Mixed Problems for the Wave Equation III
Exponential Decay of Solutions

By

Mitsuru IKAWA*

§ 1. Introduction

Let $\mathcal{O}$ be a bounded object in $\boldsymbol{R}^3$ whose boundary Γ is sufficiently smooth. Concerning the exponential decay of solutions of the wave equation in the exterior of $\mathcal{O}$ it seems to us that the cases with the Dirichlet boundary condition and with the Neumann, or the third boundary condition are studied. Besides the case with the Dirichlet boundary condition, we know only a few works, for example, Morawetz [11] on the case with the Neumann condition for a convex object and Tokita [13] on the case with the third boundary condition for $\mathcal{O} = \{x; |x| < 1\}$.

In this paper we suppose the strict convexity of $\mathcal{O}$, which is an assumption stronger than that of Morawetz [11], and treat the exponential decay of solutions of problems for a very general boundary condition.

Set $\Omega = \boldsymbol{R}^3 - \mathcal{O} - \Gamma$. Let

$$B = \sum_{j=1}^{3} b_j(x) \frac{\partial}{\partial x_j} + c(x) \frac{\partial}{\partial t} + d(x)$$

be a differential operator with C^∞ coefficients defined in a neighborhood of Γ. We pose the following assumptions:

(A-I) the Gaussian curvature of Γ is strictly positive.

(A-II) $b_j(x)$, $j = 1, 2, 3$ and $c(x)$ are real valued.

(A-III) $\sum_{j=1}^{3} b_j(x) n_j(x) = 1$ on Γ

where $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the unit outer (with respect to $\mathcal{O}$) normal of Γ at $x \in \Gamma$.

Under these assumptions a mixed problem

$$\text{(P)} \quad \begin{cases} \Box u = \dfrac{\partial^2 u}{\partial t^2} - \displaystyle\sum_{j=1}^{3} \dfrac{\partial^2 u}{\partial x_j^2} = 0 & \text{in} \quad \Omega \times (0, \infty) \\[4mm] Bu = 0 & \text{on} \quad \Gamma \times (0, \infty) \\[4mm] u(x, 0) = u_0(x) \\[4mm] \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \end{cases}$$

is well posed in the sense of C^∞ if and only if

(A-IV) $c(x) < 1$ for all $x \in \Gamma$.

This is the main result of [6].

   About the asymptotic behavior of the solution of (P) under the assumptions A-I∼IV, when $\mathrm{Re}(-d(x))$ is large in some extend the solution $u(x, t)$ for initial data u_0, u_1 with compact support decays exponentially.   Namely we have

   Theorem 1. *Suppose that*

$$\mathrm{Re}\left(\frac{1}{2} \sum_{j=1}^{3} \frac{\partial (b_j(x) - n_j(x))}{\partial x_j} - d(x)\right) \geq d_0$$

where d_0 is a certain constant.   Let u_0, u_1 be initial data satisfying the compatibility condition and

$$\bigcup_{j=0}^{1} \mathrm{supp}\ u_j \subset \{x;\ x \in \bar{\Omega},\ |x| \leq \kappa\}.$$

Then the solution $u(x, t)$ of (P) has an estimate

$$E_1(u, r_0, t) \leq \frac{C}{\varepsilon} \exp\{3\delta_0(r_0 + 2\kappa)\}$$

$$\times \exp\{-2(\delta_0 - \varepsilon)t\} \cdot E_3(u, \infty, 0),\quad \forall t \geq 0$$

where $\delta_0 = 12\rho^{-1}e^{-1}$, $\rho = $ diameter of $\mathcal{O}$, ε is an arbitrary positive constant, C a constant independent of r_0, u_0, u_1 and ε, and E_m is defined by

$$E_m(u, r_0, t) = \sum_{|r| \leq m} \int_{\Omega \cap \{|x'| \leq r_0\}} |D_{x,t}^r u(x, t)|^2 dx.$$

   In the case where $b(x) = n(x)$ and $c(x) = 0$, that is, the third boundary value problem we can study the condition on $d(x)$ in detail.

Theorem 2. *Suppose that*

$$B = \frac{\partial}{\partial n} + \sigma(x), \quad \sigma(x) \in C^\infty(\Gamma).$$

For any $M > 0$ there exist constants $\delta_1 > 0$, $\delta_2 > 0$ such that, if $\sigma(x)$ satisfies

$$|\operatorname{Im} \sigma(x)| \leq \delta_1, \quad -M \leq \operatorname{Re} \sigma(x) \leq \delta_1 \text{ on } \Gamma.$$

the solution $u(x,t)$ of (P) for u_0, u_1 satisfying the compatibility condition and

$$\bigcup_{j=0}^{1} \operatorname{supp} u_j \subset \{x; x \in \bar{\Omega}, |x| \leq \kappa\}$$

decays exponentially, namely

$$E_1(u, r_0, t) \leq C \exp\{3\delta_0(r_0 + 2\kappa)\} \exp(-\delta_2 t) \cdot E_3(u, \infty, 0).$$

We should like to remark that the mixed problem (P) is not necessarily L^2-well posed under the assumptions A-I$\sim$IV. Indeed, if there exists a point $x_0 \in \Gamma$ such that

$$(1.1) \qquad -(\sum_{j=1}^{3}(b_j(x_0) - n_j(x_0))^2)^{1/2} < c(x_0),$$

the problem (P) is not L^2-well posed for any $d(x)$ (see, for example, [1], [8]). This means that for any $T > 0$ there is no constant C_T such that the energy estimate

$$\|u(x,t)\|^2_{1, L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(x,t)\right\|^2_{L^2(\Omega)}$$

$$\leq C_T\{\|u_0(x)\|^2_{1, L^2(\Omega)} + \|u_1(x)\|^2_{L^2(\Omega)}\}, \quad \forall t \in [0, T]$$

holds for any u_0, $u_1 \in \mathcal{D}(\Omega_R)$, $\Omega_R = \{x \in \Omega; |x| < R\}$. Therefore Theorem 1 says that, when (1.1) holds, however large the total energy may increase the energy in a bounded region must decrease exponentially.

The proof relies essentially on the idea of [5] and [6] reducing the problem (P) to one with the Dirichlet boundary condition.

The author wishes to express his sincere gratitude to Prof. A. Inoue, who pointed out the efficacy of the considerations in [5] and [6] to investigate the exponential decay of solution. We could not obtain the

results presented here without the discussion with him.

§ 2. Reduction of the Problem

In the previous paper we showed that the mixed problem

$$\begin{cases} \square u(x, t) = f(x, t) & \text{in} \quad \varOmega \times (0, \infty) \\ Bu(x, t) = g(x, t) & \text{on} \quad \varGamma \times (0, \infty) \\ u(x, 0) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \end{cases}$$

is well posed in the sense of C^∞ and has a finite propagation speed when the assumptions A-I$\sim$IV are fulfilled. Consider the solution $u(x, t)$ of the problem (P). Let $\tilde{u}_j(x)$, $j=0, 1$ be functions in $C^\infty(\boldsymbol{R}^3)$ such that $u_j(x) = \tilde{u}_j(x)$ in $\varOmega$. Denote by $F(x, t)$ the solution of the Cauchy problem

$$\begin{cases} \square F(x, t) = 0 & \text{in} \quad \boldsymbol{R}^3 \times (0, \infty) \\ F(x, 0) = \tilde{u}_0(x) \\ \dfrac{\partial F}{\partial t}(x, 0) = \tilde{u}_1(x). \end{cases}$$

Then a function

$$w(x, t) = \begin{cases} u(x, t) - F(x, t) & \text{for} \quad (x, t) \in \varOmega \times (0, \infty) \\ 0 & \text{for} \quad (x, t) \in \varOmega \times (-\infty, 0] \end{cases}$$

satisfies

$$(\text{P}_0) \qquad \begin{cases} \square w(x, t) = 0 & \text{in} \quad \varOmega \times \boldsymbol{R}^1 \\ Bw(x, t) = h(x, t) & \text{on} \quad \varGamma \times \boldsymbol{R}^1 \\ \text{supp } w \subset \bar{\varOmega} \times [0, \infty), \end{cases}$$

where $h(x, t) = -BF(x, t)|_\varGamma$. The compatibility condition for u_0, u_1 is none other than

$$h(x, t) \in C^\infty(\varGamma \times \boldsymbol{R}^1).$$

Note that by Huygens' principle the assumption supp $\tilde{u}_j \subset \{x; |x| \leq \kappa\}$ im-

plies

(2. 1) $\operatorname{supp} h(x, t) \subset \Gamma \times [0, 2\kappa].$

In order to investigate the behavior of the solution $u(x, t)$ of (P) it suffices to consider the solution $w(x, t)$ of (P_0) for a boundary data $h(x, t)$ satisfying (2. 1), because we know well the properties of solutions of the Cauchy problem, that is, $\operatorname{supp} F \subset \{(x, t) ; t - \kappa \leq |x| \leq t + \kappa\}.$

Suppose that the origin of $\boldsymbol{R}^3$ is contained in $\mathcal{O}$. Set for $\delta > 0$

$$\Delta_\delta = \Delta + 2\delta \frac{\partial}{\partial |x|} + \delta^2 + \frac{2\delta}{|x|}$$

$$B^{(\delta)} = B + \delta \frac{1}{|x|} \sum_{j=1}^{3} b_j(x) x_j$$

and we have for all $u(x, t) \in C^\infty(\bar{\Omega} \times \boldsymbol{R}^1)$

$$e^{-\delta |x|} \square u(x, t) = \left(\frac{\partial^2}{\partial t^2} - \Delta_\delta\right)(e^{-\delta |x|} u)$$

$$e^{-\delta |x|} B u(x, t) = B^{(\delta)}(e^{-\delta |x|} u).$$

Then if we pose

$$v(x, t) = e^{-\delta |x|} w(x, t)$$

for $w(x, t)$ the solution of (P_0), it holds that

(P$_\delta$)
$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta_\delta\right) v(x, t) = 0 & \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] B^{(\delta)} v(x, t) = e^{-\delta |x|} h(x, t) & \text{on} \quad \Gamma \times \boldsymbol{R}^1 \\[2mm] \operatorname{supp} v \subset \bar{\Omega} \times [0, \infty). \end{cases}$$

We consider a decay of $v(x, t)$ in place of that of $w(x, t)$. As for a boundary value problem with a parameter $p \in \boldsymbol{C}$

(2. 2)
$$\begin{cases} (p^2 - \Delta_\delta) u(x) = 0 & \text{in} \quad \Omega \\[2mm] u(x) = g(x) & \text{on} \quad \Gamma \end{cases}$$

it is known that there exists a constant $\mu_0 > 0$ such that for any $g(x)$ $\in H^m(\Gamma)$ (2. 2) has a unique solution $u(x)$ in $H^m(\Omega)$ if $\operatorname{Re} p \geq \mu_0$, where $m = 1, 2, 3, \cdots$. Denote the solution $u(x)$ by $U^{(\delta)}(p, g; x)$. We see at once that $U^{(\delta)}(p, g; x)$ is analytic in $\operatorname{Re} p \geq \mu_0$ as $H^m(\Omega)$-valued function.

Moreover with the aid of consideration of Morawetz [10] we obtain the following theorem, whose proof will be given in the next section.

Theorem 2. 1. *Suppose that* $\delta \geq \delta_0 + 1$. *For every* $g(x) \in H^m(\Gamma)$, *an analytic* $H^m(\Omega)$-*valued function* $U^{(\delta)}(p, g; x)$ *in* $\mathrm{Re}\, p \geq \mu_0$ *can be prolonged analytically into* $\mathrm{Re}\, p > -\delta_0$, *and an estimate*

$$(2.3) \qquad \| U^{(\delta)}(p, g; x) \|_m \leq \frac{C_m}{\mathrm{Re}\, p + \delta_0} \| g \|_m, \quad \forall \mathrm{Re}\, p > -\delta_0{}^{1)}$$

holds.

Define an operator $\mathscr{B}^{(\delta)}(p)$ from $C^\infty(\Gamma)$ into $C^\infty(\Gamma)$ by

$$\mathscr{B}^{(\delta)}(p) g = B^{(\delta)}(p) U^{(\delta)}(p, g; x)|_\Gamma \quad \text{for} \quad g \in C^\infty(\Gamma)$$

where

$$B^{(\delta)}(p) = \sum_{j=1}^{3} b_j(x) \frac{\partial}{\partial x_j} + p c(x) + d(x) + \delta \frac{1}{|x|} \sum_{j=1}^{3} b_j(x) x_j .$$

In fact since $U^{(\delta)}(p, g; x) \in C^\infty(\bar{\Omega})$ for $g \in C^\infty(\Gamma)$ by the regularity theorem for the boundary value problem (2.2) we have $\mathscr{B}^{(\delta)}(p) g \in C^\infty(\Gamma)$. Concerning the operator $\mathscr{B}^{(\delta)}(p)$ we have

Theorem 2. 2. *For every positive number* m

$$(2.4) \qquad \| \mathscr{B}^{(\delta)}(p) g \|_m \leq C_m (\| g \|_{m+1} + |p| \| g \|_m), \quad \forall g \in C^\infty(\Gamma)$$

holds where C_m *is a positive constant. And we have for all* $p = ik + \mu$, $\mu > -\delta_0$ *and* $g \in C^\infty(\Gamma)$,

$$(2.5) \qquad -\mathrm{Re}\, (\mathscr{B}^{(\delta)}(p) g, g)_m \geq (c_0 \mu - C) \| g \|_m^2 + a \| g \|_m^2 - C'_m \| g \|_{m-1}^2$$

where $c_0 = 1 - \sup\limits_{x \in \Gamma} c(x)$, C *is a constant independent of* b_j, c, d *and* m,

$$a = \inf_{x \in \Gamma} \mathrm{Re} \left\{ \frac{1}{2} \sum_{j=1}^{3} \frac{\partial (b_j - n_j)}{\partial x_j} - d(x) \right\}$$

and C'_m *is a constant depending on* m. *When* $d(x)$ *satisfies*

$$(2.6) \qquad a \geq (c_0 \delta_0 + C + C'_0) + 1 ,$$

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[1] Hereafter $\| \cdot \|_m$ denotes the norm of $H^m(\Gamma)$ and $\| \cdot \|_m$ the norm of $H^m(\Omega)$.

if we consider by (2.4) $\mathscr{B}^{(\delta)}(p)$ *as an operator from* $H^{m-1}(\Gamma)$ *into* $H^m(\Gamma)$ *for every* $m \geq 0$, $\mathscr{B}^{(\delta)}(p)$ *is bijective from* $H^{m+1}(\Gamma)$ *onto* $H^m(\Gamma)$ *and we have for all* $\operatorname{Re} p > -\delta_0$

$$(2.7) \qquad \|\mathscr{B}^{(\delta)}(p)^{-1}g\|_m \leq C''_m \|g\|_m$$

$$(2.8) \qquad \|\mathscr{B}^{(\delta)}(p)^{-1}g\|_{m+1} \leq C''_m |p| \|g\|_m$$

where C'' is a positive constant.

A proof of the above theorem will be given in Section 4 and 5. For a while admit Theorem 2.2. Since $\mathscr{B}^{(\delta)}(p)g$ is analytic in $\operatorname{Re} p > -\delta_0$ as $H^m(\Gamma)$-valued function for every $g \in H^{m+1}(\Gamma)$ the estimates (2.7) and (2.8) imply that for all $f \in H^m(\Gamma)$

$$(2.9) \quad \begin{cases} \mathscr{B}^{(\delta)}(p)^{-1}f \text{ is analytic in } \operatorname{Re} p > -\delta_0 \text{ as } H^{m+1}(\Gamma)\text{-valued function} \\ \text{when } d(x) \text{ satisfies } (2.6). \end{cases}$$

Let $h(x,t) \in C^\infty(\Gamma \times \mathbf{R}^1)$, $\operatorname{supp} h \subset \Gamma \times [0, 2\kappa]$. If we pose

$$\hat{h}(x,p) = \int_{-\infty}^{\infty} e^{-pt} h(x,t) \, dt$$

$\hat{h}(x,p)$ is analytic in $\mathbf{C}$ as $H^m(\Gamma)$-valued function for any $m \geq 0$ and it holds that

$$(2.10) \quad |p|^j \|\hat{h}(x,p)\|_m \leq (1 + e^{-2\kappa \cdot \operatorname{Re} p}) \int_{-\infty}^{\infty} \left\| \left(\frac{\partial}{\partial t}\right)^j h(x,t)\right\|_m dt.$$

Let us set

$$(2.11) \quad v(x,t) = \int_{\operatorname{Re} p = \mu} e^{pt} U^{(\delta)}(p, \mathscr{B}^{(\delta)}(p)^{-1} e^{-\delta|x|} \hat{h}(\cdot, p); x) \, dp$$

where $\mu > -\delta_0$. The estimates (2.3) and (2.7) give

$$(2.12) \quad \|U^{(\delta)}(p, \mathscr{B}^{(\delta)}(p)^{-1} e^{-\delta|x|} \hat{h}(\cdot, p); x)\|_m$$

$$\leq |p|^{-2-j} \frac{C_m}{\delta_0 + \mu} \|p^{2+j} \hat{h}(x,p)\|_m$$

and $U^{(\delta)}(p, \mathscr{B}^{(\delta)}(p)^{-1} e^{-\delta|x|} \hat{h}(\cdot, p); x)$ is analytic in $\operatorname{Re} p > -\delta_0$ as $H^m(\Omega)$-valued function. Then from (2.10) and (2.12) the right-hand side of (2.11) converges for all $\mu > -\delta_0$, therefore it is independent of $\mu > -\delta_0$. And we have at once for all $\mu > -\delta_0$

$$\|v(x,t)\|_{m+1} + \left\|\frac{\partial v}{\partial t}(x,t)\right\|_m$$

$$\leq C_m \cdot e^{\mu t} \frac{e^{-\mu 2\kappa}+1}{\delta_0 + \mu} \sum_{j=0}^1 \int_{-\infty}^\infty \left\|\left(\frac{\partial}{\partial t}\right)^{2+j} e^{-\delta|x|} h(x,t)\right\|_{m+1-j} dt .$$

This shows that

$$v(x,t) = 0 \quad \text{when} \quad t < 0 .$$

Since we have

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta_\delta\right) v(x,t) = 0 & \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] B^{(\delta)} v(x,t) = h(x,t) e^{-\delta|x|} & \text{on} \quad \Gamma \times \boldsymbol{R}^1 \end{cases}$$

$v(x,t)$ defined by (2.11) is the desired solution of (P_δ). The solution $w(x,t)$ of (P_0) is represented as

$$w(x,t) = e^{\delta|x|} v(x,t),$$

from which it follows for any $\mu > -\delta_0$ that

(2.13) $$\|w(x,t)\|_{m+1,\Omega_R} + \left\|\frac{\partial w}{\partial t}(x,t)\right\|_{m,\Omega_R}$$

$$\leq C_m \cdot e^{\mu t} e^{\delta R} \frac{e^{-\mu \cdot 2\kappa}+1}{\delta_0 + \mu} \sum_{j=0}^1 \int_{-\infty}^\infty \left\|\left(\frac{\partial}{\partial t}\right)^{j+2} h(x,t)\right\|_{m+1-j} dt ,$$

where $\|\cdot\|_{m,\Omega_R}$ denotes the norm of the space $H^m(\Omega_R)$. And Theorem 1 follows from (2.13) by using the estimate

$$\sum_{j=0}^1 \int_{-\infty}^\infty \left\|\left(\frac{\partial}{\partial t}\right)^{j+2} h(x,t)\right\|_{m+1-j} dt \leq C\{\|\tilde{u}_0\|_{m+2} + \|\tilde{u}_1\|_{m+1}\}$$

$$\leq C'\{\|u_0\|_{m+2} + \|u_1\|_{m+1}\} .$$

§ 3. Proof of Theorem 2.1

Consider the problem

(D) $$\begin{cases} \square u = 0 & \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[1mm] u(x,t) = h(x,t) & \text{on} \quad \Gamma \times \boldsymbol{R}^1 \\[1mm] \text{supp } u(x,t) \subset \bar{\Omega} \times [0,\infty) \end{cases}$$

for a boundary data $h(x,t) \in C^\infty(\Gamma \times \boldsymbol{R}^1)$ such that supp $h \subset \Gamma \times [0,\infty)$.

It is well known that there exists a unique solution $u(x, t) \in C^\infty(\bar{\Omega} \times \mathbf{R}^1)$ satisfying (D) for any data $h(x, t)$. Concerning the asymptotic behavior of the solution for $t \to \infty$, we can derive the following proposition from the consideration of Morawetz [10].

Proposition 3. 1. *Suppose that* $h(x, t) \in C^\infty(\Gamma \times \mathbf{R}^1)$ *satisfies*

$$\operatorname{supp} h \subset \Gamma \times [0, 2\kappa].$$

Then for any δ, β *such that* $\delta - 1 \geq \delta_0 \geq \beta > 0$ *it holds that for* $m \geq 0$

$$(3.1) \qquad \left\| e^{-\delta|x|} u(x, t) \right\|_{m+1}^2 + \left\| e^{-\delta|x|} \frac{\partial u}{\partial t}(x, t) \right\|_m^2$$

$$\leq C_m e^{-2\beta t} \sum_{j=0}^{1} \int_0^{2\kappa} \left\| \left(\frac{\partial}{\partial t} \right)^j h(x, t) \right\|_{m+1-j}^2 dt, \quad \forall t \geq 0$$

where C_m *is a constant independent of* δ, β.

A proof of this proposition will be given in the last of this section. Take $\delta \geq \delta_0 + 1$ and consider the problem

$$(\mathrm{D}_\delta) \qquad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta_\delta \right) v(x, t) = 0 & \text{in} \quad \Omega \times \mathbf{R}^1 \\[2mm] v(x, t) = f(x, t) & \text{on} \quad \Gamma \times \mathbf{R}^1 \\[2mm] \operatorname{supp} v \subset \bar{\Omega} \times [0, \infty) \end{cases}$$

for the boundary data $f(x, t) \in C^\infty(\Gamma \times \mathbf{R}^1)$ such that $\operatorname{supp} f \subset \Gamma \times [0, \infty)$. Through the uniqueness of the solutions of (D) and (D_δ), when we denote by $u(x, t)$ the solution of (D) for a boundary data $h(x, t) = e^{\delta|x|} \cdot f(x, t)$, we have $v(x, t) = e^{-\delta|x|} u(x, t)$. Then the following follows immediately from Proposition 3. 1.

Proposition 3. 2. *Suppose that* $f(x, t) \in C^\infty(\Gamma \times \mathbf{R}^1)$ *and*

$$\operatorname{supp} f \subset \Gamma \times [0, \rho].$$

Then for any $\beta \leq \delta_0$, *the solution* $v(x, t)$ *of* (D_δ) *satisfies an inequality*

$$\left\| v(x, t) \right\|_{m+1}^2 + \left\| \frac{\partial v}{\partial t}(x, t) \right\|_m^2$$

$$\leq C_m e^{-2\beta t} \sum_{j=0}^{m+1} \int_0^{\rho} \left\| \left(\frac{\partial}{\partial t} \right)^j f(x,t) \right\|_{m+1-j}^2 dt, \quad \forall t \geq 0 .$$

Note that the boundary value problem for Δ_δ with a parameter $p \in C$

(3.2)
$$\begin{cases} (p^2 - \Delta_\delta) w(x) = f(x) & \text{in} \quad \Omega \\ \qquad\qquad w(x) = g(x) & \text{on} \quad \Gamma \end{cases}$$

has an apriori estimate

(3.3)
$$\mu \{ \| w(x) \|_{m+1}^2 + \| pw \|_m^2 \} + \| w \|_{m+1}^2 + |p|^2 \| w \|_m^2$$

$$\leq C_m \left\{ \frac{1}{\mu} \| f \|_m^2 + \| g \|_{m+1}^2 \right\}$$

for all $w(x) \in H^{m+1}(\Omega)$ when $\operatorname{Re} p = \mu \geq \mu_0$ where μ_0 is a positive constant determined by δ. Indeed, since the operator $\partial_t^2 - \Delta_\delta$ and the Dirichlet boundary condition satisfy the uniform Lopatinski condition we obtain the apriori estimate (3.3) through the considerations of Balaban [2], Kreiss [7] and Sakamoto [12].

Denote $U^{(\delta)}(p, g; x)$ the solution $w(x)$ of (3.2) for $f=0$ and $g \in H^m(\Gamma)$. We have

$$U^{(\delta)}(p, g; x) \in \bigcap_{m=1}^{\infty} H^m(\Omega) \quad \text{when} \quad g(x) \in C^\infty(\Gamma)$$

from the regularity theorem of an elliptic operator. And the following follows immediately from the apriori estimate (3.3).

Lemma 3.3. *For every* $g \in H^m(\Gamma)$ $U^{(\delta)}(p, g; x)$ *is analytic in* $\operatorname{Re} p \geq \mu_0$ *as* $H^m(\Omega)$*-valued function.*

Next, we will show that $U^{(\delta)}(p, g; x)$ can be prolonged analytically up to $\operatorname{Re} p > -\delta_0$. Take $q(t) \in C^\infty(\mathbf{R}^1)$ such that

(3.4)
$$q(t) \not\equiv 0 \text{ and } \operatorname{supp} q \subset [0, 2\rho]$$

and set $f(x, t) = g(x) q(t)$. Using Proposition 3.2 we have that the solution $v(x, t)$ of (D_δ) for a boundary data $f(x, t)$ satisfies

(3.5)
$$\| v(x, t) \|_m \leq C_m \cdot e^{-t\beta} \sum_{l=0}^{1} \int_0^{2\rho} \left| \left(\frac{\partial}{\partial t} \right)^l f(x, t) \right|_{m+1-l} dt, \quad \forall t \geq 0$$

for any $0 \leq \beta \leq \delta_0$. This inequality assures the summability of the integral

$$\int_0^\infty e^{-pt} v(x, t)\, dt \quad \text{for} \quad \mathrm{Re}\, p > -\delta_0$$

as a $H^m(\Omega)$-valued function. Then

$$\hat{v}(x, p) = \int_0^\infty e^{-pt} v(x, t)\, dt$$

is analytic in $\mathrm{Re}\, p > -\delta_0$ as $H^m(\Omega)$-valued function and it satisfies

$$(p^2 - \varDelta_\delta)\hat{v}(x, p) = 0 \quad \text{in} \quad \Omega$$

because $v(x, t)$ is a solution of (D_δ). On the other hand, by the choice of the boundary data we have

$$\hat{v}(x, p) = \hat{f}(x, p) = g(x)\hat{q}(p) \quad \text{on} \quad \varGamma$$

where $q(p) = \int_0^\infty e^{-pt} q(t)\, dt$. Note that $\hat{q}(p)$ is analytic in C and that $\{p; \hat{q}(p) = 0\}$ has no accumulation point in any bounded set. If we set for $p \in \mathcal{R}(q) = \{p; \mathrm{Re}\, p > -\delta_0, \hat{q}(p) \neq 0\}$

$$\tilde{U}_q^{(\delta)}(p, g; x) = \hat{v}(x, p)\hat{q}(p)^{-1}$$

$\tilde{U}_q^{(\delta)}$ is analytic in $\mathcal{R}(q)$ as $H^m(\Omega)$-valued function and it satisfies

(3. 6)
$$\begin{cases} (p^2 - \Delta_\delta)\tilde{U}_q^{(\delta)}(p, g; x) = 0 & \text{in} \quad \Omega \\ U_q^{(\delta)}(p, g; x) = g(x) & \text{on} \quad \varGamma. \end{cases}$$

For any q_1, q_2 satisfying (3. 4) we have

(3. 7) $\qquad \tilde{U}_{q_1}^{(\delta)}(p, g; x) = \tilde{U}_{q_2}^{(\delta)}(p, g; x), \quad \forall p \in \mathcal{R}(q_1) \cap \mathcal{R}(q_2).$

In fact, since $\tilde{U}_{q_j}^{(\delta)}, j = 1, 2$ satisfy (3. 6) for all $p \in \mathcal{R}(q_j)$ and the solution of (3. 2) is unique when $\mathrm{Re}\, p \geq \mu_0$, we have

$$\tilde{U}_{q_1}^{(\delta)}(p, g; x) = \tilde{U}_{q_2}^{(\delta)}(p, g; x), \quad \forall p \in \mathcal{R}(q_1) \cap \mathcal{R}(q_2) \cap \{p; \mathrm{Re}\, p \geq \mu_0\}.$$

(3. 7) follows from this fact by taking account that $\{p; Re\, p > -\delta_0, p \notin (\mathcal{R}(q_1) \cap \mathcal{R}(q_2)), |p| \leq M\}$ is a finite set for every $M < +\infty$.

We define $U^{(\delta)}(p, g; x)$ for $\mathrm{Re}\, p > -\delta_0$ by

$$U^{(\delta)}(p, g; x) = \tilde{U}_q^{(\delta)}(p, g; x)$$

by choosing a function $q(t)$ satisfying (3. 4) and $\hat{q}(p) \neq 0$. Then the right hand side is independent of choice of $q(t)$ and $U^{(\delta)}(p, g; x)$ is an-

alytic in $\mathrm{Re}\, p > -\delta_0$.   Evidently for all $\mathrm{Re}\, p > -\delta_0$ $U^{(\delta)}$ belongs to $H^m(\Omega)$ and satisfies

$$\begin{cases} (p^2 - \Delta_\partial)\, U^{(\delta)}(p, g; x) = 0 & \text{in}\quad \Omega \\ U^{(\delta)}(p, g; x) = g & \text{on}\quad \varGamma\,. \end{cases}$$

And by applying Proposition 3. 2 for $f(x, t) = q(x)q(t)$ we have

$$\|U^{(\delta)}(p, g; x)\|_m = \left\| \frac{1}{\hat{q}(p)} \int_0^\infty e^{-pt} v(x, t)\, dt \right\|_m$$

$$\leq \frac{1}{|\hat{q}(p)|} \frac{C_m}{(\delta_0 + \mu)} \|g(x)\|_m \left(\sum_{l=0}^m \int_0^{2\rho} \left| \left(\frac{\partial}{\partial t} \right)^l q(t) \right|^2 dt \right)^{1/2},$$

where a function $q(t)$ is arbitrary if only it satisfies (3. 4) and $\hat{q}(p) \neq 0$. Since we can find for every $p \in C$ a function $q(t)$ satisfying (3. 4) and

$$m|\hat{q}(p)| \geq \left(\sum_{l=0}^m \int_0^{2\rho} \left| \left(\frac{\partial}{\partial t} \right)^l q(t) \right|^2 dt \right)^{1/2},$$

the estimate (2. 3) is proved.

Now we set about to prove Proposition 3. 1.   In the first place consider the problem in the free space, that is to say, the Cauchy problem.

Lemma 3. 4.   *Suppose that* $u_j \in H^{1-j}(\mathbf{R}^3)$, $j = 0, 1$ *and that*

$$\operatorname{supp} u_j \subset \{x;\, |x| \leq \kappa\}.$$

Then the solution of the Cauchy problem with initial plane $t = \tau$

$$(3. 8) \qquad \begin{cases} \Box u(x, t) = 0 & \text{in}\quad \mathbf{R}^3 \times (\tau, \infty) \\ u(x, \tau) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x, \tau) = u_1(x) \end{cases}$$

satisfies an estimate

$$(3. 9) \quad e^{2\beta(t-\tau)} \left\{ \|e^{-\delta|x|} u(x, t)\|_1^2 + \|e^{-\delta|x|} \frac{\partial u}{\partial t}(x, t)\|_0^2 \right\}$$

$$\leq 6(1 + \delta^2) \{1 + (t-\tau)^2\} \cdot e^{2(\beta-\delta)(t-\tau)} e^{2\delta\kappa} \{\|\nabla u_0\|_0^2 + \|u_1\|_0^2\}$$

for all $t \geq \tau$, *where* δ, β *are arbitrary positive constants.*

Proof. The Huygens' principle says that if supp $u_j \subset \{x ; |x| \leq \kappa\}$ for $j = 0, 1$, the support of the solution $u(x, t)$ of the Cauchy problem (3.8) is contained in

$$\{(x, t) ; (t - \tau) - \kappa \leq |x| \leq (t - \tau) + \kappa\}.$$

Then we have

$$e^{\beta(t-\tau)} e^{-\delta|x|} \leq e^{(\beta-\delta)(t-\tau)} e^{\delta\kappa},$$

from which it follows that

$$e^{2\beta(t-\tau)} \||e^{-\delta|x|} u(x, t)\||_0^2 \leq e^{2(\beta-\delta)(t-\tau)} e^{2\delta\kappa} \||u(x, t)\||_0^2.$$

And also we have

$$e^{2\beta(t-\tau)} \left\|\left| \frac{\partial}{\partial x_j} (e^{-\delta|x|} u(x, t)) \right\|\right|_0^2$$

$$\leq 2 \cdot e^{2\beta(t-\tau)} \left\{ \||\delta e^{-\delta|x|} u(x, t)\||_0^2 + \left\|\left| e^{-\delta|x|} \frac{\partial u}{\partial x_j}(x, t) \right\|\right|_0^2 \right\}$$

$$\leq 2 \cdot e^{2\beta(t-\tau)} (1 + \delta^2) \left\{ \||\nabla u(x, t)\||_0^2 + \||u(x, t)\||_0^2 \right\}.$$

Then the left hand side of (3.9) is majorated by

$$6(1 + \delta^2) \cdot e^{2(\beta-\delta)(t-\tau)} \cdot e^{2\delta\kappa} \left\{ \||u(x, t)\||_1^2 + \left\|\left| \frac{\partial u}{\partial t}(x, t) \right\|\right|_0^2 \right\}.$$

On the other hand the Cauchy problem has an energy equality

$$\||\nabla u(x, t)\||_0^2 + \left\|\left| \frac{\partial u}{\partial t}(x, t) \right\|\right|_0^2 = \||\nabla u_0(x)\||_0^2 + \||u_1(x)\||_0^2.$$

Using an estimate

$$\||u(x, t)\||_0 \leq \int_\tau^t \left\|\left| \frac{\partial u}{\partial t}(x, t) \right\|\right|_0 dt + \||u(x, \tau)\||_0$$

$$\leq (t - \tau) \left\{ \||u_0(x)\||_1^2 + \||u_1(x)\||_0^2 \right\}^{1/2}$$

we have

$$\||u(x, t)\||_1^2 + \left\|\left| \frac{\partial u}{\partial t}(x, t) \right\|\right|_0^2$$

$$\leq \left\{ 1 + (t - \tau)^2 \right\} \left\{ \||u_0(x)\||_1^2 + \||u_1(x)\||_0^2 \right\},$$

which shows (3.9).   Q.E.D.

The idea of Morawetz [10] to prove the exponential decay of the
solution with the Dirichlet boundary condition is as follows: The solution
$u(x, t)$ of (D) for a boundary data $h(x, t)$ satisfying the condition of
Proposition 3.1 can be decomposed as

$$u(x, t) = \sum_{j=1}^{\infty} F_j(x, t) + R_\infty(x, t).$$

Here $F_j(x, t) = 0$ for $t < jT$ and for $t \geq jT$ it satisfies

$$\begin{cases} \Box F_j = 0 & \text{in} \quad \mathbf{R}^3 \times (jT, \infty) \\ F_j(x, jT) = F_{j0}(x) \\ \dfrac{\partial F_j}{\partial t}(x, jT) = F_{j1}(x), \end{cases}$$

where F_{j0} and F_{j1} have properties

(3.10) $\operatorname{supp} F_{j1} \subset \{x; |x| \leq T + \rho\}$

(3.11) $\|F_{j0}\|_1^2 + \|F_{j1}\|_0^2 \leq C \cdot \exp(-2\delta_0 \cdot jT) \sum_{j=0}^{1} \int_0^{2\pi} \left\| \left(\frac{\partial}{\partial t} \right)^j h \right\|_{1-j}^2 dt$

for a certain positive constant T.   And R_∞ satisfies

$$\operatorname{supp} R_\infty(x, t) \subset \{x; |x| \leq T + \rho\} \times \overline{\mathbf{R}_+^1}$$

(3.12) $\|R_\infty(x, t)\|_1^\infty + \left\| \frac{\partial R_\infty}{\partial t}(x, t) \right\|_0^2$

$$\leq C \cdot \exp(-2\delta_0 jT) \sum_{j=0}^{1} \int_0^{2\pi} \left\| \left(\frac{\partial}{\partial t} \right)^j h(x, t) \right\|_{1-j}^2 dt .$$

Applying Lemma 3.4 to each $F_j(x, t)$ and using (3.11) we have

$$e^{2\beta(t-jT)} \left\{ \|e^{-\delta|x|} F_j(x, t)\|_1^2 + \left\| e^{-\delta|x|} \frac{\partial F_j}{\partial t}(x, t) \right\|_0^2 \right\}$$

$$\leq 6(1+\delta^2) \{1 + (t-jT)^2\} \cdot e^{2(\beta-\delta)(t-jT)} e^{2\delta(T+\rho)}$$

$$\cdot C \ e^{-2\delta_0 T j} \sum_{l=0}^{1} \int_0^{2\pi} \left\| \left(\frac{\partial}{\partial t} \right)^l h(x, t) \right\|_{1-l}^2 dt ,$$

from which it follows at once that

$$\|e^{-\delta|x|} F_j(x, t)\|_1^2 + \left\| e^{-\delta|x|} \frac{\partial F_j}{\partial t}(x, t) \right\|_0^2$$

$$\leq 6C \left(1+\delta^2\right) \left\{1+ (t-jT)^2\right\}$$

$$\cdot e^{2(\beta-\delta)(t-jT)} \cdot e^{-2\beta t} \sum_{l=0}^{1} \int_0^{2\kappa} \left\| \left(\frac{\partial}{\partial t}\right)^l h \right\|_{1-l}^2 dt .$$

Note that

$$\sum_{j=0}^{t/T+1} e^{2(\beta-\delta)(t-jT)} \left\{1+ (t-jT)^2\right\} \leq C_{\delta, \beta}, \quad t \geq 0$$

where $C_{\delta, \beta}$ is a constant depending on $\delta - \beta$. The combination of the above estimates and (3. 12) gives

$$\left\| e^{-\delta |x|} u(x, t) \right\|_1^2 + \left\| e^{-\delta |x|} \frac{\partial u}{\partial t}(x, t) \right\|_0^2$$

$$\leq C_{\delta, \beta} e^{-2\beta t} \sum_{l=0}^{1} \int_0^{2\kappa} \left\| \left(\frac{\partial}{\partial t}\right)^l h(x, t) \right\|_{1-l}^2 dt ,$$

which proves Proposition 3. 1 for $m=1$.

Differentiate (D) l-times in t and we have

$$\begin{cases} \square \left(\dfrac{\partial}{\partial t}\right)^l u = 0 & \text{in } \ \Omega \times \boldsymbol{R}^1 \\[2mm] \left(\dfrac{\partial}{\partial t}\right)^l u(x, t) = \left(\dfrac{\partial}{\partial t}\right)^l h(x, t) & \text{on } \ \Gamma \times \boldsymbol{R}^1 . \end{cases}$$

Then we have

(3. 13) $$\left\| e^{-\delta |x|} \frac{\partial^l u}{\partial t^l} \right\|_1^2 + \left\| e^{-\delta |x|} \frac{\partial^{l+1} u}{\partial t^{l+1}} \right\|_0^2$$

$$\leq C \cdot e^{-2\beta t} \sum_{j=0}^{1} \int_0^{2\kappa} \left\| \left(\frac{\partial}{\partial t}\right)^{l+j} h \right\|_{1-j}^2 dt .$$

Since for $t \geq 0$

$$\begin{cases} \Delta_\delta \left(e^{-\delta |x|} u\right) = \dfrac{\partial^2}{\partial t^2} \left(e^{-\delta |x|} u\right) & \text{in } \ \Omega \\[2mm] \quad\quad e^{-\delta |x|} u = e^{-\delta |x|} h & \text{on } \ \Gamma \end{cases}$$

we have

(3. 14) $$\left\| e^{-\delta |x|} u(x, t) \right\|_2^2 \leq C \left\{ \left\| \frac{\partial^2}{\partial t^2} e^{-\delta |x|} u(x, t) \right\|_0^2 + \left\| e^{-\delta |x|} h(x, t) \right\|_{3/2}^2 \right\}.$$

Combining (3. 13) and (3. 14) we have (3. 1) for $m=2$. Repeating this

process we obtain (3.1) for any $m \geq 0$.

§4. Explicit Representation of the Solution of (D_δ)

In order to consider the operator $\mathcal{B}^{(\delta)}(p)$ it is necessary in the first place to obtain detailed properties of $U^{(\delta)}$. And in an attempt to do it we will construct explicitly the solution $\mathcal{W}(\mu, h; x, t)$ of the problem

(4.1)
$$\begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu \right)^2 - \Delta \right) \mathcal{W} = 0 \qquad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] \mathcal{W}(\mu, h; x, t) = h(x, t) \quad \text{on} \quad \Gamma \times \boldsymbol{R}^1 \\[2mm] \text{supp } \mathcal{W} \subset \bar{\Omega} \times [0, \infty) \end{cases}$$

for $-\delta_0 \leq \mu \leq 2\mu_0$ and $h(x, t) \in C^\infty(\Gamma \times \boldsymbol{R}^1)$ such that supp $h(x, t) \subset \Gamma \times (0, \infty)$.

The functions and notations used in this section without explanations are found in [5] and [6].

Let $s_0 \in \Gamma$ and Γ_0 be a neighborhood of s_0 such that Γ is represented in Γ_0 by parameters $\sigma = (\sigma_1, \sigma_2)$ as

$$s(\sigma) = (s_1(\sigma_1, \sigma_2), s_2(\sigma_1, \sigma_2), s_3(\sigma_1, \sigma_2)),$$

$\sigma \in I_\sigma = [-\sigma_{10}, \sigma_{10}] \times [-\sigma_{20}, \sigma_{20}], \sigma_{10}, \sigma_{20} > 0$. Take Γ_1 a neighborhood of s_0 such that $\bar{\Gamma}_1 \subset \Gamma_0$. Let $\lambda(s), \tilde{\lambda}(s)$ be functions on Γ such that

$$\lambda(s) \in \mathcal{D}(\Gamma_0), \quad \lambda(s) = 1 \quad \text{on} \quad \Gamma_1$$

$$\tilde{\lambda}(s) \in \mathcal{D}(\Gamma_0), \quad \tilde{\lambda}(s) = 1 \quad \text{on} \quad \Gamma_1$$

$$\text{supp } \tilde{\lambda} \Subset \{s; \lambda(s) = 1\}.$$

And let $0 < t_1 < t_0$, $I_t = [-t_0, t_0]$,

$$\tau(t) \in \mathcal{D}(-t_0, t_0), \quad \tau(t) = 1 \quad \text{on} \quad [-t_1, t_1]$$

$$\tilde{\tau}(t) \in \mathcal{D}(-t_0, t_0), \quad \tilde{\tau}(t) = 1 \quad \text{on} \quad [-t_1, t_1]$$

$$\text{supp } \tilde{\tau}(t) \Subset \{t; \tau(t) = 1\}.$$

Let us pose $\omega(s, t) = \lambda(s) \cdot \tau(t)$, $\tilde{\omega}(s, t) = \tilde{\lambda}(s) \cdot \tilde{\tau}(t)$. Then we have for any $h(x, t) \in \mathcal{D}(\Gamma_1 \times (0, t_1))$

$$h(s(\sigma), t) = \omega(s(\sigma), t) \int_{\boldsymbol{R}^1} dk \int_{\boldsymbol{R}^2} d\tilde{\xi} \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{i(\langle \sigma - \sigma', \xi \rangle + (t - t')k)\} \cdot \tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t').$$

Take $\chi_j(l) \in C^\infty(\boldsymbol{R}^1)$, $j = 1, 2, 3, 4$, as

$$\chi_1(l) = \begin{cases} 1 & l < 1 - 2\alpha_0 \\ 0 & l > 1 - \alpha_0 \end{cases}$$

$$\chi_2(l) = \begin{cases} 1 & |l - 1| \leqq \alpha_0 \\ 0 & |l - 1| \geqq 2\alpha_0 \end{cases}$$

$$\chi_3(l) = \begin{cases} 1 & 2 > l > 1 + 2\alpha_0 \\ 0 & l < 1 + \alpha_0, \quad \text{or} \quad l > 3 \end{cases}$$

$$\chi_4(l) = \begin{cases} 1 & l > 3 \\ 0 & l < 2 \end{cases}$$

and

(4.2) $$\sum_{j=1}^{4} \chi_j(l)^2 = 1 \quad \text{for all} \quad l \in \boldsymbol{R}^1.$$

Define operators $\mathcal{V}_j$, $j = 1, 2, 3, 4$, by

$$(\mathcal{V}_j h)(s(\sigma), t) = \omega(s(\sigma), t) \int_{\boldsymbol{R}^1} dk \int_{-1}^{\infty} d\alpha \int_{\Sigma} d\xi' \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{ik(t - t' + (1 + \alpha)\langle \sigma - \sigma', \xi' \rangle)\} \chi_j(1 + \alpha)^2 k^2(1 + \alpha)$$

$$\tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t'),$$

where $\Sigma = \{(\xi_1, \xi_2); \xi_1^2 + \xi_2^2 = 1\}$. We have at once

(4.3) $$\sum_{j=1}^{4} (\mathcal{V}_j h)(s, t) = h(s, t), \quad \forall h(s, t) \in \mathcal{D}(\Gamma_1 \times (-t_1, t_1))$$

by using (4.2) and a change of variables $\xi = k(1 + \alpha)\xi'$. Let $\theta(x, \xi, \alpha)$, $\rho(x, \xi, \alpha)$ be functions satisfying (3.6) of [6] and $\eta(\sigma, \sigma', \xi', \alpha)$, $\beta(\sigma, \sigma', \xi', \alpha)$, $v_j(l), j = 1, 2, 3$, be functions used in Section 3 of [6]. Define $\mathcal{V}_{2j}$ by

$$(\mathcal{V}_{2j} h)(s(\sigma), t) = \omega(s(\sigma), t) \cdot \int_{\boldsymbol{R}^1} dk \int_{\Sigma} d\eta \int_{|\beta| \leqq \beta_0} d\beta \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{ik(\theta(s(\sigma), \eta, \beta) - \theta(s(\sigma'), \eta, \beta) + t - t')\}$$

$$\cdot \chi_2(1 + \alpha(\sigma, \sigma', \eta, \beta))^2 k^2(1 + \alpha(\sigma, \sigma', \eta, \beta))$$

$$\cdot v_j (k^\varepsilon \beta)^2 \frac{D(\xi', \alpha)}{D(\eta, \beta)} \widetilde{\omega}(s(\sigma'), t') h(s(\sigma'), t')$$

where $0 < \varepsilon < 1/10$. Evidently we have

$$(4.4) \qquad \sum_{j=1}^3 \mathcal{V}_{2j} h = \mathcal{V}_2 h, \quad \forall h \in \mathcal{D}(\Gamma_1 \times (-t_1, t_2)).$$

We construct $\mathcal{W}$ satisfying (4.1) for $h(s, t) \in \mathcal{D}(\Gamma_1 \times (-t_1, t_1))$ through the above decomposition of h. First let us construct a function $\mathcal{W}_{2j}(\mu, h; x, t)$ satisfying almost the relation

$$(4.5) \qquad \begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu \right)^2 - \Delta \right) \mathcal{W}_{2j} = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] \mathcal{W}_{2j}(\mu, h; x, t) = \mathcal{V}_{2j} h \quad \text{on} \quad \Gamma \times \boldsymbol{R}^1 \\[2mm] \operatorname{supp} \mathcal{W}_{2j} \subset \bar{\Omega} \times [-t_0, \infty). \end{cases}$$

For this purpose we consider a problem with an oscillatory boundary data

$$(4.6) \qquad \begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu \right)^2 - \Delta \right) w(x, t) = 0 \quad\quad\quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] w(x, t) = \exp\{ik(\theta(x, \eta, \beta) + t)\} \cdot v(s, t) \quad \text{on} \quad \Gamma \times \boldsymbol{R}^1 \\[2mm] \operatorname{supp} w(x, t) \subset \bar{\Omega} \times [-t_0, \infty) \end{cases}$$

for $v(s, t) \in \mathcal{D}(\Gamma_1 \times (0, t_1))$, and ask for an asymptotic solution in the form

$$(4.7) \quad w(x, t; \eta, \beta, k, \mu) = \exp\{ik(\theta(x, \eta, \beta) + t)\}$$

$$\cdot \frac{1}{H(-k^{2/3}\beta)} \Big\{ H(k^{2/3}\rho(x, \eta, \beta)) g_0(x, t; \eta, \beta, k, \mu)$$

$$+ \frac{1}{ik^{1/3}} H'(k^{2/3}\rho(x, \eta, \beta)) g_1(x, t; \eta, \beta, k, \mu) \Big\}.$$

In order that $w(x, t)$ of (4.7) satisfies (4.5) in a neighborhood of $\Gamma_0 \times \boldsymbol{R}^1$ asymptotically it suffices to hold

$$\begin{cases} 2\dfrac{\partial g_0}{\partial t} - 2\nabla\theta \cdot \nabla g_0 - \Delta\theta \cdot g_0 - 2\rho\nabla\rho \cdot \nabla g_1 - (\nabla\rho)^2 g_1 \\[3mm] \quad - \rho\Delta\rho \cdot g_1 + 2\mu \cdot g_0 + \dfrac{1}{ik} \left(\left(\dfrac{\partial}{\partial t} + \mu \right)^2 - \Delta \right) g_0 \equiv 0 \\[3mm] \qquad\qquad (\operatorname{mod}(k^{-1}+\beta)^\infty) \quad \text{in} \quad (\bar{\Omega} \cap \mathcal{U}) \times \boldsymbol{R}^1 \end{cases}$$

$$(4.8) \quad \left\{ \begin{array}{l} 2\dfrac{\partial g_1}{\partial t} - 2\nabla\rho\cdot\nabla g_0 - \Delta\rho\cdot g_0 - 2\nabla\theta\cdot\nabla g_1 - \Delta\theta\cdot g_1 \\[2mm] \qquad + 2\mu g_1 + \dfrac{1}{ik}\left(\left(\dfrac{\partial}{\partial t}+\mu\right)^2 - \Delta\right)g_1 \equiv 0 \\[2mm] \qquad\qquad (\mathrm{mod}\,(k^{-1}+\beta)^\infty) \quad \text{in} \quad (\bar{\Omega}\cap\mathcal{U})\times\boldsymbol{R}^1 \\[3mm] g_0 + \dfrac{1}{ik^{1/3}}R(-\beta k^{2/3})\,g_1 \equiv v \\[2mm] \qquad\qquad (\mathrm{mod}\,(k^{-1}+\beta)^\infty) \quad \text{on} \quad \Gamma_0\times\boldsymbol{R}^1, \end{array} \right.$$

where $\mathcal{U}$ is a neighborhood of Γ_0 in $\boldsymbol{R}^3$.

The construction of g_0, g_1 satisfying (4.8) can be carried out by using the method in § 4 of [6] and the process of the construction shows that

$$\operatorname{supp} g_j \subset \bigcup_{(s,t)\in\operatorname{supp} v} \mathcal{L}^+(s,t)$$

where $\mathcal{L}^+(s,t) = \{(x',t'); \exists\nu=(\nu_1,\nu_2,\nu_3)$ such that $\sum_{j=1}^3 \nu_j^2 = 1$ $\sum_{j=1}^3 \nu_j n_j \geq 0$, $x'=s+l\nu$, $t'=t+l$, $l\geq 0\}$. The function $w(x,t;\eta,\beta,k,\mu)$ defined by (4.7) with g_0, g_1 satisfying (4.8) has the form

$$(4.9) \quad w(x,t;\eta,\beta,p) = \exp\{ik(\psi^+(x,\eta,\beta)+t)\}\cdot G(x,t;\eta,\beta,p)$$

in $\{x; r_0/2 \leq r \leq r_0\}$ for a certain $r_0>0$, if we denote by r the distance from $x\in\bar{\Omega}$ to Γ. And by the consideration of § 4 of [6] we see that $w(x,t)$ of (4.9) can be prolonged up to $\{x; r\geq r_0/2\}$ satisfying asymptotically $((\partial/\partial t+\mu)^2 - \Delta)w(x,t)=0$. Then

$$w(x,t;\eta,\beta,p) = \exp\{ik(\theta(x,\eta,\beta)+t)\}\,\frac{1}{H(-k^{2/3}\beta)}$$

$$\cdot\left\{H(k^{2/3}\rho)g_0 + \frac{1}{ik^{1/3}}H'(k^{2/3}\rho)g_1\right\}v_2\left(\frac{r}{r_0}\right)^2$$

$$+ \exp\{ik(\psi^+ + t)\}\cdot G^+\cdot v_3\left(\frac{r}{r_0}\right)^2$$

satisfies for any $\gamma=(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$, and N

$$\left|D_{x,t}^\gamma\left(\left(\frac{\partial}{\partial t}+\mu\right)^2 - \Delta\right)w\right| \leq C_{\gamma,N}\exp(cr)\,(|\beta|^N + |k|^{-N})$$

$$\forall(x,t)\in\bar{\Omega}\times\boldsymbol{R}^1,$$

(4.10) $|D^r_{s,t}(w-v)| \leq C_{r',N}(|\beta|^N + |k|^{-N}),\quad \forall(s,t) \in \Gamma \times \mathbf{R}^1$

and

(4.11) $\mathrm{supp}\, w \in \bigcup_{(s,t) \in \mathrm{supp}\, v} \mathcal{L}^+(s,t).$

Denote by $f(x,t;\eta,\beta,p)$ a prolongation of $((\partial/\partial t + \mu)^2 - \varDelta)w$ to the whole space $\mathbf{R}^3 \times \mathbf{R}^1$ as C^∞-function with properties

(4.12) $|D^r_{x,t} f(x,t;\eta,\beta,p)| \leq C_{r,N} \exp(cr)\,(|\beta|^N + |k|^{-N})$

(4.13) $\mathrm{supp}\, f(x,t;\eta,\beta,p) \subset \mathbf{R}^3 \times [-t_0,\infty).$

Let $z(x,t;\eta,\beta,p)$ be the solution of the problem in the free space

$$\begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \varDelta\right) z(x,t;\eta,\beta,p) = -f(x,t;\eta,\beta,p) \quad \text{in}\quad \mathbf{R}^3 \times \mathbf{R}^1 \\ \mathrm{supp}\, z(x,t;\eta,\beta,p) \subset \mathbf{R}^3 \times [-t_0,\infty). \end{cases}$$

Set $\widetilde{w}(x,t;\eta,\beta,p)$ as

$$\widetilde{w} = \begin{cases} w & \text{in}\quad \varOmega \times \mathbf{R}^1 \\ 0 & \text{in}\quad \mathcal{O} \times \mathbf{R}^1. \end{cases}$$

Taking account of (4.11) we have

$$\left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \varDelta\right)(\widetilde{w} + z) = 0 \quad \text{in}\quad \mathbf{R}^3 \times [t_0,\infty).$$

On the other hand from (4.11) and the location of the support of $f(x, t;\eta,\beta,p)$ we have that

$$\mathrm{supp}\,(\widetilde{w} + z)|_{t=t_0} \subset \{x;\, |x| \leq \rho + t_0\},$$

from which it follows with the aid of the Huygens' principle that

$$(\widetilde{w} + z)|_\Gamma = 0 \quad \text{for}\quad t \geq 2\rho + t_0.$$

By using (4.11) once more we have

(4.14) $\mathrm{supp}\, z|_\Gamma \subset \Gamma \times [-t_0, 2\rho + t_0].$

And from (4.12) we have

$$|D^r_{s,t}[z(x,t;\eta,\beta,p)]_\Gamma| \leq C_{r',N}(|\beta|^N + |k|^{-N}).$$

We now carry out the above construction of $w(x,t;\eta,\beta,p)$ and $z(x,t;\eta,\beta,p)$ taking as $v(s,t) \in \mathcal{D}(\Gamma_1 \times (0,t_1))$ a function with param-

eters σ', η, β

$$\omega(s(\sigma), t)\,(1+\alpha(\sigma, \sigma', \eta, \beta))\,\frac{D(\alpha, \hat{\xi}')}{D(\beta, \eta)}\,(\sigma, \sigma', \eta, \beta),$$

and by using these $g_j(x, t; \sigma', \eta, \beta, p)$, $j=0$, 1, $G^+(x, t; \sigma', \eta, \beta, p)$ and $z(x, t; \sigma', \eta, \beta, p)$ we define operators $\mathcal{W}_{22}$ and $\mathcal{Z}_{22}$ by the following

$$\mathcal{W}_{22}(\mu, h; x, t) = \int_{\boldsymbol{R}^1} dk \int_{\Sigma} d\eta \int_{|\beta| \leq \beta_0} d\beta \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{-ik(\theta(s(\sigma'), \eta, \beta) + t')\} \cdot \Big[\exp\{ik(\theta(x, \eta, \beta) + t)\}$$

$$\cdot \frac{1}{H(-\beta k^{2/3})} \Big\{ H(k^{2/3}\rho(x, \eta, \beta))\,g_0 + \frac{1}{ik^{1/3}} H'(k^{2/3}\rho(x, \eta, \beta))\,g_1 \Big\}$$

$$\cdot \upsilon_2\Big(\frac{r}{r_0}\Big)^2 + \exp\{ik(\psi^+(x, \eta, \beta) + t)\} \cdot G^+ \cdot \upsilon_3\Big(\frac{r}{r_0}\Big)^2 \Big]$$

$$\cdot k^2 \cdot \upsilon_2(k^\varepsilon \beta)^2 \tilde{\omega}(s(\sigma'), t')\,h(s(\sigma'), t'),$$

$$\mathcal{Z}_{22}(\mu, h; x, t) = \int_{\boldsymbol{R}^1} dk \int_{\Sigma} d\eta \int_{|\beta| \leq \beta_0} d\beta \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{-ik(\theta(s(\sigma'), \eta, \beta) + t')\} \cdot z(x, t; \sigma', \eta, \beta, p)$$

$$\cdot k^2 \cdot \upsilon_2(k^\varepsilon \beta)^2 \cdot \tilde{\omega}(s(\sigma'), t')\,h(s(\sigma'), t').$$

The properties of g_j and z assure at once

(4.15) $$\begin{cases} \left(\left(\frac{\partial}{\partial t} + \mu\right)^2 - \Delta\right)(\mathcal{W}_{22} + \mathcal{Z}_{22}) = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\ \text{supp}\,(\mathcal{W}_{22} + \mathcal{Z}_{22}) \subset \bar{\Omega} \times [-t_0, \infty), \end{cases}$$

(4.16) $$\sum_{l=0}^{m} \int_{-\infty}^{\infty} \|D_t^l(\mathcal{W}_{22}(\mu, h; x, t)|_\Gamma - \mathcal{V}_{22}h)\|_m^2\,dt$$

$$\leq C_m \int_{-\infty}^{\infty} \|h(s, t)\|_0^2\,dt,$$

(4.17) $$\text{supp}\,\mathcal{Z}_{22}|_\Gamma \subset \Gamma \times [-t_0, 2\rho + t_0]$$

(4.18) $$\sum_{l=0}^{m} \int_{-\infty}^{\infty} \|D_t^l \mathcal{Z}_{22}(\mu, h; x, t)|_\Gamma\|_m^2\,dt \leq C_m \int_{-\infty}^{\infty} \|h(s, t)\|_0^2\,dt.$$

Next, for β, k such that $\beta_0 \geq |\beta| \geq k^{-\varepsilon}$, $0 < \varepsilon < 1/10$, we can find an asymptotic solution of the problem (4.6) in the form

(4.19) $$w(x, t; \eta, \beta, p) = \exp\{ik(\psi(x, \eta, \beta) + t)\} \cdot G(x, t; \eta, \beta, p).$$

It suffices that ψ satisfies

$$\begin{cases} (\nabla\psi)^2 = 1 \quad \text{in} \quad \bar{\Omega} \quad (\text{mod } \beta^\infty) \\[1mm] \psi|_\Gamma = \theta(s, \eta, \beta) \\[1mm] \dfrac{\partial\psi}{\partial n} < 0 \quad \text{for } \beta < 0 \quad \text{and} \\[1mm] \text{Im}\,\dfrac{\partial\psi}{\partial n} > 0 \quad \text{for } \beta > 0 \end{cases}$$

and G satisfies

$$\begin{cases} 2\dfrac{\partial G}{\partial t} - 2\nabla\psi \cdot \nabla G - \Delta\psi \cdot G + 2\mu G + \dfrac{1}{ik}\left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \Delta\right)G \equiv 0 \\[2mm] \qquad\qquad (\text{mod } k^{-\infty}) \quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] G|_\Gamma = v(s, t), \end{cases}$$

and we can construct such ψ and G with the aid of the consideration of § 6 of [6]. Then $w(x, t)$ of (4.19) defined using such ψ and G has properties

$$\left| D_{x,t}^\tau\left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \Delta\right)w \right| \leqq C_{\tau, N} \cdot \exp(cr) \cdot k^{-N},$$

$$w|_\Gamma = \exp\{ik(\theta(s, \eta, \beta) + t)\} \cdot v(s, t)$$

and

$$\text{supp } w \subset \bigcup_{(s, t) \in \text{supp } v} \mathcal{L}^+(s, t).$$

Secondly we define $z(x, t; \eta, \beta, p)$ in the same manner. Taking as $v(s, t) \in \mathscr{D}(\Gamma_1 \times (0, t_1))$ a function with parameters σ', η, β

$$\omega(s(\sigma), t)(1 + \alpha(\sigma, \sigma', \eta, \beta))\dfrac{D(\alpha, \xi')}{D(\beta, \eta)}(\sigma, \sigma', \eta, \beta)\chi_2(1 + \alpha(\sigma, \sigma', \eta, \beta))^2$$

we construct $G(x, t; \sigma', \eta, \beta, p), z(x, t; \sigma', \eta, \beta, p)$ according to the above consideration, with which operators $\mathscr{W}_{2j}, \mathscr{Z}_{2j}, j = 1, 3$ are defined by

$$(4.20) \quad \mathscr{W}_{2j}(\mu, h; x, t) = \int_{\boldsymbol{R}^1} dk \int_\Sigma d\eta \int_{|\beta| \leqq \beta_0} d\beta \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{-ik(\theta(s(\sigma'), \eta, \beta) + t')\} \cdot \exp\{ik(\psi(x, \eta, \beta) + t)\}$$

$$\cdot G(x, t; \sigma', \eta, \beta, p)k^2 v_j(k^\varepsilon \beta)^2 \tilde{\omega}(s(\sigma'), t')h(s(\sigma'), t'),$$

$$(4.21) \quad \mathcal{Z}_{2j}(\mu, h; x, t) = \int_{\boldsymbol{R}^1} dk \int_{\Sigma} d\eta \int_{|\beta| \leq \beta_0} d\beta \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{-ik(\theta(s(\sigma'), \eta, \beta) + t')\} z(x, t; \sigma', \eta, \beta, p)$$

$$\cdot k^2 v_j(k^\varepsilon \beta)^2 \widetilde{\omega}(s(\sigma'), t') h(s(\sigma'), t').$$

Then we have

$$(4.22) \quad \begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \Delta\right)(\mathcal{W}_{2j} + \mathcal{Z}_{2j}) = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] \mathrm{supp}(\mathcal{W}_{2j} + \mathcal{Z}_{2j}) \subset \bar{\Omega} \times [-t_0, \infty), \end{cases}$$

$$(4.23) \quad \mathcal{W}_{2j}(\mu, h; x, t)|_\Gamma = \mathcal{V}_{2j}h,$$

$$(4.24) \quad \mathrm{supp}\, \mathcal{Z}_{2j}|_\Gamma \subset \Gamma \times [-t_0, 2\rho + t_0],$$

$$(4.25) \quad \sum_{l=0}^m \int_{-\infty}^\infty \|D_t^l \mathcal{Z}_{2j}(\mu, h; x, t)|_\Gamma\|_m^2 dt \leq C_m \int_0^\infty \|h(s, t)\|_0^2 dt.$$

Using the method of construction of $\mathcal{W}_{2j}$ and $\mathcal{Z}_{2j}$, $j = 1, 3$, we can construct $\mathcal{W}_j$, $\mathcal{Z}_j$, $j = 1, 3, 4$, in an almost same form as (4.20) and (4.21), satisfying

$$(4.26) \quad \begin{cases} \left(\left(\dfrac{\partial}{\partial t} + \mu\right)^2 - \Delta\right)(\mathcal{W}_j + \mathcal{Z}_j) = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] \mathrm{supp}(\mathcal{W}_j + \mathcal{Z}_j) \subset \bar{\Omega} \times [-t_0, \infty) \end{cases}$$

$$(4.27) \quad \mathcal{W}_j(\mu, h; x, t)|_\Gamma = \mathcal{V}_j h$$

$$(4.28) \quad \sum_{l=0}^m \int_{-\infty}^\infty \|D_t^l \mathcal{Z}_j(\mu, h; x, t)|_\Gamma\|_m^2 dt \leq C_m \int_{-\infty}^\infty \|h(s, t)\|_0^2 dt$$

$$(4.29) \quad \mathrm{supp}\, \mathcal{Z}_j(\mu, h; x, t)|_\Gamma \subset \Gamma \times [-t_0, 2\rho + t_0].$$

Let us pose

$$\sum_{j=1}^4 (\mathcal{W}_j(\mu, h; x, t) + \mathcal{Z}_j(\mu, h; x, t))|_\Gamma - h$$

$$= \int_{\boldsymbol{R}^1} dk \int_{\boldsymbol{R}^2} d\xi \int_{I_\sigma} d\sigma' \int_{I_t} dt' \cdot \exp\{-i(\langle \sigma', \xi \rangle + t'k)\}$$

$$\cdot y_0(s, t; \sigma', k, \xi, \mu) \widetilde{\omega}(s(\sigma'), t') h(s(\sigma'), t'),$$

where $\mathcal{W}_2 = \sum_{j=1}^3 \mathcal{W}_{2j}$. Then we have

$$\mathrm{supp}\, y_0(s, t; \sigma', k, \xi, \mu) \subset \Gamma \times [-t_0, 2\rho + t_0] \quad \text{for all } \sigma', k, \xi, \mu,$$

$$|D_{k,\xi}^{\tau}D_{\sigma,t}^{r'}D_{\sigma'}^{r''}y_0(s(\sigma),t;\sigma',k,\xi,\mu)|\leqq C_{\tau,N}(|\xi|^2+k^2)^{-N}.$$

Now denote by $y(x,t;\sigma',k,\xi,\mu)$ the solution of

$$\begin{cases} \left(\left(\dfrac{\partial}{\partial t}+\mu\right)^2-\Delta\right)y(x,t;\sigma',k,\xi,\mu)=0 \qquad \text{in} \quad \Omega\times\boldsymbol{R}^1 \\[2mm] \text{supp } y(x,t;\sigma',k,\xi,\mu)\subset\bar{\Omega}\times[-t_0,\infty) \\[2mm] y(x,t;\sigma',k,\xi,\mu)=-y_0(s,t;\sigma',k,\xi,\mu) \quad \text{on} \quad \Gamma\times\boldsymbol{R}^1. \end{cases}$$

Then the result of Morawetz [10] gives an estimate

$$|D_{x,t}^{\tau}D_{\sigma'}^{r'}D_{k,\xi}^{r''}y(x,t;\sigma',k,\xi,\mu)|$$

$$\leqq C_{\tau,N}\exp\{-(\delta_0+\mu)t\}\cdot(1+k^2+|\xi|^2)^{-N}, \quad \text{on} \ \Gamma\times\boldsymbol{R}^1.$$

Define an operator $\mathcal{Y}$ by

$$\mathcal{Y}(\mu,h;x,t)=\int_{\boldsymbol{R}^1}dk\int_{\boldsymbol{R}^2}d\xi\int_{I_\sigma}d\sigma'\int_{I_t}dt'\cdot\exp\{-i(\langle\xi,\sigma'\rangle+kt')\}$$

$$\cdot y(x,t;\sigma',\xi,k,\mu)\tilde{\omega}(s(\sigma'),t')h(s(\sigma'),t')$$

and we have

$$(4.30) \qquad \int_{-\infty}^{\infty}\|\mathcal{Y}(\mu,h;x,t)|_{\Gamma}\|_m dt\leqq C_m\left(\int_{-\infty}^{\infty}\|h(s,t)\|_0^2 dt\right)^{1/2}.$$

Let us pose

$$\mathcal{W}(\mu,h;x,t)=\sum_{j=1}^4(\mathcal{W}_j(\mu,h;x,t)+\mathcal{Z}_j(\mu,h;x,t))+\mathcal{Y}(\mu,h;x,t)$$

and we have for all $h\in\mathcal{D}(\Gamma_1\times(0,t_1))$

$$\begin{cases} \left(\left(\dfrac{\partial}{\partial t}+\mu\right)^2-\Delta\right)\mathcal{W}=0 \quad \text{in} \quad \Omega\times\boldsymbol{R}^1 \\[2mm] \text{supp } \mathcal{W}\subset\Omega\times[-t_0,\infty) \\[2mm] \mathcal{W}|_{\Gamma}=h. \end{cases}$$

Up to the present we assumed that the support of a boundary data h is small enough. To consider a boundary data $h\in\mathcal{D}(\Gamma\times(0,t_1))$, we introduce $\beta_j(s)\in\mathcal{D}(\Gamma)$, $j=1, 2, \cdots, N$, by the way of

(i) $\displaystyle\sum_{j=1}^N\beta_j(s)^2=1$ on Γ,

(ii)   for each j supp β_j is small so that we may carry on the

construction of $\mathcal{W}$ for $\beta_j(s)h(s,t)$ according to the process prescribed in this section. Since

$$h(s,t) = \sum_{j=1}^{N} \beta_j(s)^2 h(s,t),$$

if we define $\mathcal{W}(\mu, h; x, t)$ by

$$\mathcal{W}(\mu, h; x, t) = \sum_{j=1}^{N} \mathcal{W}(\mu, \beta_j(s)^2 h; x, t),$$

we have

$$\left(\left(\frac{\partial}{\partial t}+\mu\right)^2 - \Delta\right)\mathcal{W} = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R}^1$$

$$\mathcal{W}(\mu, h; x, t)|_\Gamma = \sum_{j=1}^{N} \beta_j(s)^2 h = h$$

$$\text{supp } \mathcal{W} \subset \bar{\Omega} \times [-t_0, \infty).$$

Then this operator $\mathcal{W}$ is the desired one of (3.1).

§5. Proof of Theorem 2.2

The operator $\mathcal{B}(p)$ is defined for Re $p > 0$ and we have a relation

$$\mathcal{B}^{(\delta)}(p)g = e^{-\delta|x|}\mathcal{B}(p)(e^{\delta|x|}g), \quad \forall g \in \mathcal{D}(\Gamma),$$

which shows

$$(e^{2\delta|x|}\mathcal{B}^{(\delta)}(p)g, g)_0 = (\mathcal{B}(p)(e^{\delta|x|}g), e^{\delta|x|}g)_0.$$

Taking account of the boundedness of $e^{\delta|x|}$ on Γ, Theorem 2 of [6] shows that (2.5) holds for Re $p > 0$.

From now on we show (2.5) for $-\delta_0 < \text{Re } p \leq 2\mu_0$. The solution of the problem

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - \Delta_\delta\right)w(x,t) = 0 \quad \text{in} \quad \Omega \times \boldsymbol{R}^1 \\[2mm] w(x,t) = \tilde{h}(s,t) \qquad \text{on} \quad \Gamma \times \boldsymbol{R}^1 \\[2mm] \text{supp } w \subset \bar{\Omega} \times [-t_0, \infty) \end{cases}$$

is expressed by the operator $\mathcal{W}$ as

$$w(x,t) = e^{-\delta|x|} \cdot e^{\mu t}\mathcal{W}(\mu, e^{\delta|x|} \cdot e^{-\mu t}\tilde{h}; x, t).$$

First we show (2.5) for $g(s) \in \mathscr{D}(\varGamma_1)$. Let $m(t) \in \mathscr{D}(0, t_1)$ and pose $\tilde{h}(s, t) = g(s) m(t)$. Suppose that $\widehat{m}(p) \neq 0$ for $p = \mu + ik$. Then

$$U^{(\delta)}(p, g; x) = \frac{1}{\widehat{m}(p)} \int_{-\infty}^{\infty} e^{-(\mu+ik)t} w(x, t) dt$$

$$= \frac{1}{\widehat{m}(p)} e^{-\delta|x|} \int_{-\infty}^{\infty} e^{-ikt} \mathscr{W}(\mu, e^{\delta|x|} g \cdot e^{-\mu t} m; x, t) dt .$$

Hence we have

$$(5.1) \quad \frac{\partial U^{(\delta)}}{\partial n}\Big|_{\varGamma} = -\delta\left(n \cdot \frac{x}{|x|}\right) \frac{e^{-\delta|x|}}{\widehat{m}(p)} e^{\delta|x|} \cdot g \cdot \int_{-\infty}^{\infty} e^{-pt} m(t) dt$$

$$+ \frac{1}{\widehat{m}(p)} e^{-\delta|x|} \int_{-\infty}^{\infty} e^{-ikt} \frac{\partial \mathscr{W}}{\partial n} (\mu, e^{\delta|x|} g \cdot e^{-\mu t} m; x, t) dt$$

$$= -\delta\left(n \frac{x}{|x|}\right) g + \frac{e^{-\delta|x|}}{\widehat{m}(p)} \int_{-\infty}^{\infty} e^{-ikt} \frac{\partial \mathscr{W}}{\partial n} (\mu, e^{\delta|x|} g \cdot e^{-\mu t} m; x, t) dt .$$

Begin with the estimate of $\partial \mathscr{W}_{22}/\partial n$. From its definition

$$\frac{\partial \mathscr{W}_{22}}{\partial n}(\mu, h; x, t)|_{\varGamma} = \int_{\mathbf{R}^1} dk \int_{\Sigma} d\eta \int_{|\beta| \leq \beta_0} d\beta \int_{I_\sigma} d\sigma' \int_{I_t} dt'$$

$$\cdot \exp\{ik(\theta(s(\sigma), \eta, \beta) - \theta(s(\sigma'), \eta, \beta) + t - t')\}$$

$$\cdot ik\left\{\frac{1}{ik^{1/3}} R(-\beta k^{2/3}) \frac{\partial \rho}{\partial n} + \frac{\partial \theta}{\partial n}\right\} \lambda(s(\sigma)) (1+\alpha) \frac{D(\alpha, \hat{\xi}')}{D(\beta, \eta)}$$

$$\cdot k^2 v_2(k^\varepsilon \beta)^2 \tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t')$$

$$+ \int \cdots \int \exp\{ik(\theta(s(\sigma), \eta, \beta) - \theta(s(\sigma'), \eta, \beta) + t - t')\}$$

$$\cdot (1 - \tau(t)) ik\left\{\frac{1}{ik^{1/3}} R(-\beta k^{2/3}) \frac{\partial \rho}{\partial n} + \frac{\partial \theta}{\partial n}\right\} \lambda(s(\sigma))(1+\alpha) \frac{D(\alpha, \xi')}{D(\beta, \eta)}$$

$$\cdot k^2 v_2(k^\varepsilon \beta)^2 \tilde{\omega}(s(\sigma'), t') h(s(\sigma'), t')$$

$$+ \int \cdots \int \exp\{ik(\theta(s(\sigma), \eta, \beta) - \theta(s(\sigma'), \eta, \beta) + t - t')\}$$

$$\cdot ik\left\{\frac{1}{ik^{1/3}} R(-\beta k^{2/3}) \frac{\partial \rho}{\partial n} + \frac{\partial \theta}{\partial n}\right\} \left\{\lambda(s(\sigma), t)(1+\alpha) \frac{D(\alpha, \xi')}{D(\beta, \eta)}\right.$$

$$\left. - \left(g_0 + \frac{1}{ik^{1/3}} R(-\beta k^{2/3}) g_1\right)|_{\varGamma}\right\} v_2(k^\varepsilon \beta)^2 k^2 \tilde{\omega} h$$

$$+ \int \cdots \int \exp\{ik(\theta(s(\sigma),\eta,\beta) - \theta(s(\sigma'),\eta,\beta) + t - t')\}$$

$$\cdot \left[ik\left\{(-\beta) - \left(\frac{1}{ik^{1/3}}R(-\beta k^{2/3})\right)^2\right\}\frac{\partial\rho}{\partial n}g_1 + \frac{\partial g_0}{\partial n}\right.$$

$$\left. + \frac{1}{ik^{1/3}}R(-\beta k^{2/3})\frac{\partial g_1}{\partial n}\right] k^2 \upsilon_2 (k^\varepsilon \beta)^2 \tilde{\omega} h$$

$$= I_{22} + t II_{22} + III_{22} + IV_{22} .$$

Define an operator $\mathcal{A}_{22}(k)$ from $\mathscr{D}(\Gamma_1)$ into $\mathscr{D}(\Gamma_0)$ by

$$\mathcal{A}_{22}(k) f = \int_\Sigma d\eta \int_{|\beta|\leq\beta_0} d\beta \int_{I_\sigma} d\sigma' \cdot \exp\{ik(\theta(s(\sigma),\eta,\beta) - \theta(s(\sigma'),\eta,\beta))\}$$

$$\cdot ik\left\{\frac{1}{ik^{1/3}}R(-\beta k^{2/3})\frac{\partial\rho}{\partial n} + \frac{\partial\theta}{\partial n}\right\} \lambda(s(\sigma)) \frac{D(\alpha,\xi')}{D(\beta,\eta)}(1+\alpha)$$

$$\cdot k^2 \upsilon_2(k^\varepsilon\beta)^2 f(s(\sigma')), \quad \text{for} \quad f \in \mathscr{D}(\Gamma_1).$$

Then we have

$$I_{22}(\mu, e^{\delta|x|}g \cdot e^{-\mu t}m(t); s, t)$$

$$= \int_{R^1} e^{ikt} \mathcal{A}_{22}(k)(e^{\delta|x|}g) \cdot \widehat{m}(ik+\mu) dk,$$

from which it follows that

(5. 2) $$\int_{-\infty}^{\infty} e^{-ikt} I_{22}(\mu, e^{\delta|x|}g \cdot e^{-\mu t}m(t); s, t) dt$$

$$= \widehat{m}(ik+\mu) \cdot \mathcal{A}_{22}(k)(e^{\delta|x|}g).$$

Next we consider II_{22}. Since $\overline{\text{supp}(\tau(t)-1)} \cap \overline{\text{supp}\,\tilde{\tau}(t)} = \phi$ II_{22} is a pseudo-differential operator belonging to $S^{-\infty}(\Gamma \times R^1)$. Then for any m, $m' > 0$ we have

$$\|II_{22}(\mu, h; \cdot, t)\|_m \leq C_{m,m'}(1+t^2)^{-m'}\|e^{\delta|x|}g\|_0$$

$$\times \left(\int_0^\infty |e^{-\mu t}m(t)|^2 dt\right)^{1/2} \quad \text{for all} \quad t \in R^1.$$

Concerning III_{22} recall that g_0 and g_1 satisfy

$$\left|D_{s,t}^r\left\{\omega(s,t)(1+\alpha)\frac{D(\alpha,\xi')}{D(\beta,\eta)} - \left(g_0 + \frac{1}{ik^{1/3}}R(-\beta k^{2/3})g_0\right)\Big|_r\right\}\right|$$

$$\leq C_{r,N}(|k|^{-N} + |\beta|^N)$$

which implies

$$\| III_{22} (\mu, e^{\delta|x|}g \cdot e^{-\mu t}m; \cdot, t) \|_m$$

$$\leq C_{m,m'} (t^2+1)^{-m'} \| e^{\delta|x|}g \|_0 \Big(\int_0^\infty |e^{-\mu t}m(t)|^2 dt \Big)^{1/2} .$$

About IV_{22} remark that it is an operator of the class $S_{1/3,0}^0(\Gamma \times \boldsymbol{R}^1)$ and that for any h

$$\text{supp } IV_{22} (\mu, h; x, t) \subset \Gamma_1 \times [-t_0, t_0].$$

Then we have

$$\int_{-\infty}^\infty \Big(\frac{\partial}{\partial\sigma} \Big)^r IV_{22} (\mu, e^{\delta|x|}g \cdot e^{-\mu t}m; x, t)_0 dt$$

$$\leq \sqrt{2t_0} \Big(\int_{-\infty}^\infty \Big(\frac{\partial}{\partial\sigma} \Big)^r IV_{22}^2 dt \Big)^{1/2}$$

$$\leq C_r \|g\|_{|r|} \Big(\int_0^\infty |e^{-\mu t}m(t)|^2 dt \Big)^{1/2} .$$

Combining the above estimates we have

$$(5.3) \quad \Big(e^{2\delta|x|} \frac{e^{-\delta|x|}}{\widehat{m}(p)} \int_{-\infty}^\infty e^{-ikt} \frac{\partial \mathcal{W}_{22}}{\partial n} (\mu, e^{\delta|x|}g \cdot e^{-\mu t}m; x, t) dt, g \Big)_m$$

$$- (\mathcal{A}_{22}(k) e^{\delta|x|}g, e^{\delta|x|}g)_{m'}$$

$$\leq C_{m} \frac{1}{|\widehat{m}(p)|} \| e^{\delta|x|}g \|_m \Big(\int_0^\infty |e^{-\mu t}m(t)|^2 dt \Big)^{1/2} .$$

Consider $\partial \mathcal{W}_{2j}/\partial n$, $j=1, 3$.

$$\frac{\partial \mathcal{W}_{2j}}{\partial n}_\Gamma = \int \cdots \int \exp\{ik(\theta(s(\sigma), \eta, \beta) - \theta(s(\sigma'), \eta, \beta) + t - t')\}$$

$$\cdot \Big\{ ik \frac{\partial\psi}{\partial n} G(s(\sigma), t; \sigma', \eta, \beta, p) + \frac{\partial G}{\partial n}(s(\sigma), t; \sigma', \eta, \beta, p) \Big\}$$

$$\cdot k^2 \upsilon_j (k^\varepsilon\beta)^2 \widetilde{\omega}(s(\sigma'), t') h(s(\sigma'), t').$$

From the equation which G satisfies it follows that

$$\frac{\partial G}{\partial n} \equiv \Big(\frac{\partial\psi}{\partial n} \Big)^{-1} \Big\{ \frac{\partial G}{\partial n} - \psi_s \cdot G_s - \frac{1}{2}\Delta\psi \cdot G + \mu G$$

$$- \frac{1}{2ik} \Big(\Big(\frac{\partial}{\partial t} + \mu \Big)^2 - \Delta \Big) G \Big\} \quad (\text{mod } k^{-\infty}),$$

where

$$\psi_s = \nabla\psi - n \cdot \frac{\partial\psi}{\partial n}\,, \quad G_s = \nabla G - n\frac{\partial G}{\partial n}\,.$$

By using

$$G|_{\Gamma} = \omega(s,t)(1+\alpha)\frac{D(\alpha,\xi')}{D(\beta,\eta)}\chi_2(1+\alpha)^2$$

we have

$$\frac{\partial\mathscr{W}_{2j}}{\partial n} = \int\cdots\int \exp\{ik(\theta(s(\sigma),\eta,\beta) - \theta(s(\sigma'),\eta,\beta) + t - t')\}$$

$$\cdot\left\{\left(ik\frac{\partial\psi}{\partial n} - \frac{1}{2}\left(\frac{\partial\psi}{\partial n}\right)^{-1}\Delta\psi\right)\lambda(s(\sigma))(1+\alpha)\cdot\frac{D(\alpha,\xi')}{D(\beta,\eta)}\chi_2(1+\alpha)^2\right.$$

$$\left. -\left(\frac{\partial\psi}{\partial n}\right)^{-1}\cdot\psi_s\cdot\left(\left(\lambda(s(\sigma))(1+\alpha)\frac{D(\alpha,\xi')}{D(\beta,\eta)}\chi_2(1+\alpha)^2\right)_s\right\}$$

$$\cdot k^2\cdot\upsilon_j(k^\varepsilon\beta)^2\widetilde{\omega}(s(\sigma'),t')h(s(\sigma'),t')$$

$$+ \int\cdots\int\exp\{ik(\theta(s(\sigma),\eta,\beta) - \theta(s(\sigma'),\eta,\beta) + t - t')\}$$

$$\cdot(1-\tau(t))\left\{\left(ik\frac{\partial\psi}{\partial n} - \frac{1}{2}\left(\frac{\partial\psi}{\partial n}\right)^{-1}\Delta\psi\right)\cdot G - \left(\frac{\partial\psi}{\partial n}\right)^{-1}\psi_s\cdot G_s\right\}$$

$$\cdot k^2\cdot\upsilon_j(k^\varepsilon\beta)^2\widetilde{\omega}(s(\sigma'),t')h(s(\sigma'),t')$$

$$+ \int\cdots\int\exp\{ik(\theta(s(\sigma),\eta,\beta) - \theta(s(\sigma'),\eta,\beta) + t - t')\}$$

$$\cdot\left\{\frac{\partial G}{\partial n} - \left(\frac{\partial\psi}{\partial n}\right)^{-1}\left(-\frac{1}{2}\Delta\psi\cdot G - \psi_s\cdot G_s\right)\right\}k^2\upsilon_j(k^\varepsilon\beta)^2$$

$$\cdot\widetilde{\omega}(s(\sigma'),t')h(s(\sigma'),t')$$

$$= I_{2j} + II_{2j} + III_{2j}\,.$$

Let us define an operator $\mathscr{A}_{2j}(k)$ from $\mathscr{D}(\Gamma_1)$ into $\mathscr{D}(\Gamma_0)$ by

$$\mathscr{A}_{2j}(k)f = \int_{\Sigma}d\eta\int d\beta\int_{I_\sigma}d\sigma'\exp\{ik(\theta(s(\sigma),\eta,\beta) - \theta(s(\sigma'),\eta,\beta))\}$$

$$\cdot\left\{\left(ik\frac{\partial\psi}{\partial n} - \left(\frac{\partial\psi}{\partial n}\right)^{-1}\frac{1}{2}\Delta\psi\right)\lambda(s(\sigma))(1+\alpha)\frac{D(\alpha,\xi')}{D(\beta,\eta)}\chi_2(1+\alpha)^2\right.$$

$$\left. -\left(\frac{\partial\psi}{\partial n}\right)^{-1}\cdot\psi_s\cdot\left(\lambda(s(\sigma))(1+\alpha)\frac{D(\alpha,\xi')}{D(\beta,\eta)}\chi_2(1+\alpha)^2\right)_s\right\}$$

$$\cdot k^2 v_j (k^\varepsilon \beta)^2 \tilde{\lambda} (s(\sigma')) f(s(\sigma')) .$$

Then for $h(s, t) = e^{\delta|x|} g(s) \cdot e^{-\mu t} m(t)$ we have

$$I_{2j}(\mu, h; s, t) = \int_{\mathbf{R}^1} e^{ikt} \mathcal{A}_{2j}(k) e^{\delta|x|} g(s) \cdot \widehat{m}(\mu + ik) dk .$$

By taking account of the fact

$$\overline{\mathrm{supp}\,(1 - \tau(t))} \cap \overline{\mathrm{supp}\,\tilde{\tau}(t)} = \phi$$

we have an estimate for all t

$$\| II_{2j}(\mu, h; s, t) \|_m \leq C_{m, m'} (1 + t^2)^{-m'} \| e^{\delta|x|} g \|_0 \cdot \left(\int_{-\infty}^\infty |e^{-\mu t} m(t)|^2 dt \right)^{1/2} .$$

Concerning III_{2j} recall that

$$\mathrm{supp}\, III_{2j} \subset \Gamma \times [-t_0, t_0]$$

and that III_{2j} is a pseudo-differential operator of the class $S_{1-\varepsilon, 0}^{-1+3\varepsilon}(\Gamma \times \mathbf{R}^1)$. Then we have at once

$$\int_{-\infty}^\infty \| D_\sigma^r III_{2j}(\mu, h; \cdot, t) \|_0 dt$$

$$\leq \sqrt{2t_0} \left(\int_{-\infty}^\infty \| D_\sigma^r III_{2j}(\mu, h; \cdot, t) \|_0^2 dt \right)^{1/2}$$

$$\leq C_r \cdot \| e^{\delta|x|} g \|_{|r|} \cdot \left(\int_{-\infty}^\infty |e^{-\mu t} m(t)|^2 dt \right)^{1/2} .$$

By the above estimate we have

(5.4) $$\left| \left(e^{2\delta|x|} \frac{1}{\widehat{m}(p)} \int_{-\infty}^\infty e^{-ikt} \frac{\partial \mathcal{W}_{2j}}{\partial n}(\mu, h; s, t) dt, g \right)_m \right.$$

$$- (\mathcal{A}_{2j}(k) e^{\delta|x|} g, e^{\delta|x|} g)_m$$

$$\leq C_m \| e^{\delta|x|} g \|_m^2 \cdot \frac{1}{|\widehat{m}(p)|} \left(\int_{-\infty}^\infty |e^{-\mu t} m(t)|^2 dt \right)^{1/2} .$$

In the same manner we have for $j = 1, 3, 4$

(5.5) $$\left| \left(e^{2\delta|x|} \frac{1}{\widehat{m}(p)} \int_{-\infty}^\infty e^{-ikt} \frac{\partial \mathcal{W}_j}{\partial n}(\mu, h; s, t) dt, g \right)_m \right.$$

$$- (\mathcal{A}_j(k) e^{\delta|x|} g, e^{\delta|x|} g)_m \bigg|$$

$$\leq C_m \|e^{\delta|x|}g\|_m^2 \frac{1}{|\widehat{m}(p)|}\Big(\int_{-\infty}^{\infty}|e^{-\mu t}m(t)|^2dt\Big)^{1/2}.$$

Here $\mathcal{A}_1(k)$ is an operator in the form

$$(\mathcal{A}_1(k)g)(s(\sigma)) = \int_{-1}^{a_0}d\alpha \int_{\Sigma}d\xi' \int_{I_\sigma}d\sigma' \exp\{ik(1+\alpha)\langle\sigma-\sigma',\xi'\rangle\}$$

$$\cdot ik(-\sqrt{1-(1+\alpha)^2})\,a_1(\sigma,\sigma',\xi,k)\chi_1(1+\alpha)^2k^2g(s(\sigma')),$$

where $a_1(\sigma,\sigma',\xi,k)\in S_{1,0}^0(I_\sigma)$, real valued and

$$a_1(\sigma,\sigma',\xi,k)\geq c>0\,.$$

And $\mathcal{A}_3(k)$ is given by

$$(\mathcal{A}_3(k)g)(s(\sigma)) = \int_{\alpha_0}^{3}d\alpha \int_{\Sigma}d\xi' \int_{I_\sigma}d\sigma'\cdot\exp\{ik(1+\alpha)\langle\sigma-\sigma',\xi'\rangle\}$$

$$\cdot k\cdot(-\sqrt{(1+\alpha)^2-1})\,a_3(\sigma,\sigma',\xi,k)\chi_3(1+\alpha)^2k^2g(s(\sigma')),$$

where $a_3(\sigma,\sigma',\xi,k)\in S_{1,0}^0(I_\sigma)$, real valued and

$$a_3(\sigma,\sigma',\xi,k)\geq c>0\,.$$

$\mathcal{A}_4(k)$ is in the form

$$\mathcal{A}_4(k)g = \int_{|\xi|\geq 2k}d\xi \int_{I_\sigma}d\sigma'\cdot\exp(i\langle\sigma-\sigma',\xi\rangle)$$

$$\cdot(-|\xi|)\,a_4(\sigma,\sigma',\xi,k)\chi_4(|\xi|/k)^2g(s(\sigma')),$$

where $a_4(\sigma,\sigma',\xi,k)\in S_{1,0}^0(I_\sigma)$ and

$$a_4(\sigma,\sigma',\xi,k)\geq c>0\,.$$

From the estimates (4.18), (4.25), (4.28) and (4.30) we have

$$(5.6)\qquad \Big\|\frac{1}{\widehat{m}(p)}\int_{-\infty}^{\infty}e^{-ikt}\frac{\partial}{\partial n}\big(\sum_{j=1}^{4}\mathcal{Z}_j+\mathcal{Y}\big)dt\Big\|_m$$

$$\leq C_m\|e^{\delta|x|}g\|_0\Big(\int_{-\infty}^{\infty}|e^{-\mu t}m(t)|^2dt\Big)^{1/2}\frac{1}{|\widehat{m}(p)|}\,.$$

Combining the estimates (5.3), (5.4), (5.5) and (5.6) we have

$$\Big|\Big(e^{2\delta|x|}\frac{\partial U^{(\delta)}(p,g;x)}{\partial n},g\Big)_m-\sum_{j=1}^{4}(\mathcal{A}_j(k)e^{\delta|x|}g,e^{\delta|x|}g)_m\Big|$$

$$\leq C_m\|e^{\delta|x|}g\|_m^2\Big(\int_{-\infty}^{\infty}|e^{-\mu t}m(t)|^2dt\Big)^{1/2}\frac{1}{|\widehat{m}(p)|}\,.$$

Remark that the left hand side is independent of $m(t)$ and that it is possible to find for any p a function $m(t) \in \mathcal{D}(-t_1, t_1)$ such that

$$|\widehat{m}(p)|^{-1}\left(\int_{-\infty}^{\infty}|e^{-\mu t}m(t)|^2 dt\right)^{1/2} \leqq 1 .$$

Then we have

$$(5.7) \qquad \left(e^{2\delta|x|}\frac{\partial U^{(\delta)}(p, g; x)}{\partial n}, g\right)_m - (\mathcal{A}(k)e^{\delta|x|}g, e^{\delta(x)}g)_m$$

$$\leqq C_m\|e^{\delta|x|}g\|_m^2, \quad \forall g \in \mathcal{D}(\Gamma_1), \mathrm{Re}\, p > -\delta_0$$

where we set $\mathcal{A}(k) = \sum_{j=1}^{4} \mathcal{A}_j(k)$.

We investigate the operator $\mathcal{A}_j(k)$.

$$\mathcal{A}_{22}(k)g = \int_{\Sigma}d\hat{\xi}' \int d\alpha \int_{I_\sigma}d\sigma' \cdot \exp\{ik\langle\sigma-\sigma', \hat{\xi}'\rangle(1+\alpha)\}$$

$$\cdot ik\left\{\frac{1}{ik^{1/3}}R(-\beta k^{2/3})\frac{\partial\rho}{\partial n}(s, \beta, \eta) + \frac{\partial\theta}{\partial n}(s, \beta, \eta)\right\}$$

$$\cdot\lambda(s(\sigma))(1+\alpha)k^2 v_2(k^\varepsilon\beta)^2 g(s(\sigma'))$$

$$= \int d\hat{\xi} \int_{I_\sigma}d\sigma' \cdot \exp(i\langle\sigma-\sigma', \hat{\xi}\rangle) \cdot A_{22}(s(\sigma), s(\sigma'), \hat{\xi}, k)$$

$$\cdot v_2(k^\varepsilon\beta(\sigma, \sigma', \mu, \hat{\xi}'))^2 g(s(\sigma')) .$$

Let us set

$$b_{22}(s, \eta, \beta, k) = k^{2/3}R(-\beta k^{2/3})\frac{\partial\rho}{\partial n}(s, \eta, 0)$$

$$= b_{221} + ib_{222} .$$

From the asymptotic behavior of $R(z)$ we have

Lemma 5. 1. *For a positive constant C sufficiently large, we have that for $c_0 > 0$*

$$-b_{221}(s, \eta, \beta, k) \geqq c_0\left(-\frac{1}{\beta}\right) \quad when \quad -k^{2/3}\beta \geqq C$$

$$-b_{221}(s, \eta, \beta, k) \geqq c_0 k^{2/3} \quad when \quad |k^{2/3}\beta| \leqq C$$

$$-b_{221}(s, \eta, \beta, k) \geqq c_0 k\sqrt{\beta} \quad when \quad k^{2/3}\beta \geqq C .$$

And we have

$$-b_{222}(s, \eta, \beta, k) \geq c_0 k \sqrt{-\beta} \quad when \quad -k^{2/3}\beta \geq C$$

$$-b_{222}(s, \eta, \beta, k) \geq c_0 k^{2/3} \quad when \quad |k^{2/3}\beta| \leq C.$$

Let us pose

$$B_{22j}(s, \xi, s', k) = b_{22j}(s, \eta(s, s', \xi', \alpha), \beta(s, s', \xi', \alpha), k)$$

$$\check{B}_{22j}(s, \xi, k) = B_{22j}(s, \xi, s, k).$$

By calculus of § 6 of [5] and § 5 of [6] we have

(5. 8) $$-\operatorname{Re}(\mathcal{A}_{22}(k)f, f)_m \geq (1 - ck^{-\varepsilon/2})([-B_{221}]_F \Upsilon_2 f, \Upsilon_2 f)_m$$

$$-\operatorname{Re}(\mathcal{P}_{22}f, f)_m - C_m \|f\|_m^2.$$

Similarly it follows that

(5. 9) $$-\operatorname{Im}(\mathcal{A}_{22}(k)f, f)_m \geq ([-\check{B}_{222}]_F \Upsilon_2 f, \Upsilon_2 f)_m$$

$$-([-\check{B}_{221}]_F \Upsilon_2 f, \Upsilon_2 f)_m - \operatorname{Im}(\mathcal{P}_{22}f, f)_m - C_m \|f\|_m^2.$$

For $j = 1, 3$

$$\mathcal{A}_{2j}(k)f = \int_\Sigma d\xi' \int_{|\alpha| \leq \alpha_0} d\alpha \int_{I_\sigma} d\sigma' \cdot \exp\{i(1+\alpha)\langle \sigma - \sigma', \xi'\rangle k\}$$

$$\cdot \left[\left(ik \frac{\partial \psi}{\partial n} - \frac{1}{2} \left(\frac{\partial \psi}{\partial n} \right)^{-1} \Delta \psi \right) \lambda(s(\sigma)) \chi_j (1+\alpha)^2 (1+\alpha) \right.$$

$$\left. - \left(\frac{\partial \psi}{\partial n} \right)^{-1} \psi_s \cdot \left((1+\alpha) \frac{D(\alpha, \xi')}{D(\beta, \eta)} \chi_j (1+\alpha)^2 \right)_s \right]$$

$$\cdot k^2 \upsilon_j (k^\varepsilon \beta)^2 \tilde{\lambda}(s(\sigma')) f(s(\sigma')).$$

We set

$$b_{21}(s, \eta, \beta, k) = -\frac{1}{2} \left(\frac{\partial \psi}{\partial n} \right)^{-1} \Delta \psi \cdot \lambda(s) + ik \frac{\partial \psi}{\partial n}$$

$$= b_{211} + i b_{212},$$

$$B_{21j}(s, \xi, s', k) = b_{21j}(s, \eta(s, s', \xi', \alpha), \beta(s, s', \xi', \alpha), k)$$

$$\check{B}_{21j}(s, \xi, k) = B_{21j}(s, \xi, s, k).$$

Recall that for $-\beta_0 \leq \beta \leq -k^{-\varepsilon}$, $\partial \psi/\partial n|_\Gamma < 0$ and

$$-b_{211}(s, \eta, \beta, k) \leqq \frac{c_0}{-\beta}$$

$$-b_{212}(s, \eta, \beta, k) \geqq c_0 k \sqrt{-\beta}.$$

Since $\partial \psi / \partial n|_\Gamma$ is purely imaginary for $\beta_0 \geqq \beta \geqq k^{-\varepsilon}$ we pose

$$b_{23} = ik \frac{\partial \psi}{\partial n} = b_{231} + i b_{232}, \quad b_{232} = 0$$

$$B_{231}(s, \xi, s', k) = b_{231}(s, \eta(s, s', \xi', \alpha), \beta(s, s', \xi', \alpha), k)$$

$$\check{B}_{231}(s, \xi, k) = B_{231}(s, \xi, s, k).$$

Then we have

$$-b_{231} \leqq c_0 k \sqrt{\beta}.$$

It holds that

$$(5.10) \quad -\operatorname{Re}(\mathcal{A}_{2j}(k)f, f)_m \geqq (1 - ck^{-\varepsilon/2})([-\check{B}_{2j1}]_F \Upsilon_j X_2 f, \Upsilon_j X_2 f)_m$$
$$-\operatorname{Re}(\mathcal{P}_{2j}f, f) - C_m \|f\|_m^2,$$

$$(5.11) \quad -\operatorname{Im}(\mathcal{A}_{21}(k)f, f)_m \geqq (1 - ck^{-\varepsilon/2})([-\check{B}_{212}]_F \Upsilon_1 X_2 f, \Upsilon_1 X_2 f)_m$$
$$-\operatorname{Im}(\mathcal{P}_{21}f, f)_m - C_m \|f\|_m^2.$$

Set

$$B_{2F} = \sum_{j=1}^3 \Upsilon_j^* [-B_{2j1}]_F \Upsilon_j$$

and we have

$$(5.12) \quad -\operatorname{Re}(\mathcal{A}_2(k)f, f)_m \geqq (1 - ck^{-\varepsilon/2})(B_{2F} X_2 f, X_2 f)_m - C_m \|f\|_m^2,$$

because it holds that

$$|\sum_{j=1}^3 (\mathcal{P}_{2j}f, f)_m| \leqq C \cdot k^{-\varepsilon/2}(B_{2F} X_2 f, X_2 f)_m.$$

Set

$$\widetilde{B}_{2F} = \sum_{j=1}^3 \Upsilon_j^* [-\check{B}_{2j2}]_F \Upsilon_j$$

and we have

$$(5.13) \quad -\operatorname{Im}(\mathcal{A}_2(k)f, f)_m \geqq (1 - c \cdot k^{-\varepsilon/2})(\widetilde{B}_{2F} X_2 f, X_2 f)_m$$
$$-C \cdot (B_{2F} X_2 f, X_2 f)_m - C_m \|f\|_m^2.$$

From the form of $\mathcal{A}_j(k)$ and the properties of their symbols we have at once

$$(5.14) \qquad - \mathrm{Re}\,(\mathcal{A}_1(k)f,f)_m \geqq - C_m \|f\|_m^2$$

$$(5.15) \qquad - \mathrm{Im}\,(\mathcal{A}_1(k)f,f)_m \geqq c_0 k \|X_1 f\|_m^2 - C_m \|f\|_m^2$$

$$(5.16) \qquad - \mathrm{Re}\,(\mathcal{A}_3(k)f,f)_m \geqq c_0 k \|X_3 f\|_m^2 - C_m \|f\|_m^2$$

$$(5.17) \qquad - \mathrm{Re}\,(\mathcal{A}_4(k)f,f)_m \geqq c_0 (|D|X_4 f, X_4 f)_m - C_m \|f\|_m^2.$$

From (5.12), (5.15), (5.16) and (5.17) it follows that

$$(5.18) \qquad - \mathrm{Re}\,(\mathcal{A}(k)f,f)_m \geqq ((X_2^* B_{2F} X_2 + c_0 k X_3^* X_3 + X_4^* |D| X_4) f, f)_m$$
$$- C_m \|f\|_m^2\,,$$

and from (5.13) and (5.15)

$$(5.19) \qquad - \mathrm{Im}\,(\mathcal{A}(k)f,f)_m \geqq ((c_0 k X_1^* X_1 + X_2^* \widetilde{B}_{2F} X_2) f, f)_m$$
$$- C_m \|f\|_m^2 - (B_{2F} X_2 f, X_2 f)_m\,.$$

By Lemma 5.1 we have

$$X_2^* (B_{2F} + \widetilde{B}_{2F}) X_2 \geqq c_0 k^{2/3} X_2^* X_2\,,$$

from which it follows that

$$(5.20) \qquad 2|(\mathcal{A}f,f)_m| \geqq c_0 k^{2/3} \|f\|_m^2 - C_m \|f\|_m^2\,.$$

$$\left(e^{2\delta|x|} \mathcal{B}^{(\delta)}(p) g, g\right)_m = \left(e^{2\delta|x|} \frac{\partial U^{(\delta)}(p,g;x)}{\partial n}\Big|_\Gamma, g\right)_m$$
$$+ \left(e^{2\delta|x|} \left(\sum_{j=1}^{3}(b_j - n_j)\frac{\partial}{\partial x_j} + \frac{1}{2} \frac{\partial (b_j - n_j)}{\partial x_j}\right) g, g\right)_m$$
$$+ (ik + l)\,(e^{2\delta|x|} c(x) g, g)_m + (e^{2\delta|x|} d(x) g, g)_m\,.$$

Then using (5.7) and (5.18) we have for a real

$$(5.21) \qquad - \mathrm{Re}\,(e^{2\delta|x|}(\mathcal{B}^{(\delta)}(p) - a) g, g)_m$$
$$\geqq ((X_2^* B_{2F} X_2 + c_0 k X_3^* X_3 + X_4^* |D| X_4) e^{\delta|x|} g, e^{\delta|x|} g)_m$$
$$- C_m \|e^{\delta|x|} g\|_m^2 + \mathrm{Re}\,((a - d(x)) e^{\delta|x|} g, e^{\delta|x|} g)_m$$
$$\geqq (|D|X_4 e^{\delta|x|} g, X_4 e^{\delta|x|} g)_m + (a + \inf_{x \in \Gamma} \mathrm{Re}\,(-d(x)) - C_m) \|e^{\delta|x|} g\|_m^2$$
$$- C_m' \|e^{\delta|x|} g\|_{m-1}^2\,.$$

Till now we have supposed that $g \in \mathscr{D}(\Gamma_1)$. In order to show that (5.21) is valid for all $g \in C^\infty(\Gamma)$ we use $\beta_j(s), j=1, 2, \cdots, N$ introduced at the end of § 4. Applying the above result for each $\beta_j(s)g(s)$ we see that (5.21) is also valid for all $g \in C^\infty(\Gamma)$.

Suppose that

$$(5.22) \qquad \inf_{x \in \Gamma} \mathrm{Re}\,(-d(x)) \geq C_0 + C_0' + 1 .$$

Then we have for any $a \geq 0$

$$(5.23) \qquad \| e^{\delta|x|}g \|_0 \leq \| e^{\delta|x|}(\mathscr{B}^{(\delta)}(p) - a)g \|_0 .$$

Therefore for (5.1) and (5.23) it follows for $c_m > 0$ that

$$(5.24) \qquad \| e^{\delta|x|}g \|_m \leq c_m \| e^{\delta|x|}(\mathscr{B}^{(\delta)}(p) - a)g \|_m .$$

Since it holds

$$\| e^{\delta|x|}g \|_{m+1} \leq \| |D| X_4 e^{\delta|x|}g \|_m + C|k| \| e^{\delta|x|}g \|_m$$

we obtain from (5.21) for any $a \geq 0$

$$(5.25) \qquad \| e^{\delta|x|}g \|_{m+1} \leq C_m |k| \| e^{\delta|x|}(\mathscr{B}^{(\delta)}(p) - a)g \|_m .$$

We will show the existence of $\mathscr{B}^{(\delta)}(p)^{-1}$ under the assumption (5.22). Let us set

$$e^{\delta|x|} \frac{\partial}{\partial n} U^{(\delta)}(p, g; x)|_\Gamma - \mathscr{A}(k) e^{\delta|x|}g = \mathscr{C}(k, \mu) e^{\delta|x|}g .$$

Then we have from (5.7) that $\mathscr{C}(k, \mu)$ is a bounded operator from $H^m(\Gamma)$ into $H^m(\Gamma)$. For a real constant a

$$(e^{2\delta|x|}(\mathscr{B}^{(\delta)}(p) - a)g, f)_0$$

$$= \left(e^{\delta|x|}g, \left(\mathscr{A}(k)^* + \mathscr{C}(k, \mu)^* - \sum_{j=1}^{3} (b_j - n_j) \frac{\partial}{\partial x_j} \right. \right.$$

$$\left. \left. + \sum_{j=1}^{3} \left(\frac{1}{2} \frac{\partial(b_j - n_j)}{\partial x_j} - \delta(b_j - n_j) \frac{x_j}{|x|} \right) + \overline{d(x)} - a \right) e^{\delta|x|}f \right)_0 .$$

Concerning $\mathscr{A}(k)^*$ we can show

$$\| (\mathscr{A}(-k) - \mathscr{A}(k)^*)f \|_0^2 \leq -\frac{1}{2} \mathrm{Re}\,(\mathscr{A}(k)f, f)_0 + C \| f \|_0^2$$

by using the properties of the symbol. Therefore it holds that for a sufficiently large $a > 0$

$$\| (\mathscr{B}^{(\delta)}(p)^* - a) e^{\delta|x|} f \|_0 \geq \| e^{\delta|x|} f \|_0 .$$

Then $\{ (\mathscr{B}^{(\delta)}(p) - a) g; g \in C^\infty(\Gamma) \}$ is dense in $L^2(\Gamma)$. For any $h(s)$ $\in L^2(\Gamma)$ there exists a sequence $g_j \in C^\infty(\Gamma)$ such that

$$(\mathscr{B}^{(\delta)}(p) - a) g_j \to h \quad \text{as } j \to \infty .$$

From (5. 24) and (5. 25) g_j converges in $H^1(\Gamma)$. Let us denote the limit by $e^{\delta|x|} g$. We have

$$(\mathscr{B}^{(\delta)}(p) - a) g = h .$$

That is, $\mathscr{B}^{(\delta)}(p) - a$ is a bijection from $H^1(\Gamma)$ onto $L^2(\Gamma)$. Therefore $(\mathscr{B}^{(\delta)}(p) - a)^{-1}$ is continuous from $L^2(\Gamma)$ onto $H^1(\Gamma)$. Taking account of Γ is compact, the operator $(\mathscr{B}^{(\delta)}(p) - a)^{-1}$ is a completely continuous operator in $L^2(\Gamma)$. The equation

(5. 26)
$$\mathscr{B}^{(\delta)}(p) g = h$$

is equivalent to

(5. 27)
$$g + a (\mathscr{B}^{(\delta)}(p) - a)^{-1} g = (\mathscr{B}^{(\delta)}(p) - a)^{-1} h .$$

Now we know the uniqueness of solutions of (5. 26) from (5. 24), which assures the solvability of (5. 27). Then we see that $\mathscr{B}^{(\delta)}(p)^{-1}$ exists and continuous from $L^2(\Gamma)$ onto $H^1(\Gamma)$.

Similarly we see for any $m \geq 0$ $\mathscr{B}^{(\delta)}(p)^{-1}$ is continuous from $H^m(\Gamma)$ onto $H^{m+1}(\Gamma)$. Thus Theorem 2. 1 is proved.

§ 6. Case of the Third Boundary Condition
(Proof of Theorem 2)

When $B = \dfrac{\partial}{\partial n}$ we denote $\mathscr{B}^{(\delta)}(p)$ by $\mathscr{N}^{(\delta)}(p)$ specially. The estimate (5. 20) shows that for $-\delta_0 < \operatorname{Re} p \leq 2\mu_0$

(6. 1)
$$(c_0 k^{2/3} - C_0) \| e^{\delta|x|} g \|_0 \leq \| e^{\delta|x|} \mathscr{N}^{(\delta)}(p) g \|_0 .$$

Then for any $\sigma(s) \in C^\infty(\Gamma)$ such that $|\sigma(s)| \leq M$, there exists $k_0 > 0$ such that for $p = ik + \mu$, $-\delta_0 < \mu \leq 2\mu_0$, $|k| \geq k_0$

$$\| e^{\delta|x|} g \|_0 \leq \| e^{\delta|x|} (\mathscr{N}^{(\delta)}(p) + \sigma) g \|_0 .$$

Then $(\mathscr{N}^{(\delta)}(p) + \sigma)^{-1}$ exists for $-\delta_0 < \mu \leq 2\mu_0$, $|k| \geq k_0$. Let $\sigma_0(s)$ be real valued and

(6. 2) $-M \leqq \sigma_0(s) \leqq 0$.

Suppose that for $\operatorname{Re} p \geqq 0$ there exists $g(s) \in L^2(\Gamma)$ such that

(6. 3) $(\mathscr{N}^{(\delta)}(p) + \sigma_0(s)) g(s) = 0$.

From the regularity theorem it follows $g(s) \in C^\infty(\Gamma)$.   A function

$$u(x) = e^{\delta|x|} U^{(\delta)}(p, g; x)$$

satisfies

$$\begin{cases} (\Delta - p^2) u(x) = 0 & \text{in} \quad \Omega \\[2mm] u(x) = e^{\delta|x|} g & \text{on} \quad \Gamma \\[2mm] \dfrac{\partial u}{\partial n} + \sigma_0(s) u = 0 & \text{on} \quad \Gamma . \end{cases}$$

When $\operatorname{Re} p > 0$ it holds that $u(x) \in H^2(\Omega)$ since we have

$$e^{\delta|x|} U^{(\delta)}(p, g; x) = U(p, e^{\delta|x|} g; x) .$$

$$((p^2 - \Delta) u, u)_0 = p^2 \|u\|_0^2 + \int_\Gamma \frac{\partial u}{\partial n} \overline{u} \, dS + \|\nabla u\|_0^2$$

and

$$\int_\Gamma \frac{\partial u}{\partial n} \overline{u} \, dS = - \int_\Gamma \sigma_0(s) |u|^2 dS \geqq 0 .$$

Then we have $u \equiv 0$ and $g = 0$.

Next consider $p = ik \neq 0$. Since $u(x)$ satisfies the radiation condition of Sommerfeld we have

$$\int_{\Omega_R} \{(\Delta + k^2) u \cdot \overline{u} - u \overline{(\Delta + k^2) u}\} \, dx$$

$$= - ik \int_\Gamma |u|^2 dS + O(R^{-1}) .$$

The application of the uniqueness theorem of Rellich implies $u \equiv 0$. For $k = 0$ by the consideration of Mizohata [9] $u(x)$ is written by potential of double layer. Therefore we have that $|x| u(x)$ and $|x|^2 \partial u / \partial x_j$ rest bounded. Then

$$0 = \int_{\Omega_R} u \cdot \overline{\Delta u} \, dx = \int_\Gamma u \left(-\frac{\overline{\partial u}}{\partial n} \right) dS - \int_{\Omega_R} |\nabla u|^2 dx + \int_{|x|=R} u \frac{\overline{\partial u}}{\partial |x|} dx$$

$$= \int_\Gamma \sigma_0 |u|^2 dS - \int_{\Omega_R} |\nabla u|^2 dx + \int_{|x|=R} u \, \frac{\overline{\partial u}}{\partial |x|} dS .$$

Taking account of $\sigma_0 \leqq 0$ we have $\nabla u \equiv 0$ by $R \to \infty$, then $u \equiv 0$.

Thus we obtain for $\mathrm{Re}\, p \geqq 0$ the uniqueness of the solution of (6.3). This fact assures that $(\mathcal{N}^{(\delta)}(p) + \sigma_0)^{-1}$ exists and that it is continuous from $L^2(\Gamma)$ onto $H^1(\Gamma)$ and also $H^m(\Gamma)$ onto $H^{m+1}(\Gamma)$.

Let us set

$$\inf \| (\mathcal{N}^{(\delta)}(p) + \sigma_0) g \|_0 = J ,$$

where inferior is taken with respect to $\|g\|_1 = 1$, $0 \leqq \mathrm{Re}\, p \leqq 2\mu_0$, $|k| \leqq k_0$ and $\sigma_0 \in \{\sigma ; \text{ real valued } C^\infty \text{ function such that } -M \leqq \sigma \leqq 0\}$. Then we have $J > 0$. Since

$$\mathcal{N}^{(\delta)}(p) + \sigma_0 + \tilde{\sigma}$$

$$= \mathcal{N}^{(\delta)}(ik) + \sigma_0 + \mathcal{N}^{(\delta)}(p) - \mathcal{N}^{(\delta)}(ik) + \tilde{\sigma}$$

$$= (\mathcal{N}^{(\delta)}(ik) + \sigma_0) \{ I + (\mathcal{N}^{(\delta)}(ik) + \sigma_0)^{-1} (\mathcal{N}^{(\delta)}(p) - \mathcal{N}^{(\delta)}(ik) + \tilde{\sigma}) \}$$

we know that

(6.4) $\| (\mathcal{N}^{(\delta)}(ik + \mu) - \mathcal{N}^{(\delta)}(ik) + \tilde{\sigma}) g \|_0 \leqq (1 - \varepsilon) J \|g\|_1, \quad \forall g \in C^\infty(\Gamma)$

implies the existence of

$$(\mathcal{N}^{(\delta)}(p) + \sigma_0 + \tilde{\sigma})^{-1} .$$

We see at once that there exist positive constants $\delta_1, \delta_2 > 0$ such that (6.4) holds for any $|k| \leqq k_0$ when

(6.5) $-\delta_0 < -\delta_2 \leqq \mu \leqq 0$ and $|\tilde{\sigma}| \leqq \delta_1$

are fulfilled.

Thus we have shown that, for $\sigma = \sigma_0 + \tilde{\sigma}$ and $p = ik + \mu$ such that (6.5) holds, $(\mathcal{N}^{(\delta)}(p) + \sigma)^{-1}$ exists. On the other hand for $\mathrm{Re}\, p \geqq \mu_0$ and p such that $|k| \geqq k_0$, $-\delta_0 < \mu \leqq 2\mu_0$ the existence of $(\mathcal{N}^{(\delta)}(p) + \sigma)^{-1}$ has already proved. Thus if $|\tilde{\sigma}| \leqq \delta_1$ $(\mathcal{N}^{(\delta)}(p) + \sigma)^{-1}$ exists for all $\mathrm{Re}\, p \geqq -\delta_2$, and it follows that

(6.6) $\| (\mathcal{N}^{(\delta)}(p) + \sigma)^{-1} g \|_m \leqq C_m |k|^{-2/3} \|g\|_m, \quad \forall \mathrm{Re}\, p \geqq -\delta_2 .$

Thus Theorem 2 is proved.

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