representation as a direct factor, $\Omega^\omega Y$ is a Hopf-G-space for any G-space Y by compositions along the parameter on which G acts trivially. Thus, if $\boldsymbol{E}$ is an Ω-G-spectrum, each term of it can be regarded as a Hopf-G-space.

In Theorem 3.3, putting

$$E_n = Y_{n\omega}$$

and

$$\varepsilon_n' = f_{n\omega, \omega}: E_n \simeq_G \Omega^\omega E_{n+1}$$

for each $n \in \boldsymbol{Z}$, we obtain an Ω-G-spectrum $\boldsymbol{E} = \{E_n, \varepsilon_n; n \in \boldsymbol{Z}\}$. And we obtain

Theorem 3.4. *Every reduced G-cohomology theory* $\tilde{h}_G^* = \{\tilde{h}^\alpha; \alpha \in RO(G)\}$ *can be represented by an Ω-G-spectrum* $\boldsymbol{E} = \{E_n; n \in \boldsymbol{Z}\}$, i.e., *we have a natural isomorphism*

$$\tilde{h}^\alpha(X) \approx [X, \Omega^V E_n]^G$$

[2] The referree remarked the authors that this notion was defined in somewhat wide sense by C. Kosniowski, *Math. Ann.*, **210** (1974), 83-104.

for each $\alpha \in RO(G)$, where V is a finite G-module such that $\alpha + V = n\omega$.

Remark 1. A similar representation theory was discussed by Matumoto [6], Theorem 6. 1, for certain equivariant cohomology theories defined on the category of his G-CW-complexes, where he obtained representations of his cohomology theories by *weak* Ω-spectra.

Remark 2. As observed by Segal [9], stable G-cohomotopy ϖ_G^* is universal for G-cohomology theories, or equivalently, we can say that every reduced G-cohomology theory is an ϖ_G^*-module. Then a result of Segal [9], Corollary to Proposition 1, suggests that every $\tilde{h}^\alpha$ should be treated as an $A(G)$-module-valued functor and the suspension σ^V as an $A(G)$-module isomorphism, where $A(G)$ denotes the Burnside ring of G. Such an $A(G)$-module structure would be important if we want to discuss further structures of G-cohomologies such an multiplicative structures, in which units of $A(G)$ might play an important role in sign conventions. Even though it seems to be difficult to discuss the general case, we will discuss the case of $G = \mathbf{Z}/2\mathbf{Z}$, i.e., spaces-with-involutions, in a subsequent paper in details.

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