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Holonomic Quantum Fields I

By

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In this series of papers we expound a treatise on these subjects: (1) Deformation theory for linear (ordinary and partial) differential equations, (2) Quantum fields with critical strength, and (3) Theory of Clifford group (Theory of 'rotations'). Indeed, our principal aim as well as idea here is to reveal a deep link between these apparently independent concepts.

Naturally we can and do exploit this connection in both ways, namely: On the one hand we exploit it toward the study of (1) and see that deformation theory can be constructively analysed in terms of field operators; and on the other hand we utilize it for construction of exact n-point τ functions (causal Green functions) for (2) in a closed form in terms of solutions to a system of non-linear differential equations (which appear as equations of deformation of linear differential equations).

Our present work has been evolved from L. Onsager [9] who discovered in effect that field operators on 2-dimensional Ising lattice are elements of a Clifford group (a link between (2) and (3)), and exploited this fact toward exact computation of the free energy of the Ising model, and from T. T. Wu et al [8] who discovered that the 2-point function for the Ising model admits an exact expression in terms of Painlevé transcendent of the third kind. Brief accounts of our theory are given in [2], [3], [4], [15].

Chapter 1 is devoted to the theory of rotations in an orthogonal vector space [1], [2], [3], [4], which plays a fundamental role in subsequent chapters. After a preliminary review on Clifford algebras we introduce the notion of the norm map Nr, and give an explicit formula expressing the norm of an element g of the Clifford group $G(W)$ in

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terms of the rotation T_g it induces in the underlying orthogonal space W. Then we characterize the closure $\overline{G}(W)$ of $G(W)$, and give a formula for the norm of a product $g^{(1)}\cdots g^{(n)}$ of elements $g^{(i)}$ of $\overline{G}(W)$. Finally we define the generalized notion of the norm (κ-norm) Nr_κ and study the transformation property under a change of the norm maps.

The construction of our operators of holonomic quantum fields is achieved in chapter 2 [2]. We let W be the space of solutions to the Dirac equation with positive mass in the 2-dimensional Minkowski space-time, and equip it with a non-degenerate inner product to make it an orthogonal vector space. By specifying a rotation in $W\otimes\boldsymbol{C}^n$, we construct a field operator $\varphi(a)$ in the normal product form of auxiliary free fermi fields $\psi(x)$ so that the rotation induced by $\varphi(a)$ coincides with the specified one. Analogous construction is performed when the space-time is only one dimensional (i.e. no time dimensions); in this case the construction of $\varphi(a)$ from the rotation leads one to the Riemann-Hilbert problem [5]. Also we derive an operator expansion formula for the product $\psi(x)\varphi(a)$ in the region where $x-a$ is small.

Chapter 3 is concerned with deformation theory of a holonomic system [3], [15]. We consider the space $W^{\mathrm{strict}}_{a_1,\cdots,a_n}$ consisting of double-valued solutions to the 2-dimensional Euclidean Dirac equation satisfying suitable growth order conditions at the branch points $a_1,\cdots,a_n$ and at ∞. After establishing its finite dimensionality we derive a holonomic system of first order linear differential equations satisfied by a basis of $W^{\mathrm{strict}}_{a_1,\cdots,a_n}$. The coefficients appearing in this system are functions of $a_1,\cdots,a_n$ and are shown to satisfy a completely integrable system of total differential equations (the deformation equations). These results are extended to the case of the space $W^{\mathrm{strict}}_{a_1,\cdots,a_n}(\Lambda)$ which consists of multi-valued solutions with the prescribed monodromy property and the same growth order conditions. The holonomic systems and the deformation equations have their 1-dimensional analogues, which are the Fuchsian systems of first order ordinary differential equations and the celebrated Schlesinger's equations [6], respectively.

In chapter 4 we construct solutions to these holonomic systems and the deformation equations in terms of the κ-expectation values $\langle\ \rangle_\kappa$ introduced in chapter 1 of the products of our field operators φ's and

ψ's [3], [15]. It is shown that by a suitable choice of $\langle\ \rangle_\kappa$ and various combinations of the operators we obtain a basis of solutions to the relevant holonomic system. This provides the relation between the coefficients of the system, which are solutions to the deformation equations, and the n-point functions $\langle\varphi(a_1)\cdots\varphi(a_n)\rangle_\kappa$ of the φ's. In this way we obtain, in one hand, an $n(n-1)/2$ parameter family of solutions to our 2-dimensional deformation equations (cf. [7]), and in terms of their solutions, on the other hand, closed expressions for n-point functions of the φ's [3], [8]. Moreover the monodromy structure of the basis constructed in this way is apparent in the rotations $T_{\varphi(a)}$ prescribed beforehand. Thus the above scheme gives, in the one-dimensional case, a constructive solution to the Riemann's problem on the complex sphere and also to the Schlesinger's deformation equations.

In subsequent chapters we study the holonomy structure of the n-point functions [2] and the lattice field theory. We shall give the norm of the spin operator of the 2-dimensional Ising model below and above the critical temperature, and exact expressions of their n-point correlation functions [9], [8], [10], [11], [12]. Their scaling limits φ_F and φ^F are shown to be obtained as special cases of the construction in chapter 2. Also we shall calculate the asymptotic fields and the S-matrix for φ^F [2].

A Summary of Results in Chapter 1

§ 1. 1. Generalities on Clifford algebra.

Let W be a vector space over $\boldsymbol{C}$ equipped with a non-degenerate inner product $\langle w, w'\rangle$. We call W an orthogonal vector space. We denote by $A(W)$ the Clifford algebra over W; an associative algebra generated by W with defining relations $ww'+w'w=\langle w, w'\rangle$. Let ε denote the automorphism of $A(W)$ characterized by $\varepsilon(w)=-w$ for $w\in W$. We denote by $G(W)$ the Clifford group $\{g\in A(W)|\exists g^{-1}\in A(W), gW\varepsilon(g)^{-1}=W\}$. g belongs to $G(W)$ if and only if $g=w_1\cdots w_l$ where $w_k\in W$ and $w_k^2\neq 0$ $(k=1,\cdots,l)$. The spinorial norm $\mathrm{nr}(g)$ is given by $\mathrm{nr}(g)=\prod_{k=1}^{l}(-w_k^2)$. T_g: $W\rightarrow W$, $w\mapsto T_g w=gw\varepsilon(g)^{-1}$, belongs to the orthogonal group $O(W)$, and we have an exact sequence

$$(1.1.3)\qquad 1 \to GL(1, \boldsymbol{C}) \xrightarrow{id} G(W) \xrightarrow{\mathfrak{T}} O(W) \to 1.$$

A holonomic decomposition of a $2r$ dimensional orthogonal space W is a decomposition $W = V^{\dagger} \oplus V$ into two holonomic (=maximal isotropic) subspaces $V^{\dagger}$ and V. There is a unique isomorphism

$$(1.1.4)\qquad \mathrm{Nr}: A(W) \to \Lambda(W)$$

of left $\Lambda(V^{\dagger})$ and right $\Lambda(V)$ bi-modules such that $\mathrm{Nr}(1) = 1$. Here $\Lambda(W)$ denotes the exterior algebra over W. $\mathrm{Nr}(a)$ is called the norm of $a \in A(W)$. In physicist's terminology an element of $V^{\dagger}$ (resp. V) is called a creation operator (resp. an annihilation operator), the inverse of (1.1.4) Nr^{-1} is called "the normal ordering" of $\mathrm{Nr}(a)$ and a is called "the normal product" of $\mathrm{Nr}(a)$. In notations $a = :\mathrm{Nr}(a):$.

The residue class of 1 in $A(W)/A(W)V$ (resp. $A(W)/V^{\dagger}A(W)$) is called the vacuum and is denoted by $|\mathrm{vac}\rangle$ (resp. $\langle\mathrm{vac}|$). We have $A(W) \cong \mathrm{End}\,(A(W)|\mathrm{vac}\rangle) \cong \mathrm{End}\,(\langle\mathrm{vac}|A(W))$. The constant term of $\mathrm{Nr}(a) \in \Lambda(W)$ is called the vacuum expectation value of a and is denoted by $\langle\mathrm{vac}|a|\mathrm{vac}\rangle$ or in short $\langle a\rangle$.

§ 1.2. Norms and rotations.

Let $W = V^{\dagger} \oplus V$ be a holonomic decomposition, and let $(v_1^{\dagger}, \cdots, v_r^{\dagger}, v_1, \cdots, v_r)$ be a dual basis: $\langle v_\mu^{\dagger}, v_\nu^{\dagger}\rangle = 0$, $\langle v_\mu, v_\nu\rangle = 0$ and $\langle v_\mu^{\dagger}, v_\nu\rangle = \delta_{\mu\nu}$. We set $(T_g v_1^{\dagger}, \cdots, T_g v_r^{\dagger}, T_g v_1, \cdots, T_g v_r) = (v_1^{\dagger}, \cdots, v_r^{\dagger}, v_1, \cdots, v_r)\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$, and set $\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} 1 & -T_2 \\ & 1 \end{pmatrix}\begin{pmatrix} T_1 & \\ & T_4^{-1} \end{pmatrix}\begin{pmatrix} 1 & \\ T_3 & 1 \end{pmatrix}$, assuming that T_4 is invertible. Then $\langle g\rangle$ and $\mathrm{Nr}(g)$ are given by

$$(1.2.7)\qquad \langle g\rangle^2 = \mathrm{nr}(g)\det T_4$$

$$(1.2.8)\qquad \mathrm{Nr}(g) = \langle g\rangle e^{\rho/2}$$

with $\rho = (v_1^{\dagger}, \cdots, v_r^{\dagger}, v_1, \cdots, v_r) R \begin{pmatrix} v_1^{\dagger} \\ \vdots \\ v_r^{\dagger} \\ v_1 \\ \vdots \\ v_r \end{pmatrix}$

where the skew-symmetric matrix R is given by $R = \begin{pmatrix} -S_2 & S_1 - 1 \\ 1 - S_4 & S_3 \end{pmatrix}$.

In general the norm of an element g in $G(W)$ is of the form $cw_1 \cdots w_k e^{\rho/2}$ with $c \in \boldsymbol{C}$, $w_1, \cdots, w_k \in W$ and $\rho \in \Lambda^2(W)$ (Theorem 1.2.8), and the relation between $\mathrm{Nr}(g)$ and T_g is given in Proposition 1. 2. 6 and Proposition 1. 2. 7.

As an example $c^{\mathsf{N}} = :e^{(c-1)\mathsf{N}}:$ is discussed where N is the number operator defined by $\mathrm{Nr}(\mathsf{N}) = \sum_{\mu=1}^{r} v_\mu^\dagger v_\mu$. Finally we shall give the relation between the norm of a and the matrix elements $\langle \nu_1, \cdots, \nu_n | a | \mu_1, \cdots, \mu_m \rangle \underset{\mathrm{def}}{=} \langle \mathrm{vac} | v_{\nu_1} \cdots v_{\nu_n} a v_{\mu_1}^\dagger \cdots v_{\mu_m}^\dagger | \mathrm{vac} \rangle$ (Proposition 1. 2. 11).

§ 1. 3. The closure of $G(W)$.

Let W be a vector space of N dimensions. We shall prove that $G = \bigcup_{k=0}^{N} G^k$ where $G^k = \{ c w_1 \cdots w_k e^{\rho/2} | c \in \boldsymbol{C}, w_1, \cdots, w_k \in W \text{ and } \rho \in \Lambda^2(W) \}$ is closed in $\Lambda(W)$ (Theorem 1. 3. 2). This immediately implies that the closure $\overline{G}(W)$ of $G(W)$ coincides with $\{ : c w_1 \cdots w_k e^{\rho/2} : | c \in \boldsymbol{C}, w_1, \cdots, w_k \in W, \rho \in \Lambda^2(W) \text{ and } k = 0, 1, 2, \cdots \}$ (Theorem 1. 3. 1).

Let $(v_1, \cdots, v_N)$ be a basis of W and let $w_1 \cdots w_k e^{\rho/2} = \sum_{m=0}^{N} \frac{1}{m!} \sum_{\mu_1, \cdots \mu_m = 1}^{N} \rho_m(\mu_1, \cdots, \mu_m) v_{\mu_m} \cdots v_{\mu_1}$. An explicit form of $\rho_m(\mu_1, \cdots, \mu_m)$ in terms of the Pfaffian of a skew-symmetric matrix is given in Proposition 1. 3. 8.

§ 1. 4. Product in $\overline{G}(W)$.

Let $W^{(\nu)} (\nu = 1, \cdots, n)$ denote a copy of an orthogonal vector space W, and let Λ denote an $n \times n$ symmetric matrix $(\lambda_{\mu\nu})_{\mu,\nu=1,\cdots,n}$ with $\lambda_{\nu\nu} = 1$ $(\nu = 1, \cdots, n)$. Let $W(\Lambda)$ denote the orthogonal vector space $\bigoplus_{\nu=1}^{n} W^{(\nu)}$ equipped with the inner product $\langle (w^{(1)}, \cdots, w^{(n)}), (w'^{(1)}, \cdots, w'^{(n)}) \rangle_\Lambda = \sum_{\mu,\nu=1}^{n} \lambda_{\mu\nu} \langle w^{(\mu)}, w'^{(\nu)} \rangle$. Let $g^{(\nu)}$ be an element in $\overline{G}(W^{(\nu)}) \subset \overline{G}(W(\Lambda))$. An explicit formula for $\mathrm{Nr}(g^{(1)} \cdots g^{(n)})$ is given in Theorem 1. 4. 3 and Theorem 1. 4. 4.

§ 1. 5. κ-norms and transformation law.

Let W^* denote the dual space of W and let $\widehat{W} = W \oplus W^*$ be an orthogonal vector space equipped with the inner product $\langle (w, \eta), (w', \eta') \rangle = \eta(w') + \eta'(w)$. Let $\kappa: W \to W^*$ be a linear homomorphism such that $\kappa(w)(w') + \kappa(w')(w) = \langle w, w' \rangle$, and set $\tilde{\kappa}: W \to \widehat{W}$, $w \mapsto (w, \kappa(w))$. Then the κ-norm is defined by

$$
(1.5.1) \qquad \mathrm{Nr}_\kappa : A(W) \xrightarrow{\cong} A(\tilde{\kappa}(W)) \xrightarrow{\cong} A(\tilde{\kappa}(W)) | \mathrm{vac} \rangle \\
\xrightarrow{\cong} A(\widehat{W}) | \mathrm{vac} \rangle \xrightarrow{\cong} \Lambda(W) | \mathrm{vac} \rangle \xrightarrow{\cong} \Lambda(W).
$$

This is a generalization of the norm Nr.

The formulas (1. 2. 7) and (1. 2. 8) are restated in terms of κ-norms (Theorem 1. 5. 3). The constant term of $\mathrm{Nr}_\kappa(a)$ is denoted by $\langle a\rangle_\kappa$. There exists a unique element $g_\kappa\in\overline{G}(W)$ such that trace $g_\kappa=1$ and $\langle a\rangle_\kappa=\mathrm{trace}\, g_\kappa a$.

A transformation law of κ-norms are given in Theorem 1. 5. 7, which gives a natural proof of the product formula in § 1. 4.

Chapter 1. Theory of Rotations in an Orthogonal Vector Space

§ 1. 1. Generalities on Clifford Algebra

Let W be an orthogonal vector space over $\boldsymbol{C}$, a vector space over $\boldsymbol{C}$ equipped with a non-degenerate symmetric inner product $\langle w, w'\rangle$. The dimension of W is denoted by N.

Let $T(W)$ be the contravariant tensor algebra over W and let $I(W)$ be the two-sided ideal in $T(W)$ generated by $[w, w']_+-\langle w, w'\rangle$ $(w, w'\in W)$. Here we denote by $[w, w']_+$ the anti-commutator $ww'+w'w$. $A(W)\underset{\mathrm{def}}{=}T(W)/I(W)$ is called the Clifford algebra over W.

There is a unique automorphism $a\mapsto\varepsilon(a)$ of $A(W)$ characterized by $\varepsilon(w)=-w$ for $w\in W$. We set $A^{\pm}(W)=\{a\in A(W)|\varepsilon(a)=\pm a\}$. An element of $A^+(W)$ (resp. $A^-(W)$) is called an even (resp. odd) element.

There is a unique anti-automorphism $g\mapsto g^*$ of $A(W)$ characterized by $w^*=w$ for $w\in W$.

We define trace: $A(W)\to\boldsymbol{C}$ to be the linear map characterized by the following conditions: For any $x, y\in A(W)$ (i) trace $xy=$ trace yx, (ii) trace $\varepsilon(x)=$ trace x and (iii) trace $1=2^{N/2}$.

We denote by $G(W)$ the Clifford group $\{g\in A(W)|\exists g^{-1}\in A(W), gW\varepsilon(g)^{-1}=W\}$. For $g\in G(W)$ we denote by T_g the linear transformation of W induced by g:

$$(1.1.1)^{*)}\qquad \begin{array}{rccl} & W & \longrightarrow & W \\ & \Psi & & \Psi \\ T_g: & w & \longmapsto & T_g w=gw\varepsilon(g)^{-1}\end{array}$$

*) In [2], [13], [14] the representation of $G(W)$ is defined by $w\mapsto gwg^{-1}$. (1. 1. 3) is not valid for odd N under this definition.

$\mathfrak{T}: g \mapsto T_g$ defines a representation of $G(W)$ on W. Since $(T_g(w))^2 = -T_g(w)\varepsilon(T_g(w)) = -gw\varepsilon(g)^{-1}\varepsilon(g)(-w)g^{-1} = gw^2g^{-1} = w^2$, T_g is a rotation of W, i.e. T_g belongs to the orthogonal group $O(W)$ over W.

$w \in W$ belongs to $G(W)$ if and only if $\langle w, w\rangle \neq 0$. In fact $\varepsilon(w)^{-1} = -w^{-1} = 2\frac{-w}{\langle w, w\rangle}$ and

$$T_w w' = w' - 2\frac{\langle w, w'\rangle}{\langle w, w\rangle} w . \tag{1.1.2}$$

T_w fixes the hyperplane $\{w' \in W | \langle w, w'\rangle = 0\}$ and transforms w to $-w$, hence it is a reflection with respect to the above hyperplane.

Proposition 1.1.1. *We have the following exact sequence of group homomorphisms:*

$$1 \to GL(1, \boldsymbol{C}) \overset{id}{\to} G(W) \overset{\mathfrak{T}}{\to} O(W) \to 1 . \tag{1.1.3}$$

If we set $G^{\pm}(W) = G(W) \cap A^{\pm}(W)$, *we have* $G(W) = G^+(W) \cup G^-(W)$, $\mathfrak{T}(G^+(W)) = SO(W)$ *and* $\mathfrak{T}(G^-(W)) = O(W) - SO(W)$.

Lemma. If an odd element a anti-commutes with any $w \in W$, then $a = 0$.

Proof. If N is even, an element of $A(W)$ which anti-commutes with any $w \in W$ is a constant multiple of $v_1 \cdots v_N$, where $(v_1, \cdots, v_N)$ is an orthonormal basis of W. Since a is odd, this implies $a = 0$. If N is odd, we embed W into an $N+1$ dimensional orthogonal space $W_1 = \boldsymbol{C}v_1 + \cdots + \boldsymbol{C}v_{N+1}$. Here $(v_1, \cdots, v_N)$ is a basis of W and $\langle v_{N+1}, v_j\rangle = 0$ $(j = 1, \cdots, N)$. Then $a \in A(W) \subset A(W_1)$ anti-commutes with any $w \in W_1$. Hence $a = cv_1 \cdots v_{N+1}$. This is possible only if $c = 0$. (The assumption that a is odd is unnecessary in the case N is odd. If a anti-commutes with any $w \in W$ the even part of a must vanish.)

Proof of Proposition 1.1.1. Since $O(W)$ is generated by reflections with respect to hyperplanes, $\mathfrak{T}$ is surjective. Now assume that $T_g w = w$ for any $w \in W$. This implies that $g + \varepsilon(g) \in A^+(W)$ belongs to the center of $A(W)$. Hence $g + \varepsilon(g) \in \boldsymbol{C}$, or equivalently, $g = c + a$

where $c\in\boldsymbol{C}$ and $a\in A^-(W)$. Moreover a must anti-commute with any $w\in W$. This implies that $a=0$. Thus we have proved that the kernel of $\mathfrak{T}$ is $GL(1,\boldsymbol{C})$. Now let $g\in G(W)$. Since T_g is a product of reflections, g itself is a product of elements in W. Thus we have proved the proposition.

Let $g=w_1\cdots w_l$ be an element of $G(W)$. Then we have $g\varepsilon(g)^*=\varepsilon(g)^*g=\prod_{j=1}^{l}(-w_j^2)\in GL(1,\boldsymbol{C})$. $\mathrm{nr}(g)\underset{\mathrm{def}}{=}g\varepsilon(g)^*=\varepsilon(g)^*g$ is called the spinorial norm(*) of g, and $g\mapsto\mathrm{nr}(g)$ defines a group homomorphism $G(W)\to GL(1,\boldsymbol{C})$. We note that, for $c\in GL(1,\boldsymbol{C})\subset G(W)$, $\mathrm{nr}(c)=c^2$ and that $\varepsilon(g)^{-1}=\dfrac{1}{\mathrm{nr}(g)}g^*$. The definition of the spinorial norm is extended to the closure $\overline{G}(W)$ of $G(W)$; in fact for $g\in\overline{G}(W)-G(W)$ $\mathrm{nr}(g)=0$. Also we have $\mathrm{nr}(g)=\mathrm{nr}(\varepsilon(g))$ and $T_g=T_{\varepsilon(g)}$.

Let $A^\mu(W)$ denote the subspace of $A(W)$ generated by elements of the form $\sum_{j_1,\cdots,j_\mu=1}^{\mu}c_{j_1\cdots j_\mu}w_{j_1}\cdots w_{j_\mu}$, where $w_1,\cdots,w_\mu\in W$ and, for $\mu\geqq 2$, $c_{j_1\cdots j_\mu}\in\boldsymbol{C}$ is skew-symmetric with respect to $j_1,\cdots,j_\mu$. We have $A(W)=\bigoplus_{\mu=0}^{N}A^\mu(W)$. This is the irreducible decomposition of the representation of $G(W)$ on $A(W)$, which is induced by T; namely $T_g(a)=gag^{-1}$ $(a\in A^+(W))$, $=ga\varepsilon(g)^{-1}(a\in A^-(W))$. We denote by σ^μ the projection $A(W)\to A^\mu(W)$. In particular trace $a=2^{N/2}\sigma^0(a)$ for $a\in A(W)$.

trace $(a\varepsilon(a)^*)$ $(a\in A(W))$ is a quadratic form invariant under the above representation. The Schur's lemma implies that trace $(a\varepsilon(a)^*)=\sum_{\mu=0}^{N}(-)^\mu$ trace $(\sigma^\mu(a)\sigma^\mu(a)^*)$. (Note that $\sigma^\mu(a)^*=(-)^{\frac{\mu(\mu-1)}{2}}\sigma^\mu(a)$.)

Proposition 1.1.2. *Let g_1, g_2 be elements of $G(W)$ such that $T_{g_1}T_{g_2}=T_{g_2}T_{g_1}$. Then g_1 and g_2 either commute or anti-commute.*

Proof. From (1.1.3) it follows that $g_1g_2=cg_2g_1$ for some constant c. Taking the spinorial norm of both sides, we have $\mathrm{nr}(g_1)\mathrm{nr}(g_2)=\mathrm{nr}(g_1g_2)=\mathrm{nr}(cg_2g_1)=c^2\,\mathrm{nr}(g_2)\mathrm{nr}(g_1)$. Hence $c^2=1$ and either $g_1g_2=g_2g_1$ or $g_1g_2=-g_2g_1$.

A subspace V of W is called isotropic (resp. orthogonal), if $\langle w,w\rangle$

(*) Note that our definition here differs from [2], [13], [14] by sign.

$=0$ for $w\in V$ (resp. if the inner product is non-degenerate in V). A maximal isotropic subspace is called holonomic.

Now assume that N is even: $N=2r$.

Let $W=V^{\dagger}\oplus V$ be a decomposition into two holonomic subspaces $V^{\dagger}$ and V. $A(W)$ is a semi-direct product of two exterior algebras $\Lambda(V^{\dagger})$ and $\Lambda(V)$. More precisely, $A(W)$ is generated by 1 as a left $\Lambda(V^{\dagger})$ and right $\Lambda(V)$ bi-module.

Definition 1.1.3. *There is a unique isomorphism*

$$\mathrm{Nr}: A(W)=\Lambda(V^{\dagger})\cdot\Lambda(V)\to\Lambda(W)=\Lambda(V^{\dagger})\wedge\Lambda(V) \tag{1.1.4}$$

of $(\Lambda(V^{\dagger}),\Lambda(V))$ *bi-modules such that* $\mathrm{Nr}(1)=1$. *We call* $\mathrm{Nr}(a)$ *the norm of* $a\in A(W)$.

V (resp. $V^{\dagger}$) generates a maximal left (resp. right)ideal $A(W)V$ (resp. $V^{\dagger}A(W)$) of $A(W)$. The quotient module $A(W)/A(W)V$ (resp. $A(W)/V^{\dagger}A(W)$) is generated by the residue class of 1, which is called the vacuum and is denoted by $|\mathrm{vac}\rangle$ (resp. $\langle\mathrm{vac}|$). We have $\Lambda(V)|\mathrm{vac}\rangle=0$ and $A(W)|\mathrm{vac}\rangle=\Lambda(V^{\dagger})|\mathrm{vac}\rangle\cong\Lambda(V^{\dagger})$, in particular, $\dim A(W)|\mathrm{vac}\rangle=2^{r}$. The representation of $A(W)$ on $A(W)|\mathrm{vac}\rangle$ induces an isomorphism: $A(W)\cong\mathrm{End}(A(W)|\mathrm{vac}\rangle)$. Similarly we have $\langle\mathrm{vac}|\Lambda(V^{\dagger})=0$ and $\langle\mathrm{vac}|A(W)=\langle\mathrm{vac}|\Lambda(V)\cong\Lambda(V)$.

$\Lambda(W)$ is the direct sum of subspaces $\Lambda^{p}(W)$ $(p=0,1,\cdots,2r)$: $\Lambda(W)=\bigoplus_{p=0}^{2r}\Lambda^{p}(W)$. The projection of $\mathrm{Nr}(a)\in\Lambda(W)$ to the summand $\Lambda^{0}(W)=\boldsymbol{C}$ is called the vacuum expectation value of a and is denoted by $\langle\mathrm{vac}|a|\mathrm{vac}\rangle$. The bilinear form

$$\begin{array}{ccc}\langle\mathrm{vac}|A(W)\times A(W)|\mathrm{vac}\rangle & \to & \boldsymbol{C}\\ \Cup & & \Cup\\ (\langle\mathrm{vac}|a_1,\ a_2|\mathrm{vac}\rangle) & \mapsto & \langle\mathrm{vac}|a_1a_2|\mathrm{vac}\rangle\end{array}$$

is well-defined and non-degenerate. Thus $\langle\mathrm{vac}|A(W)$ and $A(W)|\mathrm{vac}\rangle$ are canonically dual to each other. We often abbreviate $\langle\mathrm{vac}|a|\mathrm{vac}\rangle$ by $\langle a\rangle$.

In general, let W be a vector space of N (even or odd) dimensions and let $(v_1,\cdots,v_k,v_{k+1},\cdots,v_N)$ be a basis of W. We set $W_1=\sum_{j=1}^{k}\boldsymbol{C}v_j$

and $W_2=\sum_{j=k+1}^{N} Cv_j$. We denote by $a|_{v_1=\cdots=v_k=0}$ the image of $a\in A(W)$ by the natural projection $A(W)\to A(W)/A(W)W_1\cong A(W_2)$.

Let $(v_1^\dagger, \cdots, v_r^\dagger)$ and $(v_1, \cdots, v_r)$ be mutually dual basis of $V^\dagger$ and V, respectively. This means that $\langle v_\mu^\dagger, v_\nu\rangle=\delta_{\mu\nu}$. Since $V^\dagger$ and V are holonomic, $\langle v_\mu^\dagger, v_\nu^\dagger\rangle=0$ and $\langle v_\mu, v_\nu\rangle=0$. Hence the table J of the inner product of the basis $(v_1^\dagger, \cdots, v_r^\dagger, v_1, \cdots, v_r)$ of W reads as follows:

$$\begin{pmatrix} \langle v_1^\dagger, v_1^\dagger\rangle\cdots\langle v_1^\dagger, v_r^\dagger\rangle & \langle v_1^\dagger, v_1\rangle\cdots\langle v_1^\dagger, v_r\rangle \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \langle v_r^\dagger, v_1^\dagger\rangle\cdots\langle v_r^\dagger, v_r^\dagger\rangle & \langle v_r^\dagger, v_1\rangle\cdots\langle v_r^\dagger, v_r\rangle \\ \langle v_1, v_1^\dagger\rangle\cdots\langle v_1, v_r^\dagger\rangle & \langle v_1, v_1\rangle\cdots\langle v_1, v_r\rangle \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \langle v_r, v_1^\dagger\rangle\cdots\langle v_r, v_r^\dagger\rangle & \langle v_r, v_1\rangle\cdots\langle v_r, v_r\rangle \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$A(W)|\mathrm{vac}\rangle$ (resp. $\langle\mathrm{vac}|A(W)$) is spanned by elements of the form $|\nu_n, \cdots, \nu_1\rangle \underset{\mathrm{def}}{=} v_{\nu_n}^\dagger\cdots v_{\nu_1}^\dagger|\mathrm{vac}\rangle$ (resp. $\langle\nu_1, \cdots, \nu_n| \underset{\mathrm{def}}{=} \langle\mathrm{vac}|v_{\nu_1}\cdots v_{\nu_n}$) $(n=0, 1, 2, \cdots)$ and these elements constitute a mutually dual basis:

$$\langle\mu_1, \cdots, \mu_m|\nu_n, \cdots, \nu_1\rangle=\begin{cases} 0 & \text{if } m\neq n\,, \\ \det(\delta_{\mu_i\nu_j}) & \text{if } m=n\,. \end{cases} \tag{1.1.5}$$

The identity transformation $1\in\mathrm{End}(A(W)|\mathrm{vac}\rangle)$ is decomposable as follows:

$$1=|\mathrm{vac}\rangle\langle\mathrm{vac}|+\sum_{k=1}^{r}\frac{1}{k!}\sum_{\nu_1,\cdots,\nu_k=1}^{r}|\nu_1, \cdots, \nu_k\rangle\langle\nu_k, \cdots, \nu_1| \tag{1.1.6}$$

where $|\nu_1, \cdots, \nu_k\rangle\langle\nu_k, \cdots, \nu_1|$ denote the element $|\nu_1, \cdots, \nu_k\rangle\otimes\langle\nu_k, \cdots, \nu_1|$ in $A(W)|\mathrm{vac}\rangle\otimes\langle\mathrm{vac}|A(W)\cong\mathrm{End}(A(W)|\mathrm{vac}\rangle)\cong\mathrm{End}(\langle\mathrm{vac}|A(W))$.

§ 1.2. Norms and Rotations

The explicit formula for $\mathrm{Nr}(g)$ expressed by the rotation T_g was obtained by the first author [1]. Before explaining it we prepare the notion of conjugate transformation.

Let W be a vector space and let (V_+, V_-) be an ordered pair of its subspaces such that $W=V_+\oplus V_-$. We denote by E_+ (resp. E_-) the projection operator onto V_+ (resp. V_-).

Definition 1.2.1. *Let* $T\in\mathrm{End}(W)$ *and assume* E_++TE_- *is* *invertible. We define the conjugate transformation S to T with respect to (V_+, V_-) by*

$$S=(E_+ + TE_-)^{-1}(E_- + TE_+). \tag{1.2.1}$$

Remark. We note that if we rewrite (1.2.1) as

$$(E_+ + TE_-)S = E_- + TE_+ \tag{1.2.2}$$

the invertibility of $E_+ + TE_-$ follows. In fact, we have $(E_+ + TE_-) \times (E_+ + SE_-) = E_+ + (E_- + TE_+)E_- = 1$. Moreover this means that S is conjugate to T if and only if T is conjugate to S. It is also easy to see that if T and T^{-1} have the conjugates S and S' respectively then $S' = S^{-1}$.

Set $T=1-2P$ and $S=1-2Q$. The following proposition gives an alternative characterization of conjugate transformations in the special case $T^2=1$.

Proposition 1.2.2. *Set $E=E_+ - E_-$. The following are equivalent.*

(i) *T and S are mutually conjugate and $T^2=1$.*
(ii) *T and S are mutually conjugate and $S^2=1$.*
(iii) *$PQ=Q$ and $PEQ=PE$.*
(iv) *$QP=P$ and $QEP=QE$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from the above remark. (iii) in terms of T and S reads

$$(1+T)(E_+ + E_-)(1-S)=0, \quad (1-T)(E_+ - E_-)(1+S)=0,$$

or equivalently

$$(E_+ + TE_-)S = E_- + TE_+, \quad (E_- + TE_+)S = E_+ + TE_-.$$

Then it is easy to see from the above remark that this is equivalent to (ii). Hence we have also (ii) $\Leftrightarrow$ (iii). Similarly we have (i) $\Leftrightarrow$ (iv), hence we have proved the proposition.

Proposition 1.2.3. *Take a basis* $(v_1^+, \cdots, v_{k_1}^+)$ (resp. $(v_1^-, \cdots, v_{k_2}^-)$) *of* V_+ (resp. V_-) *and represent* T *and* S *as block matrices;*

$$(Tv_1^+, \cdots, Tv_{k_1}^+, Tv_1^-, \cdots, Tv_{k_2}^-) = (v_1^+, \cdots, v_{k_1}^+, v_1^-, \cdots, v_{k_2}^-) \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$

$$(Sv_1^+, \cdots, Sv_{k_1}^+, Sv_1^-, \cdots, Sv_{k_2}^-) = (v_1^+, \cdots, v_{k_1}^+, v_1^-, \cdots, v_{k_2}^-) \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}.$$

Then T *and* S *are conjugate to each other if and only if*

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} 1 & -S_2 \\ & 1 \end{pmatrix} \begin{pmatrix} S_1 & \\ & S_4^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ S_3 & 1 \end{pmatrix} = \begin{pmatrix} S_1 - S_2 S_4^{-1} S_3 & -S_2 S_4^{-1} \\ S_4^{-1} S_3 & S_4^{-1} \end{pmatrix}, \tag{1.2.3}$$

or equivalently

$$\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} 1 & -T_2 \\ & 1 \end{pmatrix} \begin{pmatrix} T_1 & \\ & T_4^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ T_3 & 1 \end{pmatrix} \tag{1.2.4}$$

$$= \begin{pmatrix} T_1 - T_2 T_4^{-1} T_3 & -T_2 T_4^{-1} \\ T_4^{-1} T_3 & T_4^{-1} \end{pmatrix}.$$

Proof. (1.2.4) is rewritten as

$$\begin{pmatrix} 1 & T_2 \\ & T_4 \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \begin{pmatrix} T_1 & \\ T_3 & 1 \end{pmatrix}. \tag{1.2.5}$$

This is nothing but the matrix form of (1.2.2).

Remark 1. The assumption that $E_+ + TE_-$ is invertible is equivalent to that T_4 is invertible.

Remark 2. The decomposition (1.2.3) is unique and independent of the choice of a basis, hence it defines a canonical decomposition of T: $T = T'T''T'''$ where T', T'', T''' correspond to $\begin{pmatrix} 1 & -S_2 \\ & 1 \end{pmatrix}$, $\begin{pmatrix} S_1 & \\ & S_4^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & \\ S_3 & 1 \end{pmatrix}$, respectively. In fact, $T' = 1 - E_+ SE_-$, $T'' = (E_+ + E_- SE_-)^{-1} \times (E_- + E_+ SE_+)$ and $T''' = 1 + E_- SE_+$.

Remark 3. From the decomposition (1.2.3) it follows that $\det \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \det T_4 \det (T_1 - T_2 T_4^{-1} T_3)$
and

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \\ -S_3 & 1 \end{pmatrix} \begin{pmatrix} S_1^{-1} & \\ & S_4 \end{pmatrix} \begin{pmatrix} 1 & S_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} S_1^{-1} & S_1^{-1}S_2 \\ -S_3S_1^{-1} & S_4 - S_3S_1^{-1}S_2 \end{pmatrix}.$$

Similar formulas are available when T_1 or T_2 or T_3 is a non-singular matrix.

Now we go back to the orthogonal vector space W and its holonomic decomposition $W = V^\dagger \oplus V$. The conjugate transformation will be used with respect to $(V^\dagger, V)$. We take a mutually dual basis $(v_1^\dagger, \cdots, v_r^\dagger, v_1, \cdots, v_r)$ and represent a linear transformation as in Proposition 1. 2. 3. We denote by E_+ (resp. E_-) the projection onto $V^\dagger$ (resp. V) and by $w^{(+)}$ (resp. $w^{(-)}$) the image E_+w (resp. E_-w) for $w \in W$.

Proposition 1. 2. 4. *Let T and S be mutually conjugate linear transformations of W. Then T is orthogonal if and only if*

(1. 2. 6) $$S_1 = {}^tS_4,\ {}^tS_2 = -S_2,\ {}^tS_3 = -S_3\,.$$

Proof. Denote by T^* the adjoint transformation of T with respect to the inner product in W. T is orthogonal if and only if $(T^*)^{-1} = T$. The uniqueness of the decomposition (1. 2. 3) yields that $(T^*)^{-1} = T$ if and only if $(T'^*)^{-1} = T'$, $(T''^*)^{-1} = T''$ and $(T'''^*)^{-1} = T'''$. Since in matrix representations $\begin{pmatrix} (T^*)_1 & (T^*)_2 \\ (T^*)_3 & (T^*)_4 \end{pmatrix} = J^{-1} \begin{pmatrix} {}^tT_1 & {}^tT_3 \\ {}^tT_2 & {}^tT_4 \end{pmatrix} J$, we have the proposition.

Theorem 1. 2. 5. *Let g be an element in $G(W)$ and let T_g and $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be the induced orthogonal transformation* (1. 1. 1) *and its matrix representation, respectively. We assume that T_4 is invertible. Denote by S_g and $\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$ the conjugate to T_g and its matrix representation, respectively. Then we have*

(1. 2. 7) $$\langle g\rangle^2 = \mathrm{nr}(g) \det T_4$$

and

(1. 2. 8) $$\mathrm{Nr}(g) = \langle g\rangle e^{\rho/2}$$

with $\rho=(v_1^\dagger, \cdots, v_r^\dagger, v_1, \cdots, v_r)\begin{pmatrix}-S_2 & S_1-1\\ 1-S_4 & S_3\end{pmatrix}\begin{pmatrix}v_1^\dagger\\ \vdots\\ v_r^\dagger\\ v_1\\ \vdots\\ v_r\end{pmatrix}$.

Conversely, if $g\in A(W)$ *is given by* (1.2.8) *with* S_1, S_2, S_3 *and* S_4 *satisfying* (1.2.6) *and if* S_4 *is invertible, then* g *belongs to* $G(W)$ *and its induced orthogonal transformation* T_g *is given by the conjugate to* S_g.

Lemma 1. For a, $b\in A(W)$, we have

$$e^a b=\Big(b+\frac{1}{1!}[a, b]+\frac{1}{2!}[a, [a, b]]+\cdots\Big)e^a . \tag{1.2.9}$$

The proof is straightforward.

Lemma 2. If $\mathrm{Nr}(g_1)=\exp\Big(\frac{1}{2}\sum_{\mu,\nu=1}^{r}(-S_2)_{\mu\nu}v_\mu^\dagger v_\nu^\dagger\Big)$ with ${}^tS_2=-S_2$, $g_1\in G(W)$ and the matrix representation of T_{g_1} is $\begin{pmatrix}1 & -S_2\\ & 1\end{pmatrix}$.

Proof. Since $\Big[\frac{1}{2}\sum_{\mu,\nu=1}^{r}(-S_2)_{\mu\nu}v_\mu^\dagger v_\nu^\dagger, v_\lambda^\dagger\Big]=0$ and $\Big[\frac{1}{2}\sum_{\mu,\nu=1}^{r}(-S_2)_{\mu\nu}v_\mu^\dagger v_\nu^\dagger, v_\lambda\Big]=\sum_\mu(-S_2)_{\mu\lambda}v_\mu^\dagger$, the lemma follows from Lemma 1.

Similarly we have the following lemma.

Lemma 3. If $\mathrm{Nr}(g_3)=\exp\Big(\frac{1}{2}\sum_{\mu,\nu=1}^{r}(S_3)_{\mu\nu}v_\mu v_\nu\Big)$ with ${}^tS_3=-S_3$, $g_3\in G(W)$ and the matrix representation of T_{g_3} is $\begin{pmatrix}1 & \\ S_3 & 1\end{pmatrix}$.

Lemma 4. If $\mathrm{Nr}(g_2)=\exp\Big(\frac{1}{2}\sum_{\mu,\nu=1}^{r}(S_1-1)_{\mu\nu}v_\mu^\dagger v_\nu+\frac{1}{2}\sum_{\mu,\nu=1}^{r}(1-S_4)_{\mu\nu}\times v_\mu v_\nu^\dagger\Big)$ where ${}^tS_1=S_4$ is invertible, then $g_2\in G(W)$ and the matrix representation of T_{g_2} is $\begin{pmatrix}S_1 & \\ & S_4^{-1}\end{pmatrix}$. Moreover $\mathrm{nr}(g_2)=\det S_4=\det T_4^{-1}$.

Proof. If we prove the lemma when S_1 is diagonalizable, the non-diagonalizable case follows from the continuity of Nr. Assume that PS_1P^{-1} is diagonal. If we define $(\bar{v}_1^\dagger, \cdots, \bar{v}_r^\dagger, \bar{v}_1, \cdots, \bar{v}_r)$ by $v_\nu^\dagger = \sum_{\mu=1}^{r} \bar{v}_\mu^\dagger P_{\mu\nu}$ and $v_\nu = \sum_{\mu=1}^{r} \bar{v}_\mu ({}^tP^{-1})_{\mu\nu}$, it is also a mutually dual basis and the lemma reduces to the case $r=1$. Now a direct computation shows the lemma. (See Proposition 1. 2. 9.)

Proof of Theorem 1. 2. 5. If we set $\bar{g} = g_1g_2g_3$, we have $\bar{g} \in G(W)$, $\mathrm{Nr}(\bar{g}) = e^{\rho/2}$ and the matrix representation of $T_{\bar{g}} = T_{g_1}T_{g_2}T_{g_3}$ is $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. This proves the second half of the theorem. Since $T_{\bar{g}} = T_g$, $g = c\bar{g}$ with some constant c. Taking the vacuum expectation value, we have $\langle g \rangle = c\langle \bar{g} \rangle = c$. Taking the spinorial norm, we have $\mathrm{nr}(g) = c^2 \,\mathrm{nr}(\bar{g}) = \langle g \rangle^2 \det T_4^{-1}$. This proves the first half of the theorem.

Remark. We often use the skew-symmetric matrix $R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} = \begin{pmatrix} -S_2 & S_1 - 1 \\ 1 - S_4 & S_3 \end{pmatrix}$ instead of S.

Proposition 1. 2. 6. *Let g be an element of $G(W)$ and let $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be the matrix representation of T_g. We assume that* $\mathrm{Ker}\, T_4 \neq \{0\}$. *Take $w = \sum_{\mu=1}^{r} v_\mu^\dagger c_\mu^\dagger + \sum_{\mu=1}^{r} v_\mu c_\mu \in W$ such that $w^2 = \sum_{\mu=1}^{r} c_\mu^\dagger c_\mu \neq 0$ and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} \notin \mathrm{Im}\, T_4$.**) *Set $g' = wg$ and denote by $\begin{pmatrix} T_1' & T_2' \\ T_3' & T_4' \end{pmatrix}$ the matrix representation of $T_{g'}$. Then we have $\mathrm{Ker}\, T_4 \supset \mathrm{Ker}\, T_4'$ and $\dim \mathrm{Ker}\, T_4/\mathrm{Ker}\, T_4' = 1$. There exist $v \in V$ and $v^\dagger \in V^\dagger$ such that $T_g v = -v^\dagger$ and $\langle w, v^\dagger \rangle = 1$. Then*

$$g = v^\dagger g' + \varepsilon(g')\, v\,. \tag{1.2.10}$$

Lemma. If $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ is orthogonal in the sense

$${}^t\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.2.11}$$

*) $c_\mu^\dagger$ does not mean the complex conjugate of c_μ.

we have

$$\operatorname{Im} T_4 = (T_2(\operatorname{Ker} T_4))^{\perp}. \tag{1.2.12}$$

Proof. Since $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ is non-singular, $\operatorname{Ker} T_2 \cap \operatorname{Ker} T_4 = \{0\}$. From (1.2.11) it follows that ${}^tT_2T_4 + {}^tT_4T_2 = 0$, hence, $\operatorname{Ker} {}^tT_4 = T_2(\operatorname{Ker} T_4)$. This yields that $\operatorname{Im} T_4 = (\operatorname{Ker} {}^tT_4)^{\perp} = (T_2(\operatorname{Ker} T_4))^{\perp}$.

Proof of Proposition 1.2.6. Take a vector $\boldsymbol{c}' = \begin{pmatrix} c'_1 \\ \vdots \\ c'_r \end{pmatrix} \in \boldsymbol{C}^r$ and set $v' = \sum_{\mu=1}^{r} v_\mu c'_\mu \in V$. Then we have $(T_{g'}v')^{(-)} = (T_w T_g v')^{(-)} = \frac{1}{-w^2} \times \langle w, T_g v' \rangle w^{(-)} + \sum_{\mu=1}^{r} v_\mu (T_4 \boldsymbol{c}')_\mu$. Hence $T'_4 \boldsymbol{c}' = \frac{1}{-w^2} \langle w, T_g v' \rangle \boldsymbol{c} + T_4 \boldsymbol{c}'$. Since $\boldsymbol{c} \notin \operatorname{Im} T_4$, $T'_4 \boldsymbol{c}' = 0$ if and only if $T_4 \boldsymbol{c}' = 0$ and ${}^t\boldsymbol{c} T_2 \boldsymbol{c}' = 0$. From the lemma it follows that ${}^t\boldsymbol{c} T_2(\operatorname{Ker} T_4) \neq 0$. Thus we have proved that $\operatorname{Ker} T_4 \supset \operatorname{Ker} T'_4$ and $\dim \operatorname{Ker} T_4 / \operatorname{Ker} T'_4 = 1$. $T_g v = -v^\dagger$ means $gv = -v^\dagger \varepsilon(g)$. Hence if $\langle w, v^\dagger \rangle = 1$, $\varepsilon(g')v = \varepsilon(w)\varepsilon(g)v = wv^\dagger g = \langle w, v^\dagger \rangle g - v^\dagger w g = g - v^\dagger g'$.

Proposition 1.2.7. *Let* $\operatorname{Nr}(g') = w_1 \cdots w_k \operatorname{Nr}(g)$ *where* $w_j = \sum_{\mu=1}^{r} v^\dagger_\mu c^\dagger_{j\mu} + \sum_{\mu=1}^{r} v_\mu c_{j\mu} \in W$ $(j=1, \cdots, k)$ *and* $g \in G(W)$. *Then* g' *belongs to* $\overline{G}(W)$. *Let* $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ *denote the matrix representation of* T_g, *and set* $\boldsymbol{c}_j = \begin{pmatrix} c_{j1} \\ \vdots \\ c_{jr} \end{pmatrix}$ *and* $\boldsymbol{c}^\dagger_j = \begin{pmatrix} c^\dagger_{j1} \\ \vdots \\ c^\dagger_{jr} \end{pmatrix}$. *Then we have*

$$\operatorname{nr}(g') = \det \begin{pmatrix} -{}^t\boldsymbol{c}^\dagger_1 T_4 \boldsymbol{c}_1 & \cdots & -{}^t\boldsymbol{c}^\dagger_1 T_4 \boldsymbol{c}_k \\ \vdots & & \vdots \\ -{}^t\boldsymbol{c}^\dagger_k T_4 \boldsymbol{c}_1 & \cdots & -{}^t\boldsymbol{c}^\dagger_k T_4 \boldsymbol{c}_k \end{pmatrix} \operatorname{nr}(g). \tag{1.2.13}$$

Now assume that $\operatorname{nr}(g') \neq 0$. *Then* $g' \in G(W)$, *and if we denote by* $\begin{pmatrix} T'_1 & T'_2 \\ T'_3 & T'_4 \end{pmatrix}$ *the matrix representation of* $T_{g'}$, *we have*

$$\begin{pmatrix} T'_1 & T'_2 \\ T'_3 & T'_4 \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} - \begin{pmatrix} -\boldsymbol{c}^\dagger_1 - T_2\boldsymbol{c}_1, & \cdots, & -\boldsymbol{c}^\dagger_k - T_2 \boldsymbol{c}_k \\ -T_4 \boldsymbol{c}_1, & \cdots, & -T_4 \boldsymbol{c}_k \end{pmatrix} \tag{1.2.14}$$

$$\times \begin{pmatrix} -{}^t\boldsymbol{c}^\dagger_1 T_4 \boldsymbol{c}_1 \cdots -{}^t\boldsymbol{c}^\dagger_1 T_4 \boldsymbol{c}_k \\ \vdots \qquad \vdots \\ -{}^t\boldsymbol{c}^\dagger_k T_4 \boldsymbol{c}_1 \cdots -{}^t\boldsymbol{c}^\dagger_k T_4 \boldsymbol{c}_k \end{pmatrix}^{-1} \begin{pmatrix} {}^t\boldsymbol{c}_1 + {}^t\boldsymbol{c}^\dagger_1 T_3, & {}^t\boldsymbol{c}^\dagger_1 T_4 \\ \vdots & \vdots \\ {}^t\boldsymbol{c}_k + {}^t\boldsymbol{c}^\dagger_k T_3, & {}^t\boldsymbol{c}^\dagger_k T_4 \end{pmatrix}.$$

In particular

(1.2.15) $$T_{g'}w_j^{(-)} = -w_j^{(+)}$$

and

(1.2.16) $$\mathrm{Ker}\ T_4' = \sum_{j=1}^{k} \boldsymbol{C}\boldsymbol{c}_j \oplus \mathrm{Ker}\ T_4 .$$

Lemma 1. Assume that $\mathrm{Nr}(g') = w\,\mathrm{Nr}(g)$ where $g, g' \in G(W)$ and $w \in W$. Then $T_{g'}(w^{(-)}) = -w^{(+)}$.

Proof. $g' = w^{(+)}g + \varepsilon(g)w^{(-)} = (w^{(+)} + T_g(w^{(-)}))g$. Hence we have $T_{g'}(w^{(-)}) = T_{w^{(+)}+T_g(w^{(-)})}T_g(w^{(-)}) = \frac{-1}{\langle w^{(+)}, T_g(w^{(-)})\rangle}(w^{(+)} + T_g(w^{(-)})) \times T_g(w^{(-)})(w^{(+)} + T_g(w^{(-)})) = -w^{(+)}$.

Lemma 2. Let g, g' be as in the proposition, then $h = g'g^{-1}$ is a polynomial of $w_j{}^{(+)} + T_g(w_j{}^{(-)})$ $(j = 1, \cdots, k)$.

Proof. We prove the lemma by induction on k. $k = 0$ is trivial. We assume that the lemma is proved for $k-1$. Then we have

$$\begin{aligned} g' &= w_1^{(+)} : w_2 \cdots w_k \,\mathrm{Nr}(g) : + \varepsilon(: w_2 \cdots w_k\,\mathrm{Nr}(g) :)\, w_1^{(-)} \\ &= w_1^{(+)} h' g + \varepsilon(h'g)\, w_1^{(-)}, \end{aligned}$$

where h' is a polynomial of $w_j^{(+)} + T_g(w_j^{(-)})$ $(j = 2, \cdots, k)$. Hence $g'g^{-1} = w_1^{(+)}h' + \varepsilon(h')T_g(w_j^{(-)})$ is a polynomial of $w_j^{(+)} + T_g(w_j^{(-)})$ $(j = 1, \cdots, k)$.

Proof of Proposition 1.2.7. By embedding W into a higher dimensional orthogonal space, we may assume that $r \geqq k$. We prove the proposition by induction on k. $k = 0$ is trivial. Suppose that the proposition is proved for $k-1$. Without loss of generality we may assume that $\det\begin{pmatrix} -{}^t\boldsymbol{c}_2^\dagger T_4\boldsymbol{c}_2 \cdots -{}^t\boldsymbol{c}_2^\dagger T_4\boldsymbol{c}_k \\ \vdots \quad\quad \vdots \\ -{}^t\boldsymbol{c}_k^\dagger T_4\boldsymbol{c}_2 \cdots -{}^t\boldsymbol{c}_k^\dagger T_4\boldsymbol{c}_k \end{pmatrix} \neq 0$ and $\det\begin{pmatrix} -{}^t\boldsymbol{c}_1^\dagger T_4\boldsymbol{c}_1 \cdots -{}^t\boldsymbol{c}_1^\dagger T_4\boldsymbol{c}_k \\ \vdots \quad\quad \vdots \\ -{}^t\boldsymbol{c}_k^\dagger T_4\boldsymbol{c}_1 \cdots -{}^t\boldsymbol{c}_k^\dagger T_4\boldsymbol{c}_k \end{pmatrix} \neq 0$. Then we have

$$\begin{aligned} \mathrm{nr}(g') = &-(w_1^{(+)} : w_2 \cdots w_k\,\mathrm{Nr}(g) : + \varepsilon(: w_2 \cdots w_k\,\mathrm{Nr}(g) :)\, w_1^{(-)}) \\ &\times (\varepsilon(: w_2 \cdots w_k\,\mathrm{Nr}(g) :)^* w_1^{(+)} + w_1^{(-)} : w_2 \cdots w_k\,\mathrm{Nr}(g) :^*) \end{aligned}$$

$$= -\mathrm{nr}(:w_2\cdots w_k\,\mathrm{Nr}(g):)\langle w_1^{(+)},\ T_{:w_2\cdots w_k \mathrm{Nr}(g):}(w_1^{(-)})\rangle$$

$$= -\mathrm{nr}(:w_2\cdots w_k\,\mathrm{Nr}(g):)$$

$$\times (0, {}^t c_1^\dagger)\left\{-\begin{pmatrix} -c_2^\dagger - T_2 c_2, \cdots, & -c_k^\dagger - T_2 c_k \\ -T_4 c_2, \cdots, & -T_4 c_k \end{pmatrix}\right.$$

$$\times \begin{pmatrix} -{}^t c_2^\dagger T_4 c_2 \cdots -{}^t c_2^\dagger T_4 c_k \\ \vdots \qquad\qquad \vdots \\ -{}^t c_k^\dagger T_4 c_2 \cdots -{}^t c_k^\dagger T_4 c_k \end{pmatrix}^{-1} \begin{pmatrix} {}^t c_2 + {}^t c_2^\dagger T_3, & {}^t c_2^\dagger T_4 \\ \vdots & \vdots \\ {}^t c_k + {}^t c_k^\dagger T_3, & {}^t c_k^\dagger T_4 \end{pmatrix}$$

$$\left. + \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}\right\} \begin{pmatrix} 0 \\ c_1 \end{pmatrix}$$

$$= \det \begin{pmatrix} -{}^t c_2^\dagger T_4 c_2 \cdots -{}^t c_2^\dagger T_4 c_k \\ \vdots \qquad\qquad \vdots \\ -{}^t c_k^\dagger T_4 c_2 \cdots -{}^t c_k^\dagger T_4 c_k \end{pmatrix} \cdot \mathrm{nr}(g)$$

$$\times \left\{ -{}^t c_1^\dagger T_4 c_1 - (-{}^t c_1^\dagger T_4 c_2, \cdots, -{}^t c_1^\dagger T_4 c_k)\right.$$

$$\left. \times \begin{pmatrix} -{}^t c_2^\dagger T_4 c_2 \cdots -{}^t c_2^\dagger T_4 c_k \\ \vdots \qquad\qquad \vdots \\ -{}^t c_k^\dagger T_4 c_2 \cdots -{}^t c_k^\dagger T_4 c_k \end{pmatrix}^{-1} \begin{pmatrix} -{}^t c_2^\dagger T_4 c_1 \\ \vdots \\ -{}^t c_k^\dagger T_4 c_1 \end{pmatrix} \right\}$$

$$= \det \begin{pmatrix} -{}^t c_1^\dagger T_4 c_1 & -{}^t c_1^\dagger T_4 c_2 \cdots -{}^t c_1^\dagger T_4 c_k \\ -{}^t c_2^\dagger T_4 c_1 & -{}^t c_2^\dagger T_4 c_2 \cdots -{}^t c_2^\dagger T_4 c_k \\ \vdots & \vdots \qquad\qquad \vdots \\ -{}^t c_k^\dagger T_4 c_1 & -{}^t c_k^\dagger T_4 c_2 \cdots -{}^t c_k^\dagger T_4 c_k \end{pmatrix} \cdot \mathrm{nr}(g).$$

Now we shall prove (1. 2. 14). Lemma 1 implies that when restricted to $\boldsymbol{C}\begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \cdots + \boldsymbol{C}\begin{pmatrix} 0 \\ c_k \end{pmatrix}$, (1. 2. 14) is valid. We set $W_1 = \{w \in W \mid \langle w_j^{(+)} + T_g(w_j^{(-)}), T_g w\rangle = 0$ for $j = 1, \cdots, k\}$. Then $w_1^{(-)}, \cdots, w_k^{(-)}$ and W_1 span W. From Lemma 2 it follows that when restricted to W_1 (1. 2. 14) is valid. Thus we have proved the proposition.

Summing up, we have the following theorem.

Theorem 1. 2. 8. *Let* $g \in G(W)$ *and set* $k = \dim \mathrm{Ker}\, T_4$. k *is even (resp. odd) if* g *is even (resp. odd).* $\mathrm{Nr}(g)$ *is of the form* $c w_1 \cdots w_k e^{\rho/2}$ *with* $c \in \boldsymbol{C}$, $w_1, \cdots, w_k \in W$ *and* $\rho \in \Lambda^2(W)$. *Moreover we have*

$$\sum_{j=1}^{k} \boldsymbol{C} w_j^{(-)} = \{v \in V \mid T_g v \in V^\dagger\} \tag{1. 2. 17}$$

and $T_g w_j^{(-)} = -w_j^{(+)}$. $cw_1 \cdots w_k$ *is determined uniquely by* g. L *is unique up to modulo* $\sum_{j=1}^{k} w_j W$. *Conversely if* $\mathrm{Nr}(g)$ *is of the form as above and if* $\mathrm{nr}(g) \neq 0$, *then* g *belongs to* $G(W)$, $\dim \mathrm{Ker}\, T_g = k$ *and* (1.2.17) *is valid.*

We shall give several examples.

There exists a unique operator $\mathsf{N} \in A(W)$ satisfying

$$[\mathsf{N}, v^\dagger] = v^\dagger \quad \text{for} \quad v^\dagger \in V^\dagger \tag{1.2.18}$$

$$[\mathsf{N}, v] = -v \quad \text{for} \quad v \in V$$

and

$$\langle \mathsf{N} \rangle = 0 .$$

If we take a dual basis $(v_1^\dagger, \cdots, v_r^\dagger, v_1, \cdots, v_r)$ of $W = V^\dagger \oplus V$, we have

$$\mathrm{Nr}(\mathsf{N}) = \sum_{\mu=1}^{r} v_\mu^\dagger v_\mu . \tag{1.2.19}$$

N is diagonalizable and its eigenvalues are $0, 1, \cdots, r$. The eigenspace of N with eigenvalue k is $\sum_{1 \le \mu_1 < \cdots < \mu_k \le r} \boldsymbol{C} |\mu_1, \cdots, \mu_k\rangle$. This means that

$$\mathsf{N} |\mu_1, \cdots, \mu_k\rangle = k |\mu_1, \cdots, \mu_k\rangle . \tag{1.2.20}$$

In this sense we call N the number operator.

The number operator depends on the choice of the holonomic decomposition $W = V^\dagger \oplus V$, but $T_{(-)^{\mathsf{N}}}$ does not. In general, we have the following proposition.

Proposition 1.2.9. $\mathrm{Nr}(c^{\mathsf{N}}) = e^{(c-1)\mathsf{N}}$, *and if* $c \neq 0$ *the matrix representation of* $T_{c^{\mathsf{N}}}$ *is* $\begin{pmatrix} c & \\ & c^{-1} \end{pmatrix}$.

Lemma 1. Let $\binom{\mathsf{N}}{m}$ denote the operator $\frac{1}{m!}\mathsf{N}(\mathsf{N}-1)\cdots(\mathsf{N}-m+1)$. Then we have $\mathrm{Nr}\left(\binom{\mathsf{N}}{m}\right) = \frac{1}{m!}\mathsf{N}^m$.

Proof. We prove the lemma by induction on m. $m=0$ is trivial. Assume that the lemma is proved for m. Then (1.2.18) yields

$$\binom{\mathsf{N}}{m+1}=\binom{\mathsf{N}}{m}\frac{\mathsf{N}-m}{m+1}=\frac{1}{(m+1)!}:\mathsf{N}^m:(\mathsf{N}-m)$$

$$=\frac{1}{(m+1)!}\sum_{\mu_1,\cdots,\mu_m=1}^{r}\psi^\dagger_{\mu_1}\cdots\psi^\dagger_{\mu_m}\psi_{\mu_m}\cdots\psi_{\mu_1}(\mathsf{N}-m)$$

$$=\frac{1}{(m+1)!}\sum_{\mu_1,\cdots,\mu_m=1}^{r}\psi^\dagger_{\mu_1}\cdots\psi^\dagger_{\mu_m}\mathsf{N}\psi_{\mu_m}\cdots\psi_{\mu_1}$$

$$=\frac{1}{(m+1)!}:\mathsf{N}^{m+1}:.$$

Lemma 2. Let $a, b\in A(W)$ satisfy $[a, b]=cb$ with $c\in \boldsymbol{C}$, and let $P(a)$ be a polynomial in a. Then $P(a)b=b\ P(a+c)$.

The proof is straightforward.

Proof of Proposition 1.2.9. By Lemma 1 we have $c^{\mathsf{N}}=(1+(c-1))^{\mathsf{N}}$ $=\sum_{m=0}^{r}\binom{\mathsf{N}}{m}(c-1)^m=\sum_{m=0}^{r}\frac{(c-1)^m}{m!}:\mathsf{N}^m:=:e^{(c-1)\mathsf{N}}:$ Applying Lemma 2 for $a=\mathsf{N}$, $b=w\in W$ and $P(\mathsf{N})=c^{\mathsf{N}}=\sum_{m=0}^{r}\binom{\mathsf{N}}{m}(c-1)^m$, we have $c^{\mathsf{N}}w$ $=\begin{cases}wc^{\mathsf{N}+1} & \text{if } w\in V^\dagger,\\ wc^{\mathsf{N}-1} & \text{if } w\in V.\end{cases}$

$(-)^{\mathsf{N}}$ is characterized up to the signature by the following conditions: $T_{\varepsilon_W}=-1$ and $\varepsilon_W^2=1$. We call such ε_W an orientation of W. (dim W is even or odd.) The pair (W, ε_W) of an orthogonal space W and its orientation ε_W is called an oriented orthogonal space. A holonomic decomposition $W=V^\dagger\oplus V$ is called positive (resp. negative) if $(-)^{\mathsf{N}}=\varepsilon_W$ (resp. $(-)^{\mathsf{N}}=-\varepsilon_W$). We have $\mathrm{nr}(\varepsilon_W)=(-)^r$ and trace $\varepsilon_W=0$. (See (1.5.19) below.)

Taking an orthonormal basis $v_1, \cdots, v_N$ we have $\varepsilon_W=\pm v_1\cdots v_N$.

Proposition 1.2.10. $|\mathrm{vac}\rangle\langle\mathrm{vac}|=\prod_{\mu=1}^{r}v_\mu v_\mu^\dagger=0^{\mathsf{N}}=:e^{-\mathsf{N}}:.$

Proof. Comparing the action on $A(V^\dagger)|\mathrm{vac}\rangle$ of both sides, we have the first equality. The second one follows from (1.2.20) and the last one follows from Proposition 1. 2. 9.

We shall give here the relation between the norm and the matrix elements of an element a of $A(W)$. We mean the matrix elements those in matrix representation in $A(V^{\dagger})|\mathrm{vac}\rangle$. Let $\varphi_{m,n}(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n)$ denote the matrix element $\langle \nu_1, \cdots, \nu_n|a|\mu_1, \cdots, \mu_m\rangle$. Then a is written as follows:

$$a=\sum_{m,n=0}^{r}\frac{1}{m!}\frac{1}{n!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n=1}}^{r}\varphi_{m,n}(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n)$$

$$\times v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}|\mathrm{vac}\rangle\langle\mathrm{vac}|v_{\mu_m}\cdots v_{\mu_1}.$$

Proposition 1. 2. 11. *Let $a\in A(W)$ be written in two different forms*;

$$a=\sum_{m,n=0}^{r}\frac{1}{m!}\frac{1}{n!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n=1}}^{r}\rho_{m,n}(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n) \tag{1.2.21}$$

$$\times v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}\, v_{\mu_m}\cdots v_{\mu_1},$$

$$a=\sum_{m,n=0}^{r}\frac{1}{m!}\frac{1}{n!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n=1}}^{r}\varphi_{m,n}(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n) \tag{1.2.22}$$

$$\times v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}|\mathrm{vac}\rangle\langle\mathrm{vac}|v_{\mu_m}\cdots v_{\mu_1}.$$

Then the relation between $\rho_{m,n}$'s and $\varphi_{m,n}$'s is as follows:

$$\varphi_{m,n}(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n) \tag{1.2.23}$$

$$=\sum_{l=0}^{\min(m,n)}\sum_{\sigma,\tau}{}' \operatorname{sgn}\sigma \operatorname{sgn}\tau\, \rho_{m-l,n-l}(\mu_{\sigma(1)}, \cdots, \mu_{\sigma(m-l)}; \nu_{\tau(l+1)}, \cdots, \nu_{\tau(n)})$$

$$\times\langle\mu_{\sigma(m)}, \cdots, \mu_{\sigma(m-l+1)}|\nu_{\tau(l)}, \cdots, \nu_{\tau(1)}\rangle,$$

where the summation $\sum'_{\sigma,\tau}$ is over $\sigma\in\mathfrak{S}_m$ and $\tau\in\mathfrak{S}_n$ with the restriction $\sigma(1)<\cdots<\sigma(m-l)$, $\sigma(m)>\cdots>\sigma(m-l+1)$, $\tau(l+1)<\cdots<\tau(n)$ and $\tau(l)>\cdots>\tau(1)$.

$$\rho_{m,n}(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n) \tag{1.2.24}$$

$$=\sum_{l=0}^{\min(m,n)}(-)^{l}\sum_{\sigma,\tau}{}' \operatorname{sgn}\sigma \operatorname{sgn}\tau\, \varphi_{m-l\ n-l}(\mu_{\sigma(1)}, \cdots, \mu_{\sigma(m-l)};$$

$$\nu_{\tau(l+1)}, \cdots, \nu_{\tau(n)})\langle\mu_{\sigma(m)}, \cdots, \mu_{\sigma(m-l+1)}|\nu_{\tau(l)}, \cdots, \nu_{\tau(1)}\rangle$$

where the summation $\sum\limits_{\sigma,\tau}'$ *is as above.*

Proof of (1.2.23). We apply (1.1.6).

$$a=\sum_{m,n=0}^{r}\frac{1}{m!}\frac{1}{n!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n=1}}^{r}\rho_{m,n}(\mu_1,\cdots,\mu_m;\nu_1,\cdots,\nu_n)\,v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}v_{\mu_m}\cdots v_{\mu_1}$$

$$=\sum_{l,m,n=0}^{r}\frac{1}{l!}\frac{1}{m!}\frac{1}{n!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n,\\ \lambda_1,\cdots,\lambda_l=1}}^{r}\rho_{m,n}(\mu_1,\cdots,\mu_m;\nu_1,\cdots,\nu_n)$$

$$\times v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}v_{\lambda_l}^{\dagger}\cdots v_{\lambda_1}^{\dagger}|\mathrm{vac}\rangle\langle\mathrm{vac}|v_{\lambda_1}\cdots v_{\lambda_l}v_{\mu_m}\cdots v_{\mu_1}$$

$$=\sum_{l,m,n=0}^{r}\frac{1}{(l!)^2}\frac{1}{m!}\frac{1}{n!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n,\\ \lambda_1,\cdots,\lambda_l,\\ \kappa_1,\cdots,\kappa_l=1}}^{r}\rho_{m,n}(\mu_1,\cdots,\mu_m;\nu_1,\cdots,\nu_n)$$

$$\times\langle\kappa_1,\cdots,\kappa_l|\lambda_l,\cdots,\lambda_1\rangle v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}v_{\lambda_l}^{\dagger}\cdots v_{\lambda_1}^{\dagger}|\mathrm{vac}\rangle$$

$$\times\langle\mathrm{vac}|v_{\kappa_1}\cdots v_{\kappa_l}v_{\mu_m}\cdots v_{\mu_1}$$

$$=\sum_{l,m,n=0}^{r}\frac{1}{(l!)^2}\frac{1}{m!}\frac{1}{n!}\frac{1}{\binom{m+l}{l}}\frac{1}{\binom{n+l}{l}}\sum_{\substack{\mu_1,\cdots,\mu_{m+l},\\ \nu_1,\cdots,\nu_{n+l}=1}}^{r}\sum_{\sigma,\tau}{}'\,\mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau$$

$$\times\rho_{m,n}(\mu_{\sigma(1)},\cdots,\mu_{\sigma(m)};\nu_{\tau(l+1)},\cdots,\nu_{\tau(l+n)})$$

$$\times\langle\mu_{\sigma(m+l)},\cdots,\mu_{\sigma(m+1)}|\nu_{\tau(l)},\cdots,\nu_{\tau(1)}\rangle v_{\nu_{n+l}}^{\dagger}\cdots v_{\nu_1}^{\dagger}|\mathrm{vac}\rangle$$

$$\times\langle\mathrm{vac}|v_{\mu_{m+l}}\cdots v_{\mu_1}$$

$$=\sum_{m,n=0}^{r}\frac{1}{m!}\frac{1}{n!}\Bigg[\sum_{l=0}^{\min(m,n)}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_n=1}}^{r}\sum_{\sigma,\tau}{}'\,\mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau$$

$$\times\rho_{m-l,n-l}(\mu_{\sigma(1)},\cdots,\mu_{\sigma(m-l)};\nu_{\tau(l+1)},\cdots,\nu_{\tau(n)})$$

$$\times\langle\mu_{\sigma(m)},\cdots,\mu_{\sigma(m-l+1)}|\nu_{\tau(l)},\cdots,\nu_{\tau(1)}\rangle\Bigg]$$

$$\times v_{\nu_n}^{\dagger}\cdots v_{\nu_1}^{\dagger}|\mathrm{vac}\rangle\langle\mathrm{vac}|v_{\mu_m}\cdots v_{\mu_1}\,.$$

The proof of (1.2.24) is analogous to that of (1.2.23). We apply Proposition 1.2.10 instead of (1.1.6).

§ 1.3. The Closure of $G(W)$

The closure $\overline{G}(W)$ of $G(W)$ is characterized by the following theorem.

Theorem 1.3.1. *$\overline{G}(W)$ coincides with $\{: cw_1\cdots w_k e^{\rho/2}: | c\in \boldsymbol{C}, w_1, \cdots, w_k \in W, \rho\in \Lambda^2(W)$ and $k=0, 1, 2, \cdots\}$.*

Since Nr is a linear isomorphism, it is sufficient to prove the following theorem. (See § 1.5 as for the case N is odd.)

Theorem 1.3.2. *Let W be a vector space of N dimensions and $\Lambda(W)$ be the exterior algebra over W. Denote by G^k the set $\{cw_1\cdots w_k e^{\rho/2} | c\in \boldsymbol{C}, w_1, \cdots, w_k\in W$ and $\rho\in \Lambda^2(W)\}$ and set $G=\bigcup_{k=0}^{N} G^k$. Then the Zariski closure $\overline{G}$ in $\Lambda(W)$ coincides with G. We set $\Lambda^+(W) = \bigoplus_{k:\text{even}} \Lambda^k(W)$, $\Lambda^-(W) = \bigoplus_{k:\text{odd}} \Lambda^k(W)$ and $G^\pm = G\cap \Lambda^\pm(W)$. $P(G^\pm) = (G^\pm - \{0\})/GL(1, \boldsymbol{C})$ is a non-singular projective variety in $P(\Lambda^\pm(W))$ of $\frac{1}{2}N(N-1)$ dimensions. $\{P(G^k)\}$ $(k=0, 1, \cdots, N)$ gives a stratification of $P(G)$. $P(G^k)$ is a fiber bundle over $M_{N,k}(\boldsymbol{C})$ with the fiber $\Lambda^2(\boldsymbol{C}^{N-k})$. Here we denote by $M_{N,k}(\boldsymbol{C})$ the Grassmann manifold consisting of k dimensional subspaces in $\boldsymbol{C}^N$. In particular, we have* $\dim P(G^k) = \frac{1}{2}N(N-1) - \frac{1}{2}k(k-1)$.

Before the proof we prepare some notation.

Let $W^* = \mathrm{Hom}_{\boldsymbol{C}}(W, \boldsymbol{C}) = \left\{\eta \,\middle|\, \eta : \begin{array}{ccc} W & \to & \boldsymbol{C} \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ w & \mapsto & \eta(w) \end{array}\right\}$ be the dual space of W. An orthogonal structure is introduced to $W\oplus W^*$ so that W and W^* are holonomic and $\langle w, \eta\rangle = \eta(w)$ for $w\in W$ and $\eta\in W^*$. We denote by $\widehat{W}$ the orthogonal space thus obtained.

$\widehat{W} = W\oplus W^*$ gives a holonomic decomposition. We identify $\Lambda(W)$ with $\Lambda(W)|\text{vac}\rangle$ where $|\text{vac}\rangle$ means the vacuum of $A(\widehat{W})/A(\widehat{W})W^*$. $A(\widehat{W})$ acts on $\Lambda(W)$ in this sense.

Proof of Theorem 1.3.2. Let $(v_1, \cdots, v_N, \xi_1, \cdots, \xi_N)$ denote a dual basis of $\widehat{W}$. An element a of $\Lambda(W)$ is written as $a = \sum_{m=0}^{N} \frac{1}{m!} \sum_{\mu_1, \cdots, \mu_m=1}^{N} \rho_m(\mu_1, \cdots, \mu_m) v_{\mu_m}\cdots v_{\mu_1}$ where $\rho_m(\mu_1, \cdots, \mu_m)$ is skew symmetric. Assume that $\rho_k(\nu_1, \cdots, \nu_k) \neq 0$, and let $\widehat{W} = W_1\oplus W_1^*$ be a holonomic decomposition given by $W_1 = \sum_{\mu \neq \nu_1, \cdots, \nu_k} \boldsymbol{C} v_\mu + \sum_{\nu = \nu_1, \cdots, \nu_k} \boldsymbol{C} \xi_\nu$ and $W_1^* = \sum_{\mu \neq \nu_1, \cdots, \nu_k} \boldsymbol{C} \xi_\mu + \sum_{\nu = \nu_1, \cdots, \nu_k} \boldsymbol{C} v_\nu$. Let Nr_1, $: :_1$ and $\langle\ \rangle_1$ denote the norm, the normal product and the vacuum expectation value with respect to this decomposition.

We set (cf. p. 10, lines $1 \sim 2$)

$$a_1 \underset{\text{def}}{=} \mathrm{Nr}_1 (a \xi_{\nu_1} \cdots \xi_{\nu_k})|_{v_{\nu_1} = \cdots = v_{\nu_k} = 0},$$

$$= \sum_{m=0}^{N} \frac{1}{m!} \sum_{\mu_1, \cdots, \mu_m = 1}^{N} \rho'_m (\mu_1, \cdots, \mu_m) v'_{\mu_m} \cdots v'_{\mu_1}$$

where

$$v'_\mu = \begin{cases} v_\mu & \text{if } \mu \neq \nu_1, \cdots, \nu_k, \\ \xi_\mu & \text{if } \mu = \nu_1, \cdots, \nu_k. \end{cases}$$

We have $\rho'_0 = \langle a \xi_{\nu_1} \cdots \xi_{\nu_k} \rangle_1 = \rho_k (\nu_1, \cdots, \nu_k) \neq 0$. Reciprocally we have $a = \mathrm{Nr} (a_1 v_{\nu_k} \cdots v_{\nu_1})|_{\xi_{\nu_1} = \cdots = \xi_{\nu_k} = 0}$. From Theorem 1. 5. 7 below it follows that if $g \in G$, $\mathrm{Nr}_1 (g \xi_{\nu_1} \cdots \xi_{\nu_k})|_{v_{\nu_1} = \cdots = v_{\nu_k} = 0} = \langle g \xi_{\nu_1} \cdots \xi_{\nu_k} \rangle e^{\rho_1/2}$ with $\rho_1 \in \Lambda^2 (W_1)$. Hence we have

$$\rho'_m (\mu_1, \cdots, \mu_m) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \dfrac{1}{(\rho'_0)^{(m/2)-1}} \mathrm{Pfaffian} (\rho'_2 (\mu, \nu))_{\mu, \nu = \mu_1, \cdots, \mu_m} & \text{if } m \text{ is even.} \end{cases}$$

(See (1. 3. 7) below.) This implies that in a neighborhood of $g \in \overline{G}$ such that $\rho_k (\nu_1, \cdots, \nu_k) \neq 0$ $(\nu_1 < \cdots < \nu_k)$, $\frac{1}{2} N(N-1)$ elements $\{\rho_m (\mu_1, \cdots, \mu_m) | \mu_1 < \cdots < \mu_m$ and $\{\mu_1, \cdots, \mu_m\} = \{\nu_1, \cdots, \nu_k\} \cup \{\lambda_1, \lambda_2\} - \{\nu_1, \cdots, \nu_k\} \cap \{\lambda_1, \lambda_2\}$ for some λ_1 and $\lambda_2\}$ together with $\rho_k (\nu_1, \cdots, \nu_k)$ constitute a set of local parameters of $\overline{G}$.

Now let $g = \sum_{m=0}^{N} \frac{1}{m!} \sum_{\mu_1, \cdots, \mu_m = 1}^{N} \rho_m (\mu_1, \cdots, \mu_m) v_{\mu_m} \cdots v_{\mu_1}$ be an element of $\overline{G}$ such that $\rho_k (\nu_1, \cdots, \nu_k) \neq 0$ and $\rho_m (\mu_1, \cdots, \mu_m) = 0$ if $m < k$. Then we have

$$g_1 = \rho'_0 \left\{ \exp \left(\frac{1}{\rho'_0} \sum_{\substack{\mu_1 < \mu_2 \\ \mu_1, \mu_2 \neq \nu_1, \cdots, \nu_k}} \rho'_2 (\mu_1, \mu_2) v_{\mu_2} v_{\mu_1} \right) \right\} g''_1$$

where

$$g''_1 = : \left\{ \exp \left(\frac{1}{\rho'_0} \sum_{\substack{\mu \neq \nu_1, \cdots, \nu_k \\ \nu = \nu_1, \cdots, \nu_k}} \rho'_2 (\nu, \mu) v_\mu \xi_\nu \right) \right\} :_1 .$$

Hence

$$g=\rho_0'\left\{\exp\left(\frac{1}{\rho_0'}\sum_{\substack{\mu_1<\mu_2\\ \mu_1,\mu_2\neq\nu_1,\cdots,\nu_k}}\rho_2'(\mu_1,\mu_2)\,v_{\mu_2}v_{\mu_1}\right)\right\}w_k\cdots w_1$$

where

$$w_j=g_1''v_{\nu_j}g_1''^{-1}=v_{\nu_j}+\sum_{\mu\neq\nu_1,\cdots,\nu_k}\frac{\rho_2'(\nu_j,\mu)}{\rho_0'}v_\mu.$$

$\left\{\frac{\rho_2'(\nu,\mu)}{\rho_0'}|\nu=\nu_1,\cdots,\nu_k;\,\mu\neq\nu_1,\cdots,\nu_k\right\}$ and $\left\{\frac{\rho_2'(\mu_1,\mu_2)}{\rho_2'}|\mu_1,\mu_2\neq\nu_1,\cdots,\nu_k\right\}$ constitute a set of local parameters of the base space $M_{N,k}(\boldsymbol{C})$ and of the fibre $\Lambda^2(\boldsymbol{C}^{N-k})$, respectively.

Theorem 1.3.3. *Let W be an orthogonal space of dimensions N. For an element $g\in\overline{G}(W)$ we define*

$$\sigma_t(g)=\sum_{\mu=0}^{N}(1+t)^{(N-\mu)/2}(1-t)^{\mu/2}\sigma^\mu(g). \tag{1.3.1}$$

If $g\in G(W)$, we have

$$\mathrm{nr}(\sigma_t(g))\det T_g=\mathrm{nr}(g)\det(t+T_g). \tag{1.3.2}$$

Proof. We shall prove the case where $N=2r$ is even and $g\in G^+(W)$. Other cases are proved in a similar way.

Without loss of generality we may assume that the eigenvalues $\lambda_1,\cdots,\lambda_{2r}$ of T_g are distinct. We label them so that $\lambda_1\lambda_2=1,\cdots,\lambda_{2r-1}\lambda_{2r}=1$. Let $w_\mu\neq0$ denote an eigenvector: $T_gw_\mu=\lambda_\mu w_\mu$, and set $W_\mu=\boldsymbol{C}w_{2\mu-1}+\boldsymbol{C}w_{2\mu}$. Then W is the direct sum of these orthogonal subspaces: $W=W_1\oplus\cdots\oplus W_r$ and $W_\mu\perp W_\nu$ if $\mu\neq\nu$.

g is decomposable as $g=g_1\cdots g_r$ so that $g_\mu\in G^+(W_\mu)$. Let $\alpha_\mu^\pm$ denote the eigenvalues of g_μ as an element of $G^+(W_\mu)\cong GL(1,\boldsymbol{C})^2\subset GL(2,\boldsymbol{C})$ such that $\lambda_{2\mu-1}=\alpha_\mu^+/\alpha_\mu^-$.

Note that $\mathrm{nr}(g)=\mathrm{nr}(g_1)\cdots\mathrm{nr}(g_r)$ and $\det(t+T_g)=\det(t+T_{g_1})\cdots\det(t+T_{g_r})$. On the other hand we have

$$\begin{aligned}\sigma_t(g)&=\sum_{\mu=0}^{r}(1+t)^{r-\mu}(1-t)^\mu\sigma^{2\mu}(g)\\&=\sum_{\mu=0}^{r}(1+t)^{r-\mu}(1-t)^\mu\sum_{\substack{\varepsilon_\lambda=0\text{ or }2\\ \Sigma\varepsilon_\lambda=2\mu}}\sigma^{\varepsilon_1}(g_1)\cdots\sigma^{\varepsilon_r}(g_r)\\&=\sum_{\varepsilon_\lambda=0\text{ or }2}\prod_{\lambda=1}^{r}(1+(1-\varepsilon_\lambda)t)\sigma^{\varepsilon_\lambda}(g_\lambda)\end{aligned}$$

$$= \prod_{\lambda=1}^{r} \sigma_t(g_\lambda).$$

Therefore (1. 3. 2) reduces to the case $r=1$, which is then obvious.

Remark 1. Here we quote the results in § 1. 5. Let κ_0 denote the linear homomorphism $\kappa: W \to W^*$ such that $K=\frac{1}{2}J$. Then $\mathrm{Nr}_{\kappa_0}=\bigoplus_{\mu=0}^{N} id^\mu$, where $id^\mu: \Lambda^\mu(W) \to \Lambda^\mu(W)$ is given by $id^\mu(\sum_{j_1,\cdots,j_\mu=1}^{\mu} c_{j_1\cdots j_\mu} w_{j_1}\cdots w_{j_\mu}) = \sum_{j_1,\cdots,j_\mu=1}^{\mu} c_{j_1\cdots j_\mu} w_{j_1}\cdots w_{j_\mu}$ when $c_{j_1\cdots j_\mu}$ is skew symmetric with respect to $j_1, \cdots, j_\mu$ $(\mu \geqq 2)$. (If we take a basis $(v_1, \cdots, v_{2r})$ of W so that $J=1$, (1. 5. 8) reads $R=2\frac{T-1}{T+1}$.) Hence, if $\mathrm{Nr}_{\kappa_0}(g)=w_1\cdots w_k e^{\rho/2}$ with $w_1, \cdots, w_k \in W$ and $\rho \in \Lambda^2(W)$, we have

$$\text{(1. 3. 3)} \quad \mathrm{Nr}_{\kappa_0}(\sigma_t(g)) = (1+t)^{(N-k)/2}(1-t)^{k/2} w_1\cdots w_k \exp\left(\frac{1}{2}\frac{1-t}{1+t}\rho\right).$$

If $g \in G(W)$ and $\det(1+T_g t) \neq 0$, $\sigma_t(g)$ also belongs to $G(W)$. More precisely we have

$$\text{(1. 3. 4)} \qquad T_{\sigma_t(g)} = \frac{T_g+t}{1+T_g t}.$$

The proof of (1. 3. 4) also reduces to the case when $N=1$ or 2 and then it is easy. In particular, we have

$$\sigma_0(g)=g,$$
$$\sigma_1(g)=2^{N/2}\sigma^0(g)=\text{trace } g$$
$$\sigma_{-1}(g)=2^{N/2}\sigma^N(g)$$

and

$$g^*=\begin{cases} \lim\limits_{t\to\infty}\dfrac{\sigma_t(g)}{t^N} & \text{if } g\in \overline{G}^+(W), \\ \lim\limits_{t\to\infty}\dfrac{\sigma_t(g)}{\sqrt{-t}\cdot t^{(N-1)/2}} & \text{if } g\in \overline{G}^-(W). \end{cases}$$

Proposition 1. 3. 4. *$T_g^2=1$ if and only if $g \in \Lambda^k(W)$, where k is the multiplicity of the eigenvalue -1 of T_g.*

Proof. $T_g^2=1$ implies that the eigenvalues of T_g are ± 1. Then from (1.3.7) it follows that $T_{\sigma_t(g)}$ is independent of t, or equivalently $\sigma_t(g)$ is a constant multiple of g. Hence g belongs to $A^k(W)$ for some k.

Conversely, if $g\in G(W)\cap A^k(W)$, $\mathrm{Nr}_{K_0}(g)=cw_1\cdots w_k$ with $c\in \boldsymbol{C}-\{0\}$, $w_1,\cdots,w_k\in W$. (See the above remark.) We may assume that $(w_1,\cdots,w_k,w_{k+1},\cdots,w_N)$ constitutes an orthonormal basis. Then w_j commutes or anti-commutes with g according as $(-)^k\varepsilon_j$ is positive or negative. Here $\varepsilon_j=1$ if $j\notin\{1,\cdots,k\}$, $=-1$ if $j\in\{1,\cdots,k\}$. Hence we have the proposition.

Proposition 1.3.5. *The coefficients of* $\mathrm{nr}(\sigma_t(g))$ *belongs to the coordinate ring* $A_{\overline{G}(W)}$ of $\overline{G}(W)$.

Let $N=2r$ *be even. If* $g\in\overline{G}^+(W)$, $t^{-r}\,\mathrm{nr}(\sigma_t(g))$ *is written as* $t^{-r}\,\mathrm{nr}(\sigma_t(g))=a_0(g)s^r+\cdots+a_{r-1}(g)s+a_r(g)$ *where* $s=t-2+t^{-1}$. *In particular,* $a_0(g)=\mathrm{nr}(g)$ *and* $a_r(g)=(\mathrm{trace}\ g)^2$. $a_0(g),\cdots,a_{r-1}(g)$, $\sqrt{a_r(g)}$ *are algebraically independent and the polynomial ring* $\boldsymbol{C}[a_0(g),\cdots,a_{r-1}(g),\sqrt{a_r(g)}]$ *coincides with the subring* $\{f(g)\in A_{\overline{G}(W)}|f(g)=f(hgh^{-1})$ *for* $\forall h\in G(W)\}$. *If* $g\in\overline{G}^-(W)$, $t^{1-r}(1-t^2)^{-1}\,\mathrm{nr}(\sigma_t(g))$ *is written as* $t^{1-r}(1-t^2)^{-1}\,\mathrm{nr}(\sigma_t(g))=a_0(g)s^{r-1}+\cdots+a_{r-1}(g)$ *where* $s=t-2+t^{-1}$ *and* $a_0(g)=\mathrm{nr}(g)$. $a_0(g),\cdots,a_{r-1}(g)$ *are algebraically independent and the polynomial ring* $\boldsymbol{C}[a_0(g),\cdots,a_{r-1}(g)]$ *coincides with the subring* $\{f(g)\in A_{\overline{G}^-(W)}|f(g)=f(hgh^{-1})$ *for* $\forall h\in G(W)\}$.

If $N=2r+1$ *is odd,* $t^{-r}(1\pm t)^{-1}\,\mathrm{nr}(\sigma_t(g))$ *has similar properties according as* $g\in\overline{G}^\pm(W)$.

Proof. If $N=2r$ and $g\in\overline{G}^+(W)$, $\mathrm{nr}(\sigma_t(g))=\frac{1}{2^r}\mathrm{trace}(\sigma_t(g)\sigma_t(g)^*)$ $=\frac{1}{2^r}\sum_{\mu=0}^{r}(1+t)^{2(r-\mu)}(1-t)^{2\mu}\,\mathrm{trace}\,\sigma^{2\mu}(g)\sigma^{2\mu}(g)^*$. Hence the coefficients of $\mathrm{nr}(\sigma_t(g))$ belong to $A_{\overline{G}^+(W)}$.

Using the notations in the proof of Theorem 1.3.3, we have $\mathrm{nr}(g)=\prod_{\mu=1}^{r}\alpha_\mu^+\alpha_\mu^-$, $\mathrm{trace}\ g=\prod_{\mu=1}^{r}(\alpha_\mu^++\alpha_\mu^-)$ and $\det(t+T_g)=\prod_{\mu=1}^{r}(t+\alpha_\mu^+\alpha_\mu^{-\,-1})$ $(t+\alpha_\mu^-\alpha_\mu^{+\,-1})$. Thus we have $t^{-r}\,\mathrm{nr}(\sigma_t(g))=\prod_{\mu=1}^{r}[(\alpha_\mu^++\alpha_\mu^-)^2+\alpha_\mu^+\alpha_\mu^-s]$. In particular $a_0(g)=\mathrm{nr}(g)$ and $a_r(g)=(\mathrm{trace}\ g)^2$.

Next we shall show that

$$\begin{array}{ccc}\overline{G}^+(W) & \longrightarrow & \boldsymbol{C}^{r+1}\\ \Cup & & \Cup\\ g & \mapsto & (a_0(g),\cdots,a_{r-1}(g),\mathrm{trace}\ g)\end{array}$$

is surjective. This implies that $a_0(g), \cdots, a_{r-1}(g)$, trace g are algebraically independent. Let $(b_0, \cdots, b_{r-1}, b_r) \in \boldsymbol{C}^{r+1}$ and assume that $b_0 = \cdots = b_{s-1} = 0$ and $b_s \neq 0$. Taking $\alpha_1^+ = \cdots = \alpha_{s-1}^+ = 0$, we have $a_0(g) = \cdots = a_{s-1}(g) = 0$. Hence we may assume that $s=0$. Since we can give arbitrary values to $\frac{(\alpha_\mu^+ + \alpha_\mu^-)^2}{\alpha_\mu^+ \alpha_\mu^-}$, there exists an element $g \in \overline{G}^+(W)$ such that $a_0(g) = b_0, \cdots, a_{r-1}(g) = b_{r-1}$ and $a_r(g) = b_r^2$. Then g or $-g$ is the one we need.

Now let $f(g) \in A_{\overline{G}^+(W)}$ satisfy $f(g) = f(hgh^{-1})$. The polynomial $f(tg)$ on $\boldsymbol{C} \times \overline{G}^+(W)$ is written as $f(tg) = \sum_{j=0}^{M} t^j f_j(g)$, where $f_j(g)$ satisfying $f_j(g) = f_j(hgh^{-1})$ is homogeneous of degree j. Then $f_j(g)/(\text{trace } g)^j$ is determined by the eigenvalues of T_g, hence it is a rational function of $\frac{a_1(g)}{a_0(g)}, \cdots, \frac{a_r(g)}{a_0(g)}$. Thus $f(g)$ is a rational function of $a_0(g)$, $\cdots, a_{r-1}(g)$ and trace g. Since

$$\begin{array}{ccc} \overline{G}(W) & \longrightarrow & \boldsymbol{C}^{r+1} \\ \Downarrow & & \Downarrow \\ g & \mapsto & (a_0(g), \cdots, a_{r-1}(g), \text{trace } g) \end{array}$$

is surjective, $f(g)$ belongs to $\boldsymbol{C}[a_0(g), \cdots, a_{r-1}(g), \text{trace } g]$.

Other cases are proved similarly.

Proposition 1.3.6. *Let W, W^*, $\widehat{W}$, etc. be as in Theorem 1.3.2. We take a dual basis $(v_1, \cdots, v_N, \xi_1, \cdots, \xi_N)$ of $\widehat{W}$. Let R be an $N \times N$ skew-symmetric matrix, and set $g = e^{\rho/2}$ where $\rho = \frac{1}{2}(v_1, \cdots, v_N) R \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$. Let $\eta_j = \sum_{\mu=1}^{N} c_{j,\mu} \xi_\mu$ $(j=1, \cdots, K)$ be elements of W^*, and set $r = \begin{pmatrix} c_{11} & \cdots & c_{K1} \\ \vdots & & \vdots \\ c_{1N} & \cdots & c_{KN} \end{pmatrix}$. Then the constant term of $\eta_1 \cdots \eta_K g$ is equal to* Pfaffian$(-{}^t r R r)$.

Proof. If $\eta_1 \cdots \eta_K = 0$ or K is odd, the above statement is trivial. If $\eta_1 \cdots \eta_K \neq 0$, without loss of generality we may assume that $\eta_1 = \xi_1, \cdots, \eta_K = \xi_K$. The constant term of $\xi_1 \cdots \xi_K g$ is equal to the coefficient of $v_K \cdots v_1$ in the expansion of $\frac{1}{(K/2)!} (\sum_{j_1 < k_1} (-R_{j_1 k_1}) v_{k_1} v_{j_1}) \cdots (\sum_{j_{K/2} < k_{K/2}} (-R_{j_{K/2} k_{K/2}}) v_{k_{K/2}} v_{j_{K/2}})$. Hence we have the proposition.

Ler R be a $2r \times 2r$ skew-symmetric matrix, and consider an element $g(R)$ of $A(\widehat{W})$ defined by

$$(1.3.5)\qquad g(R)=\exp\Big(\frac{1}{2}\rho+\sum_{\mu=1}^{N}\xi_\mu v_\mu\Big)=\sum_{m=0}^{N}\frac{1}{m!}\sum_{\mu_1,\cdots,\mu_m=1}^{N}\xi_{\mu_1}\cdots\xi_{\mu_m}v_{\mu_m}\cdots v_{\mu_1}e^{\rho/2}$$

where $\rho=(v_1,\cdots,v_N)R\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}$.

Proposition 1.3.7. *Let ι denote the canonical isomorphism $\iota\colon \widehat{W}\to\widehat{W}^*$, $\iota(w\oplus\eta)(w'\oplus\eta')=\langle w\oplus\eta, w'\oplus\eta'\rangle$. Then we have for $w_1,\cdots,w_k\in W$*

$$(1.3.6)\qquad w_1\cdots w_k\exp\frac{\rho}{2}=\iota(w_1)\cdots\iota(w_k)\exp\Big(\frac{1}{2}\rho+\sum_{\mu=1}^{N}\xi_\mu v_\mu\Big)\Big|_{\xi=0}.$$

The proof is straightforward.

Proposition 1.3.8. *Let $g\in A(W)$ be given by $g=w_1\cdots w_k e^{\rho/2}$. Set $\boldsymbol{r}=\begin{pmatrix}c_{1,1}\cdots c_{k,1}\\ \vdots \quad\quad \vdots\\ c_{1,N}\cdots c_{k,N}\end{pmatrix}$ where $w_j=\sum_{\mu=1}^{N}v_\mu c_{j,\mu}$. Let e_μ denote the N component column vector $(\delta_{\nu\mu})_{\nu=1,\cdots,N}$.*

If we write $g=\sum_{m=0}^{N}\frac{1}{m!}\sum_{\mu_1,\cdots,\mu_m=1}^{N}\rho_m(\mu_1,\cdots,\mu_m)v_{\mu_m}\cdots v_{\mu_1}$, the coefficient $\rho_m(\mu_1,\cdots,\mu_m)$ is given by

$$(1.3.7)\qquad \rho_m(\mu_1,\cdots,\mu_m)=\mathrm{Pfaffian}\begin{pmatrix}{}^t\boldsymbol{e} & \\ & {}^t\boldsymbol{r}\end{pmatrix}\begin{pmatrix}-R & 1\\ -1 & \end{pmatrix}\begin{pmatrix}\boldsymbol{e} & \\ & \boldsymbol{r}\end{pmatrix},$$

$$=(-)^{(m+k)/2}\,\mathrm{Pfaffian}\left(\begin{array}{cc|cc} & & {}^t\boldsymbol{e} & \\ & & & {}^t\boldsymbol{r}\\ \hline -\boldsymbol{e} & & & 1\\ & -\boldsymbol{r} & -1 & R\end{array}\right)\Big/\mathrm{Pfaffian}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$$

where $\boldsymbol{e}=(e_{\mu_1},\cdots,e_{\mu_m})$.

Proof. We apply Proposition 1.3.6 taking $\widehat{W}$, $\exp\Big(\frac{1}{2}\rho+\sum_{\mu=1}^{r}\xi_\mu v_\mu\Big)$ and $\begin{pmatrix}\boldsymbol{e} & \\ & \boldsymbol{r}\end{pmatrix}$ as W, g and $\boldsymbol{r}$, respectively. Then from (1.3.6) follows (1.3.7).

§ 1.4. Product in $\overline{G}(W)$

Since $G(W)$ is an algebraic subgroup in $A(W)$, $\overline{G}(W)$ is a semigroup. We shall give a formula to compute $\mathrm{Nr}(g^{(1)}\cdots g^{(n)})$ for $g^{(1)},\cdots,$

$g^{(n)}\in\overline{G}(W)$ when $\mathrm{Nr}(g^{(1)}),\cdots,\mathrm{Nr}(g^{(n)})$ are given. (Though we treat only the even dimensional case, the results in § 1. 5 enables us to give analogous formulas in odd dimensional case.)

First we consider a simple case. Let $w=\sum_{\mu=1}^{r}v_\mu^\dagger c_\mu^\dagger+\sum_{\mu=1}^{r}v_\mu c_\mu$ and $w_1,\cdots,w_k$ be elements of W and let $\rho=(v_1^\dagger,\cdots,v_r^\dagger,v_1,\cdots,v_r)\begin{pmatrix}R_1 & R_2\\ R_3 & R_4\end{pmatrix}\begin{pmatrix}v_1^\dagger\\ \vdots\\ v_r^\dagger\\ v_1\\ \vdots\\ v_r\end{pmatrix}$ be an element of $\Lambda^2(W)$. We assume that $R=\begin{pmatrix}R_1 & R_2\\ R_3 & R_4\end{pmatrix}$ is skew-symmetric.

Theorem 1.4.1. *Let* $\mathrm{Nr}(g)=w_1\cdots w_k e^{\rho/2}$ *with* $w_1,\cdots,w_k$ *and* ρ *as above. Then we have*

$$\mathrm{Nr}(wg)=\Big(\sum_{j=1}^{k}(-)^{j-1}w_1\cdots w_{j-1}\langle ww_j\rangle w_{j+1}\cdots w_k + w^{(1)}w_1\cdots w_k\Big)e^{\rho/2} \tag{1.4.1}$$

where $w^{(1)}=w-\sum_{\mu=1}^{r}v_\mu^\dagger(R_1\boldsymbol{c})_\mu-\sum_{\mu=1}^{r}v_\mu(R_3\boldsymbol{c})_\mu$, *and*

$$\mathrm{Nr}(gw)=\Big(\sum_{j=1}^{k}(-)^{k-j}w_1\cdots w_{j-1}\langle w_jw\rangle w_{j+1}\cdots w_k + w_1\cdots w_k w^{(2)}\Big)e^{\rho/2} \tag{1.4.2}$$

where $w^{(2)}=w+\sum_{\mu=1}^{r}v_\mu^\dagger(R_2\boldsymbol{c}^\dagger)_\mu+\sum_{\mu=1}^{r}v_\mu(R_4\boldsymbol{c}^\dagger)_\mu$. *Here we have set* $\boldsymbol{c}^\dagger=\begin{pmatrix}c_1^\dagger\\ \vdots\\ c_r^\dagger\end{pmatrix}$ *and* $\boldsymbol{c}=\begin{pmatrix}c_1\\ \vdots\\ c_r\end{pmatrix}$.

Remark. Since $\eta^{(1)}=\sum_{j=1}^{k}(-)^{j-1}w_1\cdots w_{j-1}\langle ww_j\rangle w_{j+1}\cdots w_k$ or $\eta^{(2)}=\sum_{j=1}^{k}(-)^{k-j}w_1\cdots w_{j-1}\langle w_jw\rangle w_{j+1}\cdots w_k$ is a $k-1$ form in $w_1,\cdots,w_k$, it is a monomial. We have

$$\begin{cases}\mathrm{Nr}(wg)=\eta^{(1)}e^{\rho/2+w^{(1)}w_0^{(1)}}, \\ \qquad\text{if there exists } w_0^{(1)}\in\sum_{j=1}^{k}\boldsymbol{C}w_j \text{ such that } \langle ww_0^{(1)}\rangle=1, \\ \mathrm{Nr}(wg)=w^{(1)}w_1\cdots w_k e^{\rho/2}, \quad\text{if } \langle ww_j\rangle=0 \text{ for } j=1,\cdots,k.\end{cases} \tag{1.4.3}$$

$$(1.4.4)\quad \begin{cases} \mathrm{Nr}(gw) = \eta^{(2)} e^{\rho/2 + w_0^{(2)} w^{(2)}}, \\ \qquad \text{if there exists } w_0^{(2)} \in \sum_{j=1}^{k} Cw_j \text{ such that } \langle w_0^{(2)} w \rangle = 1, \\ \mathrm{Nr}(gw) = w_1 \cdots w_k w^{(2)} e^{\rho/2}, \quad \text{if } \langle w_j w \rangle = 0 \text{ for } j = 1, \cdots, k. \end{cases}$$

The proof of Theorem 1.4.1 is given in § 1.5.

Let $W^{(\nu)} = V^{\dagger(\nu)} \oplus V^{(\nu)}$ $(\nu = 1, \cdots, n)$ denote copies of an orthogonal space W with holonomic decomposition $W = V^{\dagger} \oplus V$. Let Λ denote an $n \times n$ symmetric matrix $(\lambda_{\mu\nu})_{\mu,\nu=1,\cdots,n}$ with $\lambda_{\nu\nu} = 1$ $(\nu = 1, \cdots, n)$. Let $W(\Lambda)$ denote the vector space $\bigoplus_{\nu=1}^{n} W^{(\nu)}$ equipped with the inner product $\langle (w^{(1)}, \cdots, w^{(n)}), (w'^{(1)}, \cdots, w'^{(n)}) \rangle_{\Lambda} = \sum_{\mu,\nu=1}^{n} \lambda_{\mu\nu} \langle w^{(\mu)}, w'^{(\nu)} \rangle$. If $\det \Lambda \neq 0$, $W(\Lambda)$ is an orthogonal space. $W(\Lambda) = (\bigoplus_{\nu=1}^{n} V^{\dagger(\nu)}) \oplus (\bigoplus_{\nu=1}^{n} V^{(\nu)})$ gives a holonomic decomposition of $W(\Lambda)$. We denote by $\langle\ \rangle_{\Lambda}$ the vacuum expectation with respect to this holonomic decomposition. We note that the natural inclusion $\iota^{(\nu)} : W^{(\nu)} \to W(\Lambda)$ preserves not only the inner product but also the holonomic decomposition.

Since $\langle \iota^{(\nu)}(w) - \lambda_{\mu\nu} \iota^{(\mu)}(w), \iota^{(\mu)}(w') \rangle_{\Lambda} = 0$ for any $w, w' \in W$, we have the following proposition.

Proposition 1.4.2. *If $g^{(\mu)} \in G(W^{(\mu)})$, for any $w \in W$ we have*

$$(1.4.5)\qquad T_{g^{(\mu)}}(\iota^{(\nu)}(w) - \lambda_{\mu\nu} \iota^{(\mu)}(w)) = \iota^{(\nu)}(w) - \lambda_{\mu\nu} \iota^{(\mu)}(w).$$

Hence, in particular, $g^{(\mu)}$ belongs to $G(W(\Lambda))$.

Take a dual basis $(v_1^{\dagger}, \cdots, v_r^{\dagger}, v_1, \cdots, v_r)$ of W. We denote by $(v_1^{\dagger(\nu)}, \cdots, v_r^{\dagger(\nu)}, v_1^{(\nu)}, \cdots, v_r^{(\nu)})$ the corresponding dual basis of $W^{(\nu)}$.

Let $g^{(\nu)}$ be an element of $\overline{G}(W^{(\nu)})$ with the norm $\mathrm{Nr}(g^{(\nu)}) = \langle g^{(\nu)} \rangle e^{\rho^{(\nu)}/2}$ where

$$\rho^{(\nu)} = (v_1^{\dagger(\nu)}, \cdots, v_r^{\dagger(\nu)}, v_1^{(\nu)}, \cdots, v_r^{(\nu)}) \begin{pmatrix} R_1^{(\nu)} & R_2^{(\nu)} \\ R_3^{(\nu)} & R_4^{(\nu)} \end{pmatrix} \begin{pmatrix} v_1^{\dagger(\nu)} \\ \vdots \\ v_r^{\dagger(\nu)} \\ v_1^{(\nu)} \\ \vdots \\ v_r^{(\nu)} \end{pmatrix}.$$

We assume that $R^{(\nu)}=\begin{pmatrix} R_1^{(\nu)} & R_2^{(\nu)} \\ R_3^{(\nu)} & R_4^{(\nu)} \end{pmatrix}$ is skew-symmetric. Let R and $A(\Lambda)$ denote $2nr\times 2nr$ skew-symmetric matrices

$$\begin{pmatrix} R^{(1)} & & \\ & \ddots & \\ & & R^{(n)} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \lambda_{12}K_1 & \cdots & \lambda_{1n}K_1 \\ -\lambda_{21}{}^tK_1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_{n-1\,n}K_1 \\ -\lambda_{n1}{}^tK_1 & \cdots & -\lambda_{n\,n-1}{}^tK_1 & 0 \end{pmatrix} \text{ where } K_1=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Theorem 1.4.3. *Under the above assumptions, we have*

$$(1.4.6)\quad \langle g^{(1)}\cdots g^{(n)}\rangle_\Lambda=\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle(\det(1-A(\Lambda)R))^{1/2}$$

$$=\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle \operatorname{Pfaffian}\begin{pmatrix} -A(\Lambda) & 1 \\ -1 & R \end{pmatrix}\Big/\operatorname{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

If $\langle g^{(1)}\cdots g^{(n)}\rangle_\Lambda\neq 0$,

$$(1.4.7)\qquad \operatorname{Nr}(g^{(1)}\cdots g^{(n)})=\langle g^{(1)}\cdots g^{(n)}\rangle_\Lambda e^{\rho(\Lambda)/2}.$$

with $\rho(\Lambda)=\sum_{\mu,\nu=1}^{n}(v_1^{\dagger(\mu)},\cdots,v_r^{\dagger(\mu)},v_1^{(\mu)},\cdots,v_r^{(\mu)})R(\Lambda)_{\mu\nu}\begin{pmatrix} v_1^{\dagger(\nu)} \\ \vdots \\ v_r^{\dagger(\nu)} \\ v_1^{(\nu)} \\ \vdots \\ v_r^{(\nu)} \end{pmatrix}$ *where* $R(\Lambda)=\begin{pmatrix} R(\Lambda)_{11}\cdots R(\Lambda)_{1n} \\ \vdots \qquad \vdots \\ R(\Lambda)_{n1}\cdots R(\Lambda)_{nn} \end{pmatrix}$ *is given by* $R(\Lambda)=R(1-A(\Lambda)R)^{-1}$.

The proof of this theorem is given in § 1.5.

Let $w_j^{(\nu)}=\sum_{\mu=1}^{r} v_\mu^\dagger c_{j,\mu}^{\dagger(\nu)}+\sum_{\mu=1}^{r} v_\mu c_{j,\mu}^{(\nu)}$ ($\nu=1,\cdots,n$; $j=1,\cdots,k^{(\nu)}$) be elements of $W^{(\nu)}$. We set $\operatorname{Nr}(\bar{g}^{(\nu)})=\langle g^{(\nu)}\rangle w_1^{(\nu)}\cdots w_{k^{(\nu)}}^{(\nu)}e^{\rho^{(\nu)}/2}$. Let $\boldsymbol{c}_j^{\dagger(\nu)}$ and $\boldsymbol{c}_j^{(\nu)}$ denote $\begin{pmatrix} c_{j,1}^{\dagger(\nu)} \\ \vdots \\ c_{j,r}^{\dagger(\nu)} \end{pmatrix}$ and $\begin{pmatrix} c_{j,1}^{(\nu)} \\ \vdots \\ c_{j,r}^{(\nu)} \end{pmatrix}$, respectively. Let $\boldsymbol{r}$ denote the $2rn\times k$ matrix

$$\begin{pmatrix} \boldsymbol{c}_1^{\dagger(1)}\cdots\boldsymbol{c}_{k^{(1)}}^{\dagger(1)} & & \\ \boldsymbol{c}_1^{(1)}\cdots\boldsymbol{c}_{k^{(1)}}^{(1)} & & \\ & \ddots & \\ & & \boldsymbol{c}_1^{\dagger(n)}\cdots\boldsymbol{c}_{k^{(n)}}^{\dagger(n)} \\ & & \boldsymbol{c}_1^{(n)}\cdots\boldsymbol{c}_{k^{(n)}}^{(n)} \end{pmatrix},$$

where $k=\sum_{\nu=1}^{n}k^{(\nu)}$.

Let $(\hat{v}_1^\dagger, \cdots, \hat{v}_{rn}^\dagger, \hat{v}_1, \cdots, \hat{v}_{rn})$ denote the basis $(v_1^{\dagger(1)}, \cdots, v_r^{\dagger(1)}, \cdots, v_1^{\dagger(n)}, \cdots, v_r^{\dagger(n)}, v_1^{(1)}, \cdots, v_r^{(1)}, \cdots, v_1^{(n)}, \cdots, v_r^{(n)})$. Let $\hat{e}_1^\dagger, \cdots, \hat{e}_r^\dagger, \hat{e}_1, \cdots, \hat{e}_r, \cdots, \hat{e}_{r(n-1)+1}^\dagger, \cdots, \hat{e}_{rn}^\dagger, \hat{e}_{r(n-1)+1}, \cdots, \hat{e}_{rn}$ denote the $2rn$ component column vectors, $\begin{pmatrix}1\\0\\0\\\vdots\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\\vdots\\0\end{pmatrix}$, etc., respectively.

Theorem 1.4.4. *If we write*

$$\mathrm{Nr}(\bar{g}^{(1)}\cdots\bar{g}^{(n)}) = \sum_{m,m'=0}^{rn}\frac{1}{m!}\frac{1}{m'!}\sum_{\substack{\mu_1,\cdots,\mu_m,\\ \nu_1,\cdots,\nu_{m'}=1}}^{rn} \times\rho_{mm'}(\mu_1,\cdots,\mu_m;\nu_1,\cdots,\nu_{m'})\,\hat{v}_{\nu_{m'}}^\dagger\cdots\hat{v}_{\nu_1}^\dagger\hat{v}_{\mu_m}\cdots\hat{v}_{\mu_1},$$

the coefficient is given by

$$(1.4.8)\quad \rho_{mm'}(\mu_1,\cdots,\mu_m;\nu_1,\cdots,\nu_{m'})$$

$$=(-)^{(m+m'+k)/2}\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle\left(\begin{array}{cc|cc} & & {}^t\boldsymbol{e} & \\ & & & {}^t\boldsymbol{r} \\ \hline -\boldsymbol{e} & & -A(\Lambda) & 1 \\ & -\boldsymbol{r} & -1 & -R\end{array}\right)\Big/\mathrm{Pfaffian}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$$

where $\boldsymbol{e}=(e_{\mu_1},\cdots,e_{\mu_m},e_{\nu_1}^\dagger,\cdots,e_{\nu_{m'}}^\dagger)$.

If $\langle g^{(1)}\cdots g^{(n)}\rangle_\Lambda\neq 0$, *we have*

$$(1.4.9)\quad \rho_{mm'}(\mu_1,\cdots,\mu_m;\nu_1,\cdots,\nu_{m'})=\langle g^{(1)}\cdots g^{(n)}\rangle_\Lambda$$

$$\times\mathrm{Pfaffian}\begin{pmatrix}{}^t\boldsymbol{e} & \\ & {}^t\boldsymbol{r}\end{pmatrix}\begin{pmatrix}-R(1-A(\Lambda)R)^{-1} & (1-RA(\Lambda))^{-1}\\ -(1-A(\Lambda)R)^{-1} & (1-A(\Lambda)R)^{-1}A(\Lambda)\end{pmatrix}\begin{pmatrix}\boldsymbol{e} & \\ & \boldsymbol{r}\end{pmatrix}.$$

In particular

$$(1.4.10)\quad \langle\bar{g}^{(1)}\cdots\bar{g}^{(n)}\rangle=\langle g^{(1)}\cdots g^{(n)}\rangle\ \mathrm{Pfaffian}\ {}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}A(\Lambda)\boldsymbol{r}.$$

Set $k-k'=\mathrm{rank}\ {}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}A(\Lambda)\boldsymbol{r}$, *and choose a* $k\times k$ *non-singular matrix* (X_1, X_2) (X_1 *is a* $k\times k'$ *matrix and* X_2 *is a* $k\times(k-k')$ *matrix*) *so that*

$$\begin{pmatrix}{}^tX_1\\ {}^tX_2\end{pmatrix}{}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}A(\Lambda)\boldsymbol{r}(X_1,X_2)=\begin{pmatrix}0 & 0\\ 0 & {}^tX_2{}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}A(\Lambda)\boldsymbol{r}X_2\end{pmatrix}.$$

Then we have

(1.4.11) $\mathrm{Nr}(\bar{g}^{(1)}\cdots\bar{g}^{(n)})$

$$=\langle g^{(1)}\cdots g^{(n)}\rangle\frac{\text{Pfaffian }{}^tX_2{}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}A(\Lambda)\boldsymbol{r}X_2}{\det X}$$

$$\times\hat{w}_1\cdots\hat{w}_{k'}e^{\hat{\rho}/2}$$

where $\hat{w}_j=\sum_{\mu=1}^{rn}\hat{v}_\mu^\dagger\hat{c}_{j,\mu}^\dagger+\sum_{\mu=1}^{r}\hat{v}_\mu\hat{c}_{j,\mu}$ *is given by* $\begin{pmatrix}\hat{c}_{1,1}^\dagger & \cdots & \hat{c}_{k',1}^\dagger\\ \vdots & & \vdots\\ \hat{c}_{1,rn}^\dagger & \cdots & \hat{c}_{k',rn}^\dagger\\ \hat{c}_{1,1} & \cdots & \hat{c}_{k',1}\\ \vdots & & \vdots\\ \hat{c}_{1,rn} & \cdots & \hat{c}_{k',rn}\end{pmatrix}$

$=(1-RA(\Lambda))^{-1}\boldsymbol{r}X_1$, *and* $\hat{\rho}=(v_1^{\dagger(1)},\cdots,v_r^{\dagger(1)},v_1^{(1)},\cdots,v_r^{(1)},\cdots)\hat{R}\begin{pmatrix}v_1^{\dagger(1)}\\ \vdots\\ v_r^{\dagger(1)}\\ v_1^{(1)}\\ \vdots\\ v_r^{(1)}\\ \vdots\end{pmatrix}$

with $\hat{R}=R(1-A(\Lambda)R)^{-1}-(1-RA(\Lambda))^{-1}\boldsymbol{r}X_2[{}^tX_2{}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}$
$\times A(\Lambda)\boldsymbol{r}X_2]^{-1}{}^tX_2{}^t\boldsymbol{r}(1-A(\Lambda)R)^{-1}$.

The proof is given in § 1.5.

Remark 1. If we set $\lambda_{\mu\nu}=1$ and identify all the $W^{(\nu)}$ with W in Theorem 1.4.4, we have a formula to compute products in $\overline{G}(W)$.

Remark 2. Let us denote by $\|X\|$ the norm of an $m\times m$ matrix X; namely $\|X\|=\sum_{i,j=1}^{m}|x_{ij}|$. If $\|A(\Lambda)R\|<1$, (1.4.10) is rewritten as

(1.4.12) $\langle\bar{g}^{(1)}\cdots\bar{g}^{(n)}\rangle=\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle$ Pfaffian ${}^t\boldsymbol{r}\sum_{l=0}^{\infty}(A(\Lambda)R)^lA(\Lambda)\boldsymbol{r}$

$$\times\exp\left\{-\sum_{l=2}^{\infty}\frac{1}{2l}\,\mathrm{trace}\,(A(\Lambda)R)^l\right\}.$$

We shall give an example. Let $\mathrm{Nr}(g)=w_1\cdots w_k\,e^{\rho/2}$ with

$$w_j=\sum_{\mu=1}^{r}v_\mu^\dagger c_{j,\mu}^\dagger+\sum_{\mu=1}^{r}v_\mu c_{j,\mu}\quad(j=1,\cdots,k)$$

and

$$\rho=(v_1^\dagger,\cdots,v_r^\dagger,v_1,\cdots,v_r)\begin{pmatrix}-S_2 & S_1-1\\ 1-S_4 & S_3\end{pmatrix}\begin{pmatrix}v_1^\dagger\\ \vdots\\ v_r^\dagger\\ v_1\\ \vdots\\ v_r\end{pmatrix}.$$

Then we have

$$\mathrm{Nr}\,(c^N g)=w_1^{(1)}\cdots w_k^{(1)}e^{\rho^{(1)}/2}$$

with $$w_j^{(1)}=\sum_{\mu=1}^{r}v_\mu^\dagger c_{j,\mu}^{(1)\dagger}+\sum_{\mu=1}^{r}v_\mu c_{j,\mu}^{(1)}\quad(j=1,\cdots,k)$$

where $$\begin{pmatrix}c_1^{(1)\dagger}\cdots c_k^{(1)\dagger}\\ c_1^{(1)}\cdots c_k^{(1)}\end{pmatrix}=\begin{pmatrix}c & \\ & 1\end{pmatrix}\begin{pmatrix}c_1^\dagger\cdots c_k^\dagger\\ c_1\cdots c_k\end{pmatrix}$$

and $$\rho^{(1)}=(v_1^\dagger,\cdots,v_r^\dagger,v_1,\cdots,v_r)\begin{pmatrix}-c^2S_2 & cS_1-1\\ 1-cS_4 & S_3\end{pmatrix}\begin{pmatrix}v_1^\dagger\\ \vdots\\ v_r^\dagger\\ v_1\\ \vdots\\ v_r\end{pmatrix},$$

$$\mathrm{Nr}\,(gc^N)=w_1^{(2)}\cdots w_k^{(2)}e^{\rho^{(2)}/2}$$

with $$w_j^{(2)}=\sum_{\mu=1}^{r}v_\mu^\dagger c_{j,\mu}^{(2)\dagger}+\sum_{\mu=1}^{r}v_\mu c_{j,\mu}^{(2)}\quad(j=1,\cdots,k)$$

where $$\begin{pmatrix}c_1^{(2)\dagger}\cdots c_k^{(2)\dagger}\\ c_1^{(2)}\cdots c_k^{(2)}\end{pmatrix}=\begin{pmatrix}1 & \\ & c\end{pmatrix}\begin{pmatrix}c_1^\dagger\cdots c_k^\dagger\\ c_1\cdots c_k\end{pmatrix}$$

and $$\rho^{(2)}=(v_1^\dagger,\cdots,v_r^\dagger,v_1,\cdots,v_r)\begin{pmatrix}-S_2 & cS_1-1\\ 1-cS_4 & c^2S_3\end{pmatrix}\begin{pmatrix}v_1^\dagger\\ \vdots\\ v_r^\dagger\\ v_1\\ \vdots\\ v_r\end{pmatrix}.$$

§ 1.5. κ-Norms and Transformation Law

Let W, W^* and $\widehat{W}$ be as in the proof of Theorem 1.3.2. Let $\kappa: W\to W^*$ be a linear homomorphism and denote by $\bar{\kappa}$ the induced injective homomorphism

$$\begin{array}{rcl}\bar{\kappa}: W & \longrightarrow & \widehat{W}\\ \Uparrow & & \Uparrow\\ w & \mapsto & (w,\kappa(w)).\end{array}$$

We assume that the inner product in $\widehat{W}$ is non-degenerate when it is restricted to $W_\kappa \underset{\text{def}}{=} \bar{\kappa}(W)$, and identify W with the orthogonal space W_κ.

Definition 1.5.1. *The κ-norm is the following linear isomorphism*:

$$(1.5.1)\qquad \mathrm{Nr}_\kappa: A(W) \overset{\cong}{\to} A(W_\kappa) \overset{\cong}{\to} A(W_\kappa)|\mathrm{vac}\rangle \overset{\cong}{\to} A(\widehat{W})|\mathrm{vac}\rangle \overset{\cong}{\to} A(W)|\mathrm{vac}\rangle \overset{\cong}{\to} A(W).$$

We denote by $:\ :_\kappa$ the inverse of Nr_κ: $a = :\mathrm{Nr}_\kappa(a):_\kappa$. The constant term of $\mathrm{Nr}_\kappa(a)$ is called the κ-expectation value and is denoted by $\langle a\rangle_\kappa$.

Take a basis $(v_1, \cdots, v_N)$ of W. We denote by K the matrix $\begin{pmatrix} \langle v_1v_1\rangle_\kappa \cdots \langle v_1v_N\rangle_\kappa \\ \vdots \qquad \vdots \\ \langle v_Nv_1\rangle_\kappa \cdots \langle v_Nv_N\rangle_\kappa \end{pmatrix}$, and set $J = K + {}^tK$ and $H = K - {}^tK$.

Remark 1. Let $W = V^\dagger \oplus V$ be an orthogonal space and its holonomic decomposition. We define κ by $\kappa(w)(w') = \langle ww'\rangle$. Then the orthogonal structure on W induced by κ coincides with the original one and the κ-norm coincides with (1.1.4). If we take a dual basis $(v_1^\dagger, \cdots, v_r^\dagger, v_1, \cdots, v_r)$, we have $J = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $K = \begin{pmatrix} & \\ 1 & \end{pmatrix}$.

Remark 2. Let ${}^t\kappa$ denote the adjoint of κ, i.e. ${}^t\kappa(w)(w') = \kappa(w')(w)$. Then the following alternative coincides with (1.5.1).

$$\mathrm{Nr}_\kappa: A(W) \overset{\cong}{\to} A(W_{{}^t\kappa}) \overset{\cong}{\to} \langle \mathrm{vac}^*|A(W_{{}^t\kappa}) \overset{\cong}{\to} \langle \mathrm{vac}^*|A(\widehat{W}) \overset{\cong}{\to} \langle \mathrm{vac}^*|A(W) \overset{\cong}{\to} A(W),$$

where $\langle \mathrm{vac}^*|$ denotes the residue class of 1 in $A(\widehat{W})/W^*A(\widehat{W})$.

Proposition 1.5.2. *If $w \in W$ and $a \in A(W)$,*

$$(1.5.2)\qquad \mathrm{Nr}_\kappa(wa) = w\,\mathrm{Nr}_\kappa(a) + \kappa(w)\mathrm{Nr}_\kappa(a),$$

and

$$(1.5.3)\qquad \mathrm{Nr}_\kappa(aw) = \mathrm{Nr}_\kappa(a)\,w + \mathrm{Nr}_\kappa(a){}^t\kappa(w).$$

The action of $\kappa(w)$ (resp. ${}^t\kappa(w)$) is understood on $A(\widehat{W})|\mathrm{vac}\rangle$ (resp.

$\langle \mathrm{vac}^* | A(\widehat{W}))$.

In particular we have

$$(1.5.4) \qquad w_1 \cdots w_k = \sum_{m=0}^{[k/2]} \sum_{\substack{\mu_1<\mu_2, \cdots, \mu_{2m-1}<\mu_{2m}, \\ \mu_1<\mu_3 \cdots <\mu_{2m-1} \\ \{\mu_1, \cdots, \mu_{2m}\} \cup \{\nu_1, \cdots, \nu_{k-2m}\} = \{1, \cdots, k\}}} \operatorname{sgn}\begin{pmatrix} 1 \cdots\cdots\cdots\cdots k \\ \mu_1 \cdots \mu_{2m} \nu_1 \cdots \nu_{k-2m} \end{pmatrix}$$

$$\times \langle w_{\mu_1} w_{\mu_2} \rangle_K \cdots \langle w_{\mu_{2m-1}} w_{\mu_{2m}} \rangle_K : w_{\nu_1} \cdots w_{\nu_{k-2m}} :_K ,$$

for $w_1, \cdots, w_k \in W$.

If we set $\mathrm{Nr}_K(g) = w_1 \cdots w_k \exp\left\{\frac{1}{2}(v_1 \cdots v_N) R \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}\right\}$, $w = \sum_{\mu=1}^{N} v_\mu c_\mu$, *where* ${}^tR = -R$ *and* $\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix}$, *we have*

$$(1.5.5) \qquad \mathrm{Nr}_K(wg) = \Big(\sum_{j=1}^{k} (-)^{j-1} w_1 \cdots w_{j-1} \langle w w_j \rangle_K w_{j+1} \cdots w_k$$

$$+ w^{(1)} w_1 \cdots w_k\Big) e^{\rho/2},$$

$$(1.5.6) \qquad \mathrm{Nr}_K(gw) = \Big(\sum_{j=1}^{k} (-)^{k-j} w_1 \cdots w_{j-1} \langle w_j w \rangle_K w_{j+1} \cdots w_k$$

$$+ w_1 \cdots w_k w^{(2)}\Big) e^{\rho/2}.$$

Here $w^{(1)} = \sum_{\mu=1}^{N} v_\mu \{(1 - R\,{}^tK)\boldsymbol{c}\}_\mu$, $w^{(2)} = \sum_{\mu=1}^{N} v_\mu \{(1 + RK)\boldsymbol{c}\}_\mu$.

The proof is straightforward.

Theorem 1.5.3. *Take an element* g *of* $G(W)$. *Let* T *denote the matrix representation of* T_g *with respect to the basis* $(v_1, \cdots, v_N)$. *Then we have*

$$(1.5.7) \qquad \langle g \rangle_K^2 = \mathrm{nr}(g) \det(({}^tKT + K) J^{-1}).$$

If $\langle g \rangle_K \neq 0$, *we have*

$$(1.5.8) \qquad \mathrm{Nr}_K(g) = \langle g \rangle_K \, e^{\rho/2}$$

with $\rho = (v_1, \cdots, v_N) R \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$ *where* $R = (T-1)({}^tKT + K)^{-1}$.

Remark. If $\langle g \rangle_K \neq 0$, we have

$$(1.5.9) \qquad \langle g \rangle_K^2 = \mathrm{nr}(g) \det(1 + KR)^{-1},$$

$$(1.5.10)\qquad T=(1-R^{t}K)^{-1}(1+RK).$$

Lemma. $\tilde{\kappa}$ induces a natural inclusion $\tilde{\kappa}: G(W)\hookrightarrow G(\hat{W})$.

Proof. We have $\hat{W}=W_{\kappa}\oplus W_{-{}^t\kappa}$. Since $W_{\kappa}\perp W_{-{}^t\kappa}$, $\tilde{\kappa}(g)w\varepsilon(\tilde{\kappa}(g))^{-1}=w$ for $w\in W_{-{}^t\kappa}$. This implies $\tilde{\kappa}(g)\hat{W}\varepsilon(\tilde{\kappa}(g))^{-1}=\hat{W}$.

Proof of Theorem 1.5.3. If we take a basis $(\hat{v}_1,\cdots,\hat{v}_N,\hat{v}_1^*,\cdots,\hat{v}_N^*)$ where $\hat{v}_\mu=(v_\mu,\kappa(v_\mu))$ and $\hat{v}_\mu^*=(v_\mu,-{}^t\kappa(v_\mu))$ $(\mu=1,\cdots,N)$, the matrix representation of $T_{\tilde{\kappa}(g)}\in O(\hat{W})$ reads $\begin{pmatrix}T & \\ & 1\end{pmatrix}$. The dual basis $(v_1,\cdots,v_N,\eta_1,\cdots,\eta_N)$ is given by $(\hat{v}_1,\cdots,\hat{v}_N,\hat{v}_1^*,\cdots,\hat{v}_N^*)=(v_1,\cdots,v_N,\eta_1,\cdots,\eta_N)\begin{pmatrix}1 & 1\\ {}^tK & -K\end{pmatrix}$. Thus the matrix representation of $T_{\tilde{\kappa}(g)}$ with respect to the basis $(v_1,\cdots,v_N,\eta_1,\cdots,\eta_N)$ reads $\begin{pmatrix}1 & 1\\ {}^tK & -K\end{pmatrix}\begin{pmatrix}T & \\ & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ {}^tK & -K\end{pmatrix}^{-1}$ $\begin{pmatrix}(T-1)J^{-1}K+1 & (T-1)J^{-1}\\ ({}^tKT+K)J^{-1}K-K & ({}^tKT+K)J^{-1}\end{pmatrix}$. Since $\mathrm{nr}(\tilde{\kappa}(g))=\mathrm{nr}(g)$, $\langle g\rangle_\kappa=\langle \mathrm{vac}|\tilde{\kappa}(g)|\mathrm{vac}\rangle$ and $\mathrm{Nr}_\kappa(g)=\mathrm{Nr}(\tilde{\kappa}(g))|_{\eta=0}$, the theorem follows from (1.2.7) and (1.2.8).

The following proposition is sometimes useful in finding R from T.

Proposition 1.5.4. *Suppose there exist* $Y_\pm$ *that are invertible and satisfy*

$$(1.5.11)\qquad J^{-1t}K\cdot Y_+\cdot J^{-1}K=0,\ J^{-1}K\cdot Y_-\cdot J^{-1t}K=0,\ Y_-=Y_+T.$$

Then we have

$$(1.5.12)\qquad (K+{}^tKT)^{-1}=Y_-^{-1}(J^{-1}K\cdot Y_-+J^{-1t}K\cdot Y_+)J^{-1}$$

$$R=(Y_+^{-1}-Y_-^{-1})(J^{-1}K\cdot Y_-+J^{-1t}K\cdot Y_+)J^{-1}$$

$$=(Y_+^{-1}\cdot J^{-1}K+Y_-^{-1}\cdot J^{-1t}K)(Y_--Y_+)J^{-1}.$$

Proof. Note that $J^{-1t}K\cdot Y_+\cdot J^{-1}K=0$ (resp. $J^{-1}K\cdot Y_-\cdot J^{-1t}K=0$) is equivalent to $J^{-1t}K\cdot Y_+\cdot J^{-1t}K=J^{-1t}K\cdot Y_+$ (resp. $J^{-1}K\cdot Y_-\cdot J^{-1}K=J^{-1}K\cdot Y_-$). Hence we have

$$(J^{-1}K\cdot Y_-+J^{-1t}K\cdot Y_+)J^{-1}(K+{}^tKT)$$

$$=J^{-1}K\cdot Y_-+J^{-1t}K\cdot Y_+\cdot J^{-1t}K\cdot T$$

$$=J^{-1}K\cdot Y_-+J^{-1}{}^tK\cdot Y_+T$$

$$=J^{-1}(K+{}^tK)Y_-$$

$$=Y_-.$$

(1.5.12) follows from the above equation and (1.5.8).

Now we fix an orthogonal structure in W. Let $(v_1, \cdots, v_N)$ be a basis of W and set $J=(\langle v_\mu, v_\nu\rangle)_{\mu,\nu=1,\cdots,N}$. Let g_0 be an element of $G(W)$ such that trace $g_0\neq 0$. Let T_0 be the matrix representation of T_{g_0} with respect to $(v_1, \cdots, v_N)$. We set

$$H=J\frac{1-T_0}{1+T_0}. \tag{1.5.13}$$

Then H is skew-symmetric, and if we set $K=\frac{1}{2}(J+H)$, we have $K+{}^tK=J$. Hence there exists a unique $\kappa: W\to W^*$ such that $K=(\kappa(v_\mu)(v_\nu))_{\mu,\nu=1,\cdots,N}$.

From (1.5.7), (1.5.8) and (1.5.13) we have

$$\langle g_0\rangle_\kappa^2=\mathrm{nr}(g_0)\det\frac{1+T_0^2}{1+T_0}, \tag{1.5.14}$$

and $\mathrm{Nr}_\kappa(g_0)=\langle g_0\rangle_\kappa e^{\rho_0/2}$ with $\rho_0=(v_1, \cdots, v_N)R_0\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$ where $R_0=-\frac{1-T_0^2}{1+T_0^2}\times J^{-1}$.

Proposition 1.5.5. *Under the above assumptions, we have*

$$\langle a\rangle_\kappa=\frac{\mathrm{trace}\ g_0a}{\mathrm{trace}\ g_0}. \tag{1.5.15}$$

Remark 1. (1.5.13) is rewritten as

$$T_0=K^{-1}{}^tK. \tag{1.5.16}$$

Remark 2. In Proposition 1.5.5 we have restricted g_0 in $G(W)$. If we admit g_0 to be an element of $\overline{G}(W)$ such that trace $g_0\neq 0$, (1.5.13) gives a one to one correspondence between κ satisfying $\kappa(w)(w')+\kappa(w')(w)=\langle w, w'\rangle$ and g_0 modulo a constant factor. In fact, if we define κ by $\kappa(w)(w')=\mathrm{trace}(g_0ww')$ for given g_0, we have

$\kappa(w)(w')+\kappa(w')(w)=\operatorname{trace}(g_0ww')+\operatorname{trace}(g_0w'w)=\operatorname{trace}(g_0(ww'+w'w))=\langle w, w'\rangle$. Since (1.5.13) is an algebraic relation, it is valid even if $g_0\in\overline{G}(W)$. Conversely, taking the closure of such elements g_0 as in (1.5.12) the existence of $g_0\neq 0$, which satisfies $\langle a\rangle_\kappa \operatorname{trace} g_0=\operatorname{trace} g_0a$ for given κ, is obvious. Since $\operatorname{trace} g_0a\not\equiv 0$, we have $\operatorname{trace} g_0\neq 0$.

We shall denote by g_κ the unique element in $\overline{G}(W)$ such that $\operatorname{trace} g_\kappa=1$ and $\langle a\rangle_\kappa=\operatorname{trace} g_\kappa a$.

Remark 3. If a holonomic decomposition $W=V^\dagger\oplus V$ is given, $g_\kappa=|\mathrm{vac}\rangle\langle\mathrm{vac}|$:

(1.5.17) $$\langle\mathrm{vac}|a|\mathrm{vac}\rangle=\operatorname{trace}(|\mathrm{vac}\rangle\langle\mathrm{vac}|a).$$

Lemma. Let g be an element of $G(W)$. Then we have

(1.5.18) $$(\operatorname{trace} g)^2=\operatorname{nr}(g)\det(1+T_g^{-1}).$$

This follows from Theorem 1.3.3.

Proof of Proposition 1.5.5. It is sufficient to prove (1.5.15) when $g\in G^+(W)$ and $\operatorname{nr}(g)=1$. We also assume that $\operatorname{nr}(g_0)=1$. Let T be the matrix representation of T_g. The matrix representation of T_{g_0} is $K^{-1}\,{}^tK$. Hence we have

$$\frac{(\operatorname{trace} g_0g)^2}{(\operatorname{trace} g_0)^2}=\frac{\det(1+K^{-1}\,{}^tKT)}{\det(1+K^{-1}\,{}^tK)}=\frac{\det({}^tKT+K)}{\det({}^tK+K)}=\det({}^tKT+K)J^{-1}.$$

Now the proposition follows from (1.5.7).

Proposition 1.5.6. *If H is invertible, we have*

(1.5.19) $$\operatorname{Nr}_\kappa(\varepsilon_W g_\kappa)=\langle\varepsilon_W g_\kappa\rangle_\kappa \exp\left(-\frac{1}{2}(v_1,\cdots,v_N)H^{-1}\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}\right).$$

Proof. Without loss of generality we may assume that $g_\kappa\in G(W)$. Then the matrix representation of $\varepsilon_W g_\kappa$ is $-K^{-1}\,{}^tK$, and (1.5.19) follows from (1.5.8).

Now we shall give a transformation law to compute the κ'-norm of an element in $\overline{G}(W)$ from the κ-norm of it. Let K and K' denote the matrices $(\langle v_\mu v_\nu\rangle_\kappa)_{\mu,\nu=1,\cdots,N}$ and $(\langle v_\mu v_\nu\rangle_{\kappa'})_{\mu,\nu=1,\cdots,N}$, respectively.

Theorem 1.5.7. *Let* $g\in\overline{G}(W)$ *be given by* $\mathrm{Nr}_\kappa(g)=\langle g\rangle_\kappa e^{\rho/2}$, $\rho=(v_1,\cdots,v_N)R\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}$ *with a skew-symmetric matrix* R. *Then we have*

$$(1.5.20)\qquad \langle g\rangle_{\kappa'}=\langle g\rangle_\kappa(\det(1-(K'-K)R))^{1/2}$$

$$=\langle g\rangle_\kappa \,\mathrm{Pfaffian}\begin{pmatrix}-(K'-K) & 1\\ -1 & R\end{pmatrix}\Big/\mathrm{Pfaffian}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}.$$

If $\langle g\rangle_{\kappa'}\neq 0$, *we have*

$$(1.5.21)\qquad \mathrm{Nr}_{\kappa'}(g)=\langle g\rangle_{\kappa'}e^{\rho'/2}$$

where $\rho'=(v_1,\cdots,v_N)R'\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}$ *with* $R'=R(1-(K'-K)R)^{-1}$.

Proof. We note that $K+{}^tK=K'+{}^tK'$, hence we have

$$\begin{aligned}R(1-&(K'-K)R)^{-1}\\ &=(T-1)({}^tKT+K)^{-1}(1-(K'-K)(T-1)({}^tKT+K)^{-1})^{-1}\\ &=(T-1)({}^tKT+K-(K'-K)(T-1))^{-1}\\ &=(T-1)({}^tK'T+K')^{-1}.\end{aligned}$$

From (1.5.9) it follows that $\langle g\rangle_\kappa^2=\mathrm{nr}(g)\det(1+KR)^{-1}$ and $\langle g\rangle_{\kappa'}^2=\mathrm{nr}(g)\det(1+K'R')^{-1}$. Hence we have

$$\begin{aligned}\frac{\langle g\rangle_{\kappa'}^2}{\langle g\rangle_\kappa^2}&=\det(1+K'R')^{-1}(1+KR)\\ &=\det(1+K'R(1-(K'-K)R)^{-1})^{-1}(1+KR)\\ &=\det(1-(K'-K)R).\end{aligned}$$

Theorem 1.5.8. *Notations are as in Proposition* 1.3.5. *We set* $\mathrm{Nr}_\kappa(\overline{g})=\langle g\rangle_\kappa w_1\cdots w_k e^{\rho/2}$. *If we write*

$$\mathrm{Nr}_{K'}(\bar{g}) = \sum_{m=0}^{N} \frac{1}{m!} \sum_{\mu_1,\cdots,\mu_m=1}^{N} \rho_m(\mu_1, \cdots, \mu_m) v_{\mu_m} \cdots v_{\mu_1},$$

the coefficient $\rho_m(\mu_1, \cdots, \mu_m)$ *is given by*

(1.5.22) $\rho_m(\mu_1, \cdots, \mu_m) = (-)^{(m+k)/2} \langle g \rangle_K$

$$\times \mathrm{Pfaffian}\left(\begin{array}{cc|cc} & & {}^t e & \\ & & & {}^t r \\ \hline -e & & -(K'-K) & 1 \\ & -r & -1 & R \end{array}\right) \Big/ \mathrm{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Proof. Let ι denote the canonical isomorphism $\iota: W \to W^*$; $\iota(w)(w') = \langle w, w' \rangle$. We introduce an orthogonal structure into W^* by $\langle \iota(w), \iota(w') \rangle = \langle w, w' \rangle$. We denote by $\widehat{W}_\iota$ the orthogonal space $W \oplus W^*$ equipped with the inner product $\langle w \oplus \eta, w' \oplus \eta' \rangle = \langle w, w' \rangle + \langle \eta, \eta' \rangle$. Take a basis $(\xi_1, \cdots, \xi_N)$ of W^* so that $\xi_\mu(v_\nu) = \delta_{\mu\nu}$. Let $\hat{K}$ denote the linear homomorphism $\hat{K}: \widehat{W} \to \widehat{W}^*$ such that $\langle v_\mu v_\nu \rangle_{\hat{K}} = \langle v_\mu v_\nu \rangle_K$, $\langle v_\mu \xi_\nu \rangle_{\hat{K}} = 0$, $\langle \xi_\mu v_\nu \rangle_{\hat{K}} = 0$ and $\langle \xi_\mu \xi_\nu \rangle_{\hat{K}} = \begin{cases} 0 & \mu > \nu \\ \frac{1}{2}\langle \xi_\mu, \xi_\mu \rangle & \mu = \nu \\ \langle \xi_\mu, \xi_\nu \rangle & \mu < \nu \end{cases}$. Likewise we define $\hat{K}': \widehat{W} \to \widehat{W}^*$

Let $\hat{g}$ denote the element of $\bar{G}(\widehat{W})$ given by $\mathrm{Nr}_{\hat{K}}(\hat{g}) = g(R)$ (see (1.3.5)). Applying Theorem 1.5.7, we have

$$\langle \hat{g} \rangle_{\hat{K}'} = \mathrm{Pfaffian}\begin{pmatrix} -(K'-K) & 1 \\ -1 & R \end{pmatrix} \Big/ \mathrm{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

and if $\langle \hat{g} \rangle_{\hat{K}'} \neq 0$,

$$\mathrm{Nr}_{\hat{K}'}(\hat{g}) = \langle \hat{g} \rangle_{\hat{K}'} e^{\hat{\rho}'/2}$$

where $\hat{\rho}' = (v_1, \cdots, v_N, \xi_1, \cdots, \xi_N) \hat{R}' \begin{pmatrix} v_1 \\ \vdots \\ v_N \\ \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}$ with $\hat{R}' = \begin{pmatrix} R & -1 \\ 1 & \end{pmatrix} \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} K'-K & \\ & \end{pmatrix} \begin{pmatrix} R & -1 \\ 1 & \end{pmatrix} \right\}^{-1} = \begin{pmatrix} -(K'-K) & 1 \\ -1 & R \end{pmatrix}^{-1}$. Then using Proposition 1.3.7 and Proposition 1.3.8, we have

$$\rho_m(\mu_1, \cdots, \mu_m) = \langle g\rangle_\kappa \, \mathrm{Pfaffian}\begin{pmatrix} -(K'-K) & 1 \\ -1 & R \end{pmatrix} \Big/ \mathrm{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$\times \mathrm{Pfaffian}\left\{ -\begin{pmatrix} {}^t e & \\ & {}^t r \end{pmatrix} \begin{pmatrix} -(K'-K) & 1 \\ -1 & R \end{pmatrix}^{-1} \begin{pmatrix} e & \\ & r \end{pmatrix} \right\}$$

$$= (-)^{(m+k)/2} \langle g\rangle_\kappa \, \mathrm{Pfaffian}\left(\begin{array}{cc|cc} & & {}^t e & \\ & & & {}^t r \\ \hline -e & & -(K'-K) & 1 \\ & -r & -1 & R \end{array}\right) \Big/ \mathrm{Pfaffian}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Remark 1. Theorem 1.4.3 and Theorem 1.4.4 are special cases of Theorem 1.5.7 and Theorem 1.5.8; namely we take

$$K = \begin{pmatrix} K_1 & & & \\ \lambda_{21}(K_1 + {}^t K_1) & K_1 & & \\ \vdots & \ddots & \ddots & \\ \lambda_{n1}(K_1 + {}^t K_1) & \cdots & \lambda_{n\,n-1}(K_1 + {}^t K_1) & K_1 \end{pmatrix}$$

and
$$K' = \begin{pmatrix} K_1 & \lambda_{12}K_1 & \cdots & \lambda_{1n}K_1 \\ \lambda_{21}K_1 & K_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_{n-1\,n}K_1 \\ \lambda_{n1}K_1 & \cdots & \lambda_{n\,n-1}K_1 & K_1 \end{pmatrix}.$$

Remark 2. The analogues of (1.4.9) ~ (1.4.11) are valid in Theorem 1.5.8.

Propositions 1.2.6 and 1.2.7 are paraphrased in terms of κ-norms as follows. We omit proofs, which are only refrains.

We denote by $\underline{\kappa}$ the linear transformation $\underline{\kappa}: W \to W$ defined by

$$\langle \underline{\kappa}(w), w'\rangle \underset{\text{def}}{=} \kappa(w)(w') = \langle ww'\rangle_\kappa .$$

Note that $\underline{\kappa} + {}^t\underline{\kappa} = 1$. We have

$$(1.5.23) \qquad :\kappa(w)\mathrm{Nr}_\kappa(a):_\kappa = \underline{\kappa}(w)a - \varepsilon(a)\underline{\kappa}(w),$$

$$(1.5.24) \qquad :\mathrm{Nr}_\kappa(a){}^t\kappa(w):_\kappa = a{}^t\underline{\kappa}(w) - \varepsilon(a){}^t\underline{\kappa}(w).$$

Thus (1.5.2) and (1.5.3) read

$$(1.5.25) \qquad :w\,\mathrm{Nr}_\kappa(a):_\kappa = {}^t\underline{\kappa}(w)a + \varepsilon(a)\underline{\kappa}(w),$$

$$(1.5.26) \qquad :\mathrm{Nr}_\kappa(a)w:_\kappa = a\underline{\kappa}(w) + {}^t\underline{\kappa}(w)(a).$$

In particular, if $g'\in G(W)$ and $\mathrm{Nr}_{\underline{\kappa}}(g)=w\,\mathrm{Nr}_{\underline{\kappa}}(g')$, we have

$$(1.5.27)\qquad g=({}^t\underline{\kappa}+T_{g'}\underline{\kappa})(w)g'.$$

If g also belongs to $G(W)$, we have

$$(1.5.28)\qquad ({}^t\underline{\kappa}+T_g\underline{\kappa})(w)=0.$$

The matrix representation of $\underline{\kappa}$ is $J^{-1}\,{}^tK$. If we choose a basis so that $J=1$, the matrix representation of ${}^t\underline{\kappa}+T_g\underline{\kappa}$ is $K+T{}^tK$. Hence, if $\det(K+T{}^tK)=\det({}^tKT+K)\neq 0$, we can apply Theorem 1.5.3 to compute the $\underline{\kappa}$-norm of g.

Now assume that $\mathrm{Ker}({}^t\underline{\kappa}+T_g\underline{\kappa})\neq 0$, and set $g'=wg$, where w is a generic element of W. Then the following conditions i) and ii) for $w_1\in W$ are equivalent;

i) $({}^t\underline{\kappa}+T_{g'}\underline{\kappa})(w_1)=0$,

ii) $\begin{cases}({}^t\underline{\kappa}+T_g\underline{\kappa})(w_1)=0,\\ \langle w,{}^t\underline{\kappa}(w_1)\rangle=0.\end{cases}$

Moreover we have $\mathrm{Nr}_{\underline{\kappa}}(g)=w_1\mathrm{Nr}_{\underline{\kappa}}(g')$, where w_1 is any element of W satisfying $({}^t\underline{\kappa}+T_g\underline{\kappa})(w_1)=0$ and $\langle w,{}^t\underline{\kappa}w_1\rangle=1$.

Take a basis $(v_1,\cdots,v_n)$ of W, and assume that $\mathrm{Nr}_{\underline{\kappa}}(g)=w_1\cdots w_k e^{\rho/2}$, with $w_j=\sum_{\mu=1}^{N}v_\mu c_{j\mu}$ $(j=1,\cdots,k)$ and $\rho=(v_1,\cdots,v_N)R\begin{pmatrix}v_1\\ \vdots\\ v_N\end{pmatrix}$ where R is skew-symmetric. We set $\mathrm{Nr}_{\underline{\kappa}}(g_1)=e^{\rho/2}$. Then we have

$$(1.5.29)\qquad \mathrm{nr}(g)=\det(\langle({}^t\underline{\kappa}+T_{g_1}\underline{\kappa})(w_\mu),\ T_{g_1}(w_\nu)\rangle_{\mu,\nu=1,\cdots,k}\,\mathrm{nr}(g_1),$$

$$=\det\begin{pmatrix}{}^tc_1\\ \vdots\\ {}^tc_k\end{pmatrix}(1-{}^tKR)^{-1}\,{}^tK(c_1\cdots c_k)\det(1+KR).$$

If $\mathrm{nr}(g_1)\neq 0$, then $g_1\in G(W)$ and gg_1^{-1} is a polynomial of $({}^t\underline{\kappa}+T_{g_1}\underline{\kappa})(w_j)$ $(j=1,\cdots,k)$. If $\mathrm{nr}(g)\neq 0$, $g\in G(W)$ and we have

$$(1.5.30)\qquad \mathrm{Ker}({}^t\underline{\kappa}+T_g\underline{\kappa})=\sum_{j=1}^{k}\boldsymbol{C}w_j.$$

If we denote by T the matrix representation of g, we have

$$(1.5.31)\qquad T=(1-R{}^tK)^{-1}(1+RK)$$

$$-(1-R{}^tK)^{-1}(c_1\cdots c_k)\left\{\begin{pmatrix}{}^tc_1\\ \vdots\\ {}^tc_k\end{pmatrix}(1-{}^tKR)^{-1}\,{}^tK(c_1\cdots c_k)\right\}^{-1}$$

$$\times\begin{pmatrix}{}^tc_1\\ \vdots\\ {}^tc_k\end{pmatrix}(1-{}^tKR)^{-1}J.$$

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