Publ. RIMS, Kyoto Univ.
14 (1978), 283-307

Microlocal Parametrices for Hyperbolic Mixed Problems in the Case Where Boundary Waves Appear

By

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§ 1. Introduction

In this paper we shall construct microlocal parametrices for hyperbolic mixed problems at non-glancing points in the case where their Lopatinski's determinants have real zeros, and we shall investigate the reflection of singularities. It was proved for hyperbolic mixed problems with constant coefficients in a quarter-space that singularities corresponding to boundary waves generally appear when Lopatinski's determinant has real zeros (see [2], [11], [12]). We shall show that singularities corresponding to boundary waves appear in variable coefficients cases.

Microlocal parametrices for hyperbolic mixed problems were constructed in some cases by using the theory of Fourier integral operators. Chazarain [1] constructed microlocal parametrices for the Dirichlet problem for wave equations at non-glancing points. Microlocal parametrices for the Dirichlet problem for second order operators were constructed at diffractive points by Melrose [6] and Taylor [10]. But it seems very difficult to construct microlocal parametrices at glancing points which are not diffractive. On the other hand there is a problem of constructing microlocal parametrices when Lopatinski's determinant has real zeros. This problem has no difficulty and we can investigate the reflection of singularities corresponding to boundary waves by constructing microlocal parametrices. We can construct a microlocal parametrix as the

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composition of a microlocal parametrix for the Dirichlet problem and a microlocal parametrix for the Cauchy problem for a system of pseudo-differential operators on the boundary when Lopatinski's determinant has real zeros (see § 5). However, we shall construct microlocal parametrices more directly in that case.

Now let us state our problem and assumptions. Let $\boldsymbol{R}^{n+1}$ denote the (n+1)-dimensional Euclidean space and write $x'=(x_1,\cdots, x_n)$, $x''=(x_2,\cdots, x_n)$ for the coordinate $x=(x_0, x_1,\cdots, x_n)$ in $\boldsymbol{R}^{n+1}$ and $\xi'=(\xi_1,\cdots, \xi_n)$, $\xi''=(\xi_2,\cdots, \xi_n)$ for the dual coordinate $\xi=(\xi_0,\cdots, \xi_n)$ in $\boldsymbol{R}^{n+1}$. We shall also denote by $\boldsymbol{R}^{n+1}_+$ the half-space $\{x=(x_0, x')\in\boldsymbol{R}^{n+1};\ x_0>0\}$ and use the symbol $D=i^{-1}(\partial/\partial x_0,\cdots, \partial/\partial x_n)$. Let $P(x, \xi)$ be a polynomial of order m of $n+1$ variables ξ with C^∞ coefficients and $p(x, \xi)$ its principal part. We assume that $p(x, \xi)$ is a strictly hyperbolic polynomial with respect to ξ_1 and $p(x, 1, 0, \cdots, 0)=1$. Thus we can write

$$p(x, \xi)=\Pi_{j=1}^{l}(\xi_0-\lambda_j^+(x, \xi'))\cdot\Pi_{j=1}^{m-l}(\xi_0-\lambda_j^-(x, \xi')),$$

where the $\lambda_j^\pm(x, \xi')$ are continuous in (x, ξ') and

$$\operatorname{Im}\ \lambda_j^\pm(x, \xi')\gtrless 0 \text{ when } \operatorname{Im}\ \xi_1<0,\ \xi''\in\boldsymbol{R}^{n-1}.$$

We consider the mixed initial-boundary value problem for the hyperbolic operator $P(x, D)$ in a quarter-space

$$P(x, D)u(x)=f(x),\quad x\in\boldsymbol{R}^{n+1}_+,\ x_1>0, \tag{1.1}$$

$$D_1^{k-1}u(x)\,|_{x_1=0}=0,\quad x_0>0,\quad 1\leqq k\leqq m, \tag{1.2}$$

$$B_j(x', D)u(x)\,|_{x_0=0}=g_j(x'),\quad x_1>0,\quad 1\leqq j\leqq l. \tag{1.3}$$

Here the $B_j(x', D)$ are boundary operators with C^∞ coefficients.

Now let $(x^{0\prime}, \xi^{0\prime})$ be a fixed point in $T^*\boldsymbol{R}^n\backslash 0$ and put $x^0=(0, x^{0\prime})$. We may assume that the $\lambda_j^+(x, \xi')$ are enumerated in the following way:

$$\operatorname{Im}\ \lambda_j^+(x^0, \xi^{0\prime})=0 \quad \textit{for } 1\leqq j\leqq l_1,$$
$$\operatorname{Im}\ \lambda_j^+(x^0, \xi^{0\prime})>0 \quad \textit{for } l_1+1\leqq j\leqq l.$$

Then we put

$$L(x', \xi')=(b_j(x', \lambda_1^+(0, x', \xi'), \xi'),\cdots, b_j(x', \lambda_{l_1}^+, \xi'),$$
$$\frac{1}{2\pi i}\int_{c_{\xi'}}\frac{b_j(x', \xi)}{p(0, x', \xi)}d\xi_0,\cdots, \frac{1}{2\pi i}\int_{c_{\xi'}}\frac{b_j(x', \xi)\,\xi_0^{l-l_1-1}}{p(0, x', \xi)}d\xi_0)_{j\downarrow 1,\dots,l},$$

where $b_j(x', \xi)$ is the principal part of $B_j(x', \xi)$ and $C_{\xi'}$ is a simple closed curve enclosing only roots $\lambda^+_{l_1+1}(0, x', \xi'), \cdots, \lambda^+_l(0, x', \xi')$ of $p(0, x', \lambda, \xi')=0$, and we define Lopatinski's determinant† for $\{p, b_j\}$ by

$$R(x', \xi') = \det L(x', \xi').$$

Remark. It is easy to see that

$$R(x', \xi') = (-1)^{l_1(l-l_1)} \Pi_{1\leqq j<k\leqq l_1}(\lambda^+_j(0, x', \xi') - \lambda^+_k) / \Pi_{\substack{l_1+1\leqq j\leqq l \\ 1\leqq k\leqq m-l}}(\lambda^+_j - \lambda^-_k)$$
$$\times \det\left(\frac{1}{2\pi i}\oint \frac{b_k(x', \xi)\xi_0^{j-1}}{p_+(0, x', \xi)} d\xi_0\right)_{j,k=1,\dots,l},$$

where $p_+(x, \xi) = \Pi^l_{j=1}(\xi_0 - \lambda^+_j(x, \xi'))$.

We state the assumptions that we impose on $\{p, b_j\}$:

(A. 1) $(x^0, \xi^{0\prime})$ is not a glancing point for p, i.e., $\lambda^+_j(x^0, \xi^{0\prime})$, $1\leqq j \leqq l_1$, are simple real roots of $p(x^0, \lambda, \xi^{0\prime})=0$.

(A. 2) $$R(x', \xi') = (\xi_1 - \xi_1(x', \xi''))^\theta r(x', \xi'),$$

where $\xi_1(x', \xi'')$ and $r(x', \xi')$ are C^∞ functions defined in a conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$ in $T^*\boldsymbol{R}^n\backslash 0$, $\xi_1(x', \xi'')$ is real valued and homogeneous of degree 1 in ξ'', $\xi_1(x^{0\prime}, \xi^{0\prime\prime}) = \xi^0_1$, $r(x^{0\prime}, \xi^{0\prime}) \neq 0$ and θ is a positive integer.

(A. 3) There exist $l\times l$ matrix valued C^∞ functions $U(x', \xi')$ and $V(x', \xi')$ defined in a conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$ in $T^*\boldsymbol{R}^n\backslash 0$ such that

$$U(x', \xi')L(x', \xi')V(x', \xi') = \begin{bmatrix} (\xi_1 - \xi_1(x', \xi''))I_{\theta'} & 0 \\ 0 & L_e(x', \xi') \end{bmatrix},$$

$\det U(x^{0\prime}, \xi^{0\prime}) \neq 0$, $\det L_e(x^{0\prime}, \xi^{0\prime}) \neq 0$, the (i, j)-entry of U is homogeneous of degree $1-\rho_i-m_j$ and the (i, j)-entry of V is homogeneous of degree ρ_j for $1\leqq i\leqq l_1$ and of degree ρ_j+m+l_1-i for $l_1+1\leqq i\leqq l$, where θ' is a positive integer, $I_{\theta'}$ is the identity matrix of order θ', L_e is an $(l-\theta')\times(l-\theta')$ matrix and $\deg B_j = m_j$.

Remark. (i) If the condition (A. 2) with $\theta=1$ is satisfied then the condition (A. 3) also holds. In fact, taking $U(x', \xi')=I$ and

† Lopatinski's determinant defined above is different from original one.

$V(x', \xi') = r(x', \xi')^{-1}\, {}^t\mathrm{cof}\ L(x', \xi')$, we have $ULV = (\xi_1 - \xi_1(x', \xi''))I$.
(ii) Suppose that $\lambda_j^+(x^0, \xi^{0\prime})$, $1 \leqq j \leqq l_2$, are simple roots of $p(x^0, \lambda, \xi^{0\prime}) = 0$. If rank $(B_j(x^0, \lambda_k^+(x^0, \xi^{0\prime}), \xi^{0\prime}))_{\substack{j\downarrow 1,\dots,l \\ k\to 1,\dots,l_2}} = l_2 - \theta$ the condition (A. 3) follows from (A. 2) (see [4]).

Let Γ be a conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$ in $T^*\boldsymbol{R}^n \backslash 0$ and U a neighborhood of $x^{0\prime}$ in $\boldsymbol{R}^n$. Let us define a microlocal parametrix for the problem

(1. 1)′ $\quad P(x, D)u(x) = 0, \quad x \in \boldsymbol{R}_+^{n+1}, \quad x_1 > 0,$

(1. 2)′ $\quad D_1^{j-1}u(x)\,|_{x_1=0} = 0, \quad x_0 > 0, \quad 1 \leqq j \leqq m,$

(1. 3)′ $\quad B_j(x', D)u(x)\,|_{x_0=0} = \delta_{jk}g(x'), \quad x_1 > 0, \quad 1 \leqq j \leqq l,$

where $1 \leqq k \leqq l$.

Definition 1. 1. A right microlocal parametrix (Poisson operator) for the problem (1. 1)′ – (1. 3)′ at $(x^0, \xi^{0\prime})$ is a triple $\{E_k, \Gamma, [0, \varepsilon) \times U\}$ satisfying the conditions

(i) E_k is a continuous linear map: $\mathscr{D}'(U) \to C^\infty([0, \varepsilon)\,;\ \mathscr{D}'(U))$ †,

(ii) $PE_k(g) \in C^\infty([0, \varepsilon) \times U)$,

(iii) $B_jE_k(g)\,|_{x_0=0} - \delta_{jk}g \in C^\infty(U)$, $1 \leqq j \leqq l$, if $WF(g) \subset \Gamma$,

(iv) $E_k(g)\,|_{x_1<c}$ is smooth if $WF(g) \subset \{x_1 \geqq c\}$.

The remainder of this paper is organized as follows. In § 2 we shall formally construct a microlocal parametrix. In § 3 the procedure of § 2 will be justified and singularities of a microlocal parametrix will be studied. In § 4 we shall construct microlocal parametrices for the problem (1. 1) – (1. 3), following Melrose [6], and study reflection of singularities of solutions to the problem (1. 1) – (1. 3). A microlocal parametrix will be constructed as the composition of a microlocal parametrix for the Dirichlet problem and a microlocal parametrix for a system of pseudo differential operators on the boundary in § 5.

The author wishes to thank Professor M. Matsumura for his helpful discussions and his encouragement.

† It is easy to see that for any $f \in C^\infty([0, \varepsilon)\,;\ \mathscr{D}'(U))$ there is $F \in C^\infty((-\varepsilon, \varepsilon)\,;\ \mathscr{D}'(U))$ such that $f = F|_{x_0 \geqq 0}$.

§ 2. Formal Construction

In this section we first determine the boundary values of the phase functions and solve the eiconal equations to determine the phase functions. Then we discuss the transport equation on the boundary and the transport equations.

There exist a conic neighborhood Γ_0 of $(x^{0\prime}, \xi^{0\prime})$ in $T^*\boldsymbol{R}^n\backslash 0$ and $\tilde{\phi}(x', \xi')\in C^\infty(\Gamma_0)$ such that $\tilde{\phi}(x', \xi')$ satisfies the equations

$$(2.1)\qquad \begin{aligned}&\partial_1\tilde{\phi}(x', \xi')-\xi_1(x', \nabla_{x''}\tilde{\phi}(x', \xi'))=\xi_1-\xi_1(x^{0\prime}, \xi''),\\ &\tilde{\phi}(x_1^0, x'', \xi')=x''\cdot\xi'',\end{aligned}$$

where $\partial_j=\partial_{x_j}=\partial/\partial x_j$ and $\nabla_{x''}f=(\partial_2 f, \cdots, \partial_n f)$. Moreover $\tilde{\phi}(x', \xi')$ is homogeneous of degree 1 in ξ'. Let $\chi(x', y', \xi')$ be a C^∞ function in $\boldsymbol{R}^{3n}$ such that $\chi=1$ in $\dot{\Gamma}_2\cap\{|\xi'|\geqq 1\}$ and $\operatorname{supp}\chi\subset\dot{\Gamma}_1$, where $\Gamma_1(\subset\subset^\dagger\Gamma_0)$ and Γ_2 are conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$ in $T^*\boldsymbol{R}^n\backslash 0$ and

$$\dot{\Gamma}=\{(x', y', \xi');\ (x', \xi')\in\Gamma \text{ and } (y', \xi')\in\Gamma\}.$$

Since $(\partial^2/\partial x_j\partial\xi_k\tilde{\phi}(x^{0\prime}, \xi^{0\prime}))=I$, it follows that the operator A:

$$\begin{aligned}\mathscr{D}'(\boldsymbol{R}^n)\ni g(x')\to(Ag)(x')=\int&\exp[i(\tilde{\phi}(x', \xi')-\tilde{\phi}(y', \xi'))]\\ &\times\chi(x', y', \xi')g(y')dy'\bar{d}\xi'\in\mathscr{D}'(\boldsymbol{R}^n)\end{aligned}$$

is a properly supported pseudo-dieffrential operator, if necessary, shrinking Γ_1, where $\bar{d}\xi'=(2\pi)^{-n}d\xi'$. A is elliptic in a conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$. Thus there is a microlocal parametrix (pseudo-differential operator) B of A at $(x^{0\prime}, \xi^{0\prime})$, i.e., there exists a conic neighborhood Γ of $(x^{0\prime}, \xi^{0\prime})$ such that $ABg-g\in C^\infty(\boldsymbol{R}^n)$ if $WF(g)\subset\Gamma$.

Let us formally construct a microlocal parametrix for the problem (1.1)′−(1.3)′ with $k=1$ at $(x^0, \xi^{0\prime})$ in the form

$$\begin{aligned}(2.2)\quad E_1(g)=\textstyle\sum_{j=1}^{l_1}\displaystyle\int&\exp[i\phi_j(x, y', \xi')]a_j(x, y', \xi')(Bg)(y')dy'\bar{d}\xi'\\ &+\int\exp[i(\tilde{\phi}(x', \xi')-\tilde{\phi}(y', \xi'))]a(x, y', \xi')(Bg)(y')dy'\bar{d}\xi',\end{aligned}$$

† $\Gamma_1\subset\subset\Gamma_0$ means that the closure of Γ_1 is included in the interior of Γ_0.

$$a(x, y', \xi') = \sum_{j=1}^{l-l_1} \frac{1}{2\pi i}\int_{C_{\xi'}} c_j(x, y', \xi)\xi_0^{j-1}\exp[ix_0\xi_0]d\xi_0.$$

Then we have

$$\begin{aligned}(2.3)\qquad PE_1(g) = \sum_{j=1}^{l_1}\int&\exp[i\phi_j(x, y', \xi')]\{p(x, \nabla_x\phi_j)\\ &+\sum_{|\alpha|=1}p^{(\alpha)}(x, \nabla_x\phi_j)D^\alpha+s(\phi_j ; x)+q(\phi_j ; x, D)\}a_j(x, y', \xi')\\ &\times(Bg)(y')dy'd\xi'+\int\exp[i\tilde{\phi}(x', \xi')-\tilde{\phi}(y', \xi'))]\\ &\times[\sum_{j=1}^{l-l_1}\frac{1}{2\pi i}\int_{C_{\xi'}}\{p(x, \nabla_x\psi_0)+\sum_{|\alpha|=1}p^{(\alpha)}(x, \nabla_x\psi_0)D^\alpha+s(\psi_0 ; x)\\ &+q(\psi_0 ; x, D)\}c_j(x, y', \xi)\xi_0^{j-1}\exp[ix_0\xi_0]d\xi_0]dy'd\xi',\end{aligned}$$

where $p^1(x, \xi)=P(x, \xi)-p(x, \xi)$, $p^{(\alpha)}(x, \zeta)=\partial_\zeta^\alpha p(x, \zeta)$, $\psi_0(x, \xi) =\tilde{\phi}(x', \xi')+x_0\xi_0$,

$$s(\phi ; x)=p^1(x, \nabla_x\phi)+\sum_{|\alpha|=2}\frac{1}{\alpha!}p^{(\alpha)}(x, \nabla_x\phi)(iD^\alpha\phi),$$

$$\begin{aligned}q(\phi ; x, D)f(x)=\{&\sum_{|\alpha|=1}p^{1(\alpha)}(x, \nabla_x\phi)D^\alpha+\sum_{|\alpha|=2}\frac{1}{\alpha!}p^{1(\alpha)}(x, \nabla_x\phi)\\ &\times(iD^\alpha\phi)+\sum_{|\alpha|=2}\frac{1}{\alpha!}P^{(\alpha)}(x, \nabla_x\phi)D^\alpha\}f(x)\\ &+\sum_{|\alpha|\geqq 3}\frac{1}{\alpha!}P^{(\alpha)}(x, \nabla_x\phi)\cdot D_z\{f(z)\exp[ih(\phi ; x, z)]\}_{z=x},\end{aligned}$$

$$h(\phi ; x, z)=\phi(z)-\phi(x)-(z-x)\cdot\nabla_x\phi(x).$$

Thus $\phi_j(x, y', \xi')$, $1\leqq j\leqq l_1$, are determined by the eiconal equations

$$(2.4)\qquad \partial_0\psi_j(x, \xi')=\lambda_j^+(x, \nabla_{x'}\psi_j),\ \psi_j(0, x', \xi')=\tilde{\phi}(x', \xi'),$$

where $\phi_j(x, y', \xi')=\psi_j(x, \xi')-\psi_j(0, y', \xi')$. We easily see that $\phi_j(x, y', \xi')\in C^\infty([0, \varepsilon)\times\dot{\Gamma}_0)$ for some $\varepsilon>0$, if necessary, shrinking Γ_0. If $a_j(x, y', \xi')$, $1\leqq j\leqq l_1$, can be written as asymptotic sums

$$(2.5)\qquad a_j(x, y', \xi')\sim\sum_{\nu=0}^\infty a_j^\nu(x, y', \xi')$$

in a certain sense, we obtain the transport equations

$$\begin{aligned}(2.6)\qquad &\{\sum_{|\alpha|=1}p^{(\alpha)}(x, \nabla_x\phi_j)D^\alpha+s(\phi_j ; x)\}a_j^\nu(x, y', \xi')\\ &\quad+q(\phi_j ; x, D)a_j^{\nu-1}(x, y', \xi')=0,\\ &a_j^{-1}(x, y', \xi')\equiv 0,\quad 1\leqq j\leqq l_1,\quad \nu=0, 1, 2, \dots.\end{aligned}$$

(2.6) is an ordinary differential equation for a_j^ν along rays corresponding to (2.4). Thus we can solve the transport equations (2.6) when the boundary values of a_j^ν are given. Put

$$(2.7)\qquad a_j^\nu(0, x', y', \xi') = \tilde{a}_j^\nu(x', y', \xi') \in C^\infty(\dot{\Gamma}_0),\ 1 \leqq j \leqq l_1,\ \nu = 0, 1, \cdots.$$

Then we can assume without loss of generality that the solutions $a_j^\nu(x, y', \xi')$ belong to $C^\infty([0, \varepsilon) \times \dot{\Gamma}_0)$. Moreover from (2.6) we have

$$(2.8)\qquad D_0 a_j^\nu(0, x', y', \xi') = \textstyle\sum_{k=1}^n \partial\lambda_j^+/\partial\zeta_k(0, x', \nabla_{x'}\tilde{\phi}) D_k \tilde{a}_j^\nu(x', y', \xi') + s_j(x', \xi')\tilde{a}_j^\nu(x', y', \xi') + q_j(x', \xi', D) a_j^{\nu-1}(0, x', y', \xi'),$$

where $D_0^k f(0,x') = D_0^k f(x)|_{x_0=0}, \partial\lambda_j^+/\partial\zeta_k(0,x', \nabla_{x'}\tilde{\phi}) = \partial\lambda_j^+/\partial\zeta_k(0, x', \zeta')|_{\zeta=\nabla_{x'}\tilde{\phi}}$, $p^{(j)}(x, \zeta) = \partial p/\partial\zeta_j(x, \zeta)$,

$$s_j(x', \xi') = -p^{(0)}(0, x', \lambda_j^+(0, x', \nabla_{x'}\tilde{\phi}), \nabla_{x'}\tilde{\phi})^{-1} s(\phi_j;\ 0, x'),$$
$$q_j(x', \xi', D) = -p^{(0)}(0, x', \lambda_j^+, \nabla_{x'}\tilde{\phi})^{-1} q(\phi_j;\ 0, x', D).$$

We represent $c_j(x, y', \xi)$ as asymptotic sums

$$(2.9)\qquad c_j(x, y', \xi) \sim \textstyle\sum_{\nu,\mu=0}^\infty c_j^{\nu\mu}(x, y', \xi).$$

From (2.3) we put

$$(2.10)\qquad c_j^{\nu 0}(x, y', \xi) = \tilde{c}_j^\nu(x', y', \xi')\rho(x_0) p(x, \nabla_x\phi_0)^{-1},$$

$$(2.11)\qquad c_j^{\nu\mu+1}(x, y', \xi) = -[\{\textstyle\sum_{|\alpha|=1} p^{(\alpha)}(x, \nabla_x\phi_0) D^\alpha + s(\phi_0;\ x)\} c_j^{\nu\mu}(x, y', \xi) + q(\phi_0;\ x, D) c_j^{\nu-1\mu}(x, y', \xi)] p(x, \nabla_x\phi_0)^{-1},$$
$$c_j^{-1\mu} \equiv 0,\quad 1 \leqq j \leqq l - l_1,\quad \nu, \mu = 0, 1, 2, \cdots,$$

where $\rho(x_0) \in C_{(0)}^\infty([0, \varepsilon))^\dagger$, $\rho(x_0) = 1$ in a neighborhood of $x_0 = 0$ and the $\tilde{c}_j^\nu(x', y', \xi') \in C^\infty(\dot{\Gamma})$ will be determined by the transport equations on the boundary. Then we have $PE_1(g) \sim 0$ in some sense.

Next let us determine the $\tilde{a}_j^\nu(x', y', \xi')$ and the $\tilde{c}_j^\nu(x', y', \xi')$. We obtain formally

$$(2.12)\qquad B_k E_1(g)|_{x_0=0} = \textstyle\sum_{j=1}^{l_1} \int \exp[i(\tilde{\phi}(x', \xi') - \tilde{\phi}(y', \xi'))] \times \{b_k(x', \lambda_j^+(0, x', \nabla_{x'}\tilde{\phi}(x', \xi')), \nabla_{x'}\tilde{\phi}) + \sum_{|\alpha|=1} b_k^{(\alpha)}(x', \lambda_j^+, \nabla_{x'}\tilde{\phi}) D^\alpha + h_k(\phi_j;\ x') + t_k(\phi_j;\ x', D)\} a_j(0, x', y', \xi')(Bg)(y') dy' d\!\!{}^-\xi' + \int \exp[i(\tilde{\phi}(x', \xi') - \tilde{\phi}(y', \xi'))][\textstyle\sum_{j=1}^{l-l_1} \frac{1}{2\pi i} \int_{C_{\xi'}} \{b_k(x', \nabla_x\phi_0)$$

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† $C_{(0)}^\infty([0, \varepsilon)) = \{f(x_0) \in C^\infty([0, \varepsilon));\ \text{supp}\ f \text{ is compact}\}$.

$$+\sum_{|\alpha|=1}b_k^{(\alpha)}(x', \nabla_x\phi_0)D^\alpha+h_k(\phi_0; x')+t_k(\phi_0; x', D)\}$$
$$\times c_j(0, x', y', \xi)\xi_0^{j-1}d\xi_0](Bg)(y')dy'd\xi',$$

where $b_k^1(x', \xi)=B_k(x', \xi)-b_k(x', \xi)$,

$$h_k(\phi; x')=b_k^1(x', \nabla_x\phi(0, x'))+\sum_{|\alpha|=2}\frac{1}{\alpha!}b_k^{(\alpha)}(x', \nabla_x\phi(0, x'))$$
$$\times(iD^\alpha\phi(0, x')),$$
$$t_k(\phi; x', D)f(0, x')=\{\sum_{|\alpha|=1}b_k^{1(\alpha)}(x', \nabla_x\phi(0, x'))D^\alpha$$
$$+\sum_{|\alpha|=2}\frac{1}{\alpha!}b_k^{1(\alpha)}(x', \nabla_x\phi)(iD^\alpha\phi(0, x'))+\sum_{|\alpha|=2}\frac{1}{\alpha!}B_k^{(\alpha)}(x', \nabla_x\phi)D^\alpha\}$$
$$\times f(0, x')+\sum_{|\alpha|\geqq 3}\frac{1}{\alpha!}B_k^{(\alpha)}(x', \nabla_x\phi)D_z^\alpha\{f(z)e^{ih(\phi\ x, z)}\}_{z=x=(0, x')}.$$

Thus we have the transport equations on the boundary

$$\sum_{j=1}^{l_1}b_k(x', \lambda_j^+(0, x', \nabla_{x'}\tilde{\phi}(x', \xi')), \nabla_{x'}\tilde{\phi})\tilde{a}_j^\nu(x', y', \xi')$$
$$+\sum_{j=1}^{l-l_1}\left(\frac{1}{2\pi i}\int_{C_{\xi'}}\frac{b_k(x', \xi_0, \nabla_{x'}\tilde{\phi})\xi_0^{j-1}}{p(0, x', \xi_0, \nabla_{x'}\tilde{\phi})}d\xi_0\right)\tilde{c}_j^\nu(x', y', \xi')$$
$$+\sum_{j=1}^{l_1}\sum_{s=1}^{n}(b_k^{(s)}(x', \lambda_j^+, \nabla_{x'}\tilde{\phi})+b_k^{(0)}(x', \lambda_j^+, \nabla_{x'}\tilde{\phi})\frac{\partial\lambda_j^+}{\partial\zeta_s}(0, x', \nabla_{x'}\tilde{\phi}))$$
$$\times D_s\tilde{a}_j^\nu(x', y', \xi')+\sum_{j=1}^{l-l_1}\sum_{s=1}^{n}\left\{\frac{1}{2\pi i}\int_{C_{\xi'}}[b_k^{(s)}(x', \xi_0, \nabla_{x'}\tilde{\phi})\right.$$
$$\left.\times p(0, x', \xi_0, \nabla_{x'}\tilde{\phi})-b_kp^{(s)}]\xi_0^{j-1}p^{-2}d\xi_0\right\}D_s\tilde{c}_j^\nu(x', y', \xi')$$
$$+\sum_{j=1}^{l_1}S_{kj}(x', \xi')\tilde{a}_j^\nu(x', y', \xi')+\sum_{j=1}^{l-l_1}S_{kj}^0(x', \xi')\tilde{c}_j^\nu(x', y', \xi')$$
$$=\delta_{\nu 0}\delta_{k1}\chi(x', y', \xi')+f_k^\nu(x', y', \xi'),\quad 1\leqq k\leqq l,\ \nu=0, 1, 2, \cdots, \tag{2.13}$$

where

$$S_{kj}(x', \xi')=h_k(\phi_j; x')+b_k^{(0)}(x', \lambda_j^+, \nabla_{x'}\tilde{\phi})s_j(x', \xi'),$$
$$S_{kj}^0(x', \xi')=\sum_{j=1}^{l-l_1}\frac{1}{2\pi i}\int_{C_{\xi'}}\left\{h_k(\phi_0; x')\right.$$
$$-\sum_{|\alpha|=1}b_k^{(\alpha)}(x', \xi_0, \nabla_{x'}\tilde{\phi})\frac{D^\alpha p(0, x', \xi_0, \nabla_{x'}\tilde{\phi})}{p(0, x', \xi_0, \nabla_{x'}\tilde{\phi})}-b_k\cdot h\cdot p^{-1}$$
$$\left.+\sum_{|\alpha|=1}b_k\cdot(D^\alpha p)\cdot p^{(\alpha)}\cdot p^{-2}\right\}\xi_0^{j-1}p^{-1}d\xi_0,$$

$$f_k^\nu(x', y', \xi')\sim-\sum_{j=1}^{l_1}\{b_k^{(0)}(x', \lambda_j^+(0, x', \nabla_{x'}\tilde{\phi}), \nabla_{x'}\tilde{\phi})q_j(x', \xi', D) \tag{2.14}$$
$$+t_k(\phi_0; x', D)\}a_j^{\nu-1}(0, x', y', \xi')$$
$$-\sum_{j=1}^{l-l_1}\frac{1}{2\pi i}\int_{C_{\xi'}}\left[-\frac{b_k(x', \xi_0, \nabla_{x'}\tilde{\phi})}{p(0, x', \xi_0, \nabla_{x'}\tilde{\phi})}q(\phi_0; 0, x', D)c_j^{\nu-10}(0, x', y', \xi)\right.$$
$$+\sum_{\mu=0}^{\infty}t_k(\phi_0; x', D)c_j^{\nu-1\mu}(0, x', y', \xi)+\sum_{\mu=1}^{\infty}\{\sum_{|\alpha|=1}b_k^{(\alpha)}D^\alpha$$

$$+h_k(\psi_0\,;x')\}c_j^{\nu-1\mu}(0,x',y',\xi)+\sum_{\mu=2}^{\infty}b_k\cdot c_j^{\nu-1\mu}(0,x',y',\xi)\Big]\xi_0^{j-1}d\xi_0,$$
$$\nu=1,\ 2,\cdots,$$

(2.15) $$f_k^0(x',\ y',\ \xi')\equiv 0.$$

In fact (2.13) follows from (2.8) and (2.10)–(2.12). Putting

$$S(x',\ \xi')=(S_{k1}(x',\ \xi'),\cdots,\ S_{kl_1},S_{k1}^0(x',\ \xi'),\cdots,\ S_{kl-l_1}^0)_{k\downarrow 1,\ \ ,l},$$

(2.16) $$\boldsymbol{a}^\nu(x',\ y',\ \xi')={}^t(\tilde{a}_1^\nu(x',\ y',\ \xi'),\cdots,\ \tilde{a}_{l_1}^\nu,\ \tilde{c}_1^\nu(x',\ y',\ \xi'),\cdots,\ \tilde{c}_{l-l_1}^\nu),$$

(2.17) $$\boldsymbol{F}^\nu(x',\ y',\ \xi')={}^t(\delta_{\nu 0}\chi(x',\ y',\ \xi')+f_1^\nu,\ f_2^\nu,\cdots,\ f_l^\nu),$$

we can rewrite (2.13) in the form

(2.18) $$L(x',\ \nabla_{x'}\tilde{\phi}(x',\ \xi'))\boldsymbol{a}^\nu(x',\ y',\ \xi')+\sum_{j=1}^n\partial L/\partial\zeta_jD_j\boldsymbol{a}^\nu(x',\ y',\ \xi')$$
$$+S(x',\ \xi')\boldsymbol{a}^\nu(x',\ y',\ \xi')=\boldsymbol{F}^\nu(x',\ y',\ \xi'),\ \nu=0,\ 1,\ 2,\cdots,$$

where $\partial L/\partial\zeta_j=\partial L/\partial\zeta_j(x',\ \zeta')\,|_{\zeta'=\nabla_{x'}\tilde{\phi}}$. Put

(2.19) $$\boldsymbol{a}^\nu(x',\ y',\ \xi')=(V(x',\nabla_{x'}\tilde{\phi})+\sum_{j=1}^n\partial V/\partial\zeta_jD_j)\boldsymbol{v}^\nu(x',\ y',\ \xi').$$

Multiplying (2.18) by $U(x',\nabla_{x'}\tilde{\phi})+\sum_{k=1}^n\partial U/\partial\zeta_kD_k$ and using (A.3) and (2.1) we have

(2.20) $$(\xi_1-\xi_1(x^{0'},\ \xi''))v_k^\nu(x',y',\xi')+\{D_1-\sum_{j=2}^n\partial\xi_1/\partial\zeta_j(x',\nabla_{x''}\tilde{\phi})D_j\}$$
$$\times v_j^\nu(x',\ y',\ \xi')=G_k^\nu(x',\ y',\ \xi'),\quad 1\leqq k\leqq\theta',$$

(2.21) $$L_e(x',\nabla_{x'}\tilde{\phi}(x',\ \xi'))\boldsymbol{v}_e^\nu(x',y',\xi')+Q_e^1(x',\ \xi',\ D')\ V_{\theta'}^\nu(x',y',\xi')$$
$$+Q_e^2(x',\ \xi',\ D')\boldsymbol{v}_e^\nu(x',\ y',\ \xi')=\boldsymbol{G}_e^\nu(x',\ y',\ \xi'),$$

where $\boldsymbol{v}^\nu={}^t(v_1^\nu,\cdots,\ v_l^\nu)$, $\boldsymbol{v}_{\theta'}^\nu={}^t(v_1^\nu,\cdots,\ v_{\theta'}^\nu)$, $\boldsymbol{v}_e^\nu={}^t(v_{\theta'+1}^\nu,\cdots,\ v_l^\nu)$,

(2.22) $$T(x',\ \xi')\,(\equiv(T_{ij}(x',\ \xi')))=\sum_{k=1}^n[\partial U/\partial\zeta_j(x',\ \nabla_{x'}\tilde{\phi}(x',\ \xi'))$$
$$\times\{D_j(L(x',\ \nabla_{x'}\tilde{\phi})V(x',\ \nabla_{x'}\tilde{\phi}))\}+U\partial L/\partial\zeta_j(D_jV)]$$
$$+US(x',\xi')V,$$
$$W(x',\ \xi',\ D')\boldsymbol{f}(x')\,(\equiv(W_{ij}(x',\ \xi',\ D'))\boldsymbol{f}(x'))$$
$$=\sum_{j,k=1}^n\{\partial U/\partial\zeta_jD_j(L\partial V/\partial\zeta_kD_k\boldsymbol{f}(x'))$$
$$+U\partial L/\partial\zeta_jD_j(\partial V/\partial\zeta_kD_k\boldsymbol{f}(x'))\}$$
$$+\sum_{i,j,k=1}^n\partial U/\partial\zeta_iD_i[\partial L/\partial\zeta_jD_j\{(V+\partial V/\partial\zeta_kD_k)\boldsymbol{f}(x')\}]$$
$$+\sum_{j,k=1}^nUS\partial V/\partial\zeta_jD_j\boldsymbol{f}(x')+\sum_{j=1}^n\partial U/\partial\zeta_jD_j\{S(V$$
$$+\sum_{k=1}^n\partial V/\partial\zeta_kD_k)\boldsymbol{f}(x')\},$$

(2.23) $$\boldsymbol{G}^\nu(x',\ y',\ \xi')={}^t(G_1^\nu(x',\ y',\ \xi'),\cdots,\ G_l^\nu(x',\ y',\ \xi'))$$

$$= (U + \sum_{j=1}^{n} \partial U/\partial\zeta_j D_j) \boldsymbol{F}^{\nu}(x', y', \xi'),$$

$$T_e^1 = (T_{jk})_{\substack{j \downarrow \theta'+1,\dots,l \\ k \to 1,\dots,\theta'}}, \quad T_e^2 = (T_{jk})_{j,k=\theta'+1,\dots,l},$$

$$Q_e^1(x', \xi', D') = T_e^1(x', \xi') + W_e^1(x', \xi', D'),$$

$$Q_e^2(x', \xi', D') = \sum_{j=1}^{n} \partial L_e/\partial\zeta_j(x', \nabla_{x'}\tilde{\phi}) D_j + T_e^2 + W_e^2.$$

Representing $\boldsymbol{v}^{\nu}(x', y', \xi')$ as asymptotic sums

$$\boldsymbol{v}^{\nu}(x', y', \xi') \sim \sum_{\mu=0}^{\infty} \boldsymbol{v}^{\nu\mu}(x', y', \xi'), \quad \nu = 0, 1, 2, \cdots, \tag{2.24}$$

we have

$$\begin{aligned}(\xi_1 - \xi_1(x^{0\prime}, \xi'')) v_k^{\nu\mu}(x', y', \xi') &+ (D_1 - \sum_{j=2}^{n} \partial\xi_1/\partial\zeta_j D_j) \\ \times v_k^{\nu\mu}(x', y', \xi') &+ \sum_{j=1}^{\theta'} T_{kj}(x', \xi') v_j^{\nu\mu}(x', y', \xi') \\ = \delta_{\mu 0} G_k^{\nu}(x', y', \xi') &- \sum_{j=\theta'+1}^{l} T_{kj} v_j^{\nu\mu-1}(x', y', \xi') \\ - \sum_{j=1}^{l} W_{kj}(x', \xi', D') v_j^{\nu\mu-1}&(x', y', \xi'), \quad 1 \leqq k \leqq \theta',\end{aligned} \tag{2.25}$$

$$\begin{aligned}L_e(x', \nabla_{x'}\tilde{\phi}(x', \xi')) \, \boldsymbol{v}_e^{\nu\mu}(x', y', \xi') &= \delta_{\mu 0} \boldsymbol{G}_e^{\nu}(x', y', \xi') \\ - Q_e^1(x', \xi', D') \boldsymbol{v}_{\theta'}^{\nu\mu} - Q_e^2(x', \xi', D') \boldsymbol{v}_e^{\nu\mu-1}&, \quad \nu, \mu = 0, 1, 2, \cdots,\end{aligned} \tag{2.26}$$

where $\boldsymbol{v}^{\nu,-1}(x', y', \xi') \equiv 0$. We can solve (2.25) and (2.26) with the initial conditions

$$\boldsymbol{v}^{\nu\mu}(x', y', \xi')|_{x_1 \downarrow -\infty} = 0 \quad (\boldsymbol{v}^{\nu\mu}|_{x_1 < x_1^0 - \varepsilon_0} \equiv 0 \text{ for some } \varepsilon_0), \quad \nu, \mu = 0, 1, 2, \cdots. \tag{2.27}$$

In fact, since (2.25) is a system of ordinary differential equations for $v_k^{\nu\mu}$ along rays corresponding to (2.1), we can determine $v_k^{\nu\mu}(x', y', \xi')$, $1 \leqq k \leqq l$, by (2.25) if $\boldsymbol{G}^{\nu}$ and $\boldsymbol{v}^{\nu\mu-1}$ are determined. Then $\boldsymbol{v}_e^{\nu\mu}(x', y', \xi')$ can be determined by (2.26). Thus $\boldsymbol{v}^{\nu}(x', y', \xi')$ is obtained from (2.24) which is justified in § 3. (2.19) gives $\boldsymbol{a}^{\nu}(x', y', \xi')$ and, therefore, (2.6), (2.7), (2.10) and (2.11) give $a_j^{\nu}(x, y', \xi')$, $1 \leqq j \leqq l_1$, and $c_j^{\nu\mu}(x, y', \xi)$ $(1 \leqq j \leqq l - l_1, \mu = 0, 1, \cdots)$ if $a_j^{\nu-1}(x, y', \xi')$, $1 \leqq j \leqq l_1$, and $c_j^{\nu-1\mu}(x, y', \xi)$ $(1 \leqq j \leqq l - l_1, \mu = 0, 1, \cdots)$ are determined. From (2.14), (2.15), (2.17) and (2.23) we obtain $\boldsymbol{G}^{\nu+1}$. Therefore the above arguments give the $a_j(x, y', \xi')$ and the $c_j(x, y', \xi)$ if (2.5), (2.9) and (2.14) are justified and if the $\boldsymbol{a}^{\nu}(x', y', \xi')$ defined by (2.19) are shown to satisfy (2.18).

§ 3. Main Theorem

A ray corresponding to (2.1) is a solution of the equations

$$(3.1)\qquad dX'/ds(s;x',\xi')=(1,-\nabla_{\zeta''}\xi_1(X'(s;x',\xi'),\nabla_{x''}\tilde{\phi}(X',\xi'))),\quad X'(0;x',\xi')=x'.$$

Then $X'(s;x',\xi')$ is a C^∞ function, homogeneous of degree 0 in ξ' and defined when x' belongs to a small neighborhood of $x^{0'}$ and $(X'(s;x',\xi'),\xi')\in\Gamma_0$. Now we can assume without loss of generality that

$$\Gamma_0=U_0\times\gamma_0,\ U_0=\{x'\in\mathbf{R}^n;\ |x_j-x_j^0|<\varepsilon_0,\ 1\leqq j\leqq n\}$$

and γ_0 is a conic neighborhood of $\xi^{0'}$ in $\mathbf{R}^n\setminus\{0\}$. Thus we can assume that $X'(s;x',\xi')$ is defined and $(X'(s;x',\xi'),\xi')\in\Gamma_0$ when $(x',\xi')\in\Gamma_1$ and $x_1^0-\varepsilon_0-x_1\leqq s\leqq x_1^0+\varepsilon_0-x_1$, if necessary, shrinking Γ_1.

$T_{ij}(x',\xi')$ defined by (2.22) is homogeneous of degree $\rho_j-\rho_i$ and, therefore, $T_{ij}(x',\xi')$ belongs to a symbol class $S^{\rho_j-\rho_i}(\Gamma_0)$, i.e.,

$$|\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}T_{ij}(x',\xi')|\leqq C_{\alpha'\beta'}(1+|\xi'|)^{\rho_j-\rho_i-|\beta'|}\quad\text{when } (x',\xi')\in\Gamma_0 \text{ and } |\xi'|\geqq 1.$$

In order to determine $\boldsymbol{v}_{\theta'}^{\nu\mu}$ we consider the equations

$$(3.2)\qquad \begin{aligned}&(D_1-\textstyle\sum_{j=2}^n\partial\xi_1/\partial\zeta_j(x',\nabla_{x''}\tilde{\phi})D_j)v_k(x',y',\xi')+(\xi_1-\xi_1(x^{0'},\xi''))\\&\times v_k(x',y',\xi')+\textstyle\sum_{j=1}^{\theta'}T_{kj}(x',\xi')v_j(x',y',\xi')\\&=G_{0k}(x',y',\xi')+G_{1k}(x',y',\xi'),\quad 1\leqq k\leqq\theta',\end{aligned}$$

$$(3.3)\qquad v_k(x',y',\xi')|_{x_1\leqq x_1^0-\varepsilon_0}\equiv 0,\quad 1\leqq k\leqq\theta'.$$

Lemma 3.1. *Assume that* $G_{0k}(x',y',\xi')\in S^{1-\rho_k-m_1-\nu}(\dot{\Gamma}_0)$ *and*

$$G_{1k}(x',y',\xi')=\int_{-\infty}^0\exp[i(\xi_1-\xi_1(x^{0'},\xi''))s]g_{1k}(x',y',\xi',s)ds,$$
$$g_{1k}(x',y',\xi',s)\in S^{1-\rho_k-m_1-\nu}(\dot{\Gamma}_0\times(-\infty,0]),$$

i.e., when $(x',y',\xi',s)\in\dot{\Gamma}_0\times(-\infty,0]$ *and* $|\xi'|\geqq 1$

$$|\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}\partial_{\xi'}^{\gamma'}\partial_s^j g_{1k}(x',y',\xi',s)|\leqq C_{\alpha'\beta'\gamma'j}(1+|\xi'|)^{1-\rho_k-m_1-\nu-|\gamma'|},$$

and that supp $G_{0k}(x',y',\xi')\subset\dot{\Gamma}_1$,

$$g_{1k}(x',y',\xi',s)=0 \textit{ if } (y',\xi')\notin\Gamma_1 \textit{ or } s\leqq x_1^0-\varepsilon_0-x_1,$$

where $1\leqq k\leqq\theta'$. *Then the solutions of the equations* (3.2) *and* (3.3) *can be written in the form*

$$v_k(x', y', \xi')=\int_{-\infty}^{0}\exp[i(\xi_1-\xi_1(x^0, \xi''))s]d_k(x', y', \xi', s)ds,$$
$$1\leqq k\leqq\theta',$$

where $d_k(x', y', \xi', s)\in S^{1-\rho_k-m_1-\nu}(\dot{\Gamma}_0\times(-\infty, 0])$ *and*

$$d_k(x', y', \xi', s)=0 \quad \textit{if} \quad (y', \xi')\notin\Gamma_1 \quad \textit{or} \quad s\leqq x_1^0-\varepsilon_0-x_1.$$

Proof. Along the ray defined by (3.1) (3.2) becomes the system of ordinary differential equations

$$\frac{dv_k(X'(s;x',\xi'),y',\xi')}{ds}+i(\xi_1-\xi_1(x^{0'}, \xi''))v_k(X', y', \xi')$$
$$+i\sum_{j=1}^{\theta'}T_{kj}(X', \xi')v_j(X', y', \xi')=iG_{0k}(X', y', \xi')$$
$$+iG_{1k}(X', y', \xi'), \quad 1\leqq k\leqq\theta'.$$

Since $v_k(X'(s; x', \xi'), y', \xi')|_{s\leqq x_1^0-\varepsilon_0-x_1}=0$, we have

$$\begin{aligned}(3.4)\qquad \boldsymbol{v}_{\theta'}(X'(s;x',\xi'),y',\xi')=i\int_{-\infty}^{s}ds'\exp[-i\int_{s'}^{s}\{(\xi_1-\xi_1(x^{0'}, \xi''))I_{\theta'}\\ +T_{\theta'}(X'(q; x', \xi'), \xi')\}dq]\{\boldsymbol{G}_{0\theta'}(X'(s'; x', \xi'), y', \xi')\\ +\boldsymbol{G}_{1\theta'}(X'(s'; x', \xi'), y', \xi')\},\end{aligned}$$

where $\boldsymbol{v}_{\theta'}={}^t(v_1,\cdots, v_{\theta'})$, $\boldsymbol{G}_{j\theta'}={}^t(G_{j1},\cdots, G_{j\theta'})$, $j=0, 1$, and $T_{\theta'}=(T_{ij})_{i,j=1,\dots,\theta'}$. Thus putting $s=0$ in (3.4) we have

$$\boldsymbol{v}_{\theta'}(x', y', \xi')=i\int_{-\infty}^{0}ds\ \exp[i(\xi_1-\xi_1(x^{0'}, \xi''))s]$$
$$\times\{\exp[-i\int_{s}^{0}T_{\theta'}(X'(q; x', \xi'), \xi')dq]\boldsymbol{G}_{0\theta'}(X'(s; x', \xi'), y', \xi')$$
$$+\int_{s}^{0}ds'\exp[-i\int_{s'}^{0}T_{\theta'}(X'(q; x', \xi'), \xi')dq]\boldsymbol{g}_{1\theta'}(X'(s'; x', \xi'), y',$$
$$\xi', s-s')\},$$

where $\boldsymbol{g}_{1\theta'}={}^t(g_{11},\cdots, g_{1\theta'})$. On the other hand it is easily seen that if the (i, j)-entries of $M(x', \xi', s)$ are homogeneous of degree $\rho_j-\rho_i$ in ξ' the (i, j)-entries of $\exp[M(x',\xi',s)]$ are homogeneous of degree $\rho_j-\rho_i$ in ξ'. So the lemma easily follows. **Q. E. D.**

Lemma 3.2. *(i) Let* $v_k^{\nu\mu}(x', y', \xi')$ *be solutions of* (2.25)–(2.27). *Then we have for* $1\leqq k\leqq\theta'$

(3.5) $$v_k^{\nu\mu}(x', y', \xi') = \int_{-\infty}^{0} \exp[i(\xi_1 - \xi_1(x^{0\prime}, \xi''))s] d_k^{\nu\mu}(x', y', \xi', s)\, ds,$$

(3.6) $$d_k^{\nu\mu}(x', y', \xi', s) \in S^{1-\rho_k-m_1-\nu-\mu}(\dot{\Gamma}_0 \times (-\infty, 0]), \quad \nu, \mu = 0, 1, 2, \cdots.$$

For $\theta'+1 \leqq k \leqq l$ *we have*

(3.7) $$v_k^{\nu\mu}(x', y', \xi') = v_{0k}^{\nu\mu}(x', y', \xi') + \int_{-\infty}^{0} \exp[i(\xi_1 - \xi_1(x^{0\prime}, \xi''))s] \times d_k^{\nu\mu}(x', y', \xi', s)\, ds,$$

(3.8) $$v_{0k}^{\nu\mu}(x', y', \xi') \in S^{-\rho_k-m_1-\nu-\mu}(\dot{\Gamma}_0),$$

(3.9) $$d_k^{\nu\mu}(x', y', \xi', s) \in S^{-\rho_k-m_1-\nu-\mu}(\dot{\Gamma}_0 \times (-\infty, 0]), \quad \nu, \mu = 0, 1, 2, \cdots.$$

Moreover we have

(3.10) $$\mathrm{supp}\; v_{0k}^{\nu\mu}(x', y', \xi') \subset \dot{\Gamma}_1,$$

(3.11) $$d_k^{\nu\mu}(x', y', \xi', s) = 0 \quad \textit{if} \quad (y', \xi') \notin \Gamma_1 \quad \textit{or} \quad s \leqq -\varepsilon_1 + x_1^0 - x_1.$$

We can define $d_k^{\nu}(x', y', \xi', s) \in S^{\alpha_k-\rho_k-m_1-\nu}(\dot{\Gamma}_0 \times (-\infty, 0])$ *and* $v_{0k}^{\nu}(x', y', \xi') \in S^{-\rho_k-m_1-\nu}(\dot{\Gamma}_0)$ *by*

(3.12) $$d_k^{\nu}(x', y', \xi', s) \sim \sum_{\mu=0}^{\infty} d_k^{\nu\mu}(x', y', \xi', s),$$

(3.13) $$v_{0k}^{\nu}(x', y', \xi') \sim \sum_{\mu=0}^{\infty} v_{0k}^{\nu\mu}(x', y', \xi'),$$

respectively, and put

$$v_k^{\nu}(x', y', \xi') = v_{0k}^{\nu}(x', y', \xi') + \int_{-\infty}^{0} \exp[i(\xi_1 - \xi_1(x^{0\prime}, \xi''))s] \times d_k^{\nu}(x', y', \xi', s)\, ds,$$

where $v_{0k}^{\nu}(x', y', \xi') \equiv 0$ *for* $1 \leqq k \leqq \theta'$ *and* $\alpha_k = 1$ *if* $1 \leqq k \leqq \theta'$, $=0$ *if* $\theta'+1 \leqq k \leqq l$.

(*ii*) *Let* $\tilde{a}_k^{\nu}(x', y', \xi')$, $\tilde{c}_k^{\nu}(x', y', \xi')$ *be defined by* (2.16) *and* (2.19). *For* $1 \leqq k \leqq l_1$ *we have*

(3.14) $$a_k^{\nu}(x, y', \xi') = a_{0k}^{\nu}(x, y', \xi') + \int_{-\infty}^{0} \exp[i(\xi_1 - \xi_1(x^{0\prime}, \xi''))s] \times a_{1k}^{\nu}(x, y', \xi', s)\, ds,$$

(3.15) $$a_{0k}^{\nu}(x, y', \xi') \in S^{1-m_1-\nu}([0, \varepsilon) \times \dot{\Gamma}_0),$$

(3.16) $$a_{1k}^{\nu}(x, y', \xi', s) \in S^{1-m_1-\nu}([0, \varepsilon) \times \dot{\Gamma}_0 \times (-\infty, 0]),$$

where the $a_k^{\nu}(x, y', \xi')$ *are the solutions of* (2.5) *and* (2.6).

Moreover

(3. 17) $\operatorname{supp} D^{\alpha} a_{0k}^{\nu}(0, x', y', \xi') \subset \dot{\Gamma}_1$ *for any* α,

(3. 18) $a_{1k}^{\nu}(x, y', \xi', s) = 0$ *if* $(y', \xi') \notin \Gamma_1$ *or* $s \leqq x_1^0 - \varepsilon_1 - x_1$.

(*iii*) *For* $1 \leqq k \leqq l - l_1$

(3. 19) $$c_k^{\nu\mu}(x, y', \xi) = c_{0k}^{\nu\mu}(x, y', \xi) + \int_{-\infty}^{0} \exp[i(\xi_1 - \xi_1(x^{0'}, \xi''))s] \times c_{1k}^{\nu\mu}(x, y', \xi, s)\, ds,$$

(3. 20) $$c_{0k}^{\nu\mu}(x, y', \xi) = \sum_{\text{finite sum}} p(x, \xi_0, \nabla_{x'}\tilde{\phi}(x', \xi'))^{-j} c_{0kj}^{\nu\mu}(x, y', \xi),$$

(3. 21) $$c_{0kj}^{\nu\mu}(x, y', \xi) \in S^{1+jm-m_1-k-\nu-\mu}([0, \varepsilon) \times \dot{\Gamma}_0 \times \boldsymbol{C})$$
(*with obvious definition*),

(3. 22) $$c_{1k}^{\nu\mu}(x, y', \xi, s) = \sum_{\text{finite sum}} p(x, \xi_0, \nabla_{x'}\tilde{\phi})^{-j} c_{1kj}^{\nu\mu}(x, y', \xi, s),$$

(3. 23) $$c_{1kj}^{\nu\mu}(x, y', \xi, s) \in S^{1+jm-m_1-k-\nu-\mu}([0, \varepsilon) \times \dot{\Gamma}_0 \times \boldsymbol{C} \times (-\infty, 0]),$$

where the $c_k^{\nu\mu}(x, y', \xi)$ *are defined by* (2. 10) *and* (2. 11). *Moreover*

(3. 24) $\operatorname{supp} D^{\alpha} c_{0k}^{\nu\mu}(0, x', \xi) \subset \Gamma_1 \times \boldsymbol{C}$ *for any* α,

(3. 25) $c_{1k}^{\nu\mu}(x, y', \xi, s) = 0$ *if* $(y', \xi') \in \Gamma_1$ *or* $s \leqq x_1^0 - \varepsilon_0 - x_1$.

Proof. By induction the lemma can be proved. From Lemma 3. 1, (2. 15), (2. 17), (2. 23) and (2. 25) it follows that (3. 5), (3. 6) and (3. 11) are valid when $\nu, \mu = 0$, $1 \leqq k \leqq \theta'$. Then by (2. 26) we have (3. 7) – (3. 11), when $\nu, \mu = 0$, $\theta' + 1 \leqq k \leqq l$. Let us assume that (3. 5) – (3. 11) hold for $\nu = \nu_0$ and $\mu = 0$ and that (3. 14) – (3. 25) hold for $\nu = \nu_0 - 1$ and $\mu = 0, 1, 2, \cdots$. By the induction assumptions, Lemma 3. 1, (2. 25) and (2. 26) we can inductively obtain (3. 5) – (3. 11) for $\nu = \nu_0$ and $\mu = 1, 2, \cdots$. Thus (3. 12) and (3. 13) can be interpreted as the asymptotic sums of symbols. Thus $\tilde{a}_k^{\nu_0}(x', y', \xi')$ and $\tilde{c}_k^{\nu_0}(x', y', \xi')$ can be defined by (2. 16) and (2. 19). Then the transport equations (2. 6) with the boundary conditions (2. 7) give (3. 14) – (3. 18) for $\nu = \nu_0$. From (2. 10) and (2. 11) (3. 19) – (3. 25) also follow. From Lemma 3. 1, (2. 14), (2. 17), (2. 23), (2. 25) and (2. 26) it follows that (3. 5) – (3. 11) are valid when $\nu = \nu_0 + 1$ and $\mu = 0$. Here we have used the fact that $f_k^{\nu_0+1}(x', y', \xi')$ can be determined by (2. 14) as the sums of the asymptotic sums of symbols and the integrals of the asymptotic sums of symbols. So the induction argument is complete.

Q. E. D.

We can define $a_{0k}(x, y', \xi') \in S^{1-m_1}([0, \varepsilon) \times \dot{\Gamma}_0)$ and $a_{1k}(x, y', \xi', s) \in S^{1-m_1}([0, \varepsilon) \times \dot{\Gamma}_0 \times (-\infty, 0])$ by $a_{0k}(x, y', \xi') \sim \sum_{\nu=0}^{\infty} a_{0k}^{\nu}(x, y', \xi')$ and $a_{1k}(x, y', \xi', s) \sim \sum_{\nu=0}^{\infty} a_{1k}^{\nu}(x, y', \xi', s)$, respectively. Put

$$a_k^{\nu\mu}(x, y', \xi') = \frac{1}{2\pi i}\int_{C_{\xi'}} c_k^{\mu\nu}(x, y', \xi')\xi_0^{k-1}\exp[ix_0\xi_0]d\xi_0$$
$$\equiv a_{0k}^{\nu\mu}(x, y', \xi') + \int_{-\infty}^{0}\exp[i(\xi_1 - \xi_1(x^{0\prime}, \xi''))s]a_{1k}^{\nu\mu}(x, y', \xi', s)\ ds.$$

Then, $a_{0k}^{\nu\mu}(x, y', \xi') \in S_{11}^{1-m_1-\nu-\mu}([0, \varepsilon) \times \dot{\Gamma}_0)$, i.e.,

$$|\partial_x^{\alpha}\partial_{y'}^{\beta'}\partial_{\xi'}^{\gamma'} a_{0k}^{\nu\mu}(x, y', \xi')| \leqq C_{\alpha\beta'\gamma'}(1+|\xi'|)^{1-m_1-\nu-\mu+\alpha_0-|\gamma'|}$$
$$\text{when } (x, y', \xi') \in [0, \varepsilon) \times \dot{\Gamma}_0 \text{ and } |\xi'| \geqq 1,$$

and $a_{1k}^{\nu\mu}(x, y', \xi', s) \in S_{11}^{1-m_1-\nu-\mu}([0, \varepsilon) \times \dot{\Gamma}_0 \times (-\infty, 0])$,

$$a_{1k}^{\nu\mu}(x, y', \xi', s) = 0 \text{ if } (y', \xi') \notin \Gamma_1 \text{ or } s \leqq x_1^0 - \varepsilon_0 - x_1.$$

Therefore we can define $a(x, y', \xi')$ by

$$a(x, y', \xi') = e_0(x, y', \xi') + \int_{-\infty}^{0}\exp[i(\xi_1 - \xi_1(x^{0\prime}, \xi''))s]$$
$$\times e_1(x, y', \xi', s)ds,$$
$$e_0(x, y', \xi') \sim \textstyle\sum_{k=1}^{l-l_1}\sum_{\nu,\mu=0}^{\infty} a_{0k}^{\nu\mu}(x, y', \xi'),$$
$$e_1(x, y', \xi', s) \sim \textstyle\sum_{k=1}^{l-l_1}\sum_{\nu,\mu=0}^{\infty} a_{1k}^{\nu\mu}(x, y', \xi', s).$$

Then $e_0(x, y', \xi') \in S_{11}^{1-m_1}([0, \varepsilon) \times \dot{\Gamma}_0)$, $e_1(x, y', \xi', s) \in S_{11}^{1-m_1}([0, \varepsilon) \times \dot{\Gamma}_0 \times (-\infty, 0])$,

$$e_1(x, y', \xi', s) = 0 \text{ if } (y', \xi') \notin \Gamma_1 \text{ or } s \leqq x_1^0 - \varepsilon_0 - x_1.$$

Moreover $e_0(x, y', \xi')$ and $e_1(x, y', \xi', s)$ belong to $S^{-\infty}$ when $x_0 > 0$. So we can define the distribution

$$E_1(g) = \textstyle\sum_{j=1}^{l_1}[\int\exp[i\phi_j(x, y', \xi')]a_{0j}(x, y', \xi')(Bg)(y')dy'd\!\!\!{}^-\xi'$$
$$+\int dy'd\!\!\!{}^-\xi'\int_{-\infty}^{0} ds\ \exp[i\{\phi_j(x, y', \xi') + (\xi_1 - \xi_1(x^{0\prime}, \xi'')s\}]$$
$$\times a_{1j}(x, y', \xi', s)(Bg)(y')] + \int\exp[i(\tilde{\phi}(x', \xi') - \tilde{\phi}(y', \xi'))] \tag{3.26}$$

$$\times e_0(x, y', \xi')(Bg)(y')dy'd\!\!{}^-\xi' + \int dy'd\!\!{}^-\xi' \int_{-\infty}^{0} ds \exp[i\{\tilde{\phi}(x', \xi') - \tilde{\phi}(y', \xi') + (\xi_1 - \xi_1(x^{0\prime}, \xi''))s\}]e_1(x, y', \xi', s)(Bg)(y'),$$
$$g \in \mathscr{D}'(U),$$

as an oscillatory integral, where U is a small neighborhood of $x^{0\prime}$.

Definition 3.3. Let $x'=x'(t; y', \eta')$ and $\zeta'=\zeta'(t; y', \eta')$ be the solutions of a system of the equations

(3.27) $$dx'/dt=(1, -\nabla_{\zeta''}\xi_1(x', \zeta'')),$$

(3.28) $$d\zeta'/dt=\nabla_{x'}\xi_1(x', \zeta''),$$

(3.29) $$x'=y', \ \zeta'=\eta' \text{ and } \eta_1-\xi_1(y', \eta'')=0 \text{ when } t=0.$$

Then the curves $\{(x'(t; y', \eta'), \zeta'(t; y', \eta'))\in\Gamma_0; t\in\boldsymbol{R}\}$ are said to be boundary null-bicharacteristic strips. Let $x=x_j(t; y', \eta')$ and $\zeta=\zeta_j(t; y', \eta')$, $1\leqq j\leqq l_1$, be the solutions of a system of the equations

$$dx/dt=(1, -\nabla_{\zeta'}\lambda_j^+(x, \zeta')),$$
$$d\zeta/dt=\nabla_x\lambda_j^+(x, \zeta'),$$
$$x_0=0, \ x'=y', \ \zeta'=\eta' \text{ and } \zeta_0=\lambda_j^+(0, y', \eta') \text{ when } t=0.$$

Then the curves $\{(x_j(t; y', \eta'), \zeta_j(t; y', \eta')); t\geqq 0\}$, $1\leqq j\leqq l_1$, are said to be outgoing null-bicharacteristic strips. Further we define

$C_0(\Gamma_0)=\{(x', \zeta', y', \eta')\in\Gamma_0\times\Gamma_0; (x', \zeta')=(y', \eta')$ or $x_1>y_1$ and there exists a boundary null-bicharacteristic strip which contains both (x', ζ') and $(y', \eta')\}$,

$C_j(\Gamma_0)=\{(x, \zeta, y', \eta')\in(T^*((0, \varepsilon)\times U)\backslash 0)\times\Gamma_0$; there exists an outgoing null-bicharacteristic strip which contains both (x, ζ) and $(0, y', \lambda_j^+(0, y', \eta'), \eta')\}$, $1\leqq j\leqq l_1$.

Let us define wave front sets for $u\in C^\infty([0, \varepsilon); \mathscr{D}'(U))$. Since we can regard $u\in C^\infty([0, \varepsilon); \mathscr{D}'(U))$ as an element of $\mathscr{D}'((0, \varepsilon)\times U)$ we can define $WF(u)$ for $u\in C^\infty([0, \varepsilon); \mathscr{D}'(U))$ by regarding u as an element of $\mathscr{D}'((0, \varepsilon)\times U)$.

Definition 3.4. For $u\in C^\infty([0, \varepsilon); \mathscr{D}'(U))$ we say that a point $(x^{1\prime}, \xi^{1\prime})$ in $T^*U\backslash 0$ is not in the set $WF_0(u)$ if there exist $\phi\in C_0^\infty(U)$, a conic neighborhood γ_1 of $\xi^{1\prime}$ and a positive constant ε_1 such that $\phi(x^{1\prime})\neq 0$ and

$$|\mathscr{F}_{x'}[\phi(x')D_0^j u(x)](x_0, \xi')|\leqq C_{kj}(1+|\xi'|)^{-k}$$

when $\xi'\in\gamma_1$, $x_0\in[0, \varepsilon_1)$ and $k, j=0, 1, 2, \cdots$.

The following lemma is an immediate consequence of the definition.

Lemma 3.5. *Let* $u\in C^\infty([0, \varepsilon); \mathscr{D}'(U))$.

(i) $(x^{1\prime}, \xi^{1\prime})\notin WF_0(u)$ *if and only if there exist a properly supported pseudo-differential operator in* x' *variables elliptic at* $(x^{1\prime}, \xi^{1\prime})$ *and a positive constant* ε_1 *such that* $Au(x_0, x')\in C^\infty([0, \varepsilon_1)\times U)$.

(ii) $WF_0(Au)\subset WF_0(u)$ *for every properly supported pseudo-differential operator in* x' *variables.*

(iii) $\pi(WF_0(u))=U\setminus\{x'\in U$; *there exist a neighborhood* U_1 *of* x' *and a positive constant* ε_1 *such that* $u\in C^\infty([0, \varepsilon_1)\times U_1)\}$, *where* π: $T^*U\setminus 0\to U$ *is the natural projection, i.e.,* $\pi(x', \xi')=x'$.

Now we can state our main theorem.

Theorem 3.6. *Assume that the conditions* $(A.1)-(A.3)$ *are satisfied. Then* $\{E_1, \Gamma, [0, \varepsilon)\times U\}$ *is a right microlocal parametrix for the problem* $(1.1)'-(1.3)'$ *with* $k=1$ *at* $(x^0, \xi^{0\prime})$, *where the operator* E_1 *is defined by* (3.26) *and* $\varepsilon(>0)$ *and* U *are suitably chosen. Moreover we have*

$$WF(E_1(g))\subset \cup_{j=1}^{l_1} C_j(\Gamma_0)\circ C_0(\Gamma_0)\circ WF(g)^\dagger, \tag{3.30}$$

$$WF_0(E_1(g))\subset C_0(\Gamma_0)\circ WF(g) \ \textit{for}\ g\in\mathscr{D}'(U). \tag{3.31}$$

Remark. (i) It is obvious that we can construct microlocal parametrices for the problems $(1.1)'-(1.3)'$, $2\leqq k\leqq l$, at $(x^0, \xi^{0\prime})$ and that we have the estimates (3.30) and (3.31) where $E_1(g)$ is replaced by $E_k(g)$, $2\leqq k\leqq l$. (ii) $C_0(\Gamma_0)$ is related to a boundary

† Let A and B be sets, C a subset of $A\times B$ and D a subset of B. Then we write $C\circ D=\{a\in A$; there exists b in D such that (a, b) in $C\}$.

wave.

Proof. Let $a_j(x', y', \xi')$, $1\leqq j\leqq l$, be symbols in $S^{\nu_1-m_j}(\dot{\Gamma}_0)$ and $b_j(x', y', \xi', s)$, $1\leqq j\leqq l$, symbols in $S^{\nu_2-m_j}(\dot{\Gamma}_0\times(-\infty, 0])$ such that $b_j(x', y', \xi', s)=0$ when $s\leqq s_0$ for some $s_0<0$, and put $\boldsymbol{a}(x', y', \xi')={}^t(a_1, \cdots, a_l)$, $\boldsymbol{b}={}^t(b_1, \cdots, b_l)$. Then it follows that

$$\boldsymbol{a}(x', y', \xi')+\int_{-\infty}^{0}\exp[i(\xi_1-\xi_1(x^{0\prime}, \xi''))s]\boldsymbol{b}(x', y', \xi', s)ds \\ \in S^{-\infty}(\dot{\Gamma}_0\times(-\infty, 0])$$

if

$$(U(x', \nabla_{x'}\tilde{\phi}(x', \xi'))+\textstyle\sum_{k=1}^{n}\partial U/\partial\zeta_k\cdot D_k)\{\boldsymbol{a}(x', y', \xi') \\ +\displaystyle\int_{-\infty}^{0}\exp[i(\xi_1-\xi_1(x^{0\prime}, \xi''))s]\boldsymbol{b}(x', y', \xi', s)ds\}\equiv 0 \pmod{S^{-\infty}}.$$

Thus the arguments in § 2 and Lemma 3.2 show that $\{E_1, \Gamma, [0, \varepsilon)\times U\}$ is a microlocal parametrix at $(x^0, \xi^{0\prime})$. Applying the method of stationary phase for (3.26) we obtain

$$WF(E_1(g))\subset\cup_{j=1}^{l_1}\Lambda_j'(\Gamma_0)\circ WF(g), \\ WF_0(E_1(g))\subset\Lambda_0'(\Gamma_0)\circ WF(g),$$

where

$$\Lambda_j'(\Gamma_0)=\{(x, \zeta, y', \eta')\in(T^*((0, \varepsilon)\times U)\backslash 0)\times\Gamma_0;\ \zeta=\nabla_x\phi_j(x, y', \xi'), \\ \eta'=-\nabla_{y'}\phi_j \text{ if } \nabla_{\xi'}\phi_j=0 \text{ for some } (y', \xi')\in\Gamma_0 \text{ or if } \xi_1= \\ \xi_1(x^{0\prime}, \xi'') \text{ and } \nabla_{\xi'}\phi_j(x, y', \xi')+s\nabla_{\xi'}(\xi_1-\xi_1(x^{0\prime}, \xi''))=0 \text{ for} \\ \text{some } (y', \xi')\in\Gamma_0 \text{ and } s<0\},$$

$$\Lambda_0'(\Gamma_0)=\{(x', \zeta', y', \eta')\in\Gamma_0\times\Gamma_0;\ (x', \zeta')=(y', \eta') \text{ or } \zeta'= \\ \nabla_{x'}\tilde{\phi}(x', \xi'), \eta'=\nabla_{y'}\tilde{\phi}(y', \xi') \text{ if } \xi_1=\xi_1(x^{0\prime}, \xi'') \text{ and } \nabla_{\xi'}(\tilde{\phi}(x', \xi') \\ -\tilde{\phi}(y', \xi'))+s\nabla_{\xi'}(\xi_1-\xi_1(x^{0\prime}, \xi''))=0 \text{ for some } (y', \xi')\in\Gamma_0 \text{ and} \\ s<0\}.$$

It suffices to show that there exists a boundary null-bicharacteristic strip which contains both (x', ζ') and (y', η') if $\zeta'=\nabla_{x'}\tilde{\phi}(x', \xi')$, $\eta'=\nabla_{y'}\tilde{\phi}(y', \xi')$, $\xi_1=\xi_1(x^{0\prime}, \xi'')$ and $\nabla_{\xi'}(\tilde{\phi}(x', \xi')-\tilde{\phi}(y', \xi'))+s\nabla_{\xi'}(\xi_1-\xi_1(x^{0\prime}, \xi''))=0$ for some $(y', \xi')\in\Gamma_0$ and $s<0$. For the same argument as in a proof of the above assertion yields characterizations of $\Lambda_j'(\Gamma^0)$ and, therefore, one can prove the theorem. The uniqueness theorem for a system of ordinary differential equations and (2.1) imply that $\zeta'(t; y', \eta') = \nabla_{x'}\tilde{\phi}(x'(t; y', \eta'), \xi')$ where $x'(t; y', \eta')$ and $\zeta'(t; y', \eta')$ are the solutions of (3.27)–(3.29). Thus we have

$$\frac{d}{dt}\frac{\partial\tilde{\phi}}{\partial\xi_j}(x'(t; y', \eta'), \xi') = \frac{\partial}{\partial\xi_j}(\xi_1 - \xi_1(x^{0\prime}, \xi'')), \ 1 \leqq j \leqq n. \tag{3.32}$$

(3.32) yields

$$\nabla_{\xi'}\{\tilde{\phi}(x'(t; y', \eta'), \xi') - \tilde{\phi}(y', \xi')\} = t\nabla_{\xi'}(\xi_1 - \xi_1(x^{0\prime}, \xi'')). \tag{3.33}$$

Thus by (3.33) with $t = -s$ we have

$$\nabla_{\xi'}\tilde{\phi}(x', \xi') = \nabla_{\xi'}\tilde{\phi}(x'(-s; y', \eta'), \xi').$$

Since we may assume that $\det(\partial^2\tilde{\phi}/\partial x_j\partial\xi_k(x', \xi'))_{j,k=1,\dots,n} \neq 0$ for $(x', \xi') \in \Gamma_0$ we obtain $x' = x'(-s; y', \eta')$ and $\zeta'(-s; y', \eta') = \nabla_{x'}\tilde{\phi}(x', \xi') = \zeta'$. **Q. E. D.**

§ 4. Microlocal Parametrices and Uniqueness Theorem

It follows from Duistermaat and Hörmander [3] that there exists an operator $F: \mathscr{D}'((-\varepsilon, \varepsilon) \times U) \to \mathscr{D}'((-\varepsilon, \varepsilon) \times U)$ such that (i) $PF(f) - f \in C^\infty((-\varepsilon, \varepsilon) \times U)$ if $WF(f) \subset T^*((-\varepsilon_1, \varepsilon_1) \times U_1) \backslash 0$, (ii) $WF(F(f)) \subset \{x_1 \geqq c\}$ if $WF(f) \subset \{x_1 \geqq c\}$, where $\varepsilon_1 < \varepsilon$ and $U_1 (\subset U)$ is a neighborhood of $x^{0\prime}$ in $\boldsymbol{R}^n$. We write

$$\mathring{\mathscr{D}}'([0, \varepsilon) \times U) = \{f \in \mathscr{D}'((-\varepsilon, \varepsilon) \times U) ;\ \mathrm{supp}\, f \subset [0, \varepsilon) \times U\}.$$

$C^\infty[(0, \varepsilon) ; \mathscr{D}'(U))$ can be mapped injectively into $\mathring{\mathscr{D}}'([0, \varepsilon) \times U)$ in a natural manner:

$$C^\infty([0, \varepsilon) ; \mathscr{D}'(U)) \ni f \to \tilde{f} \in \mathring{\mathscr{D}}'([0, \varepsilon) \times U).$$

From partial hypoellipticity we can regard $F(\tilde{f})|_{x_0 \geqq 0}$ as an element of $C^\infty([0, \varepsilon) ; \mathscr{D}'(U))$ if $f \in C^\infty([0, \varepsilon) ; \mathscr{D}'(U))$. Thus we can define maps $W_j^0: C^\infty([0, \varepsilon) ; \mathscr{D}'(U)) \to \mathscr{D}'(U)$ by

$$W_j^0(f) = \lim_{x_0 \to +0} B_j(x', D) F(\tilde{f})(x_0, x'), \ 1 \leqq j \leqq l.$$

Put

$$E^0(f, g_1, \cdots, g_l) = F(\tilde{f})|_{x_0 \geqq 0} + \textstyle\sum_{j=1}^{l} E_j(g_j - W_j^0(f))$$
$$\text{for } f \in C^\infty([0, \varepsilon) ; \mathscr{D}'(U)), \ g_j \in \mathscr{D}'(U), \ 1 \leqq j \leqq l.$$

Then we have the following

Theorem 4.1. *Under the conditions* $(A.1)-(A.3)$ *the above operator* $E^0: C^\infty([0, \varepsilon); \mathscr{D}'(U)) \times \{\mathscr{D}'(U)\}^l \to C^\infty([0, \varepsilon); \mathscr{D}'(U))$ *satisfies the following properties:*

(i) $PE^0(f, g_1, \cdots, g_l)-f \in C^\infty([0, \varepsilon) \times U)$
if $Wf(\tilde{f}) \subset T^*((-\varepsilon_1, \varepsilon_1) \times U_1)\backslash 0$.

(ii) $B_j E^0(f, g_1, \cdots, g_l)|_{x_0=0}-g_j \in C^\infty(U)$
if $WF_0(f)$, $WF(g_j) \subset \Gamma$, $1 \leqq j \leqq l$, *and* $WF(f) \subset (0, \varepsilon_1) \times U_1 \times \boldsymbol{R} \times \gamma_1$.

(iii) $WF(E^0(f, g_1, \cdots, g_l)) \subset \{x_1 \geqq c\}$ *if* $WF(f) \subset \{x_1 \geqq c\}$ *and* $WF_0(f)$, $WF(g_j) \subset \{x_1 \geqq c\}$.

Here $\varepsilon_1 (<\varepsilon)$ *is a small positive constant and* $U_1 \times \gamma_1$ $(\subset \Gamma)$ *is a small conic neighborhood of* $(x^{0\prime}, \xi^{0\prime})$. *Moreover we have*

$$WF(E^0(f, g_1, \cdots, g_l)) \subset \tilde{C} \circ WF(f) \cup [\cup_{j=1}^{l_1} C_j(\Gamma_0) \circ C_0(\Gamma_0) \circ \{\cup_{k=1}^{l} WF(g_k) \cup (\tilde{C}_\partial \circ WF(f)) \cup WF_0(f)\}],$$

$$WF_0(E^0(f, g_1, \cdots, g_l)) \subset C_0(\Gamma_0) \circ [\cup_{j=1}^{l} WF(g_j) \cup (\tilde{C}_\partial \circ WF(f)) \cup WF_0(f)],$$

where

$\tilde{C} = \{(x, \xi, y, \eta) \in (T^*((0, \varepsilon) \times U)\backslash 0)^2;\ (x, \xi) = (y, \eta)$ or $x_1 > y_1$ *and there exists a null bicharacteristic strip for* p *which contains both* (x, ξ) *and* $(y, \eta)\}$,

$\tilde{C}_\partial = \{(x', \xi', y, \eta) \in (T^*U\backslash 0) \times (T^*((0, \varepsilon) \times U)\backslash 0);\ x_1 > y_1$ *and there exists a null bicharacteristic strip for* p *which contains both* $(0, x', \xi_0, \xi')$ *and* (y, η) *for some* $\xi_0 \in \boldsymbol{R}\}$.

Let us construct a microlocal parametrix for the problem (1.1) $-$(1.3) in the case where f belongs to $\mathring{\mathscr{D}}'([0, \varepsilon) \times U)$, following Melrose [6]. We assume in the remainder of this section that
(A.4) $\{B_j(x', D)\}_{1 \leqq j \leqq l}$ is a normal system on the hyperplane $x_0=0$ and $m_j \leqq m-1$, $1 \leqq j \leqq l$.
Therefore we obtain a Dirichlet system $\{B_j\}_{1 \leqq j \leqq m}$ of order m on $x_0=0$ by complementing suitably $\{B_j\}_{l+1 \leqq j \leqq m}$.

Lemma 4.2 ([5], [8], [9]). *There exists a Dirichlet system* $\{B'_j(x', D)\}_{1\leq j\leq m}$, *uniquely determined, such that for any* $u\in C^\infty([0, \varepsilon)\,;\, \mathscr{D}'(U))$

$$P\tilde{u}-(Pu)^\sim=\sum_{j=1}^m B'_j(x', D)\{\delta(x_0)\otimes B_j(x', D)u(x)|_{x_0=0}\}.$$

Moreover we have

$$\begin{aligned} i^{m-l}\det&\left(\frac{1}{2\pi i}\oint\frac{b'_j(x', \xi_0, -\xi')\xi_0^{k-1}}{p_-(0, x', -\xi_0, \xi')}d\xi_0\right)_{\substack{j=l+1,\dots,m\\ k=1,\dots,m-l}}\\ &=(\operatorname{sgn}\sigma)(-1)^{l(l-1)/2}b_1(x', 1, 0,\cdots, 0)^{-1}\dots b_m^{-1}\\ &\quad\times\det\left(\frac{1}{2\pi i}\oint\frac{b_j(x', \xi)\xi_0^{k-1}}{p_+(0, x', \xi)}d\xi_0\right)_{j,k=1,\dots,l},\ \operatorname{Im}\,\xi_1\leq 0,\end{aligned}$$

where b'_j *is the principal part of* B'_j, σ *is a permutation defined by* $m_{\sigma(j)}=j-1$, $1\leq j\leq m$, *and* $p_-(x, -\xi)=\prod_{j=1}^{m-l}(\xi_0-\lambda_j^-(x, -\xi'))$, $\operatorname{Im}\,\xi_1\geq 0$.

Put

$$\begin{aligned}\mathscr{M}&=\{f\in\mathring{\mathscr{D}}'([0, \varepsilon)\times U)\,;\, f=\textstyle\sum_{j=l+1}^m B'_j(x', D)(\delta(x_0)\otimes h_j(x')),\\ &\qquad h_j\in\mathscr{D}'(U)\},\\ \mathscr{M}_1&=\{f\in\mathring{\mathscr{D}}'([0, \varepsilon)\times U)\,;\, f=\textstyle\sum_{j=0}^{m-1}\delta^{(j)}(x_0)\otimes h_j(x'),\ h_j\in C^\infty(U)\},\\ \mathring{C}^\infty&([0, \varepsilon)\times U)=\{f\in\mathring{\mathscr{D}}'([0, \varepsilon)\times U)\,;\, f\in C^\infty([0, \varepsilon)\times U)\}.\end{aligned}$$

Lemma 4.3. $\mathscr{M}$ *is uniquely determined by* $\{P, B_1,\cdots, B_l\}$.

Proof. Assume that $\{B_1,\cdots, B_l, \tilde{B}_{l+1},\cdots, \tilde{B}_m\}$ is another Dirichlet system of order m on $x_0=0$. Then we have

$$\begin{aligned}&\textstyle\sum_{j=1}^m B'_j(x', D)(\delta(x_0)\otimes B_j(x', D)u(x)|_{x_0=0})\\ &\quad=\textstyle\sum_{j=1}^l\tilde{B}'_j(x', D)(\delta(x_0)\otimes B_ju|_{x_0=0})+\sum_{j=l+1}^m\tilde{B}'_j(x', D)(\delta(x_0)\\ &\qquad\otimes\tilde{B}_ju|_{x_0=0})\end{aligned}$$

for $u\in C^\infty([0, \varepsilon)\,;\,\mathscr{D}'(U))$, where $\{\tilde{B}'_j\}_{1\leq j\leq m}$ is defined for $\{B_1,\cdots, B_l, \tilde{B}_{l+1}, \cdots, \tilde{B}_m\}$ by Lemma 4.2. On the other hand it easily follows that for any $g_j\in\mathscr{D}'(U)$, $1\leq j\leq m$, there exists u in $C^\infty([0, \varepsilon)\,;\,\mathscr{D}'(U))$ such that $B_j(x', D)u(x)|_{x_0=0}=g_j(x')$, $1\leq j\leq m$ (see [5]). This proves the lemma. **Q. E. D.**

Definition 4.5. Let $u\in\mathring{\mathscr{D}}'([0, \varepsilon)\times U)$. We define $WF_1(u)$ by

$$WF_1(u)=\{(x', \xi')\in T^*U\backslash 0;\ \iota^{*-1}(\gamma)\cap WF(u)\neq\emptyset \text{ for any conic neighborhood } \gamma \text{ of } (x', \xi')\},$$

where the inclusion $\iota: U\to(-\varepsilon, \varepsilon)\times U$ is defined by $\iota(x')=(0, x')$.

Definition 4.6. A right microlocal parametrix for the problems (1.1)−(1.3) at $(x^0, \xi^{0\prime})$ is a triple $\{E^1, \Gamma, [0, \varepsilon)\times U\}$ satisfying the conditions

(i) Γ is a conic neighborhood of $(x^{0\prime}, \xi^{0\prime})$, U is a neighborhood of $x^{0\prime}$ and $\varepsilon>0$,

(ii) E^1 is an operator: $\mathring{\mathscr{D}}'([0, \varepsilon)\times U)\times\{\mathscr{D}'(U)\}^l\to\mathring{\mathscr{D}}'([0, \varepsilon)\times U)$,

(iii) $PE^1(f, g_1,\cdots, g_l)-f-\sum_{j=1}^l B_j'(x', D)(\delta(x_0)\otimes g_j(x'))\in\mathring{C}^\infty([0, \varepsilon)\times U)+\mathscr{M}+\mathscr{M}_1$ if $WF_1(f)$, $WF(g_j)\subset\Gamma$ and $WF(f|_{x_0>0})\subset(0, \varepsilon_1)\times U_1\times\boldsymbol{R}\times\gamma_1$, where $\Gamma=U_1\times\gamma_1$ and γ_1 is a conic neighborhood of $\xi^{0\prime}$,

(iv) $WF(E^1(f, g_1,\cdots, g_l)|_{x_0>0})\subset\{x_1\geqq c\}$ if $WF(f|_{x_0>0})\subset\{x_1\geqq c\}$ and $WF_1(f)$, $WF(g_j)\subset\{x_1\geqq c\}$.

It follows from partial hypoellipticity that we can define maps W_j^1: $\mathring{\mathscr{D}}'([0, \varepsilon)\times U)\to\mathscr{D}'(U)$ by

$$W_j^1(f)=\lim_{x_0\to-0}B_j(x', D)F(f)(x_0, x').$$

Put

$$E^1(f, g_1,\cdots, g_l)=F(f)-(F(f)|_{x_0\leqq 0})^\sim+\sum_{j=1}^l(E_j(g_j-W_j^1(f)))^\sim \quad \text{for } f\in\mathring{\mathscr{D}}'([0, \varepsilon)\times U),\ g_j\in\mathscr{D}'(U),$$

where $(F(f)|_{x_0\leqq 0})^\sim\in\mathring{\mathscr{D}}'((-\varepsilon, 0]\times U)$ is defined for $F(f)|_{x_0\leqq 0}\in C^\infty((-\varepsilon, 0];\mathscr{D}'(U))$ in a natural manner. Then we have the following

Theorem 4.7. *Assume that the conditions* $(A.1)-(A.4)$ *are satisfied. Then modifying* U, Γ *and* ε, *if necessary,* $\{E^1, \Gamma, [0, \varepsilon)\times U\}$ *is a microlocal parametrix for* (1.1)−(1.3) *at* $(x^0, \xi^{0\prime})$. *Moreover we have*

$$WF(E^1(f, g_1,\cdots, g_l)|_{x_0>0})\subset\tilde{C}\circ WF(f|_{x_0>0})\cup[\cup_{j=1}^{l_1}C_j(\Gamma_0)\circ C_0(\Gamma_0)\circ\{\cup_{k=1}^l WF(g_k)\cup(\tilde{C}_\partial\circ WF(f|_{x_0>0}))\cup WF_1(f)\}],$$

$$WF_1(E^1(f, g_1,\cdots, g_l))\subset C_0(\Gamma_0)\circ[\cup_{j=1}^l WF(g_j)\cup(\tilde{C}_\partial\circ WF(f|_{x_0>0}))\cup WF_1(f)].$$

Using the result obtained by Lax and Nirenberg [7] the same argument as in [6], § 11, yields the following

Theorem 4.8. *Let* $(x^{1\prime}, \xi^{1\prime}) \in \Gamma = U_1 \times \gamma_1$ *and* $u \in \mathring{\mathscr{D}}'([0, \varepsilon) \times U)$. *Then we have* $(x^{1\prime}, \xi^{1\prime}) \notin WF_1(u)$ *if there exists* v *in* $\mathscr{M}$ *such that* $(x^{1\prime}, \xi^{1\prime}) \notin WF_1(Pu - v)$ *and*

$$(x^{1\prime}, \xi^{1\prime}) \notin C_0(\Gamma_0) \circ [\tilde{C}_\partial \circ \{WF(u|_{x_0>0}) \cap (0, \varepsilon) \times U_1 \times \boldsymbol{R} \times \gamma_1\} \cup \{(WF_1(u) \setminus \{(x^{1\prime}, \xi^{1\prime})\}) \cap \Gamma\}].$$

Corollary 4.9. *Let* $u \in C^\infty([0, \varepsilon) ; \mathscr{D}'(U))$. *Then we have* $(x^{1\prime}, \xi^{1\prime}) \notin WF_0(u)$ *if* $(x^{1\prime}, \xi^{1\prime}) \notin WF_0(Pu)$, $(x^{1\prime}, \xi^{1\prime}) \notin WF(B_j u|_{x_0=0})$, $1 \leqq j \leqq l$, *and*

$$(x^{1\prime}, \xi^{1\prime}) \notin C_0(\Gamma_0) \circ [\tilde{C}_\partial \circ \{WF(u) \cap (0, \varepsilon) \times U_1 \times \boldsymbol{R} \times \gamma_1\} \cup \{WF_0(u) \setminus \{(x^{1\prime}, \xi^{1\prime})\}) \cap \Gamma\}].$$

§ 5. Some Remarks

Let $\mathscr{E} : \{\mathscr{D}'(U)\}^l \to C^\infty([0, \varepsilon) ; \mathscr{D}'(U))$ be a microlocal parametrix for the Dirichlet problem

$$\begin{aligned} &P(x, D)u(x) = 0, \quad x \in \boldsymbol{R}_+^{n+1}, \quad x_1 > 0, \\ &D_1^{k-1}u(x)|_{x_1=0} = 0, \quad x_0 > 0, \quad 1 \leqq k \leqq m, \\ &D_0^{j-1}u(x)|_{x_0=0} = h_j(x'), \quad x_1 > 0, \quad 1 \leqq j \leqq l \end{aligned}$$

at $(x^0, \xi^{0\prime})$. The operator $\mathscr{E}$ can be constructed in the same way as in § 2. The phase functions $\phi_j(x, y', \xi')$ are defined by the eiconal equations

$$\begin{aligned} &\partial_0 \phi_j(x, y', \xi') = \lambda_j^+(x, \nabla_{x'} \phi_j), \\ &\phi_j(0, x', y', \xi') = (x' - y') \cdot \xi', \quad 1 \leqq j \leqq l. \end{aligned}$$

Then we have the following

Lemma 5.1. *Under the condition* (*A*. 1) *a microlocal parametrix for the Dirichlet problem can be given in the form*

$$\mathscr{E}(h_1, \cdots, h_l) = \sum_{k=1}^l \sum_{j=1}^{l_1} \int \exp[i\phi_j(x, y', \xi')] a_{jk}(x, y', \xi')$$

$$\times h_k(y')dy'đ\xi' + \sum_{k=1}^{l}\int \exp[i(x'-y')\cdot\xi']a_k(x, y', \xi')h_k(y')dy'đ\xi',$$

$$(a^0_{1k}(0, x', y', \xi'), \cdots, a^0_{l_1k}, c^0_{1k}(0, x', y', \xi'), \cdots, c^0_{l-l_1k})_{k\downarrow 1,\dots,l}$$
$$= \chi(x', y', \xi')^t L_d(x', \xi')^{-1},$$

where a_{jk} *and* a_k *are symbols,* a^0_{jk} *and* a^0_k *are their principal symbols as multiple symbols,*

$$a^0_k(x, y', \xi') = \sum_{j=1}^{l-l_1} \frac{1}{2\pi i}\int_{C_{\xi'}} p(x, \xi)^{-1}\xi_0^{j-1}\exp[ix_0\xi_0]d\xi_0 \times c^0_{jk}(x, y', \xi'),$$

and

$$L_d(x', \xi') = (\lambda_1^+(0, x', \xi')^{j-1}, \cdots, \lambda_{l_1}^{+\,j-1},$$
$$\frac{1}{2\pi i}\int_{C_{\xi'}} p(0, x', \xi)^{-1}\xi_0^{j-1}d\xi_0, \cdots, \frac{1}{2\pi i}\int_{C_{\xi'}} p^{-1}\xi_0^{j+l-l_1-2}d\xi_0)_{j\downarrow 1,\dots,l}.$$

Here we note that

$$\det L_d(x', \xi') = (-1)^{(l/2+l_1)(l-1)}\prod_{1\leq j<k\leq l_1}(\lambda_j^+(0, x', \xi') - \lambda_k^+) \times \prod_{l_1+1\leq j\leq l}\prod_{1\leq k\leq m-l}(\lambda_j^+ - \lambda_k^-)^{-1}.$$

Lemma 5.2. *Assume that the condition (A.1) is satisfied. For* $h_k \in \mathscr{D}'(U)$, $1\leq k\leq l$, *we can define pseudo-differential operators* $\mathscr{B}_{jk}$ *in* x' *variables by*

$$\sum_{k=1}^{l}\mathscr{B}_{jk}(h_k) = B_j(x', D)\,\mathscr{E}\,(h_1, \cdots, h_l)\,|_{x_0=0}.$$

Moreover the principal symbols β^0_{jk} *of* $\mathscr{B}_{jk}$ *can be given in the form*

$$\beta^0_{jk}(x', y', \xi') = \chi(x', y', \xi')L(x', \xi')L_d(x', \xi')^{-1}.$$

Under the conditions (A. 1) – (A. 3) we can construct a microlocal parametrix $\mathscr{A} = (\mathscr{A}_{jk})_{j,k=1,\dots,l}$ for the problem

$$\sum_{k=1}^{l}\mathscr{B}_{jk}(h_k) = g_j(x'),\ g_j \in \mathscr{D}'(U),\ 1\leq j\leq l.$$

Thus putting $E_k(g) = \mathscr{E}\,(\mathscr{A}_{1k}(g), \cdots, \mathscr{A}_{lk}(g))$, we obtain a right microlocal parametrix for the problem (1. 1)′ – (1. 3)′ at $(x^0, \xi^{0\prime})$.

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