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On General Denting Points and the Unique Positive Extension of Certain Positive Linear Functionals

By

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Throughout this paper, let X be a completely regular Hausdorff space, $C_b(X)$ the space of all real-valued bounded and continuous functions on X with supremum norm, $M(X)$ the set of all positive linear functionals μ on $C_b(X)$ such that $\mu(1)=1$, ($M(X)$ is also the set of all positive, regular finitely additive measures, each of total mass 1, on the algebra generated by zero sets.), A a subspace of $C_b(X)$ which separates points of X and contains the constant functions, and A^* the set of all real-valued continuous linear functionals on A. If X is a non-empty closed, bounded and convex subset of an LCHTVS (locally convex Hausdorff topological vector space) E over the field R of real numbers, then we always regard A as the subspace $\{f|_X+r : f\in E^*, r\in R\}$ of $C_b(X)$, where E^* is the topological dual of E, and $f|_X$ is the restriction of f to X. Denote by $K(A)$ the set of all L in A^* such that $L(1)=1=||L||$. If we consider A^* (resp. $C_b(X)^*$) in its weak* topology, then $K(A)$ (resp. $M(X)$) is a non-empty compact convex subset of an LCHTVS A^* (resp. $C_b(X)^*$) over R. If a is in X, let $\phi(a)$ be the element of $K(A)$ defined by $\phi(a)(f)=f(a)$ for any f in A, and $\varepsilon(a)$ the element of $M(X)$ defined by $\varepsilon(a)(f)=f(a)$ for any f in $C_b(X)$. Note that ϕ is a one-to-one and continuous mapping from X into $K(A)$.

The purpose of this paper is to give a characterization of point a in X with the following property (*).

$$(*)\qquad \{\mu\in M(X) : \mu(f)=f(a) \text{ for any } f \text{ in } A\}=\{\varepsilon(a)\}.$$

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In other words, we consider some conditions of the point a under which L_0 is uniquely extended so as to become an element of $M(X)$, where $L_0: A \rightarrow R$ is defined by $L_0(f) = f(a)$ for any f in A.

Concerning this problem, Bauer ([1]) has first proved that if X is a compact convex subset of an LCHTVS E over R, then a point a in X satisfies the property (*) if and only if a is an extreme point of X. More generally, making use of this result, it is proved in [7] (Proposition 6. 2) that if X is a compact Hausdorff space, then a point a satisfies the property (*) if and only if $\phi(a) \in$ ext $K(A)$, the set of all extreme points of $K(A)$. This point a is called a Choquet point of X with respect to A. Successively, Khurana ([4]) has extended the Bauer's theorem to the non-compact case as follows: if X is a closed, bounded and convex subset of an LCH-TVS E over R, then a point a in X satisfies the property (*) if and only if $a \in$ Dent X, the set of all denting points of X. A point a in X is called a denting point of X if for every neighborhood V of a, $a \overline{\in}$ cl-conv $(X \setminus V)$, the closed convex hull of $X \setminus V$. Choquet ([2]) calls these points strongly extreme points.

In this paper, we attempt to extend above all results to the non-compact case without algebraic structures. We define a topology τ_A on X such that (X, τ_A) is a completely regular Hausdorff space for which a net $\{x_\alpha\} \subset X$ converges to $x \in X$ in the topology τ_A if and only if $\lim_\alpha f(x_\alpha) = f(x)$ for any f in A, and for which f in A is continuous on (X, τ_A). We define a general denting point of X with respect to A as follows, and denote by $D_A(X)$ the set of all general denting points of X with respect to A.

Definition. A point a in X is called a general denting point of X with respect to A if two following conditions are satisfied.

(1) $\phi(a) \in$ ext $K(A)$.

(2) For a net $\{x_\alpha\}$ in X, $x_\alpha \rightarrow a$ in the original topology if $x_\alpha \rightarrow a$ in the topology τ_A.

Then two following examples show that condition (1) and condition (2) are independent.

Example 1. Here we give an example in which condition (1) does not imply condition (2). In l_2 with canonical basis $\{e_n\}$, $0 \in X = \text{cl-conv}(\{e_n\})$. Then X is a weakly compact convex subset of l_2, and $0 \in \text{ext } X$. Then, by making use of Khurana's theorem (Theorem 2. 5 in [3]), we have that $\phi(0) \in \text{ext } K(A)$. If a point 0 in X satisfies (2), then we know that $e_n \to 0$ in the norm topology, since $e_n \to 0$ in the topology τ_A. But $\|e_n\| = 1$ for all n, which is a contradiction. Hence $(1) \not\Rightarrow (2)$.

In [5], Looney has given this example in which an exposed point need not be a denting point.

Example 2. Here we give an example in which condition (2) does not imply condition (1). Let X be a compact convex subset of an LCHTVS E over R such that $X \backslash \text{ext } X \neq \phi$. Then a point a of $X \backslash \text{ext } X$ satisfies (2), but does not satisfy (1), since a is in ext X if and only if $\phi(a)$ is in ext $K(A)$ in this case. Hence $(2) \not\Rightarrow (1)$.

Now we obtain a following result concerning the problem stated above.

Theorem. *A point a in X satisfies the property (*) if and only if $a \in D_A(X)$.*

Before we prove this theorem, we prepare a following lemma.

Lemma. *Let $\varepsilon(X) = \{\varepsilon(x) : x \in X\}$. Then*
(1) $\varepsilon(X) \subset \text{ext } M(X)$, (2) $\text{ext } M(X) \subset \overline{\varepsilon(X)}$.

Proof. To verify (1), suppose that $\varepsilon(x) = t \cdot \mu + (1-t) \cdot \nu$ where μ and ν are elements of $M(X)$ and $0 < t < 1$. Let g be in $C_b(X)$ such that $g(x) = 0$ and $0 \leqq g \leqq 1$ everywhere. Then we have that

$$0 \leqq t \cdot \mu(g) + (1-t) \cdot \nu(g) = \varepsilon(x)(g) = g(x) = 0$$

which means that $\mu(g) = 0 = g(x)$ and $\nu(g) = 0 = g(x)$. From this it easily follows that $\mu=\nu=\varepsilon(x)$. To verify (2), we recall that $M(X)$ is the closed convex hull of $\varepsilon(X)$. Then, by Milman's converse to the Krein Milman theorem (p. 9 in [7]), (2) holds.

Proof of Theorem. Suppose that $a\in D_A(X)$. Let $N=\{\mu\in M(X):\mu(f)=f(a)$ for any f in $A\}$. Then N is a non-empty compact convex subset of an LCHTVS $C_b(X)^*$ over R. Hence we are going to prove that ext $N=\{\varepsilon(a)\}$, which means that $N=\{\varepsilon(a)\}$ by the Krein-Milman theorem. Let $\lambda\in$ext N. Since $\phi(a)$ is an element of ext $K(A)$, ext $N\subset$ext $M(X)$. Hence, by the above lemma, there is a net $\{x_\alpha\}$ in X such that $\varepsilon(x_\alpha)\to\lambda$. Then $f(x_\alpha)\to\lambda(f)=f(a)$ for any f in A, that is, $x_\alpha\to a$ in the topology τ_A. Hence, by condition (2), $x_\alpha\to a$ in the original topology, that is, $\varepsilon(x_\alpha)\to\varepsilon(a)$, which shows that $\lambda=\varepsilon(a)$. This proves that ext $N=\{\varepsilon(a)\}$. Conversely, suppose that the property (*) is satisfied. We first prove that condition (1) is satisfied. Let $\phi(a)=t\cdot S+(1-t)\cdot T$ on A, where S and T are elements of $K(A)$ and $0<t<1$. Since S and T are in $K(A)$, they may be extended to the elements μ_S and μ_T of $M(X)$, respectively, by the Hahn-Banach theorem. Hence we have that $f(a)=\phi(a)(f)=\{t\cdot\mu_S+(1-t)\cdot\mu_T\}(f)$ for any f in A. From this and the assumption we get that $t\cdot\mu_S+(1-t)\cdot\mu_T=\varepsilon(a)$. By the above lemma, we have that $\mu_S=\mu_T=\varepsilon(a)$, which means that $S=T=\phi(a)$ on A, and so $\phi(a)\in$ext $K(A)$. We next prove that condition (2) is satisfied. Let $\{x_\alpha\}$ be a net in X such that $x_\alpha\to a$ in the topology τ_A and μ an arbitrary cluster point of the net $\{\varepsilon(x_\alpha)\}$. Then there is a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $\varepsilon(x_\beta)\to\mu$, and so $\mu(f)=\lim_\beta \varepsilon(x_\beta)(f)=\lim_\beta f(x_\beta)=f(a)$ for any f in A, since $x_\beta\to a$ in the topology τ_A. Hence, by the property (*), we have that $\mu=\varepsilon(a)$. Hence $\varepsilon(a)$ is the only cluster point of the net $\{\varepsilon(x_\alpha)\}$, and so $\varepsilon(x_\alpha)\to\varepsilon(a)$. It follows that $x_\alpha\to a$ in the original topology by Varadarajan's theorem (Theorem 9 of part 2 in [8]). Thus the proof is completed.

Corollary 1. *Let X be a closed, bounded and convex subset of an LCHTVS E over R. Then $a\in$ Dent X if and only if $a\in D_A(X)$.*

Proof. This follows trivially from our theorem and Khurana's theorem (Theorem 1 in [4]) stated above.

Corollary 2 (cf. Remark 4.5 in [6]). *Let X be a weakly compact convex subset of an LCHTVS E over R. Then $a \in$* Dent X *if and only if a is a point in* ext X *where the identity map: (X, weak topology) $\rightarrow$ (X, original topology) is continuous.*

Proof. We easily see that $a \in \text{ext } X$ if and only if $\phi(a) \in \text{ext } K(A)$ in this case. Hence this corollary immediately follows from our Theorem and Corollary 1.

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