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Some Remarks on C^*-Dynamical Systems with a Compact Abelian Group

By

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Abstract

A C^*-dynamical system consisting of a C^*-algebra $\mathfrak{A}$ and an action α of a compact abelian group G as a group of automorphisms of $\mathfrak{A}$ is investigated.

An explicit structure of the C^*-crossed product $C^*(\mathfrak{A};\alpha)$ of $\mathfrak{A}$ by α is given in terms of the spectral subspaces $\mathfrak{A}^\alpha(p)$, $p\in\hat{G}$ of $\mathfrak{A}$.

If $\mathfrak{A}$ has a strictly positive element and if the closed ideal of the fixed point algebra $\mathfrak{A}^\alpha$ generated by $\mathfrak{A}^\alpha(p)^*\mathfrak{A}^\alpha(p)$ is $\mathfrak{A}^\alpha$ itself for any $p\in\hat{G}$, then $C^*(\mathfrak{A};\alpha)$ is shown to be stably isomorphic to $\mathfrak{A}^\alpha\otimes\mathscr{C}(L^2(G))$.

For the Connes-Olesen invariant $\Gamma(\alpha)$, it is shown that $p\in\Gamma(\alpha)$ if and only if $\hat{\alpha}_p(I)\mathrm{I}\neq(0)$ for any non-zero closed ideal I of $C^*(\mathfrak{A};\alpha)$ where $\hat{\alpha}$ is the action of $\hat{G}$ on $C^*(\mathfrak{A};\alpha)$ dual to the action α on $\mathfrak{A}$.

The relative commutant of $\mathfrak{A}^\alpha$ in $\mathfrak{A}$ is shown to be commutative if $G=T^1$ or $Z/(p)$ and to be of type I if G is finite or the product of T^1 with a finite group.

§ 1. Introduction and Main Results

In the study of C^*-dynamical systems, one of the important tasks is the analysis of the structure of continuous C^*-algebra crossed products. At the moment our knowledge on this problem is very limited. (See [1], [7] and [9].) In this note we try to add a little more information on the structure and basic properties of the crossed product of a C^*-algebra by a compact abelian group. As a related problem we also examine the relative commutant of the fixed point algebra.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact

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abelian group G. Let $\mathfrak{A}^\alpha(p)$ be the spectral subspace of α at $p\in\hat{G}$.

For a Hilbert space $\mathscr{H}$, $\mathscr{C}(\mathscr{H})$ denotes the C^*-algebra of all compact operators on $\mathscr{H}$. We consider the group C^*-algebra $C^*(G)$ of G as a C^*-subalgebra of $\mathscr{C}(L^2(G))$ in the following:

Theorem 1. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group G. Then there is a unitary representation $\hat{u}$ of $\hat{G}$ in $L^2(G)$ such that the C^*-crossed product $C^*(\mathfrak{A}; \alpha)$ is isomorphic to the closed span of tensor products $\mathfrak{A}^\alpha(p)\otimes C^*(G)\hat{u}_p^*$ with $p\in\hat{G}$ in $\mathfrak{A}\otimes\mathscr{C}(L^2(G))$.*

The $\hat{u}$ will be defined in section 3.

Following [2], two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are called stably isomorphic if $\mathfrak{A}\otimes\mathscr{C}(\mathscr{H})$ and $\mathfrak{B}\otimes\mathscr{C}(\mathscr{H})$ are isomorphic where $\mathscr{H}$ is a separable infinite-dimensional Hilbert space.

A positive element x of a C^*-algebra $\mathfrak{A}$ is called strictly positive if $\phi(x)>0$ for any state ϕ of $\mathfrak{A}$.

Theorem 2. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group G. For each $p\in\hat{G}$ let I_p be the closed ideal of $\mathfrak{A}^\alpha=\mathfrak{A}^\alpha(1)$ generated by $\mathfrak{A}^\alpha(p)^*\mathfrak{A}^\alpha(p)$. If $\mathfrak{A}$ has a strictly positive element and if $I_p=\mathfrak{A}^\alpha$ for all $p\in\hat{G}$, then $C^*(\mathfrak{A}; \alpha)$ is stably isomorphic to $\mathfrak{A}^\alpha\otimes\mathscr{C}(L^2(G))$.*

Let $\Gamma(\alpha)$ be the Connes-Olesen invariant of the C*-dynamical system $(\mathfrak{A}, G, \alpha)$ and let $\hat{\alpha}$ be the action of $\hat{G}$ on $C^*(\mathfrak{A}; \alpha)$ dual to the action α of G.

Theorem 3. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group G. Tnen $p\in\Gamma(\alpha)$ if and only if $\hat{\alpha}_p(I)I\neq(0)$ for any non-zero closed two-sided ideal I of $C^*(\mathfrak{A}; \alpha)$.*

Theorem 4. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group G. Let $\mathfrak{A}\cap(\mathfrak{A}^\alpha)'$ be the relative commutant of $\mathfrak{A}^\alpha$ in $\mathfrak{A}$. If $\mathfrak{A}\cap(\mathfrak{A}^\alpha)'\neq(0)$, the following statements hold:*

(i) *If $G=T^1$ or $Z/(p)$, then $\mathfrak{A}\cap(\mathfrak{A}^\alpha)'$ is commutative;*

(ii) *If G is finite or the product group of T^1 with a finite group,* *then* $\mathfrak{A}\cap(\mathfrak{A}^{\alpha})'$ *is of type* I.

In general, $\mathfrak{A}\cap(\mathfrak{A}^{\alpha})'$ is not of type I even if $G=T^{2}$.

We shall give the proofs of Theorems 1 to 4 in sections 3 to 6, respectively.

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§ 2. Notation and Preliminaries

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group. The C^*-crossed product $C^*(\mathfrak{A};\alpha)$ of $\mathfrak{A}$ by α is defined as the enveloping C^*-algebra of $L^1_\alpha(G;\mathfrak{A})$, the set of all Bochner integrable $\mathfrak{A}$-valued functions on G equipped with the following Banach $*$-algebra structure:

$$(xy)(g)=\int_G x(h)\alpha_h(y(h^{-1}g))\,dh,$$
$$x^*(g)=\alpha_g(x(g^{-1}))^*,$$
$$\|x\|_1=\int_G \|x(g)\|\;dg,$$

where dg is the normalized Haar measure on G.

For a representation ρ of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\rho$, let $\mathrm{Ind}\,\rho$ be a representation of $L^1_\alpha(G;\mathfrak{A})$ on $L^2(G;\mathscr{H}_\rho)$ such that

$$(\mathrm{Ind}\,\rho(x)\eta)(g)=\int_G \rho\circ\alpha_g^{-1}(x(h))\eta(h^{-1}g)\,dh$$

for $x\in L^1_\alpha(G;\mathfrak{A})$ and $\eta\in L^2(G;\mathscr{H}_\rho)$. If ρ is faithful, $\mathrm{Ind}\,\rho$ can be extended to a faithful representation of $C^*(\mathfrak{A};\alpha)$ ([10]).

For $x\in L^1_\alpha(G;\mathfrak{A})$ and $p\in\hat{G}$ let $\hat{\alpha}_p(x)$ be an element of $L^1_\alpha(G;\mathfrak{A})$ such that $\hat{\alpha}_p(x)(g)=\overline{\langle g, p\rangle}x(g)$. The dual action of $\hat{G}$ on $C^*(\mathfrak{A};\alpha)$ is defined as the unique extension of $\hat{\alpha}_p$ on $L^1_\alpha(G;\mathfrak{A})$ to an action on $C^*(\mathfrak{A};\alpha)$ which we also denote by $\hat{\alpha}_p$.

For two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}\otimes\mathfrak{B}$ denotes the C^*-tensor product of $\mathfrak{A}$ and $\mathfrak{B}$ with respect to some C^*-(cross) norm. In most of cases $\mathfrak{A}$ or $\mathfrak{B}$ will be of type I; if this is the case, the C^*-norm is unique and hence there is no ambiguity about C^*-norms.

Let $\mathscr{C}(\mathscr{H})$ be the C^*-algebra of all compact operators on a Hilbert space $\mathscr{H}$. Let λ be the regular representation of G on $L^2(G)$ and Ad λ the adjoint action of G on $\mathscr{C}(L^2(G))$. Since G is compact abelian, it follows from [7] that $C^*(\mathfrak{A};\alpha)$ is isomorphic to the fixed point algebra of $\mathfrak{A}\otimes\mathscr{C}(L^2(G))$ under the product action $\alpha\otimes\mathrm{Ad}\,\lambda$ of G.

According to [8], let $\Gamma(\alpha)=\cap \mathrm{Sp}\,\alpha^{\mathfrak{B}}$ be the Connes invariant of $(\mathfrak{A}, G, \alpha)$ where the intersection is taken over all nonz-ero α-invariant hereditary C^*-subalgebras $\mathscr{B}$ of $\mathfrak{A}$. $\Gamma(\alpha)$ is a (closed) subgroup of the dual group $\hat{G}$ of G.

For every $p\in\hat{G}$ and $a\in\mathfrak{A}$, let

$$\varepsilon_p^\alpha(a)=\int_G \langle g,\ r\rangle \alpha_g(a)\,dg$$

where $\langle g, p\rangle$ means the value of p at g. Then ε_p^α (or simply ε_p) is a mapping from $\mathfrak{A}$ onto the spectral subspace $\mathfrak{A}^\alpha(p)$ of α at p. For $p=1\in\hat{G}$, ε_1 is a projection of norm one from $\mathfrak{A}$ onto the fixed point algebra $\mathfrak{A}^\alpha=\mathfrak{A}^\alpha(1)$. The family $\{\mathfrak{A}^\alpha(p)\,;\,p\in\hat{G}\}$ is total in $\mathfrak{A}$ and satisfies that $\mathfrak{A}^\alpha(p)\mathfrak{A}^\alpha(q)\subset\mathfrak{A}^\alpha(pq)$.

Let $M(\mathfrak{A})$ be the multiplier algebra of the C^*-algebra $\mathfrak{A}$. The strict topology of $M(\mathfrak{A})$ is the weakest topology in which the maps $x\to xa$ and $x\to ax$ from $M(\mathfrak{A})$ into $\mathfrak{A}$ are continuous for each $a\in\mathfrak{A}$.

The bitransposed action α^{**} of α on the second dual $\mathfrak{A}^{**}$ leaves $M(\mathfrak{A})$ invariant. Thus α^{**} defines the action of G on $M(\mathfrak{A})$ which is an extension of α on $\mathfrak{A}$ and will be denoted by α.

§ 3. Structure of C^*-Crossed Products

We first make the following observation about the fixed point algebra of a tensor product. In the case of periodic modular actions, a prototype of the following proposition is in [6].

Proposition 3.1. *Let* $(\mathfrak{A}, G, \alpha)$ *and* $(\mathfrak{B}, G, \beta)$ *be* C^**-dynamical systems based on a compact abelian group* G*. Then the fixed point* *algebra* $(\mathfrak{A}\otimes\mathfrak{B})^{\alpha\otimes\beta}$ *of* $\mathfrak{A}\otimes\mathfrak{B}$ *under the product action* $\alpha\otimes\beta$ *of* G *is the closed span of tensor products* $\mathfrak{A}^{\alpha}(p)\otimes\mathfrak{B}^{\beta}(p^{-1})$ *with* $p\in\hat{G}$.

Proof. Since $\mathfrak{A}^{\alpha}(p)$ and $\mathfrak{B}^{\beta}(q)$ are total in $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and since $\varepsilon_1^{\alpha\otimes\beta}$ is of norm 1, the proposition follows from the following formula for $x\in\mathfrak{A}^{\alpha}(p)\otimes\mathfrak{B}^{\beta}(q)$:

$$\varepsilon_1^{\alpha\otimes\beta}(x)=\begin{cases}0 & \text{if} \quad pq\neq 1\\ x & \text{if} \quad pq=1\end{cases}.$$

This formula follows immediately from: $\alpha_g\otimes\beta_g(x)=\langle g, pq\rangle x$.

Q. E. D.

Now we consider the C^*-dynamical system $(\mathscr{C}(L^2(G)), G, \mathrm{Ad}\,\lambda)$ based on a compact abelian group. Let $\hat{u}$ be a unitary representation of $\hat{G}$ on $L^2(G)$ such that

$$(\hat{u}_p\xi)(g)=\overline{\langle g, p\rangle}\xi(g), \quad \xi\in L^2(G).$$

Then $\mathrm{Ad}\,\lambda_g(\hat{u}_p)\equiv\lambda_g\hat{u}_p\lambda_g^*=\langle g, p\rangle\hat{u}_p$. Further for $p\in\hat{G}$ let e_p be the one-dimensional projection onto $\overline{\langle\cdot, p\rangle}$ in $L^2(G)$, which satisfies

$$e_p=\int_G\overline{\langle g, p\rangle}\lambda_g dg.$$

Then $\hat{u}_p e_q=e_{pq}\hat{u}_p$ and the closed span of $\{\hat{u}_p e_q\,;\ p, q\in\hat{G}\}$ is $\mathscr{C}^2(L(G))$.

Lemma 3.2. $\mathscr{C}(L^2(G))^{\mathrm{Ad}\,\lambda}(p)=C^*(G)\hat{u}_p$.

Proof. The group C^*-algebra $C^*(G)$ is the closed span of $\{e_p\,;\ p\in\hat{G}\}$. The lemma follows from the fact: $\mathrm{Ad}\,\lambda_g(e_p)=e_p$, $\mathrm{Ad}\,\lambda_g(\hat{u}_p)=\langle g, p\rangle\hat{u}_p$. Q. E. D.

Theorem 1 is a consequence of Proposition 3.1 and Lemma 3.2 since $C^*(\mathfrak{A}\,;\,\alpha)$ is isomorphic to $(\mathfrak{A}\otimes\mathscr{C}(L^2(G)))^{\alpha\otimes\mathrm{Ad}\,\lambda}$.

As a corollary we have

Proposition 3.3. *Let* $(\mathfrak{A}, G, \alpha)$ *be a* C^*-*dynamical system based on a compact abelian group* G. *Suppose that* $M(\mathfrak{A})^{\alpha}(p)$ *contains a* *unitary for each* $p\in\hat{G}$. *Then* $C^*(\mathfrak{A}\,;\alpha)$ *is isomorphic to* $\mathfrak{A}^\alpha\otimes\mathscr{C}(L^2(G))$.

Proof. Let v_p be a unitary in $M(\mathfrak{A})^\alpha(p)$. Then $\mathfrak{A}^\alpha(p)=\mathfrak{A}^\alpha v_p$. Let $U=\sum_{p\in\hat{G}}v_p\otimes e_p$ with convergence in the strict topology, which is a unitary in $M(\mathfrak{A}\otimes\mathscr{C}(L^2(G)))$. Then for $x\in\mathfrak{A}^\alpha$ we have

$$U(xv_p\otimes e_q\hat{u}_p^*)\,U^*=v_qxv_pv_{pq}^*\otimes e_q\hat{u}_p^*$$

which implies that $U(\mathfrak{A}^\alpha(p)\otimes C^*(G)\hat{u}_p^*)\,U^*=\mathfrak{A}^\alpha\otimes C^*(G)\hat{u}_p^*$. By Theorem 1 we have the proposition.

Finally we remark:

Proposition 3.4. *Let* $(\mathfrak{A}, G, \alpha)$ *be a* C^**-dynamical system based on a compact abelian group* G. *Suppose that* $\alpha_g=\mathrm{Ad}\,u_g$ *for a measurable representation* u *of* G *into the unitary part of* $M(\mathfrak{A})$ *equipped with the strict topology. Then* $C^*(\mathfrak{A}\,;\alpha)$ *is isomorphic to* $\mathfrak{A}\otimes C^*(G)$.

Proof. Let ρ be a non-degenerate faithful representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\rho$. Let U be a unitary on $L^2(G:\mathscr{H}_\rho)$ such that $(U\xi)(g)=\rho(u_g)\xi(g)$ for $\xi\in L^2(G\,;\mathscr{H}_\rho)$ where ρ is the unique extension of ρ to a representation of $M(\mathfrak{A})$ on $\mathscr{H}_\rho$. For $x\in L^1_\alpha(G\,;\mathfrak{A})$ and $\xi\in L^2(G\,;\mathscr{H}_\rho)$ we have

$$\begin{aligned}&[U(\mathrm{Ind}\,\rho(x))\,U^*\xi](g)\\&=\rho(u_g)\int\rho\circ\alpha_g^{-1}(x(h))\,U^*\xi(h^{-1}g)\,dh\\&=\int\rho(x(h)u_h)\,(\lambda_h\xi)\,(g)\,dh.\end{aligned}$$

Since $xu\in L^1_\alpha(G\,;\mathfrak{A})$ where $(xu)(h)=x(h)u_h$, $h\in G$, we have that $U\,\mathrm{Ind}\,\rho(x)\,U^*\in\rho(\mathfrak{A})\otimes C^*(G)$ on $\mathscr{H}_\rho\otimes L^2(G)=L^2(G\,;\mathscr{H}_\rho)$. Together with the converse computation we can conclude that $\mathrm{Ind}\,\rho(C^*(\mathfrak{A}\,;\alpha))=\rho(\mathfrak{A})\otimes C^*(G)$, which completes the proof. Q. E. D.

Remark. The above proposition holds for a general locally compact amenable group G without any change of the proof. The key point is that $\mathrm{Ind}\,\rho$ is faithful when ρ is faithful (cf. [10]).

§ 4. Stable Isomorphism of $C^*(\mathfrak{A};\alpha)$ to $\mathfrak{A}^\alpha\otimes\mathscr{C}(L^2(G))$

Let $\mathscr{K}$ be the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. For a C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ we consider $(\mathfrak{A}\otimes\mathscr{K}, G, \bar{\alpha}\equiv\alpha\otimes\iota)$ where ι is the trivial action of G on $\mathscr{K}$.

To prove Theorem 2 we first state a key lemma.

Lemma 4.1. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a C^*-algebra $\mathfrak{A}$ including a strictly positive element and a compact abelian group G. Then the following statements are equivalent:*

(i) $I_p=\mathfrak{A}^\alpha$ *for all* $p\in\hat{G}$,

(ii) $M(\mathfrak{A}\otimes\mathscr{K})^{\bar{\alpha}}(p)$ *contains a unitary for all* $p\in\hat{G}$.

Theorem 2 is an easy consequence of the above lemma (i)⇒(ii) and Proposition 3.3 as follows: $C^*(\mathfrak{A};\alpha)\otimes\mathscr{K}=C^*(\mathfrak{A}\otimes\mathscr{K}, \bar{\alpha})\simeq(\mathfrak{A}\otimes\mathscr{K})^{\bar{\alpha}}\otimes\mathscr{C}(L^2(G))=\mathfrak{A}^\alpha\otimes\mathscr{K}\otimes\mathscr{C}(L^2(G))$.

Now we have to prove Lemma 4.1. The proof depends on an idea of Brown [2] In the following let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group G.

Lemma 4.2. *$\mathfrak{A}$ has an increasing approximate identity of elements of $\mathfrak{A}^\alpha$.*

Proof. If (u_λ) is an increasing approximate identity in $\mathfrak{A}$, then so is $(\alpha_g(u_\lambda))$ for any $g\in G$. Since G is compact, $(\varepsilon_1(u_\lambda))\,(\subset\mathfrak{A}^\alpha)$ is an increasing approximate identity.

Lemma 4.3. *Suppose that $I_p=\mathfrak{A}^\alpha$. Then $\mathfrak{A}$ has an increasing approximate identity of finite sums $\sum a_i^*a_i$, $a_i\in\mathfrak{A}^\alpha(p)$.*

Proof. By using the argument of [5] we have an increasing family $(u_\lambda)_{\lambda\in\Lambda}$ of finite sums $\sum a_i^*a_i$, $a_i\in\mathfrak{A}^\alpha(p)$, with a directed set Λ such that $\|(1-u_\lambda)(b^*b)^{\frac{1}{2}}\|\to 0$ for any $b\in\mathfrak{A}^\alpha(p)$. Thus $\|(1-u_\lambda)x\|$ tends to zero for any $x=b^*b$ with $b\in\mathfrak{A}^\alpha(p)$ and hence for any $x\in\mathfrak{A}^\alpha$

due to the assumption $I_p=\mathfrak{A}^\alpha$. By Lemma 4.2 this implies that $\|(1-u_\lambda)x\|\to 0$ for any $x\in\mathfrak{A}$. Q. E. D.

Lemma 4.4. *Suppose that $\mathfrak{A}$ has a strictly positive element and that $I_p=\mathfrak{A}^\alpha$. Then there is a sequence $\{a_i; i=1, 2, \cdots\}$ in $\mathfrak{A}^\alpha(p)$ such that $\sum_1^\infty a_i^* a_i=1$ with convergence in the strict topology of $M(\mathfrak{A})$.*

The proof is quite similar to that of Lemma 2.3 in [2] and so we omit it.

Lemma 4.5. *$M(\mathfrak{A})^\alpha$ is isomorphic to $M(\mathfrak{A}^\alpha)$ by restriction.*

Proof. Let $x\in M(\mathfrak{A})^\alpha$. Then xa and ax belong to $\mathfrak{A}^\alpha$ for $a\in\mathfrak{A}^\alpha$. If $xa=0$ for any $a\in\mathfrak{A}^\alpha$, then $x=0$ by Lemma 4.2. Thus we can assume $M(\mathfrak{A})^\alpha$ as a subalgebra of $M(\mathfrak{A}^\alpha)$. An element of $M(\mathfrak{A}^\alpha)$ can be extended to a multiplier of the closed span of $\mathfrak{A}^\alpha\mathfrak{A}\mathfrak{A}^\alpha$, which is $\mathfrak{A}$. Hence $M(\mathfrak{A})^\alpha=M(\mathfrak{A}^\alpha)$. Q. E. D.

Now we identify $M(\mathfrak{A})^\alpha$ with $M(\mathfrak{A}^\alpha)$.

Let $\{e_{ij}\}$ be a family of matrix units which generate $\mathscr{K}$.

Lemma 4.6. *Suppose that $\mathfrak{A}$ has a strictly positive element and that $I_p=I_{p-1}=\mathfrak{A}^\alpha$. Then there is a partial isometry u in $M(\mathfrak{A}\otimes\mathscr{K})^\alpha(p)$ such that $u^*u=1\otimes e_{11}$ and uu^* is full in $M(\mathfrak{A}^\alpha\otimes\mathscr{K})$.*

Proof. By Lemma 4.4 we have sequences $\{a_i\}$ and $\{b_i\}$ in $\mathfrak{A}^\alpha(p)$ such that $\sum a_i^*a_i=1$ and $\sum b_ib_i^*=1$. Let $b=1+\sum_{j=1}^\infty 2^{-j}b_j^*b_j$ and set $d_{2j-1}=a_jb^{-\frac{1}{2}}$ and $d_{2j}=2^{-j/2}b_jb^{-\frac{1}{2}}$. Then $\sum d_j^*d_j=1$ with convergence in the strict topology.

Now the proof proceeds as in [2, Lemma 2.4]. Let $u=\sum d_j\otimes e_{j1}$. Then $u^*u=1\otimes e_{11}$ and $uu^*=\sum d_id_j^*\otimes e_{ij}\in M(\mathfrak{A}\otimes\mathscr{K})^\alpha=M(\mathfrak{A}^\alpha\otimes\mathscr{K})$. The norm closed ideal of $M(\mathfrak{A}^\alpha\otimes\mathscr{K})$ generated by uu^* contains $d_id_j^*\otimes e_{11}$, in particular $b_jb^{-1}b_j^*\otimes e_{11}$. Since $b^{-1}\geq 2^{-1}$ and $\sum b_jb_j^*=1$, this implies that the closed ideal generated by uu^* contains $\mathfrak{A}^\alpha\otimes\mathscr{K}$, thus by definition uu^* is full. Q. E. D.

Lemma 4.7. *With the same hypotheses as in Lemma* 4.6, *there exists a unitary in* $M(\mathfrak{A}\otimes\mathscr{K})^{\alpha}(p)$.

Proof. Let u be a partial isometry given in Lemma 4.6. Since $u^*u(=1\otimes e_{11})$ and uu^* are full in $M(\mathfrak{A}^{\alpha}\otimes\mathscr{K})$ we have v_1, v_2 in $M(\mathfrak{A}^{\alpha}\otimes\mathscr{K}\otimes\mathscr{K})$ such that $v_1^*v_1=v_2^*v_2=1$, $v_1v_1^*=u^*u\otimes 1$ and $v_2v_2^*=uu^*\otimes 1$ (Lemma 2.5 in [2]). Let $U=v_2^*(u\otimes 1)v_1$. Then $U\in M(\mathfrak{A}\otimes\mathscr{K}\otimes\mathscr{K})^{\bar{\alpha}}(p)$ with $\bar{\alpha}=\pi\otimes\iota\otimes\iota$ and U is a unitary. Since $(\mathfrak{A}\otimes\mathscr{K}\otimes\mathscr{K}, G, \bar{\alpha})$ is isomorphic to $(\mathfrak{A}\otimes\mathscr{K}, G, \bar{\alpha})$ we have a desired unitary.

Q. E. D.

Proof of Lemma 4.1. (i)$\Rightarrow$(ii) follows from Lemma 4.7. Suppose (ii) and let u be a unitary in $M(\mathfrak{A}\otimes\mathscr{K})^{\alpha}(p)$. Let $u=\sum u_{ij}\otimes e_{ij}$ with $u_{ij}\in M(\mathfrak{A})^{\alpha}(p)$. Then $\sum u_{i1}^*u_{i1}=1$ implies that $I_p=\mathfrak{A}^{\alpha}$.

Q. E. D.

§5. The Invariant $\Gamma(\alpha)$ and the Dual Action $\hat{\alpha}$

We study the relation between $\Gamma(\alpha)$ and the dual action $\hat{\alpha}$ of α. Such a relation is studied in [7] for a discrete abelian group G.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system based on a compact abelian group. Define $\hat{x}$ for $x\in L^1_{\alpha}(G;\mathfrak{A})$ by

$$\hat{x}(p)=\int_G \overline{\langle g, p\rangle} x(g)\,dg, \qquad p\in\hat{G}.$$

We make the following computation in the algebra $L^1_{\alpha}(G;\mathfrak{A})$ where $a\otimes p$ denotes the function $g\in G\longmapsto\langle g, p\rangle\, a\in M(\mathfrak{A})$ for $p\in\hat{G}$ and $a\in M(\mathfrak{A})$:

$$\begin{aligned}[x(a\otimes p)](g) &= \int_G x(h)\alpha_h((a\otimes p)(h^{-1}g))\,dh \\ &= \langle g, p\rangle \int_G \overline{\langle h, p\rangle} x(h)\alpha_h(a)\,dh.\end{aligned}$$

By putting $a=1$ we obtain that $x(1\otimes p)=\hat{x}(p)\otimes p$ for $x\in L^1_{\alpha}(G;\mathfrak{A})$ and $p\in\hat{G}$. By putting $x=1\otimes q$ we obtain that $(1\otimes q)(a\otimes p)=\varepsilon_{pq^{-1}}(a)\otimes p$. Hence we have that $(1\otimes q)x(1\otimes p)=\varepsilon_{pq^{-1}}(\hat{x}(p))\otimes p$ for any $x\in L^1_{\alpha}(G;\mathfrak{A})$ and any pair p, $q\in\hat{G}$.

Lemma 5.1. *For* $x\in C^*(\mathfrak{A};\alpha)$ *there exists a family* $(x_{p,q})_{p,q\in\hat{G}}$ *in* $\mathfrak{A}$ *such that* $x_{p,q}\in\mathfrak{A}^{\alpha}(qp^{-1})$ *and*

$$(1\otimes p)x(1\otimes q)=x_{p,q}\otimes q.$$

Proof. we have shown the assertion for $x\in L^1_\alpha(G;\mathfrak{A})$. Since $\|(1\otimes p)x(1\otimes q)\|_{C^*(\mathfrak{A};\alpha)}=\|\varepsilon_{qp^{-1}}(\hat{x}(q))\|_{\mathfrak{A}}$ for $x\in L^1_\alpha(G;\mathfrak{A})$ we have the lemma by limiting procedure. Q. E. D.

We remark the following formula which is obtained by easy computation: For $a\in\mathfrak{A}^{\alpha}(p)$, $b\in\mathfrak{A}^{\alpha}(\mathrm{q})$ and for f, $g\in L^1(G)$,

$$(a\otimes f)(b\otimes g)=ab\otimes(f\cdot q*g)$$

where $(f\cdot q)(g)=\langle g, q\rangle f(g)$.

Now we begin to prove Theorem 3. Suppose $p\notin\Gamma(\alpha)$. Then there exists a non-zero positive element $a\in\mathfrak{A}^{\alpha}$ such that $a\,\mathfrak{A}^{\alpha}(p)a=(0)$. For $b\in\mathfrak{A}^{\alpha}(\mathrm{q})$ and $f\in L^1(G)$ we heve

$$\begin{aligned}(\mathrm{a}\otimes p^{-1})(b\otimes f)(a\otimes 1)&=(a\otimes p^{-1})(ba\otimes f*1)\\&=(a\otimes p^{-1})(ba\otimes\hat{f}(1)\cdot 1)\\&=\hat{f}(1)aba\otimes p^{-1}\mathrm{q}*1\\&=\delta_{p,q}\hat{f}(1)aba\otimes 1.\end{aligned}$$

Since $a\mathfrak{A}^{\alpha}(p)a=0$, this implies that $(a\otimes p^{-1})(b\otimes f)(a\otimes 1)=0$ for all $b\in\mathfrak{A}^{\alpha}(q)$ with $q\in\hat{G}$ and all $f\in L^1(G)$. Since the family $\{\mathfrak{A}^{\alpha}(q)\otimes L^1(G)\,;q\in\hat{G}\}$ is total in $L^1_\alpha(G;\mathfrak{A})$ and hence in $C^*(\mathfrak{A};\alpha)$ we obtain $(a\otimes p^{-1})C^*(\mathfrak{A};\alpha)(a\otimes 1)=0$. Let I be the closed ideal of $C^*(\mathfrak{A};\alpha)$ generated by $a\otimes 1$. Then $\hat{\alpha}_p(I)$ is the closed ideal generated by $\hat{\alpha}_p(a\otimes 1)=a\otimes p^{-1}$. Hence we obtain $\hat{\alpha}_p(I)I=(0)$.

Conversely let $p\in\hat{G}$ and let I be a non-zero closed ideal of $C^*(\mathfrak{A};\alpha)$ such that $\hat{\alpha}_p(I)I=(0)$. Let x be a non-zero positive element of I. Since $x^{\frac{1}{2}}(1\otimes q)\neq 0$ for some $q\in\hat{G}$, we have a non-zero $a=x_{q,q}\in\mathfrak{A}^{\alpha}$ such that $(1\otimes q)x\cdot(1\otimes q)=a\otimes q\in I$ by Lemma 5.1. Since $a\otimes p^{-1}q=\hat{\alpha}_p(a\otimes q)\in\hat{\alpha}_p(I)$, the following holds for all $b\in\mathfrak{A}^{\alpha}(p)$:

$$\begin{aligned}aba\otimes 1&=(a\otimes p^{-1}q)(ba\otimes q)\\&=(a\otimes p^{-1}q)(b\otimes q)(a\otimes q)\\&\in\hat{\alpha}_p(I)I=(0).\end{aligned}$$

Thus $a\mathfrak{A}^{\alpha}(p)a=(0)$ which implies $p\notin\Gamma(\alpha)$. This completes the proof.

§6. The Relative Commutant of Fixed Point Algebra

In this section we determine the algebraic structure of the relative commutant of the fixed point algebra in a C^*-dynamical system (cf. [3], [9]).

Let $\mathscr{B}$ be a C^*-algebra and $\mathscr{C}$ be a C^*-subalgebra of the center $Z_{\mathscr{B}}$ of $\mathscr{B}$ such that the closed ideal of $\mathscr{B}$ generated by $\mathscr{C}$ is $\mathscr{B}$ itself. Let Ω be the spectrum of $\mathscr{C}$ and let I_ω for each $\omega\in\Omega$ be the maximal ideal of $\mathscr{C}$ consisting of all elements of $c\in\mathscr{C}$ with $\langle c, \omega\rangle=0$. Denote by J_ω the closed ideal of $\mathscr{B}$ generated by I_ω. Now suppose J is a primitive ideal of $\mathscr{B}$. Then there is an irreducible representation π such that $\pi^{-1}(0)=J$. Since $\mathscr{C}\subset Z_{\mathscr{B}}$, it holds that $\pi(\mathscr{C})\subset \boldsymbol{C}\cdot 1$ and hence that $\pi(\mathscr{C})=\boldsymbol{C}\cdot 1$ due to the assumption for $\mathscr{C}$. Therefore we have an element $w\in\Omega$ such that $\pi(c)=\langle c, \omega\rangle 1$ for all $c\in\mathscr{C}$. Thus $J\supset I_\omega$, which implies that $J\supset J_\omega$. Therefore we have the following lemma:

Lemma 6.1. $\bigcap_{\omega\in\Omega} J_\omega=(0)$.

Let $\mathscr{B}_\omega=\mathscr{B}/J_\omega$ and let b_ω be the image of $b\in\mathscr{B}$ by the quotient map η_ω of $\mathscr{B}$ onto $\mathscr{B}/J_\omega$. It follows from Lemma 6.1 that $\mathscr{B}$ is commutative if and only if $\mathscr{B}_\omega$ is commutative for any $\omega\in\Omega$. Moreover, let J be a primitive ideal of $\mathscr{B}$. As we saw before, there is an element $w\in\Omega$ such that $J\supset J_\omega$. Then $\eta_\omega(J)$ is a primitive ideal of $\mathscr{B}_\omega$ such that $\mathscr{B}/J$ is isomorphic to $\mathscr{B}_\omega/\eta_\omega(J)$. Conversely let $\omega\in\Omega$ and let J be a primitive ideal of $\mathscr{B}_\omega$. Then the inverse image $\eta_\omega^{-1}(J)$ of J by η_ω is a primitive ideal of $\mathscr{B}$ such that $\mathscr{B}/\eta_\omega^{-1}(J)$ is isomorphic to $\mathscr{B}_\omega/J$. By this argument we have the following lemma:

Lemma 6.2. (i) *$\mathscr{B}$ is commutative if and only if $\mathscr{B}_\omega$ is commutative for any $\omega\in\Omega$.*

(ii) *$\mathscr{B}$ is of type* I *if and only if $\mathscr{B}_\omega$ is of type* I *for any $\omega\in\Omega$.*

Remark. In Lemma 6.2 (ii), it is the theorem of Sakai that the separability assumption for $\mathscr{B}$ is removable (cf. [4]).

Let $(\mathscr{B}, G, \beta)$ be a C^*-dynamical system based on a compact abelian group G. Let $\mathscr{C} = \mathscr{B}^{\beta}$. By Lemma 4.1 the closed ideal of $\mathscr{B}$ generated by $\mathscr{C}$ is $\mathscr{B}$ itself. Now suppose $\mathscr{C} \subset Z_{\mathscr{B}}$ and let I_ω, J_ω be as before. Since $\beta_g(I_\omega) = I_\omega$, it holds that $\beta_g(J_\omega) = J_\omega$ for any $g \in G$ and $\omega \in \Omega$. For each $\omega \in \Omega$ we can define an action β^ω of G on $\mathscr{B}_\omega$ such that $\beta^\omega_g(a_\omega) = (\beta_g(a))_\omega$ for $a \in \mathscr{B}$. Then β^ω is ergodic on $\mathscr{B}_\omega$. In fact let $x \in \mathscr{B}_\omega$ with $\beta^\omega_g(x) = x$ for all $g \in G$. If $a \in \mathscr{B}$ satisfies $a_\omega = x$, then $(\beta_g(a))_\omega = \beta_g(a_\omega) = x$. Hence $(\varepsilon_1(a))_\omega = x$. Since $\varepsilon_1(a) \in \mathscr{B}^{\beta} = \mathscr{C}$, we have $x \in \boldsymbol{C}\cdot 1$. Thus we have the following lemma:

Lemma 6.3. *Let $(\mathscr{B}, G, \beta)$ be a C^*-dynamical system such that $\mathscr{C} \equiv \mathscr{B}^{\beta} \subset Z_{\mathscr{B}}$. Then $(\mathscr{B}_\omega, G, \beta^\omega)$ is ergodic for each $\omega \in \Omega = \mathrm{Spec}\, \mathscr{C}$.*

Now we begin to prove Theorem 4. We take the C^*-dynamical system $(\mathscr{B} \equiv \mathfrak{A} \cap (\mathfrak{A}^\alpha)', G, \beta)$ where we assume $\mathscr{B} \neq (0)$ and β is the restriction to $\mathscr{B}$ of the action α on $\mathfrak{A}$. Then $\mathscr{C} = \mathscr{B}^{\beta}$ is the center of $\mathfrak{A}^\alpha$ and hence $\mathscr{C} \subset Z_{\mathscr{B}}$. By the above lemma we have the ergodic system $(\mathscr{B}_\omega, G, \beta^\omega)$ for each $\omega \in \Omega$. Fix u_p in $\mathscr{B}^{\beta^\omega}_\omega(p)$ with $\|u_p\| = 1$ for each $p \in \mathrm{Sp}\, \beta^\omega$. Then the ergodicity implies that u_p is a unitary and that $\mathscr{B}^{\beta^\omega}_\omega(p) = \boldsymbol{C}\cdot u_p$. In particular $\mathrm{Sp}\, \beta^\omega$ is a subgroup of $\hat{G}$.

Case (i). Since $\mathrm{Sp}\beta^\omega$ has a generating element q we know that u_q generates $\mathscr{B}_\omega$. Thus $\mathscr{B}_\omega$ is commutative. Hence by Lemma 6.2 $\mathscr{B}$ is commutative.

Case (ii). If G is finite, then obviously $\mathscr{B}_\omega$ is finite-dimensional. Hence by Lemma 6.2 $\mathscr{B}$ is of type I.

Suppose that G is the product group of T^1 with a finite group H. Let $N = H^\perp \cap \mathrm{Sp}\beta^\omega$ where H is considered as a subgroup of G. We may assume that N is infinite, otherwise $\mathrm{Sp}\beta^\omega$ is finite and $\mathscr{B}_\omega$ is of type I. Since N is a cyclic group, the family $(u_p)_{p\in N}$ is commutative. Let m be the order of H. For any $p \in \mathrm{Sp}\beta^\omega$ we have that $u_p^m u_q = u_q u_p^m$ for any $q \in N$ since $p^m \in N$. On the other hand, since β^ω is ergodic, there exists a $\lambda \in C$ such that $u_p u_q = \lambda u_q u_p$. Since $u_p^m u_q =$

$\lambda^m u_q u_p^m$ we have $\lambda^m=1$, which implies that $u_p u_q^m = u_q^m u_p$. Since p is arbitrary we have that $(u_q)_{q\in N_1}$ are unitaries in $Z_{\mathscr{B}_\omega}$ where $N_1=\{q^m;\ q\in N\}$. Let γ be the restricion of β^ω to the subgroup $N_1^\perp$ which is finite. Since $(\mathscr{B}_\omega)^\gamma$ is the C^*-algebra generated by $(u_q)_{q\in N_1}$ we have $(\mathscr{B}_\omega)^\gamma \subset Z_{\mathscr{B}_\omega}$. Therefore the relative commutant $\mathscr{B}_\omega \cap (\mathscr{B}_\omega^\gamma)'$ is equal to $\mathscr{B}_\omega$ and must be of type I by the preceding result for finite G. Hence by Lemma 6. 2 $\mathscr{B}$ is of type I.

If $G=T^2$, we have an example of an ergodic C^*-dynamical system $(\mathfrak{A}, T^2, \alpha)$ where $\mathfrak{A}$ is separable and not of type I.

Example. Let $\mathfrak{H}$ be a countably infinite dimensional Hilbert space and let $\{e_{k,l}\}_{k,l\in Z}$ be a family of matrix units. Let $\mathfrak{R}=\mathfrak{H}\otimes\mathfrak{H}$. We define unitaries u_i and $w(s)$ on $\mathfrak{R}$ by

$$u_1=\sum_{k\in Z} e_{k+1,k}\otimes 1,\ u_2=\sum_{k\in Z} 1\otimes e_{k+1,k}$$

and

$$w(s)=w(s_1,\ s_2)=\sum_{k_1,k_2\in Z} e^{i(k_1 s_1+k_2 s_2)} e_{k_1,k_1}\otimes e_{k_2,k_2}$$

for $s=(s_1, s_2)\in R^2$. We can consider $w(\cdot)$ as a unitary representation of T^2 on $\mathfrak{R}$. Let δ be a real number such that $\delta/2\pi$ is irrational and set $v_1=w(0, \delta)u_1$, and $v_2=u_2$. Let $\mathfrak{A}$ be the C^*-algebra generated by v_1 and v_2 and α the continuous action of T^2 on $\mathfrak{A}$ such that $\alpha_s=\mathrm{Ad}\,\omega(s)$ for each $s\in T^2$. In fact, α is well-defined since $\mathrm{Ad}\,w(s)(v_j)=e^{is_j}v_j$. Thus we have defined the C^*-dynamical system $(\mathfrak{A}, T^2, \alpha)$. We assert that α is ergodic on $\mathfrak{A}$ and the weak closure $\mathfrak{A}''$ of $\mathfrak{A}$ on $\mathfrak{R}$ is a factor of type II_1. Let $\mathfrak{A}_0$ be the $*$-algebra algebraically generated by v_1 and v_2. Then we clearly have $\mathfrak{A}_0^\alpha=C\cdot 1$ which implies $\mathfrak{A}^\alpha=C\cdot 1$ by compactness. To prove that $\mathfrak{A}''$ is a factor, let $\bar{\alpha}_s=\mathrm{Ad}\,w(s)$ on $\mathfrak{A}''$ for each $s\in T^2$. Then $(\mathfrak{A}'')^{\bar{\alpha}}=(\mathfrak{A}^\alpha)''=C\cdot 1$ and $\varepsilon_n(\mathfrak{A}'')=C\cdot v_1^{n_1}v_2^{n_2}$ for all $n=(n_1, n_2)\in Z^2=\hat{T}^2$. Now we compute that

$$\begin{aligned} v_2v_1 &= u_2 w(0,\ \delta)u_1 = e^{-i\delta}w(0,\ \delta)u_2u_1 \\ &= e^{-i\delta}v_1v_2. \end{aligned}$$

Hence $v_1^{n_1}v_2^{n_2}\notin Z_{\mathfrak{A}''}$ unless $n_1=n_2=0$. Thus $Z_{\mathfrak{A}''}=(Z_{\mathfrak{A}''})^{\bar{\alpha}}=C\cdot 1$. To prove that $\mathfrak{A}''$ is of type II_1, let ξ be a unit vector of $\mathfrak{H}$ such that $e_{0,0}\xi=\xi$.

Let φ be a vector state of $\mathfrak{A}$ defined by $\xi\otimes\xi$. Since the factor $\mathfrak{A}''$ is infinite-dimensional, we can conclude that $\mathfrak{A}''$ is of type II_1 if we show that φ is a tracial state of $\mathfrak{A}$. Since φ is α-invariant, this is a special case of the following proposition:

Proposition 6.4. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system. Let $\mathscr{B}=\mathfrak{A}\cap(\mathfrak{A}^{\alpha})'$ and $\beta=\alpha|_{\mathscr{B}}$. If $\mathscr{B}\neq(0)$, any β-invariant state φ of $\mathscr{B}$ is tracial.*

Proof. We first note the following three points:

(i) Since φ is β-invariant, $\varphi(\varepsilon_p(\mathscr{B}))=0$ for all $p\neq 1$.

(ii) The linear hull of $\varepsilon_p(\mathscr{B})$, $p\in \mathrm{Sp}\beta$ is dense in $\mathscr{B}$.

(iii) $\mathscr{B}^{\beta}$ is the center of $\mathfrak{A}^{\alpha}$ and hence in the center of $\mathscr{B}$.

For $a\in\varepsilon_p(\mathscr{B})$ and $b\in\varepsilon_q(\mathscr{B})$, $\varphi([a, b])=0$ for $pq\neq 1$ due to (i) and $[a,b]=0$ for $pq=1$ due to (iii) and the lemma below. Hence $\varphi([a,b])=0$ for all $a,b\in\mathscr{B}$ due to (ii). Q. E. D.

Lemma 6.5. *If Z is the center of a C^*-algebra $\mathfrak{A}$ and if $a\in\mathfrak{A}$, $b\in\mathfrak{A}$, $ab\in Z$ and $ba\in Z$, then $ab=ba$.*

Proof. Let π be a primary representation of $\mathfrak{A}$. Then $\pi(a)\pi(b)=c\cdot 1$ and $\pi(b)\pi(a)=d\cdot 1$ for some complex numbers c and d. Then

$$d\pi(a)=\pi(a)(\pi(b)\pi(a))=(\pi(a)\pi(b))\pi(a)=c\pi(a).$$

Hence $c=d$ or $\pi(a)=0$. In the latter case $c=0=d$. Hence $\pi([a, b])=0$ for all primary representations π. Therefore $[a, b]=0$.

Q. E. D.

References

[1] Bratteli, O., A non-simple crossed product of a simple C^*-algebra by a properly outer automorphic action, Preprint, (1977).

[2] Brown, L. G., Stable isomorphism of hereditary subalgebras of C^*-algebras, Preprint, (1976).

[3] Connes, A., Un part d'un groupe integrable d'automorphismes exterieurs d'un facteurs, Preprint, (1973).

[4] Sakal, S., On type I C*-algebras, Proc. Amer. Math. Soc., 18 (1967), 681-863.

[5] Dixmier, J., Les C*-algebres et leurs representations, Gauthier-Villars, Paris (1964).

[6] Herman, R. H., The centralizer under tensor product, Math. Scand., 36 (1975), 286

-290.
[7] Kishimoto, A. and Takai, H., On the invariant $\Gamma(\alpha)$ in C^*-dynamical systems, Tohoku. Math. Jour., 30 (1978), 83-94.
[8] Olesen, D., Inner $*$-automorphisms of simple C^*-algebras, Commun. Math. Phys., **44** (1975), 175-190.
[9] Olesen, D., Pedersen, G. K. and Størmer, E. with appendix by Elliott, G. A., Compact abelian groups of automorphisms of simple C^*-algebras, Inventionse. Math., 39 (177), 55-64.
[10] Takai, H., On a duality for crossed products of C^*-algebras, Jour. Functional Analysis, **19** (1975), 25-39.