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On an Invariant Defined by Using $P(n)_*(\text{-})$ Theory

By

Nobuaki YAGITA*

Introduction

In this paper, we shall study stable homotopy invariants $s(X)$ which explain *the complexity of the torsion* of a finite complex X in such a way that $s(X)=0$ if X is torsion free, and $s(X)>0$ otherwise.

As an example of such invariants, $\text{homdim}_{BP_*}BP_*(X)$ has been studied by many authors. Especially, Johnson-Wilson [3] proved that $\text{homdim}_{BP_*}BP_*(X)\leqq n$ iff $BP\langle n\rangle_*(X)\simeq BP\langle n\rangle_*\underset{BP_*}{\otimes}BP_*(X)$ where $BP\langle n\rangle_*(-)$ is the bordism theory with the coefficient $BP\langle n\rangle_*\simeq BP_*/(v_{n+1}, v_{n+2}, \ldots)$.

Moreover, Johnson-Wilson [4] defined another invariant $t(X)$ as follows: $t(X)\leqq n$ iff there is a BP_*-module isomorphism $P(n)_*(X)\simeq P(n)_*\otimes H_*(X; Z_p)$ where $P(n)_*(-)$ is the bordism theory with the coefficient $P(n)_*\simeq BP_*/I_n=BP_*/(p, v_1, \ldots, v_{n-1})$.

This invariant $t(X)$ appears to have better properties and to be more easily computable than $\text{homdim}_{BP_*}BP_*(X)$. In this paper, we shall study it in comparison with $\text{homdim}_{BP_*}BP_*(X)$.

In § 1 we shall give the definition of $t(X)$ and consider its geometric meaning. In § 2 we shall study the properties of $t(X)$ in connection with skeletons of X, the Spanier-Whitehead duality, cohomolgy operations of $H^*(X; Z_p)$, the BP_*-module structure of $BP_*(X; Z_p)$, the smash product, and the cofibering.

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* Department of Mathematics, Tokyo Institute of Technology, Meguroku, Tokyo 152, Japan.

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§1. Definition

In this paper, we shall always assume that X, Y are *finite* complexes.

Let $BP_*(-)$ be the Brown-Peterson homology theory *at a prime* $p \geqq 3$ and denote its coefficient $BP_* \simeq \boldsymbol{Z}_{(p)}[v_1, \ldots]$. Let $S_n = (p = v_{i_0}, v_{i_1}, \ldots, v_{i_{n-1}})$ be a sequence of elements v_{i_j}. By using manifolds with singularities, we can construct the bordism theory $BP(S_n)_*(-)$ with the coefficient $BP_*/(S_n)$ [11].

Let $P(n)_*(-)$ be $BP(p, v_1, \ldots, v_{n-1})_*(-)$ [4]. By the Sullivan's (Bockstein) exact sequence, it is easily proved [4] that the following (1)−(3) are equivalent.

(1) $P(n)_*(X) \simeq P(n)_* \otimes H_*(X; \boldsymbol{Z}_p)$.

(2) $P(n)_*(X)$ is $P(n)_*$-free.

(3) The natural homomorphism i: $P(n)_*(X) \to H_*(X; Z_p)$ is epic.

Let $k(n)_*(-)$ be the bordism theory with the coefficient $k(n)_* \simeq Z_p[v_n] \simeq BP_*/(p, \ldots, \hat{v}_n, \ldots)$. Johnson-Wilson [4] proved also that the following (4)−(6) are equivalent to (1)−(3).

(4) $k(n)_*(X) \simeq k(n)_* \otimes H_*(X; Z_p)$.

(5) $k(n)_*(X)$ is $k(n)_*$-free.

(6) The natural homomorphism $i: k(n)_*(X) \to H_*(X; Z_p)$ is epic.

Now we *define* $t(X) \leqq n$ *iff* (1)−(6) *are satisfied*. This is well defined, since there is the tower of homology theories [4],

$$BP_*(-) \to P(1)_*(-) \to \cdots \to P(n)_*(-) \to P(n+1)_*(-) \to \cdots \to H_*(-; Z_p).$$

Geometrically, $t(X) \leqq n$ means that all elements of $H_*(X; Z_p)$ can be represented by manifolds with singularities type $I_n = (p, v_1, \ldots, v_{n-1})$ [11]. Let $S = (p, v_{i_1}, v_{i_2}, \ldots)$ be an infinite sequence and let $S_n = (p, \ldots, v_{i_{n-1}})$. We can analogously define an invariant $t_S(-)$, i. e., $t_S(X) \leqq n$ iff $BP(S_n)_*(X) \simeq BP(S_n)_* \otimes H_*(X; Z_p)$. However, from the following fact, $t_S(X)$ depends on $t(X)$.

Theorem 1. *Let* $S=(p, v_1, \ldots, v_{m-1}, v_{i_m}, \ldots)$ *and* $i_m \neq m$, $i_j < i_{j+1}$. *If* $n \geqq m$, *then*, $t_S(X) \leqq n$ *iff* $t(X) \leqq m$.

Proof. If $t(X) \leqq m$ then it is clear that $t_S(X) \leqq n$ by the Sullivan's exact sequence. Let $BP(S_n)_*(X) \simeq BP_*/(S_n) \otimes H_*(X; Z_p)$. Then, since $S_n \subset (p, \ldots, v_{m-1}, v_{m+1}, \ldots) = (p, \ldots, \hat{v}_m, \ldots)$, we have an isomorphism $k(m)_*(X) \simeq k(m)_* \otimes H_*(X; Z_p)$. Hence from (4) we have proved $t(X) \leqq m$.

§ 2. Properties of t(X).

Let X^q be a q-dimensional skeleton of X. When $\text{homdim}_{BP_*} BP_*(X) \leqq 2$, Johnson showed [2] that $\text{homdim}_{BP_*} BP_*(X^q) \leqq \text{homdim}_{BP_*} BP_*(X)$. However in general case, it is unknown whether the inequality holds or not. We can easily prove the following theorem by descending induction on q.

Theorem 2. $t(X^q) \leqq t(X)$ *and* $t(X/X^q) \leqq t(X)$.

It is known [3] that $\text{homdim}_{BP_*} BP_*(X)$ is not necessarily equal to $\text{homdim}_{BP_*} BP^*(X)$.

Theorem 3. *Let DX be a Spanier-Whitehead dual of X. Then* $t(X)=t(DX)$, *i.e.*, $P(n)_*(X) \simeq P(n)_* \otimes H_*(X; Z_p)$ *iff* $P(n)^*(X) \simeq P(n)^* \otimes H^*(X; Z_p)$.

Proof. If $t(X) \leqq n$, then by the definition, the Atiyah-Hirzebruch spectral sequence $H_*(X; P(n)_*) \Rightarrow P(n)_*(X)$ is trivial. Hence, by Lemma 4. 2 in [1], we have

$$P(n)^*(X) \simeq \text{Hom}_{P(n)_*}(P(n)_*(X), P(n)_*).$$

Therefore $P(n)^*(X)$ is $P(n)^*$-free. The same argument for DX shows that if $P(n)^*(X)$ is $P(n)^*$-free then $P(n)_*(X)$ is $P(n)_*$-free.

We now consider the relation of $t(X)$ to the action of cohomology operations of $H^*(-\,;\,Z_p)$. Let Q_i be the Milnor operation, i. e., Q_0 is the Bockstein operation and Q_i is defined by $Q_{i-1}\mathscr{P}^{p^i} - \mathscr{P}^{p^i} Q_{i-1}$. Conner proved (see [3]) that if $Q_{i_1}\cdots Q_{i_n}x\neq 0$ for some element $x\in H^*(X\,;\,Z_p)$ then $\mathrm{homdim}_{BP_*} BP_*(X)>n$.

Theorem 4. *If there is an element $x\in H^*(X\,;\,Z_p)$ such that $Q_n x\neq 0$ then $t(X)\geqq n+1$.*

Proof. Consider the Sullivan's exact sequence

$$\begin{array}{ccc} k(n)^*(X) & \xrightarrow{\quad v_n \quad} & k(n)^*(X) \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow{\scriptstyle i} & \\ & H^*(X\,;\,Z_p) & \end{array}$$

By [11], $i\delta=Q_n$, and since $i\delta(x)\neq 0$, i is not epic. Hence from (6) of the definition, we have $t(X)\geqq n+1$.

Remark. The geometrical meaning of Theorem 4 is as follows. First, note that we shall consider in the homology theory taking the Spanier-Whitehead duality. Let $x=[A,f]$, where $[A,f]$ is a manifold with singularities type $(p, v_1, v_2, \ldots)$ in X. By [11], $Q_n[A,f]=[A(n+1), f(n+1)]$ where $[A(n+1), f(n+1)]$ is the normal factor of $(n+1)$-th boundary, i. e., $\partial_{n+1}A\simeq v_n\times A(n+1)$ (for details see [11], [9]). Since $[A(n+1), f(n+1)]\neq 0$, $[A,f]$ has the singularity type v_n. Therefore $[A,f]$ is not representable by a manifold with singularities type $I_n=(p,\ldots, v_{n-1})$. Hence $t(X)>n$.

Examples 1.

(1) Let $L^{2m+1}(p)$ be a $2m+1$ dimensional p-Lens space. Let $p^j\leqq m<p^{j+1}$. It is well known [7] that for $0\leqq i\leqq m$, $H^{2i}(L^{2m+1}(p)\,;\,Z_p)\simeq H^{2i+1}(L^{2m+1}(p)\,;\,Z_p)\simeq Z_p$, and generators are β^i and $\alpha\beta^i$ satisfying $Q_i\alpha=\beta^{p^i}$. Hence from Theorem 4, we have $t(L^{2m+1}(p))>j$. It is also well known that $BP^*(L^{2m+1}(p))\simeq BP^*[x]/([p], x^{2m+1})$ where $[p]$ is the p-product of the formal group law of BP_*. By using the Sullivan's exact sequence, we can prove that $P(j+1)^*(L^{2m+1}(p))\simeq$

$BP^*/I_{j+1}(x\oplus x^2\oplus\ldots\oplus x^m\oplus y_1\oplus y_2\ldots\oplus y_m)$ where $Q_0 y_i = x^i$. Thus we have $t(L^{2m+1}(p)) = j+1$.

(2) Let $V(n)$ be the finite complex such that $BP_*(V(n)) \simeq P(n+1)_*$. $V(n)$ exists for the following cases: $n=0$; $n=1$ and $p\geqq 3$; $n=2$ and $p\geqq 5$; $n=3$ and $p\geqq 7$. If $V(n)$ exists then $t(V(n)) = n+1$ [10].

(3) The converse of Theorem 4 is not true. Indeed, when $X = S^0\cup_{p^2} e^1$, we have $Q_i = 0$ for all i, but $t(X) = 1$.

We next consider the relation of $t(X)$ to the BP_*/p-module structure of $P(1)_*(X) = BP_*(X; Z_p)$. Let $BP_*/p \supset Z_p[v_N, v_{N+1}, \ldots] = P[N]_*$. Then we can take N so large that $P(1)_*(X)$ is a free $P[N]_*$-module [5], [12]. And if $P(1)_*(X)$ is $P[N]_*$-free then $\text{homdim}_{P(1)_*} P(1)_*(X) \leqq N$ [4], [12].

Theorem 5. *If $P(1)_*(X)$ is a free $P[N]_*$-module then $t(X) \leqq N$.*

Proof. Consider the Sullivan's exact sequence

$$\begin{array}{ccc} P(1)_*(X) & \xrightarrow{\quad v_1 \quad} & P(1)_*(X) \\ & \nwarrow \quad \swarrow & \\ & P(2)_*(X) & \end{array}$$

Then v_1-images of $P[N]_*$-module generators of $P(1)_*(X)$ are also $P[N]_*$-module generators except the case of zero. Hence (co) ker v_1 is also $P[N]_*$-free. Thus $P(2)_*(X)$ is also $P[N]_*$-free. Continuing this argument, we see that $P(N)_*(X)$ is $P[N]_*$-free. Since $P(N)_*(X)$ is a $P(N)_*$-module, it is $P(N)_*$-free, hence the proof is completed.

Example 2. The converse of this theorem is not true. If $p\geqq 5$, by [8], there are a finite complex X and a map $v_2^p: X\to X$ such that $BP_*(X) \simeq BP_*/(p, v_1^2)$ and $(v_2^p)_* = v_2^p$. Since there is a map $v_1: X$

$\to X$ such that $(v_1)_*=v_1$, if $Y=X\cup_{v_1v_2}{}_p CX$ then $BP_*(Y)\simeq BP_*/(p, v_1^2, v_1v_2^p)$. Hence $P(1)_*(Y)$ is not $P[2]_*$-free. However, from the exact sequence

$$\longrightarrow P(2)_*(X) \xrightarrow[(v_1v_2^p)_*]{} P(2)_*(X) \longrightarrow P(2)_*(Y) \longrightarrow$$

here $(v_1v_2^p)_*=0$ in $P(2)_*(X)$, we can prove $t(Y)=2$.

The behavior of $\text{homdim}_{BP_*}BP_*(-)$ with respect to the smash product is somewhat complicated, in the following sense.

(1) $\text{homdim}_{BP_*}BP^*(L^{2p^j+1}(p))=1$ and

$\text{homdim}_{BP_*}BP^*(\underbrace{L^{2p^j+1}(p)\wedge\ldots\wedge L^{2p^j+1}(p)}_{j\text{-factors}})\geqq j.$

(2) There is a finite complex V such that $BP_*(V)\simeq BP_*/(p^2, pv_1)$ [10]. Then $\text{homdim}_{BP_*}BP_*(V)=2$ but $\text{homdim}_{BP_*}BP_*(S^0\cup_p e^1\wedge V)=1$.

Theorem 6. $t(X\wedge Y)=\max(t(X), t(Y))$.

Proof. Let $t(Y)\geqq t(X)$ and $t(X)\leqq n$. Then $P(n)_*(X)$ is a free $P(n)_*$-module. The product of $P(n)_*(-)$ theory [9] induces the following map

$$\mu\colon P(n)_*(X)\underset{P(n)_*}{\otimes}P(n)_*(Y)\longrightarrow P(n)_*(X\wedge Y).$$

By the exact functor theorem [12], $P(n)_*(X)\underset{P(n)_*}{\otimes}P(n)_*(-)$ and $P(n)_*(X\wedge -)$ are homology theories with the same coefficient $P(n)_*(X)$. Hence μ is an isomorphism. Therefore we have $t(X\wedge Y)=t(Y)$. This completes the proof.

Let $S^N\to X\to Y$ be a cofibering. Then Johnson-Wilson [3] questioned whether $\text{homdim}_{BP_*}BP_*(Y)\leqq\text{homdim}_{BP_*}BP_*(X)+1$ holds or not.

Example 3. Let X, Y be complexes defined by the following cofibering

$$\begin{array}{ccccc} S^0 & \hookrightarrow & V(1) & \longrightarrow & Y \\ X & \hookrightarrow & Y & \longrightarrow & S^{2p}. \end{array}$$

Then $BP_*(Y) \simeq \text{Ideal}(p, v_1) \simeq BP_*\sigma \oplus BP_*\tau/p\sigma = v_1\tau$ and $BP_*(X) \simeq BP_{*+2p-1} \oplus BP_{*+1}$, so we have $t(X) = t(S^{2p}) = 0$ but $t(Y) = 2$.

Theorem 7. *Let $S^N \to X \to Y$ be a cofibering. Then $t(Y) \leqq t(X) + m$ where m is the number of Z_p-basis of $H_*(X; Z_p)$.*

Proof. Using the Spanier-Whitehead duality, we shall consider in cohomology theories. Let $t(X) \leqq n$. Then we have the exact sequence

$$\longrightarrow P(n)^*(S^M) \xrightarrow[f^*]{} P(n)^* \otimes H^*(DX; Z_p) \longrightarrow P(n)^*(DY) \longrightarrow$$

Let $\{\sigma_1, \ldots, \sigma_m\}$ be a system of $P(n)^*$-basis of $P(n)^*(DX)$ with $\dim\sigma_i \leqq \dim\sigma_{i+1}$, and let τ be a $P(n)^*$-module generator of $P(n)^*(S^M)$. Let $f^*\tau = \sum k_i\sigma_i$ where $k_i \in P(n)^*$.

To prove the theorem, we may assume that $t(DY) \geqq n+m$. By induction on t for $1 \leqq t \leqq m$, we assume that

$$k_i = 0 \bmod (v_n, \ldots, v_{n+t-2}) \text{ for } 1 \leqq i \leqq t-1.$$

Then, by the Cartan formula of r_α, $\alpha > 0$ in $P(n)^*(-)$ theory [11], we have

$$\begin{aligned} 0 = f^* r_\alpha \tau = r_\alpha f^* \tau &= r_\alpha(\textstyle\sum k_i\sigma_i) \\ &= (r_\alpha k_t)\sigma_t + \sum_{j>t} k'_j\sigma_j \bmod (v_n, \ldots, v_{n+t-2}). \end{aligned}$$

Hence $r_\alpha k_t = 0 \bmod (v_n, \ldots, v_{n+t-2})$.

From Proposition 2.11 in [5], we have $k_t = \lambda v_{n+t-1}^s$ or $\lambda \bmod (v_n, \ldots, v_{n+t-2})$ where $\lambda \in Z_p$.

Suppose $k_t = \lambda \neq 0 \bmod (v_n, \ldots, v_{n+t-2})$. Then consider the exact sequence of $P(n+t)^*(-)$ theory

$$\longrightarrow P(n+t)^*(S^M) \xrightarrow[f^*]{} P(n+t)^*(DX) \longrightarrow P(n+t)^*(DY) \longrightarrow$$

Since $f^*\tau$ is a $P(n+t)^*$-module generator of $P(n+t)^*(DX)$,

$P(n+t)^*(DY)$ is also $P(n+t)^*$-free, and so $t(Y)\leqq n+t$. This contradicts to the first assumption $t(Y)\geqq n+m$. Hence $k_t=0 \bmod (v_n, \dots, v_{n+t-1})$. Therefore we have $k_i=0 \bmod(v_n, \dots, v_{n+m-1})$ for $1\leqq i\leqq m$.

Consider the exact sequence of $P(n+m)^*(-)$ theory

$$\longrightarrow P(n+m)^*(S^M)\xrightarrow[f^*]{} P(n+m)^*(DX)\longrightarrow P(n+m)^*(DY)\longrightarrow$$

Since $f^*\tau=\sum k_i\sigma_i=0$ in $P(n+m)^*(DX)$, $P(n+m)^*(DY)$ is also $P(n+m)^*$-free. This completes the proof.

Examples 4.

(1) When X is a 2-cell complex, we have $BP_*(X)\simeq BP_*\oplus BP_*$ or $BP_*/\lambda p^s$. Hence $t(X)\leqq 1$.

(2) When X is a 3-cell complex, we have $BP_*(X)\simeq BP_*\oplus BP_*/\lambda p^s$, $BP_*\oplus BP_*\oplus BP_*$, or $BP_*\sigma\oplus BP_*\tau/p^s\sigma=v_1^r\tau$. Therefore $t(X)\leqq 2$.

(3) When X is a 4-cell complex, consider a cofibering $S^N\xrightarrow[f]{} Y\longrightarrow X$ where Y is a 3-cell complex. If $BP_*(Y)\simeq BP_*\oplus BP_*\oplus BP_*$ then from Theorem 7, $t(X)\leqq 3$. Otherwise, let BP^*-module generators of $BP^*(DY)$, $BP^*(DS^N)$ be σ_1, σ_2, τ where $\dim\sigma_1\leqq\dim\sigma_2$. Let $f^*\tau=k_1\sigma_1+k_2\sigma_2$. Then take the operation r_α for $|\alpha|>0$,

$$0=r_\alpha(f^*(\tau))=(r_\alpha k_1)\sigma_1+\sum_{\alpha=\alpha_1+\alpha_2,\,|\alpha_1|>0} r_{\alpha_1}k_1\cdot r_{\alpha_2}\sigma_1+(r_\alpha k_2)\sigma_2.$$

From (2), $r_\alpha k_1=0 \bmod p$, hence $k_1=0 \bmod(p, v_1)$ or $k_1\in Z_p$, so we have $r_\alpha k_2=0 \bmod(p, v_1)$. Hence, if $t(X)\geqq 3$, $k_2=0 \bmod (p, v_1, v_2)$. Therefore, if $t(X)\geqq 3$, $f^*=0$ in $P(3)^*(DY)$. Thus we have $t(X)\leqq 3$.

Question 1. *If X is an n-cell complex, $t(X)\leqq n-1$?*

It is clear that $P(n)_*(X)$ is not necessarily decided by the BP_*-module structure of $BP_*(X)$, in fact $BP_*(V(1)^{2p-1})\simeq BP_*/I_2\oplus BP_{*+2p-1}\simeq BP_*(V(1)\vee S^{2p-1})$ but $P(n)_*(V(1)^{2p-1})\not\simeq P(n)_*(V(1)\vee S^{2p-1})$ for $n\geqq 1$.

Question 2. *Is $t(X)$ decided by the BP_*-module structure of $BP_*(X)$?*

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