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Ergodic Decomposition of Quasi-Invariant Measures

By

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§1. Introduction

In this paper we shall consider mainly ergodic decomposition of translationally quasi-invariant (simply, quasi-invariant) measures which are defined on the usual Borel σ-field $\mathfrak{B}(\boldsymbol{R}^\infty)$ of $\boldsymbol{R}^\infty$. For the notions of quasi-invariance and ergodicity, we refer [9] or [10]. The set of all probability measures on $\mathfrak{B}(\boldsymbol{R}^\infty)$ will be denoted by $M(\boldsymbol{R}^\infty)$, and the subset of $M(\boldsymbol{R}^\infty)$ which consists of all $\boldsymbol{R}_0^\infty$-quasi-invariant measures will be denoted by $M_0(\boldsymbol{R}^\infty)$. $\boldsymbol{R}_0^\infty$ is the set of all $x=(x_1,\cdots, x_n,\cdots)\in\boldsymbol{R}^\infty$ such that $x_n=0$ except for a finite number of n's. For $t\in\boldsymbol{R}^\infty$ and $\mu\in M(\boldsymbol{R}^\infty)$, we put $\mu_t\in M(\boldsymbol{R}^\infty)$ as follows, $\mu_t(B)=\mu(B-t)$ for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$.

(I) Let $\mu\in M_0(\boldsymbol{R}^\infty)$. If μ is not $\boldsymbol{R}_0^\infty$-ergodic, then we can decompose it to two mutually singular $\mu^j\in M_0(\boldsymbol{R}^\infty)$ $(j=1, 2)$ such that μ is the convex sum of μ^1 and μ^2. If at least, one of μ^j is not $\boldsymbol{R}_0^\infty$-ergodic, then we proceed in the same manner and decompose it to two measures. So the following problem arises naturally.

(P_1) *Let $\mu\in M_0(\boldsymbol{R}^\infty)$. Then is μ represented as a suitable sum of $\boldsymbol{R}_0^\infty$-quasi-invariant and $\boldsymbol{R}_0^\infty$-ergodic measures which are mutually singular?*

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The problem (P_1) was first considered by Skorohod [8]. He obtained such a decomposition of $\mu\in M_0(\boldsymbol{R}^\infty)$ using the family of conditional probability measures $\{\mu^x\}_{x\in\boldsymbol{R}^\infty}$ with respect to the sub-σ-field $\mathfrak{B}_\infty$. (See, §2.) Factor measures μ^x are $\boldsymbol{R}_0^\infty$-quasi-invariant and $\boldsymbol{R}_0^\infty$-ergodic. But μ^x is not mutually singular with each other. In this paper, we shall investigate this decomposition in more detail. Roughly speaking, it will be proved in Theorem 4. 2 that changing the parameter space from $\boldsymbol{R}^\infty$ to $\boldsymbol{R}^1$, we can choose mutually singular factor measures $\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}$. Moreover, Theorem 4. 2 assures that not only the $\boldsymbol{R}_0^\infty$-quasi-invariant measures but also the general $\mu\in M(\boldsymbol{R}^\infty)$ can be represented as a superposition of mutually singular tail-trivial measures. (For the tail-triviality, see §2.) This decomposition will be called a canonical decomposition of μ.

For $\mu\in M(\boldsymbol{R}^\infty)$, we put $T_\mu=\{t\in\boldsymbol{R}^\infty\,|\,\mu_t\simeq\mu\}$, and we shall denote the maximal vector space of T_μ by T_μ^0. Then introducing a suitable metric, T_μ^0 becomes a complete separable metric topological vector space, whose topology is stronger than the induced topology from $\boldsymbol{R}^\infty$. (See, [1].) If $\mu\in M_0(\boldsymbol{R}^\infty)$, then we always have $\boldsymbol{R}_0^\infty\subsetneqq T_\mu^0$. Therefore it is interesting to investigate the following problems.

(P_2) *Let $\mu\in M_0(\boldsymbol{R}^\infty)$. Then for the factor measures $\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}\subset M_0(\boldsymbol{R}^\infty)$ of a canonical decomposition of μ, does $T_{\mu^\tau}^0\supset T_\mu^0$ hold for all $\tau\in\boldsymbol{R}^1$?*

More generally,

(P_3) *Let $\boldsymbol{R}_0^\infty\subset\Phi\subset\boldsymbol{R}^\infty$ and Φ be a complete separable metric linear topological space, whose topology is stronger than the induced topology from $\boldsymbol{R}^\infty$. Let $\mu\in M(\boldsymbol{R}^\infty)$ be Φ-quasi-invariant. Then does $T_{\mu^\tau}^0\supset\Phi$ hold for all $\tau\in\boldsymbol{R}^1$?*

The problem (P_3) will be discussed in §5. A Φ-quasi-invariant measure $\mu\in M(\boldsymbol{R}^\infty)$ is said to be Φ-decomposable if and only if the problem (P_3) is affirmative for μ. In general, Φ-quasi-invariant measures are not necessarily Φ-decomposable. However under the assumption that Φ contains $\boldsymbol{R}_0^\infty$ densely, we do not yet know whether the problem (P_3) is always affirmative or not. But we will obtain an equivalent condition (in Theorem 5.2) for the Φ-decomposability, introducing a notion of strong-Φ-quasi-invariance.

(II) Before discussing (P_1), (P_2) and (P_3) by probabilistic method, we shall look again the problem (P_1) in view of the theory of von Neumann algebra.

Let $\mu \in M_0(\boldsymbol{R}^\infty)$. We form the set $L^2_\mu(\boldsymbol{R}^\infty)$ of all square summable complex-valued functions. Let $U_e(e\in\boldsymbol{R}^\infty_0,\ e=(e_1,\cdots,\ e_n,\cdots))$ and $V_t(t\in\boldsymbol{R}^\infty_0)$ be unitary operators on $L^2_\mu(\boldsymbol{R}^\infty)$ acting for each $f\in L^2_\mu(\boldsymbol{R}^\infty)$ as follows. U_e ; $f(x)\longmapsto \exp(ix(e))\cdot f(x)$, V_t ; $f(x)\longmapsto \sqrt{\frac{d\mu_t}{d\mu}}(x)\cdot f(x-t)$. $x(e)$ is the duality bracket, $x(e)=\sum_{j=1}^\infty e_jx_j$. Let M_μ be a von Neumann algebra generated by $\{U_e\}_{e\in\boldsymbol{R}^\infty_0}$ and $\{V_t\}_{t\in\boldsymbol{R}^\infty_0}$ and M'_μ be its commutant.

Theorem 1.1. $M'_\mu=M_\mu\cap M'_\mu$.

Proof. Since M'_μ is generated by its projection, so it is sufficient to show that any projection $P\in M'_\mu$ belongs to M_μ. Applying the function 1 (whose values are constantly 1) to the both side of $U_eP=PU_e$, we have $[P(\exp(ix(e)))](x)=\exp(ix(e))P(1)(x)$ mod μ. The family of all finite linear combinations of $\{\exp(ix(e))\}_{e\in\boldsymbol{R}^\infty_0}$ is dense in $L^2_\mu(\boldsymbol{R}^\infty)$. It follows that

(1) $Pf(x)=f(x)\cdot P(1)(x)$ mod μ for all $f\in L^2_\mu(\boldsymbol{R}^\infty)$.

As $\int U_eP(1)(x)d\mu(x)=\int U_eP^2(1)(x)d\mu(x)=\int PU_eP(1)(x)d\mu(x)=\int U_eP(1)(x)\cdot P(1)(x)d\mu(x)$, so $\{P(1)(x)\}^2=P(1)(x)$ mod μ, which shows the existence of such $A\in\mathfrak{B}(\boldsymbol{R}^\infty)$ as $P(1)(x)=\chi_A(x)$. (χ_A is the indicator function of A.) Therefore (1) becomes

(2) $Pf(x)=\chi_A(x)\cdot f(x)$ mod μ for all $f\in L^2_\mu(\boldsymbol{R}^\infty)$.

The operator $f(x)\longmapsto\chi_A(x)\cdot f(x)$ belongs to M_μ, (but the proof requires additional arguments.) so $P\in M_\mu$. Thus the assertion of Theorem 1.1 has established. But we shall investigate the set A in more detail. Applying the function 1 to the both sides of $V_tP=PV_t$, from (2) we have $\sqrt{\frac{d\mu_t}{d\mu}}(x)\cdot\chi_A(x)=\sqrt{\frac{d\mu_t}{d\mu}}(x)\chi_A(x-t)$ mod μ. Since

$\frac{d\mu_t}{d\mu}(x)>0 \mod \mu$, so

(3) $\mu(A\ominus[A-t])=0$ for all $t\in \boldsymbol{R}_0^\infty$.

Theorem 1. 2. *Let $\mu\in M_0(\boldsymbol{R}^\infty)$. Then, μ is $\boldsymbol{R}_0^\infty$-ergodic, if and only if M_μ is a factor.* (For the notion of factor, we refer [4].)

Proof. We continue the notation of Theorem 1. 1. If μ is $\boldsymbol{R}_0^\infty$-ergodic, from (3) we have $\mu(A)=1$ or 0. It implies $P=I$ or 0 and $M_\mu \cap M'_\mu = M'_\mu = \{\alpha I\}_{\alpha\in R^1}$. Conversely, let M_μ be a factor. For a set $A\in\mathfrak{B}(\boldsymbol{R}^\infty)$ which satisfies (3), we set P_A ; $f(x)\longmapsto \chi_A(x)\cdot f(x)$. Then $P_A\in M_\mu\cap M'_\mu$ and therefore $P_A=I$ or 0, because P_A is a projection. It follows that $\mu(A)=1$ or 0.

Theorem 1.2 implies that an ergodic decomposition of $\mu\in M_0(\boldsymbol{R}^\infty)$ may be derived from the factor decomposition of M_μ. (See, [5].) However in this paper we shall discuss the ergodic decomposition without using the theory of von Neumann algebra.

§ 2. Conditional Probability Measures with Respect to $\mathfrak{B}_\infty$

Theorem 2. 1. *Let X be a complete separable metric space, $\mathfrak{B}$ be the σ-field generated by its open subset and μ be a probability measure on $\mathfrak{B}$. Let $\mathfrak{A}$ be a sub-σ-field of $\mathfrak{B}$. Then there exists a family $\{\mu(x, \mathfrak{A}, \cdot)\}_{x\in X}$ which satisfies*

(c, 1.) *for any fixed $x\in X$, $\mu(x, \mathfrak{A}, \cdot)$ is a probability measure on $\mathfrak{B}$.*

(c, 2.) *for any fixed $B\in\mathfrak{B}$, $\mu(x, \mathfrak{A}, B)$ is an $\mathfrak{A}$-measurable function of $x\in X$.*

(c, 3.) *$\mu(A\cap B)=\int_A \mu(x, \mathfrak{A}, B)\,d\mu(x)$ for all $A\in\mathfrak{A}$ and for all $B\in\mathfrak{B}$.*

Moreover, if another family $\{\mu^0(x, \mathfrak{A}, \cdot)\}_{x\in X}$ exists and satisfies the above three conditions for the same μ, then $\mu^0(x, \mathfrak{A}, \cdot)=\mu(x, \mathfrak{A}, \cdot)$ for μ-a. e. x. (We shall say that $\mu(x, \mathfrak{A}, \cdot)$ is the conditional probability measure of μ at x with respect to $\mathfrak{A}$, and we shall call *the above fact the decomposition of μ with respect to $\mathfrak{A}$.)*

Proof is omitted. (See, pp. 145-146 in [6].) We remark that for any $A\in\mathfrak{A}$ and for any $f(x)\in L^1_\mu(X)$, $\int_A f(x)\,d\mu(x)=\int_A\{\int f(y)\,\mu(x,\mathfrak{A},dy)\}\,d\mu(x)$ and that $\int f(y)\,\mu(x,\mathfrak{A},dy)$ is an $\mathfrak{A}$-measurable function.

Theorem 2. 2. *Under the same notation in Theorem 2.1, let μ and ν be probability measures on $\mathfrak{B}$. Then $\mu\lesssim\nu$ implies that $\mu(x,\mathfrak{A},\cdot)\lesssim\nu(x,\mathfrak{A},\cdot)$ for μ-a. e. x.*

Proof is omitted. See, [8].

From now on we shall consider the case $X=\boldsymbol{R}^\infty$. For the general element $x\in\boldsymbol{R}^\infty$, let x_n be its nth coordinate, $x=(x_1,\cdots,x_n,\cdots)$. The minimal σ-field with which all the functions $x_1,x_2,\cdots,x_n$ (or x_j, $j=n+1,\cdots$) are measurable will be denoted by $\mathfrak{B}_n$ (or $\mathfrak{B}^n$), respectively. Put $\bigcap_{n=1}^\infty\mathfrak{B}^n=\mathfrak{B}_\infty$. The tail σ-field $\mathfrak{B}_\infty$ coincides with the sub-family of $\mathfrak{B}(\boldsymbol{R}^\infty)$ which consists of all the invariant sets under all translations by the elements of $\boldsymbol{R}_0^\infty$. We say that $\mu\in M(\boldsymbol{R}^\infty)$ is tail-trivial, if μ takes only the values 1 or 0 on $\mathfrak{B}_\infty$.

Theorem 2. 3. (*Skorohod*, [8]) *Let $\mu\in M(\boldsymbol{R}^\infty)$. Consider the decomposition of μ with respect to $\mathfrak{B}_\infty$. Then there exists $\Omega_1\in\mathfrak{B}_\infty$, $\mu(\Omega_1)=1$ such that $\mu(x,\mathfrak{B}_\infty,\cdot)$ is tail-trivial for all $x\in\Omega_1$.*

Proof. In this proof, we shall sometimes write $\mu^x(\cdot)$ instead of $\mu(x,\mathfrak{B}_\infty,\cdot)$, and $\mu(x,\mathfrak{B}^n,\cdot)$ stands for the conditional probability measure of μ with respect to $\mathfrak{B}^n$.

(I) First we shall derive the decomposition of μ^x with respect to $\mathfrak{B}_\infty$ for each $x\in\boldsymbol{R}^\infty$. Let $A\in\mathfrak{B}_\infty$, $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$ and $E\in\mathfrak{B}^n$. We shall calculate $\mu(A\cap B\cap E)$ in two ways. $\mu(A\cap B\cap E)=\int_A\mu^x(B\cap E)\,d\mu(x)$. On the other hand, $\mu(A\cap B\cap E)=\int_{A\cap E}\mu(x,\mathfrak{B}^n,B)\,d\mu(x)=\int_A\int_E\mu(t,\mathfrak{B}^n,B)\,d\mu^x(t)\,d\mu(x)$. Hence, for fixed B and E, $\mu^x(B\cap E)=\int_E\mu(t,\mathfrak{B}^n,B)\,d\mu^x(t)$

for μ-a. e. x. Since $\mathfrak{B}^n$ is countably generated, it follows that there exists $\Omega_B^n \in \mathfrak{B}_\infty$ with $\mu(\Omega_B^n)=1$ such that

(4) for all $x \in \Omega_B^n$, $\mu^x(B \cap E)=\int_E \mu(t, \mathfrak{B}^n, B)\,d\mu^x(t)$ for any $E \in \mathfrak{B}^n$.

On the other hand, $\mu^x(B \cap E)=\int_E \mu^x(t, \mathfrak{B}^n, B)\,d\mu^x(t)$, so from (4)

(5) for all $x \in \Omega_B^n$, $\mu^x(t, \mathfrak{B}^n, B)=\mu(t, \mathfrak{B}^n, B)$ for μ^x-a.e.t.

$\mathfrak{B}(\boldsymbol{R}^\infty)$ is countably generated. It follows that there exists $\Omega^n \in \mathfrak{B}_\infty$ with $\mu(\Omega^n)=1$ such that

(6) for all $x \in \Omega^n$, $\mu^x(t, \mathfrak{B}^n, \cdot)=\mu(t, \mathfrak{B}^n, \cdot)$ for μ^x-a.e.t.

Using the martingale convergence theorem,

(7) $\lim_{n\to\infty}\int |\mu(t, \mathfrak{B}^n, B)-\mu^t(B)|\,d\mu(t)=0$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$, and

(8) $\lim_{n\to\infty}\int |\mu^x(t, \mathfrak{B}^n, B)-\mu^x(t, \mathfrak{B}_\infty, B)|\,d\mu^x(t)=0$ for all $x \in \boldsymbol{R}^\infty$.

From (7) there exist a subsequence $\{n_j\}$ and $T_B \in \mathfrak{B}_\infty$ with $\mu(T_B)=1$ such that

(9) for all $x \in T_B$, $\lim_{j\to\infty}\int |\mu(t, \mathfrak{B}^{n_j}, B)-\mu^t(B)|\,d\mu^x(t)=0.$

Therefore from (6), (8), and (9) we have

(10) for all $x \in \bigcap_{n=1}^\infty \Omega^n \cap T_B$, $\mu^x(t, \mathfrak{B}_\infty, B)=\mu^t(B)$ for μ^x-a. e. t.

Using the fact that $\mathfrak{B}(\boldsymbol{R}^\infty)$ is countably generated, there exists $\Omega^0 \in \mathfrak{B}_\infty$ with $\mu(\Omega^0)=1$ such that

(11) for all $x \in \Omega^0$, $\mu^x(t, \mathfrak{B}_\infty, \cdot)=\mu(t, \mathfrak{B}_\infty, \cdot)$ for μ^x-a. e. t.

Equivalently,

(12) for all $x \in \Omega^0$, $\mu^x(A \cap B)=\int_A \mu^t(B)\,d\mu^x(t)$ for all $A \in \mathfrak{B}_\infty$ and for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$.

(II) We shall prove that for μ-a.e.x, $\mu^x=\mu^t$ for μ^x-a.e.t. Because, $\int |\mu^x(B)-\mu^t(B)|^2 d\mu^x(t)\,d\mu(x)=2\int \{\mu^x(B)\}^2 d\mu(x)-2\int \mu^x(B)\int \mu^t(B)\,d\mu^x(t)\,d\mu(x)=0$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$. It ensures the existence of $S_B \in \mathfrak{B}_\infty$ with $\mu(S_B)=1$ such that for all $x \in S_B$ $\mu^x(B)=\mu^t(B)$ for μ^x-a.e.t. Again using the countably generated property of $\mathfrak{B}(\boldsymbol{R}^\infty)$, we conclude that there exists $\Omega^1 \in \mathfrak{B}_\infty$ with $\mu(\Omega^1)=1$ such that

(13) for all $x \in \Omega^1$, $\mu^x=\mu^t$ for μ^x-a.e.t.

(III) We put $\Omega_1=\Omega^0\cap\Omega^1$. Then from (12) and (13) we have for all $x\in\Omega_1$, $\mu^x(A\cap B)=\int_A\mu^t(B)\,d\mu^x(t)=\mu^x(B)\cdot\mu^x(A)$ for all $A\in\mathfrak{B}_\infty$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. In particular, putting $B=A$, we have $\mu^x(A)=1$ or 0. Therefore μ^x is tail-trivial for all $x\in\Omega_1$. Q. E. D.

For the remainder part of this section, we shall consider the application of Theorem 2.3.

Let Σ be the set of all permutations of natural numbers which shift only finite numbers of n. For each $\sigma\in\Sigma$, we associate the map S_σ on $\boldsymbol{R}^\infty$ such that S_σ; $x=(x_1, x_2,\cdots)\longmapsto(x_{\sigma(1)}, x_{\sigma(2)},\cdots)$. We shall say that $\mu\in M(\boldsymbol{R}^\infty)$ is permutationally-invariant, if $S_\sigma\mu=\mu$ for all $\sigma\in\Sigma$. Let $O(n)$ be the orthogonal group on $\boldsymbol{R}^n$. Naturally $O(n)$ can be considered as a transformation group on $\boldsymbol{R}^\infty$. We set $O_0(\infty)=\bigcup_{n=1}^\infty O(n)$. We shall say that $\mu\in M(\boldsymbol{R}^\infty)$ is rotationally-invariant, if $T\mu=\mu$ for all $T\in O_0(\infty)$.

Proposition 2.1. *Let $\mu\in M(\boldsymbol{R}^\infty)$. Consider the decomposition of μ with respect to $\mathfrak{B}_\infty$. Then, if μ is permutationally-invariant (rotationally-invariant), μ^x is also permutationally-invariant (rotationally-invariant) for μ-a.e.x, respectively.*

Proof. The first assertion is due to the fact that $\{S_\sigma\}_{\sigma\in\Sigma}$ is a countable set. The later one is derived by taking a countable dense set of $O_0(\infty)$. Q. E. D.

Proposition 2.2. *Let $\mu\in M(\boldsymbol{R}^\infty)$ be permutationally-invariant and be tail-trivial. Then there exists a Borel measure m on $\boldsymbol{R}^1$ and μ is the product-measure of the same m.*

Proof. For each j we put $e_j=(0, 0, \cdots, 0, \overset{j}{\breve{1}}, 0, \cdots)$, and consider the integral, $\hat{\mu}(y_1e_1+\cdots+y_ne_n)=\int\exp(ix(\sum_{j=1}^n y_je_j))\,d\mu(x)$. Then for any n and k,

$$\hat{\mu}(y_1e_1+\cdots+y_ne_n)=\int\exp\ (iy_1x_1)\ \exp\ (i\textstyle\sum_{j=2}^n y_jx_{j+k-1})\,d\mu(x)=$$
$$\int E_\mu[\exp(iy_1x_1)\ |\mathfrak{B}^k](x)\cdot\exp(i\textstyle\sum_{j=2}^n y_jx_{j+k-1})\,d\mu(x),$$

where $E_\mu[f|\mathfrak{B}^k](x)$ stands for the conditional expectation of f with respect to $\mathfrak{B}^k$. From the martingale convergence theorem we have, $\int |E_\mu[\exp(iy_1x_1)|\mathfrak{B}^k](x)-E_\mu[\exp(iy_1x_1)|\mathfrak{B}_\infty](x)|d\mu(x) \longrightarrow 0 \ (k\to\infty)$. In virtue of the assumption of μ, $E_\mu[\exp(iy_1x_1)|\mathfrak{B}_\infty](x)=\int\exp(iy_1z_1)d\mu(z)$ for μ-a.e.x. It follows that

$$\begin{aligned}\hat{\mu}(y_1e_1+\cdots+y_ne_n) &= \int\exp(iy_1x_1)d\mu(x)\cdot\lim_{k\to\infty}\int\exp(i\textstyle\sum_{j=2}^n y_jx_{j+k-1})d\mu(x)\\ &= \hat{\mu}(y_1e_1)\cdot\hat{\mu}(y_2e_2+\cdots+y_ne_n).\end{aligned}$$

Repeating this procedure, we have $\hat{\mu}(y_1e_1+\cdots+y_ne_n)=\Pi_{j=1}^n\hat{\mu}(y_je_j)$ for all n and $(y_1, y_2,\cdots, y_n)$. Let m be the Borel measure on $\boldsymbol{R}^1$ such that $\hat{m}(v)=\int_{\boldsymbol{R}^1}\exp(ivu)dm(u)=\hat{\mu}(ve_1)$ for all $v\in\boldsymbol{R}^1$. Then the above formula implies that μ is the product-measure of m. Q. E. D.

Proposition 2.3. *Let $\mu\in M(\boldsymbol{R}^\infty)$ be rotationally-invariant and be tail-trivial. Then there exists 1-dimensional Gaussian measure g_c with mean 0 and variance c and μ is the product-measure of the same g_c (, which will be denoted by G_c).*

Proof. By Proposition 2.2, the proof will be complete if we show $m=g_c$. Now for any n and for $y=y_1e_1+\cdots+y_ne_n$,

$$\begin{aligned}\Pi_{j=1}^n\hat{m}(y_j) &= \int\exp(i\textstyle\sum_{j=1}^n y_jx_j)d\mu(x)=\int\exp(i\|y\|x_1)d\mu(x)\\ &= \hat{m}(\|y\|),\ \ \|y\|=(y_1^2+\cdots+y_n^2)^{\frac{1}{2}}.\end{aligned}$$

Since m is symmetric with respect to origin, so $\hat{m}$ is always real-valued and $\hat{m}(v)=\hat{m}(|v|)$ for all $v\in\boldsymbol{R}^1$. Therefore from the above equality,

(14) $\hat{m}(v)=\{\hat{m}(v/\sqrt{n})\}^n$ for all n and for all $v\in\boldsymbol{R}^1$.

Putting $n=2$ in (14), $\hat{m}(v)\geqq 0$, and taking n sufficiently large, we understand that $\hat{m}(v)>0$ for all $v\in\boldsymbol{R}^1$. It follows that $\infty> -\log\hat{m}(v)=-\lim_{n\to\infty} n\log(\hat{m}(v/\sqrt{n}))=\lim_{n\to\infty} n(1-\hat{m}(v/\sqrt{n}))=\lim_{n\to\infty} n\int_{\boldsymbol{R}^1}(1-\cos(vu/\sqrt{n}))dm(u)$.

Therefore Lebesgue-Fatou's lemma assures that $\infty>\int u^2dm(u)\equiv c$. Again applying Lebesgue's bounded convergence theorem for the above equality, we have $\log \hat{m}(v) = -cv^2/2$. Therefore $m = g_c$.

Q. E. D.

Theorem 2.4 (*Umemura*, [10], [11]). *Let $\mu \in M(\boldsymbol{R}^\infty)$ be rotationally-invariant. Then there exists a Borel probability measure ω on $\boldsymbol{R}^1$ and the following formula holds for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$ and for all $a \in \boldsymbol{R}^1$.*

$$\mu(B \cap \{x \in \boldsymbol{R}^\infty \mid r(x) \leqq a\}) = \int_{[0, a]} G_\tau(B)\, d\omega(\tau).$$

$r(x)$ is the function defined on $\boldsymbol{R}^\infty$ as follows,

$$r(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n} \sum_{j=1}^n x_j^2, & \text{if it exists} \\ 0, & \text{elsewhere.} \end{cases}$$

Proof. We decompose μ with respect to $\mathfrak{B}_\infty$. Then Proposition 2.1 and Proposition 2.3 assure the exitstence of $\alpha(x) \in \boldsymbol{R}^1$ such that $\mu(x, \mathfrak{B}_\infty, \cdot) = G_{\alpha(x)}(\cdot)$ for μ-a.e.x. Since $\alpha(x) = \int y_1^2\, dG_{\alpha(x)}(y) = \int y_1^2\, \mu(x, \mathfrak{B}_\infty, dy)$, so $\alpha(x)$ is a $\mathfrak{B}_\infty$-measurable function. Set $A_c = \{x \in \boldsymbol{R}^\infty \mid r(x) \leqq c\}$. Then $G_\tau(A_c) = 1$ if $\tau \leqq c$, and $G_\tau(A_c) = 0$ if $\tau > c$. So for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$ we have $\mu(B \cap A_c) = \int G_{\alpha(x)}(B \cap A_c)\, d\mu(x) = \int_{x \in \{z \mid \alpha(z) \leqq c\}} G_{\alpha(x)}(B)\, d\mu(x) = \mu(B \cap \{x \in \boldsymbol{R}^\infty \mid \alpha(x) \leqq c\})$.

It follows easily that $\alpha(x) = r(x)$ for μ-a.e.x. Define a measure ω on $\mathfrak{B}(\boldsymbol{R}^1)$ by $\omega(E) = \mu(\{x \in \boldsymbol{R}^\infty \mid r(x) \in E\})$ for all $E \in \mathfrak{B}(\boldsymbol{R}^1)$. Then we have,

$$\mu(B \cap A_c) = \int_{A_c} G_{r(x)}(B)\, d\mu(x) = \int_{[0, c]} G_\tau(B)\, d\omega(\tau).$$

Now the proof is complete. Q. E. D.

§ 3. $\boldsymbol{R}_0^\infty$-Quasi-Invariant Measures and the Tail σ-Field $\mathfrak{B}^\infty$

The content of this section is mainly preparations for our later discussions. But it will be turned out that $\mathfrak{B}_\infty$ plays an essential role for $\boldsymbol{R}_0^\infty$-quasi-invariant measures, since the equivalence relation for these measures are completely determined by the behaviour on $\mathfrak{B}_\infty$. (See, Proposition 3.2.) Let $\mu \in M_0(\boldsymbol{R}^\infty)$. We often refer the following fact assured in [9].

(A) *There exists a positive sequence* $\{a_n\}$ *such that* $T_\mu \supset H_a$, $H_a = \{x \in \boldsymbol{R}^\infty \mid \sum_{j=1}^\infty a_j^2 x_j^2 < \infty\}$.

(B) *There exists* $\sigma \in M_0(\boldsymbol{R}^\infty)$ *such that* $\sigma(H_a) = 1$ *and* σ *is* $\boldsymbol{R}_0^\infty$*-ergodic. (It follows easily that the convolution* $\mu * \sigma$ *of* μ *and* σ *is equivalent with* μ.)

Proposition 3.1. *Let* $\mu \in M_0(\boldsymbol{R}^\infty)$. *Suppose that* $A_0 \in \mathfrak{B}(\boldsymbol{R}^\infty)$ *satisfies the relation,* $\mu(A_0 \ominus [A_0 - t]) = 0$ *for all* $t \in \boldsymbol{R}_0^\infty$. *Then there exists* $A \in \mathfrak{B}_\infty$ *such that* $\mu(A \ominus A_0) = 0$.

Proof. If $\mu(A_0) = 1$ or 0, we take $A = \boldsymbol{R}^\infty$ or $\emptyset$. So we shall assume that $0 < \mu(A_0) < 1$. Define $\mu_j \in M(\boldsymbol{R}^\infty)$ $(j = 1, 2)$ as follows.

$$\mu_1(B) = \frac{\mu(B \cap A_0)}{\mu(A_0)} \text{ and } \mu_2(B) = \frac{\mu(B \cap A_0^c)}{\mu(A_0^c)} \text{ for all } B \in \mathfrak{B}(\boldsymbol{R}^\infty).$$

Then in virtue of the assumption, $\mu_j \in M_0(\boldsymbol{R}^\infty)$. We shall take H_a and σ for μ stated in (A) and (B). Since H_a satisfies the condition of Theorem 1.3 in [9] and contains $\boldsymbol{R}_0^\infty$ densely, so by its Remark, we have $\mu(A_0 \ominus [A_0 - t]) = 0$ for all $t \in H_a$. It follows that μ_j are H_a-quasi-invariant and $\mu_j * \sigma \simeq \mu_j$ $(j = 1, 2)$. The set $A = \{y \in \boldsymbol{R}^\infty \mid \sigma(A_0 - y) = 1\}$ belongs to $\mathfrak{B}_\infty$. As $\mu_1(A_0) = 1$, so $\mu_1 * \sigma(A_0) = 1$ and therefore $\mu_1(A) = 1$. Consequently, $\mu(A^c \cap A_0) = 0$. On the other hand, $\mu_2(A_0) = 0$. So the similar arguments derive that $\mu(A \cap A_0^c) = 0$. Combining it with the above fact, we have $\mu(A \ominus A_0) = 0$. Q. E. D.

Remark 1. Let $\mu \in M_0(\boldsymbol{R}^\infty)$. We take a $\sigma \in M_0(\boldsymbol{R}^\infty)$ which satisfies (A) and (B). Then, $\mu * \sigma(A_0 \cap B) = \int_{A_0} \sigma(B - y)\, d\mu(y)$ for all $A_0 \in \mathfrak{B}_\infty$ and for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$.

Proof. From the proof of Proposition 3.1, we understand that $\mu(A \ominus A_0) = 0$. (A has the same meaning in it.) Since σ is $\boldsymbol{R}_0^\infty$-ergodic, the function $\sigma(A_0 - y)$ of y takes only the value 1 or 0. Thus, $\mu * \sigma(A_0 \cap B) = \int_A \sigma(B - y)\, d\mu(y) = \int_{A_0} \sigma(B - y)\, d\mu(y)$. Q. E. D.

Corollary. *Let* $\mu \in M_0(\boldsymbol{R}^\infty)$. *If* μ *is tail-trivial, then it is*

$\boldsymbol{R}_0^\infty$-ergodic.

Proof. Let $A_0 \in \mathfrak{B}(\boldsymbol{R}^\infty)$ be a some set which satisfies $\mu(A_0 \ominus [A_0 - t]) = 0$ for all $t \in \boldsymbol{R}_0^\infty$. Then Proposition 3. 1 assures the existence of $A \in \mathfrak{B}_\infty$ such as $\mu(A_0 \ominus A) = 0$. In virtue of the assumption, $\mu(A) = 1$ or 0, therefore the same holds for A_0. Q. E. D.

Naturally from Proposition 3.1 a following problem (P) arises.

Let Φ be a subset of $\boldsymbol{R}^\infty$ which satisfies

(S_0) *Φ is a complete separable metric linear topological space. Φ contains $\boldsymbol{R}_0^\infty$ densely and it is continuously imbedded into $\boldsymbol{R}^\infty$.*

Then,

(P) *For any Φ-quasi-invariant-measure $\mu \in M(\boldsymbol{R}^\infty)$, $\mathfrak{B}_\infty = \mathfrak{B}_\Phi$ mod μ? That is, for any $A_0 \in \mathfrak{B}_\infty$, does there exist μ-measurable set A which is invariant under all translations by the elements of Φ and satisfies $\mu(A_0 \ominus A) = 0$?*

We do not yet know if (P) is true or not. Later we shall show that (P) concerns with the ergodic decomposition of Φ-quasi-invariant measures.

Proposition 3. 2. *Let μ, $\nu \in M_0(\boldsymbol{R}^\infty)$. For $\mu \lesssim \nu$, it is necessary and sufficient that $\mu \lesssim \nu$ on $\mathfrak{B}_\infty$.*

Proof. The necessity is obvious. For the sufficiency, we shall take a $\sigma \in M_0(\boldsymbol{R}^\infty)$ which satisfies $\mu * \sigma \simeq \mu$ and $\nu * \sigma \simeq \nu$. Suppose that $\nu(B) = 0$ for some $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$. Then $\int \sigma(B - x) d\nu(x) = 0$. The set $A = \{x \in \boldsymbol{R}^\infty \mid \sigma(B - x) = 0\}$ belongs to $\mathfrak{B}_\infty$ and $\nu(A) = 1$. By the assumption, $\mu(A) = 1$, consequently $\mu * \sigma(B) = 0$, so $\mu(B) = 0$. Q. E. D.

Corollary. *Let μ, $\nu \in M_0(\boldsymbol{R}^\infty)$ be both $\boldsymbol{R}_0^\infty$-ergodic. For $\mu \simeq \nu$, it is necessary and sufficient that $\mu = \nu$ on $\mathfrak{B}_\infty$. If μ and ν are not equivalent, then they are mutually singular.*

Proof. It is immediate from the above proposition.

Let $B_0 \in \mathfrak{B}(\boldsymbol{R}^\infty)$ and $\mu(B_0) > 0$. We define $\mu_{B_0} \in M(\boldsymbol{R}^\infty)$ such that $\mu_{B_0}(B) = \dfrac{\mu(B \cap B_0)}{\mu(B_0)}$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$.

Proposition 3.3. *Let $\mu \in M_0(\boldsymbol{R}^\infty)$. For $\mu_{B_0} \in M_0(\boldsymbol{R}^\infty)$, it is necessary and sufficient that $B_0 \in \mathfrak{B}_\infty$ mod μ. Further under the assumption that Φ satisfies the condition* (S_0), *if μ is Φ-quasi-invariant, then μ_{B_0} is also Φ-quasi-invariant.*

Proof. The sufficiency is obvious. Let $\mu_{B_0} \in M_0(\boldsymbol{R}^\infty)$. We shall take a $\sigma \in M_0(\boldsymbol{R}^\infty)$ which satisfies (A) and (B) for μ and μ_{B_0} simultaneously. Since $\mu_{B_0}(B_0) = 1$, so putting $A = \{x \in \boldsymbol{R}^\infty \mid \sigma(B_0 - x) = 1\} \in \mathfrak{B}_\infty$, we have $\mu_{B_0}(A) = 1$ and therefore $\mu(A^c \cap B_0) = 0$. On the other hand, in virtue of Remark 1, $\mu * \sigma(A \cap B_0^c) = \int_A \sigma(B_0^c - x)\, d\mu(x) = 0$, so $\mu(A \cap B_0^c) = 0$. Therefore $\mu(A \ominus B_0) = 0$ and it shows $B_0 \in \mathfrak{B}_\infty$ mod μ.

If μ is Φ-quasi-invariant, then $\mu(A \ominus [A - \varphi]) = 0$ for all $\varphi \in \Phi$. (See, [9].) It implies that if $\mu_{B_0}(B) = 0$ for some $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$, then $\mu(A \cap [B - \varphi]) = 0$. Consequently, $\mu_{B_0}(B - \varphi) = 0$ for all $\varphi \in \Phi$ and therefore μ_{B_0} is Φ-quasi-invariant. Q. E. D.

Proposition 3.4. *Let Φ satisfy* (S_0). *Suppose that $\mu, \nu \in M_0(\boldsymbol{R}^\infty)$ and μ be Φ-quasi-invariant. Then $\nu \lesssim \mu$ implies that ν is Φ-quasi-invariant.*

Proof. Since $\nu \lesssim \mu$ on $\mathfrak{B}_\infty$, so the Radon-Nikodim derivative $g(x)$ of ν with respect to μ on $\mathfrak{B}_\infty$ is defined. The set $B_0 = \{x \in \boldsymbol{R}^\infty \mid g(x) > 0\}$ belongs to $\mathfrak{B}_\infty$ and $\mu(B_0) > 0$, so we form the measure μ_{B_0}. μ_{B_0} is $\boldsymbol{R}_0^\infty$-quasi-invariant and $\mu_{B_0} \simeq \nu$ on $\mathfrak{B}_\infty$. By Proposition 3.2, it implies $\mu_{B_0} \simeq \nu$. Now the assertion is derived from Proposition 3.3.

Q. E. D.

§4. Ergodic Decomposition of $\boldsymbol{R}_0^\infty$-Quasi-Invariant Measures

Theorem 4.1. *Let $\mu \in M(\boldsymbol{R}^\infty)$. Consider the decomposition of μ*

with respect to $\mathfrak{B}_\infty$. *Suppose that* μ *is* $\boldsymbol{R}_0^\infty$*-quasi-invariant. Then there exists a set* $\Omega_2 \in \mathfrak{B}_\infty$ *with* $\mu(\Omega_2)=1$ *such that* $\mu(x, \mathfrak{B}_\infty, \cdot)$ *is* $\boldsymbol{R}_0^\infty$*-quasi-invariant and* $\boldsymbol{R}_0^\infty$*-ergodic for all* $x \in \Omega_2$.

Proof. In this proof we shall also sometimes write $\mu^x(\cdot)$ instead of $\mu(x, \mathfrak{B}_\infty, \cdot)$. Now we shall take a $\sigma \in M_0(\boldsymbol{R}^\infty)$ which satisfies (A) and (B) in §3. From the Remark 1 of §3, for any $A \in \mathfrak{B}_\infty$ and for any $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$,

$$\mu * \sigma(A \cap B) = \int_A \sigma(B-x)\, d\mu(x) = \int_A \int \sigma(B-y)\, d\mu^x(y)\, d\mu(x)$$
$$= \int_A \mu^x * \sigma(B)\, d(\mu * \sigma)(x).$$

The last equality is due to the fact $\mu = \mu * \sigma$ on $\mathfrak{B}_\infty$. The family $\{\mu^x * \sigma\}_{x \in \boldsymbol{R}^\infty}$ satisfies the three conditions of Theorem 2.1, so Theorem 2.2 assures the existence of $\Omega^1 \in \mathfrak{B}_\infty$ with $\mu(\Omega^1)=1$ such that $\mu^x * \sigma \simeq \mu^x$ for all $x \in \Omega^1$. Since σ is $\boldsymbol{R}_0^\infty$-quasi-invariant, so the same holds for $\mu^x * \sigma$ for all $x \in \boldsymbol{R}^\infty$. Putting $\Omega_2 = \Omega_1 \cap \Omega^1$ (Ω_1 is the set in Theorem 2.3), it follows that μ^x is $\boldsymbol{R}_0^\infty$-quasi-invariant and $\boldsymbol{R}_0^\infty$-ergodic (due to the Corollary of Proposition 3.1) for all $x \in \Omega_2$. Q. E. D.

Lemma 4.1. *Let* $(X, \mathfrak{B})$ *be a measurable space and* $\mathfrak{B}$ *be countably generated. Then there exists a map* p *from* X *to* $\boldsymbol{R}^1$ *such that* $\mathfrak{B}$ *coincides with* $p^{-1}(\mathfrak{B}(\boldsymbol{R}^1))$. ($\mathfrak{B}(\boldsymbol{R}^1)$ *is the usual Borel field on* $\boldsymbol{R}^1$.)

Proof. Let $\{A_j\}_{j=1,2,}$ be a countable subfamily of $\mathfrak{B}$ which generates $\mathfrak{B}$. We may assume that this family is closed for the operation of taking the complement. Put $\varepsilon_j(x)$ for the indicator function of A_j for each j and define the map p as $p(x) = \sum_{j=1}^\infty \varepsilon_j(x)/2^j$ for all $x \in \boldsymbol{R}^\infty$. Now consider the binary expansion of each $\tau \in (0, 1)$, $\tau = \sum_{j=1}^\infty \alpha_j(\tau)/2^j$ with $\alpha_j(\tau)=1$ or 0. The set $X = \{\tau \in (0, 1) \mid \alpha_n(\tau)=0$ and $\alpha_m(\tau)=1$ occur for infinitely many n and $m.\}$ belongs to $\mathfrak{B}(\boldsymbol{R}^1)$ and the range of p is the subset of X. Moreover the binary expansion is uniquely determined on X. It follows that $p^{-1}(\mathfrak{B}(\boldsymbol{R}^1)) \supset \mathfrak{B}$, because $p^{-1}\{\tau \in (0, 1) \mid \alpha_j(\tau)=1\} = A_j \in \mathfrak{B}$ for all j. On the other hand, for the set $E(\delta_1, \delta_2, \cdots, \delta_n) = \{\tau \in (0, 1) \mid \alpha_1(\tau)=\delta_1, \cdots, \alpha_n(\tau)=\delta_n.\}$ ($\delta_j=1$ or 0, $1 \leqq j \leqq n$), we have $p^{-1}(E(\delta_1, \cdots, \delta_n)) = \cap_{j=1}^n A_j^{\delta_j} \in \mathfrak{B}$. ($A^1$ stands for

A itself and A^0 stands for A^c.) Since the inverse image of any open subset G of $\boldsymbol{R}^1$ is a countable union of the above sets $p^{-1}(E(\delta_1, \cdots, \delta_n))$, so $p^{-1}(G)\in\mathfrak{B}$. Hence it follows easily that $p^{-1}(\mathfrak{B}(\boldsymbol{R}^1))=\mathfrak{B}$.

Q. E. D.

Now we shall return to the original $\mu\in M(\boldsymbol{R}^\infty)$. Since $L^2_\mu(\boldsymbol{R}^\infty)$ is separable, so there exists a sub-σ-field $\hat{\mathfrak{B}}$ which satisfies

(a) $\hat{\mathfrak{B}}\subset\mathfrak{B}_\infty$, and for any $A\in\mathfrak{B}_\infty$ there exists $\hat{A}\in\hat{\mathfrak{B}}$ such that $\mu(A\ominus\hat{A})=0$.

(b) $\hat{\mathfrak{B}}$ is countably generated, and therefore a map p from $\boldsymbol{R}^\infty$ to $\boldsymbol{R}^1$ exists such that $p^{-1}(\mathfrak{B}(\boldsymbol{R}^1))=\hat{\mathfrak{B}}$. According to Theorem 2.1, we decompose μ with respect to $\hat{\mathfrak{B}}$. Then for any $A\in\mathfrak{B}_\infty$ and for any $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu(A\cap B)=\mu(\hat{A}\cap B)=\int_{\hat{A}}\mu(x, \hat{\mathfrak{B}}, B)\,d\mu(x)=\int_{A}\mu(x, \hat{\mathfrak{B}}, B)\,d\mu(x)$. So in virtue of Theorem 2.1, a set $\Omega_3\in\mathfrak{B}_\infty$ with $\mu(\Omega_3)=1$ exists such that $\mu(x, \hat{\mathfrak{B}}, \cdot)=\mu(x, \mathfrak{B}_\infty, \cdot)$ for all $x\in\Omega_3$. Therefore for any $x\in\Omega_1\cap\Omega_3$, $\mu(x, \hat{\mathfrak{B}}, \cdot)$ is tail-trivial. Moreover if μ is $\boldsymbol{R}^\infty_0$-quasi-invariant, then for any $x\in\Omega_3\cap\Omega_2$, $\mu(x, \hat{\mathfrak{B}}, \cdot)$ is $\boldsymbol{R}^\infty_0$-quasi-invariant and $\boldsymbol{R}^\infty_0$-ergodic. Let $\{A_j\}_{j=1,2,\ldots}$ be a countable family of $\hat{\mathfrak{B}}$ which generates $\hat{\mathfrak{B}}$. Put for each j, $N^c_j=\{x\in\boldsymbol{R}^\infty\,|\,\chi_{A_j}(x)=\mu(x, \hat{\mathfrak{B}}, A_j)\}$ and $N=\cup_{j=1}N_j$. Then $N\in\hat{\mathfrak{B}}$ and $\mu(N)=0$. It is easily checked that

(*) $x\in N^c \Longleftrightarrow \mu(x, \hat{\mathfrak{B}}, p^{-1}(E))=\chi_E(p(x))$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)\Longleftrightarrow \mu(x, \hat{\mathfrak{B}}, p^{-1}(p(x)))=1$.

Now it is evident that $p(x)=p(y)$ implies $\mu(x, \hat{\mathfrak{B}}, \cdot)=\mu(y, \hat{\mathfrak{B}}, \cdot)$. Conversely, if x, $y\in N^c$ and $p(x)\neq p(y)$, then from (*) we have $\mu(x, \hat{\mathfrak{B}}, \cdot)$ and $\mu(y, \hat{\mathfrak{B}}, \cdot)$ are mutually singular. Define a measure ω on $\mathfrak{B}(\boldsymbol{R}^1)$ by $\omega(E)=\mu(p^{-1}(E))$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$. It is known that there exists a family $\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}\subset M(\boldsymbol{R}^\infty)$ which satisfies

(**) for each fixed $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu^\tau(B)$ is a $\mathfrak{B}(\boldsymbol{R}^1)$-measurable function.

(***) $\mu(B\cap p^{-1}(E))=\int_E\mu^\tau(B)\,d\omega(\tau)$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. (See, [6].)

From (***) we have for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu(B\cap p^{-1}(E))=\int_{p^{-1}(E)}\mu^{p(x)}(B)\,d\mu(x)$, so that there exists $\Omega_4\in\mathfrak{B}_\infty$ with $\mu(\Omega_4)=1$

such that $\mu^{p(x)}(\cdot)=\mu(x, \hat{\mathfrak{B}}, \cdot)$ for all $x\in\Omega_4$. We put $G=p(\Omega_1\cap\Omega_3\cap\Omega_4\cap N^c)$ in the case of $\mu\in M(\boldsymbol{R}^\infty)$ and put $G=p(\Omega_2\cap\Omega_3\cap\Omega_4\cap N^c)$ in the case of $\mu\in M_0(\boldsymbol{R}^\infty)$. Then G is an analytic set, therefore G is ω-measurable and $\omega(G)=1$. (See, [6].) Modifying μ^τ on a set with ω-measure 0, we have,

Theorem 4.2. *Let $\mu\in M(\boldsymbol{R}^\infty)$. Then there exist a family $\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}\subset M(\boldsymbol{R}^\infty)$ and a map p from $\boldsymbol{R}^\infty$ to $\boldsymbol{R}^1$ such that*
(a) $\mu^\tau(B)$ is a $\mathfrak{B}(\boldsymbol{R}^1)$-measurable function for any fixed $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$,
(b) μ^τ is a tail-trivial measure for all $\tau\in\boldsymbol{R}^1$,
(c) $p^{-1}(\mathfrak{B}(\boldsymbol{R}^1))\subset\mathfrak{B}_\infty$,
(d) putting $\omega=p\mu$, $\mu(B\cap p^{-1}(E))=\int_E\mu^\tau(B)d\omega(\tau)$ for all $E\in\mathfrak{B}(\boldsymbol{R}^1)$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$,
(e) there exists a set $E_0\in\mathfrak{B}(\boldsymbol{R}^1)$ with $\omega(E_0)=1$ such that μ^{τ_1} and μ^{τ_2} are mutually singular for all $\tau_1, \tau_2\in E_0$ $(\tau_1\neq\tau_2)$,
(f) moreover, if μ is $\boldsymbol{R}_0^\infty$-quasi-invariant, then μ^τ is $\boldsymbol{R}_0^\infty$-quasi-invariant and $\boldsymbol{R}_0^\infty$-ergodic for all $\tau\in\boldsymbol{R}^1$.
(We shall call the above fact the canonical decomposition of μ and symbolically write $\mu=[\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}, p]$.)

For the uniqueness,

Theorem 4.3. *Let $\mu\in M(\boldsymbol{R}^\infty)$. Consider two canonical decompositions of μ, $[\{\mu_1^\tau\}_{\tau\in\boldsymbol{R}^1}, p_1]$ and $[\{\mu_2^\tau\}_{\tau\in\boldsymbol{R}^1}, p_2]$ which satisfy (a), (b), (c) and (d) in Theorem 4.2. Then there exist $M_j\in\mathfrak{B}(\boldsymbol{R}^1)$ $(j=1, 2)$ and a $\mathfrak{B}(\boldsymbol{R}^1)$-measurable map $s(\tau)$ on $\boldsymbol{R}^1$ such that*
(a) $\omega_j(M_j)=1$ $(j=1, 2)$,
(b) $s(M_1)=M_2$ and s is one to one on M_1,
(c) $\mu_1^\tau=\mu_2^{s(\tau)}$ for all $\tau\in M_1$,
(d) $s\circ p_1(x)=p_2(x)$ for μ-a.e.x.

Proof. By the assumption, μ_1^τ is tail-trivial for all τ, so μ_1^τ takes only the value 0 or 1 on $p_2^{-1}(\mathfrak{B}(\boldsymbol{R}^1))$. It follows that for any $\tau\in\boldsymbol{R}^1$, there exists a unique $s(\tau)\in\boldsymbol{R}^1$ such that $\mu_1^\tau(p_2^{-1}(s(\tau)))=1$. In a similar way, for any $\tau\in\boldsymbol{R}^1$, there exists a unique $t(\tau)\in\boldsymbol{R}^1$ such that

$\mu_2^\tau(p_1^{-1}(t(\tau)))=1$. The map s and t are $\mathfrak{B}(\boldsymbol{R}^1)$-measurable, because for any $E\in\mathfrak{B}(\boldsymbol{R}^1)$,

$$s^{-1}(E)=\{\tau\in\boldsymbol{R}^1\,|\,\mu_1^\tau(p_2^{-1}(E))=1\}\ \text{and}\ t^{-1}(E)=\{\tau\in\boldsymbol{R}^1\,|\,\mu_2^\tau(p_1^{-1}(E)))=1\}.$$

We put $M_1=\{\tau\in\boldsymbol{R}^1\,|\,t\circ s(\tau)=\tau,\ \mu_1^\tau=\mu_2^{s(\tau)}\}$ and $M_2=\{\tau\in\boldsymbol{R}^1\,|\,s\circ t(\tau)=\tau,\ \mu_2^\tau=\mu_1^{t(\tau)}\}$. Then $\tau\in M_1$ implies $s\circ t\circ s(\tau)=s(\tau)$ and $\mu_2^{s(\tau)}=\mu_1^\tau=\mu_1^{t\circ s(\tau)}$, therefore $s(\tau)\in M_2$. If $\tau\in M_2$, then $t\circ s\circ t(\tau)=t(\tau)$ and $\mu_1^{t(\tau)}=\mu_2^\tau=\mu_2^{s\circ t(\tau)}$, therefore $t(\tau)\in M_1$. It follows that $\tau=s\circ t(\tau)\in s(M_1)$.

Consequently, we have $s(M_1)=M_2$ and s is one to one on M_1. Let $A\in\mathfrak{B}_\infty$, and we put $E=\{\tau\in\boldsymbol{R}^1\,|\,\mu_1^\tau(A)=1\}$. Then since $E^c=\{\tau\in\boldsymbol{R}^1\,|\,\mu_1^\tau(A)=0\}$, so $\mu(A\cap p_1^{-1}(E^c))=\int_{E^c}\mu_1^\tau(A)\,d\omega(\tau)=0$, $\mu(A^c\cap p_1^{-1}(E))=\int_E\mu_1^\tau(A^c)\,dw(\tau)=0$. Thus, $\mu(A\ominus p_1^{-1}(E))=0$. It follows from Theorem 4.2 that $\mu(A\cap B)=\mu(p_1^{-1}(E)\cap B)=\int_{p_1^{-1}(E)}\mu_1^{p_1(x)}(B)\,d\mu(x)=\int_A\mu_1^{p_1(x)}(B)\,d\mu(x)$ for all $A\in\mathfrak{B}_\infty$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. Similarly, $\mu(A\cap B)=\int_A\mu_2^{p_2(x)}(B)\,d\mu(x)$ for all $A\in\mathfrak{B}_\infty$ and for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. So in virtue of Theorem 2.1, there exists $\Omega_5\in\mathfrak{B}_\infty$ with $\mu(\Omega_5)=1$ such that

(15) $\mu_1^{p_1(x)}=\mu_2^{p_2(x)}$ for all $x\in\Omega_5$.

Now from (*) in the proof of Theorem 4.2, there exist $N_j\,(j=1,2)$ $\in\mathfrak{B}(\boldsymbol{R}^1)$ with $\omega_j(N_j)=0$ $(\omega_j=p_j\mu)$ such that

(16) $\mu_j^\tau(p_j^{-1}(\tau))=1$ for all $\tau\in N_j^c$.

Put $p_1^{-1}(N_1^c)\cap p_2^{-1}(N_2^c)\cap\Omega_5=\Omega_6$. Then $\mu(\Omega_6)=1$ and therefore $\omega_j(p_j(\Omega_6))=1$ for $j=1,2$. If $x\in\Omega_6$, then by (15), (16) and the definition of $s(\tau)$, we have

$$\mu_2^{p_2(x)}(p_2^{-1}(p_2(x)))=1=\mu_1^{p_1(x)}(p_2^{-1}(s\circ p_1(x)))=\mu_2^{p_2(x)}(p_2^{-1}(s\circ p_1(x))).$$

It follows that $p_2(x)=s\circ p_1(x)$ for all $x\in\Omega_6$. Similarly, $p_1(x)=t\circ p_2(x)$ for all $x\in\Omega_6$. From (15), if $\tau=p_1(x)\in p_1(\Omega_6)$, then $\tau=t\circ p_2(x)=t\circ s(\tau)$ and $\mu_1^\tau=\mu_2^{s(\tau)}$, so that $p_1(\Omega_6)\subset M_1$. Similarly, $p_2(\Omega_6)\subset M_2$. Therefore $\omega_j(M_j)=1$ for $j=1,2$. Q. E. D.

§ 5. Φ-Decomposable Measures

Definition 5.1. *Let* $\boldsymbol{R}_0^\infty\subset\Phi\subset\boldsymbol{R}^\infty$, *and let* $\mu\in M_0(\boldsymbol{R}^\infty)$ *be a*

Φ-quasi-invariant measure. Consider a canonical decomposition $\mu=[\{\mu^\tau\}_{\tau\in R^1}, p]$. We say that μ is Φ-decomposable if there exists $F\in\mathfrak{B}(R^1)$ with $\omega(F)=1$ $(\omega=p\mu)$ such that μ^τ is Φ-quasi-invariant for all $\tau\in F$.

The definition does not depend on a particular choice of canonical decompositions due to Theorem 4.3. In this section we shall study Φ-decomposable measures. From now on we shall demand that Φ satisfies the following condition (S).

(S) *Φ is a complete separable metric linear topological space, Φ is continuously imbedded into R^∞, and Φ contains R_0^∞.*

The following example shows that Φ-quasi-invariant measures are not necessarily Φ-decomposable. However under the assumption that Φ satisfies the condition (S_0) in §3, we do not yet know whether Φ-quasi-invariant measures are always Φ-decomposable or not.

Example. Set $e=(1, 1, \cdots, 1, \cdots)\in R^\infty$ and let $G\in M(R^\infty)$ be the product-measure of the 1-dimensional Gaussian measure with mean 0 and variance 1. We define $\lambda\in M(R^\infty)$ such that $\lambda(B)=\int_{R^1} G(B-\tau e)dm(\tau)$ for all $B\in\mathfrak{B}(R^\infty)$, where m is a definite probability measure on $\mathfrak{B}(R^1)$ which is equivalent with the Lebesgue measure. It is easily checked $\tau e\in T_\lambda$ for all $\tau\in R^1$. Moreover, from the l^2-quasi-invariance of G, we understand that λ is Φ-quasi-invariant, $\Phi=\{h+\tau e\,|\,h\in l^2 \text{ and } \tau\in R^1\}$. Φ becomes a separable Hilbert space with the norm $|||\cdot|||$ defined by $|||h+\tau e|||=\sqrt{\|h\|^2+\tau^2}$, ($\|\cdot\|$ is the l^2-norm.) and the injection from Φ to R^∞ is continuous. (But R_0^∞ is not a dense subset of Φ.) Let $n(x)$ be the function defined on R^∞ such that

$$n(x)=\begin{cases}\lim\limits_{n\to\infty}\dfrac{1}{n}\sum_{j=1}^n x_j & \text{if it exists}\\ 0 & \text{elsewhere.}\end{cases}$$

Then for any $a\in R^1$ and for any $B\in\mathfrak{B}(R^\infty)$, $\lambda(B\cap\{x\in R^\infty\,|\,n(x)\leqq a\})=\int_{-\infty}^a G(B-\tau e)dm(\tau)$. It follows that $\{G_{\tau e}\}_{\tau\in R^1}$ and $n(x)$ satisfies the conditions of Theorem 4.2. Therefore a canonical decomposition of λ is $[\{G_{\tau e}\}_{\tau\in R^1}, n(x)]$. However, $G_{\tau e}$ is strictly-l^2-quasi-invariant. So λ is not Φ-decomposable.

Let Φ satisfy (S). We shall denote the Borel σ-field generated by open subsets of Φ by $\mathfrak{B}(\Phi)$. Consider a transformation T on $\boldsymbol{R}^\infty$ which is represented as follows. $T(x)=x+\varphi(x)$ for all $x\in\boldsymbol{R}^\infty$ and $\varphi(x)$ is a measurable map from $(\boldsymbol{R}^\infty, \mathfrak{B}_\infty)$ to $(\Phi, \mathfrak{B}(\Phi))$. The set of all such T is denoted by $\mathscr{T}(\Phi)$.

Definition 5.2. *Let $\mu\in M(\boldsymbol{R}^\infty)$ and Φ satisfy (S). We shall say that μ is strongly-Φ-quasi-invariant, if and only if $\mu\lesssim T\mu$ for all $T\in\mathscr{T}(\Phi)$.*

Clearly the strong-Φ-quasi-invariance implies the usual Φ-quasi-invariance. But we do not yet know if the converse assertion is true, under the assumption that Φ satisfies the condition (S_0) in §3.

Proposition 5.1. *Let Φ satisfy* (S). *Suppose that $\mu\in M(\boldsymbol{R}^\infty)$ is strongly-Φ-quasi-invariant. Then for any $A_0\in\mathfrak{B}_\infty$ with $\mu(A_0)>0$, μ_{A_0} is also a strongly-Φ-quasi-invariant measure.*

Proof. Let $T\in\mathscr{T}(\Phi)$, $T(x)=x+\varphi(x)$, and $T\mu_{A_0}(B)=0$ for some $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. We put $\varphi_1(x)=\chi_{A_0}(x)\cdot\varphi(x)$ for all $x\in\boldsymbol{R}^\infty$ and define $T_1\in\mathscr{T}(\Phi)$ as $T_1(x)=x+\varphi_1(x)$. Then $T_1^{-1}(B\cap A_0)\subset T^{-1}(B)\cap A_0$, so $T_1\mu(B\cap A_0)=0$. From the assumption, therefore we have $\mu(B\cap A_0)=0$. It shows that $T\mu_{A_0}\gtrsim\mu_{A_0}$ for all $T\in\mathscr{T}(\Phi)$. Q. E. D.

Proposition 5.2. *Let Φ satisfy* (S). *If μ is strongly-Φ-quasi-invariant, then $\mu(T^{-1}(A)\ominus A)=0$ for all $T\in\mathscr{T}(\Phi)$ and for all $A\in\mathfrak{B}_\infty$.*

Proof. We set $A_0=T^{-1}(A)\ominus A$. Then $0=\mu_{A_0}(T^{-1}(A)\cap A)=\mu_{A_0\cap A}(T^{-1}(A))$. Therefore we have $\mu(A_0\cap A)=0$ by Proposition 5.1. The set A_0 does not change if we take A^c instead of A. So the same arguments derive that $\mu(A_0 \cap A^c)=0$, and therefore $\mu(A_0)=0$.

Q. E. D.

Proposition 5. 3. *Let Φ satisfy* (S). *Suppose that $\mu \in M_0(\boldsymbol{R}^\infty)$ and $T\mu=\mu$ on $\mathfrak{B}_\infty$ for all $T \in \mathscr{T}(\Phi)$. Then μ is strongly-Φ-quasi-invariant.*

Proof. In virtue of Proposition 3.2, we have only to check that $T\mu \in M_0(\boldsymbol{R}^\infty)$. But it is immediate from the fact, $T^{-1}(B)+t=T^{-1}(B+t)$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$ and for all $t \in \boldsymbol{R}_0^\infty$. Q. E. D.

Settling these arguments,

Theorem 5. 1. *Let Φ satisfy* (S). *Then the following conditions are all equivalent for $\mu \in M(\boldsymbol{R}^\infty)$.*

(*a*) *μ is strongly-Φ-quasi-invariant.*

(*b*) *μ is $\boldsymbol{R}_0^\infty$-quasi-invariant and $\mu(T^{-1}(A) \ominus A)=0$ for all $T \in \mathscr{T}(\Phi)$ and for all $A \in \mathfrak{B}_\infty$.*

(*c*) *μ is $\boldsymbol{R}_0^\infty$-quasi-invariant, and $T\mu=\mu$ on $\mathfrak{B}_\infty$ for all $T \in \mathscr{T}(\Phi)$.*

(*d*) *$T\mu \simeq \mu$ for all $T \in \mathscr{T}(\Phi)$.*

Proposition 5. 4. *Let Φ satisfy* (S). *Suppose that $\mu \in M(\boldsymbol{R}^\infty)$ is Φ-quasi-invariant and $\boldsymbol{R}_0^\infty$-ergodic ($\Longrightarrow$ Φ-ergodic). Then μ is strongly-Φ-quasi-invariant.*

Proof. Let $T \in \mathscr{T}(\Phi)$, $T(x)=x+\varphi(x)$. We shall denote the nth coordinate of $\varphi(x)$ by $\varphi_n(x)$ for each n. Then $\varphi_n(x)$ is a $\mathfrak{B}_\infty$-measurable function, so the ergodic assumption assures the existence of $\varphi_n \in \boldsymbol{R}^1$ such that $\varphi_n(x)=\varphi_n$ for μ-a.e.x. It follows that $(\varphi_1, \cdots, \varphi_n, \cdots) \equiv \varphi=\varphi(x)$ for μ-a.e.x and therefore $\varphi \in \Phi$. Consequently $T(x)$ may be regarded as the translation map by φ, so $T\mu=\mu_\varphi \simeq \mu$.

Q. E. D.

Proposition 5. 5. *Let Φ satisfy* (S) *and $(X, \mathfrak{B}, \lambda)$ be a measure space. Suppose that a family $\{\mu^\alpha\}_{\alpha \in X} \subset M(\boldsymbol{R}^\infty)$ is given such that $\mu^\alpha(B)$*

is a $\mathfrak{B}$-measurable function of α for any fixed $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. If μ^α is strongly-Φ-quasi-invariant for λ-a.e. $\alpha\in X$, then a measure $\mu\in M(\boldsymbol{R}^\infty)$ defined by $\mu(B)=\int_X \mu^\alpha(B)d\lambda(\alpha)$ for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$ is also strongly-Φ-quasi-invariant.

Proof. If follows easily from (*b*) of Theorem 5. 1. Q. E. D.

Thus, from Theorem 4. 2, Proposition 5. 4, and Proposition 5. 5. the necessary part of the following theorem has established.

Theorem 5. 2. *Let Φ satisfy* (S). *Then for $\mu\in M(\boldsymbol{R}^\infty)$, μ is Φ-decomposable if and only if μ is strongly-Φ-quasi-invariant.*

The proof of sufficiency will be derived from following discussions. Let μ be strongly-Φ-quasi-invariant. Consider a canonical decomposition of μ, $\mu=[\{\mu^\tau\}_{\tau\in\boldsymbol{R}^1}, p]$. For the general element μ_1, $\mu_2\in M(\boldsymbol{R}^\infty)$, we put $d(\mu_1, \mu_2)=\sup_F\{\int F(x)d\mu_1(x)-\int F(x)d\mu_2(x)\}$, where $F(x)$ is a $\mathfrak{B}(\boldsymbol{R}^\infty)$-measurable function such that $|F(x)|\leqq 1$. (The metric d is same with the Kakutani's metric.)

Lemma 5. 1. *Let Φ satisfy* (S). *Then $d(\mu^\tau_\varphi, \mu^\tau)$ is a $\mathfrak{B}(\boldsymbol{R}^1)\times\mathfrak{B}(\Phi)$-measurable function of $(\tau, \varphi)\in\boldsymbol{R}^1\times\Phi$.*

Proof. Let $\mathscr{F}=\cup_{n=1} p_n^{-1}(C_0(\boldsymbol{R}^n))$, where $p_n^{-1}(C_0(\boldsymbol{R}^n))$ is the set of functions $F(x)$ defined on $\boldsymbol{R}^\infty$ such that $F(x)=f(p_n(x))$ for some $f\in C_0(\boldsymbol{R}^n)$. $C_0(\boldsymbol{R}^n)$ is the set of all continuous functions with compact support defined on $\boldsymbol{R}^n$ and p_n is the map,

$$x=(x_1, x_2, \cdots, x_n, \cdots)\in\boldsymbol{R}^\infty\longmapsto(x_1, \cdots, x_n)\in\boldsymbol{R}^n.$$

Since $d(\mu^\tau_\varphi, \mu^\tau)=\sup_{\substack{F\in\mathscr{F}\\ |F|\leqq 1}}\int\{F(x)-F(x+\varphi)\}d\mu^\tau(x)$, and $\mathscr{F}$ is separable in the uniform norm on $\boldsymbol{R}^\infty$, so for the proof it will be sufficient that $I_F(\tau, \varphi)=\int\{F(x)-F(x+\varphi)\}d\mu^\tau(x)$ is a measurable function of (τ, φ) for each $F\in\mathscr{F}$. Since $F\in\mathscr{F}$, so there exists some n such that $I_F(\tau, \varphi)=I_F(\tau, \varphi_1, \cdots, \varphi_n)$, where $\varphi=(\varphi_1, \cdots, \varphi_n, \cdots)$. From the well-known theorem

(*) *For any fixed* $\tau\in\boldsymbol{R}^1$, $I_F(\tau, \varphi_1, \cdots, \varphi_n)$ *is a continuous function of* $(\varphi_1, \cdots, \varphi_n)\in\boldsymbol{R}^n$.

On the other hand, it is evident that

(**) *For any fixed* $\varphi\in\Phi$, $I_F(\tau, \varphi_1, \cdots, \varphi_n)$ *is a* $\mathfrak{B}(\boldsymbol{R}^1)$*-measurable function of* $\tau\in\boldsymbol{R}^1$.

Therefore from (*) and (**), it follows that $I_F(\tau, \varphi_1, \cdots, \varphi_n)$ is a $\mathfrak{B}(\boldsymbol{R}^1)\times\mathfrak{B}(\boldsymbol{R}^n)$-measurable function, consequently $I_F(\tau, \varphi)$ is a $\mathfrak{B}(\boldsymbol{R}^1)\times\mathfrak{B}(\Phi)$-measurable function. Q. E. D.

Corollary. $\hat{S}_\mu=\{(\tau, \varphi)\in\boldsymbol{R}^1\times\Phi\,|\,\mu^\tau$ *and* μ^τ_φ *are mutually singular.*$\}$ *is a* $\mathfrak{B}(\boldsymbol{R}^1)\times\mathfrak{B}(\Phi)$*-measurable set.*

Proof. Two measures $\mu_1, \mu_2\in M(\boldsymbol{R}^\infty)$ are mutually singular if and only if $d(\mu_1, \mu_2)=2$. So the proof is immediate from Lemma 5. 1. Q. E. D.

Now we shall return to the proof of Theorem 5. 2. Let q be the projection from $\boldsymbol{R}^1\times\Phi$ to $\boldsymbol{R}^1$. Then $q(\hat{S}_\mu)=S_\mu$ is an analytic set of $\boldsymbol{R}^1$ (See, [6].), and there exists a map $\hat{\varphi}(\tau)$ from S_μ to $\hat{S}_\mu$ which satisfies

(a) $q\circ\hat{\varphi}(\tau)=\tau$ for all $\tau\in S_\mu$,

(b) the set $\{\tau\in S_\mu\,|\,\hat{\varphi}(\tau)\in B\}$ is a universally-measurable set for any Borel set B of $\boldsymbol{R}^1\times\Phi$. (See, [5].)

We shall extend the domain of $\hat{\varphi}$ to $\boldsymbol{R}^1$ defining as $\hat{\varphi}(\tau)=(\tau, 0)$ for $\tau\in S^c_\mu$ and denote it by the same letter $\hat{\varphi}$. Put $\hat{\varphi}(\tau)=(\tau, \varphi'(\tau))$. Then $\varphi'(\tau)\in\Phi$ for all $\tau\in\boldsymbol{R}^1$ and the set $\{\tau\in\boldsymbol{R}^1\,|\,\varphi'(\tau)\in B\}$ is a universally-measurable set for all $B\in\mathfrak{B}(\Phi)$. It follows that a Borel map $\varphi''(\tau)$ from $\boldsymbol{R}^1$ to $\boldsymbol{R}^\infty$ exists and $\varphi'(\tau)=\varphi''(\tau)$ for ω-a.e.τ. If $\omega(S_\mu)=0$, then the proof will be complete. So we shall assume that $\omega(S_\mu)>0$ and derive a contradiction. We take a set $F_0\in\mathfrak{B}(\boldsymbol{R}^1)$ such that $F_0\subset\{\tau\in\boldsymbol{R}^1\,|\,\varphi'(\tau)=\varphi''(\tau)\}\cap S_\mu$ and $\omega(F_0)=\omega(S_\mu)$, and define a map $\varphi(\tau)$ from $\boldsymbol{R}^1$ to Φ as follows.

$$\varphi(\tau)=\begin{cases}\varphi''(\tau) & \text{if } \tau\in F_0\\ 0 & \text{if } \tau\in F^c_0.\end{cases}$$

Then $\varphi(\tau)$ is a Borel map and $(\tau, \varphi(\tau)) \in \hat{S}_\mu$(i. e., $\mu^\tau_{\varphi(\tau)}$ and μ^τ are mutually singular.) for all $\tau \in F_0$.
In this step we remark that from $(*)$ in the proof of Theorem 4. 2,

$(*)$ For ω-a.e.τ, $p(x) = \tau$ for μ^τ-a.e.x.

Now we put $T(x) = x + \varphi(p(x))$ for all $x \in \boldsymbol{R}^\infty$. Then $T \in \mathscr{T}(\Phi)$. In virtue of $(*)$, we have $T\mu^\tau = \mu^\tau_{\varphi(\tau)}$ for ω-a.e.τ. Since μ is strongly-Φ-quasi-invariant, $\mu(T^{-1}(p^{-1}(E)) \ominus p^{-1}(E)) = 0$ for all $E \in \mathfrak{B}(\boldsymbol{R}^1)$, so that $T\mu(p^{-1}(E) \cap B) = \int_E T\mu^\tau(B)\, d\omega(\tau) = \int_E \mu^\tau_{\varphi(\tau)}(B)\, d\omega(\tau)$. Since $T\mu = \mu$ on $\mathfrak{B}_\infty$, $[\{\mu^\tau_{\varphi(\tau)}\}_{\tau \in \boldsymbol{R}^1}, p]$ is a canonical decomposition of $T\mu$. Therefore from Theorem 2.2, $T\mu \simeq \mu$ implies $\mu^\tau \simeq \mu^\tau_{\varphi(\tau)}$ for ω-a.e.τ. This is a contradiction. Q. E. D.

Remark 2. We shall discuss the problem (P) in §3 concerning the above arguments.

Let Φ satisfy the condition (S_0) in §3. If the problem (P) is affirmative, we can deduce that any Φ-quasi-invariant measure $\mu \in M(\boldsymbol{R}^\infty)$ is always Φ-decomposable as follows. (Consequently, the notion of strong-Φ-quasi-invariance coincides with the notion of usual Φ-quasi-invariance.) Let $\mu \in M(\boldsymbol{R}^\infty)$ be Φ-quasi-invariant. For the Φ-decomposability, as we have seen in the proof of Theorem 5. 2, it will be sufficient that $T_p\mu \simeq \mu$ for any $T_p \in \mathscr{T}(\Phi)$ which is represented as $T_p(x) = x + \varphi(p(x))$, where $\varphi(\tau)$ is a measurable map from $(\boldsymbol{R}^1, \mathfrak{B}(\boldsymbol{R}^1))$ to $(\Phi, \mathfrak{B}(\Phi))$. Since μ is Φ-quasi-invariant, so for a fixed $\varphi_0 \in \Phi$, $\mu^\tau_{\varphi_0}$ and μ^τ are equivalent for ω-a.e.τ. Since $T_p\mu(B) = \int \mu^\tau_{\varphi(\tau)}(B)\, d\omega(\tau)$ for all $B \in \mathfrak{B}(\boldsymbol{R}^\infty)$, it follows that $T_p\mu$ is also Φ-quasi-invariant. Put $\lambda = \frac{T_p\mu + \mu}{2}$. Then using the assumption of the problem (P) for λ, for any $A \in \mathfrak{B}_\infty$, there exists λ-measurable set $\hat{A}$ which is invariant under all translations by $\varphi \in \Phi$, and satisfies $\lambda(A \ominus \hat{A}) = 0$. Therefore $\mu(A \ominus \hat{A}) = 0$ and $\mu(T_p^{-1}(A) \ominus T_p^{-1}(\hat{A})) = 0$. Since $T_p^{-1}(\hat{A}) = \hat{A}$, so $\mu(T_p^{-1}(A) \ominus A) = 0$. It shows that $T_p\mu = \mu$ on $\mathfrak{B}_\infty$, so that $T_p\mu \simeq \mu$.

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