

Some Global Properties of Symmetric Diffusion Processes¹⁾

By

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Introduction

The principal purpose of this paper is to investigate some global properties of the diffusion process associated with a second order elliptic partial differential operator of self adjoint form. More specifically recurrence, transience and the rate of escape of the process will be discussed

Let L be an elliptic operator on R^d defined by

$$L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

where a_{ij} is smooth, symmetric and the coefficients' matrix $(a_{ij}(x))_{1 \leq i, j \leq d}$ is strictly positive definite on R^d . We denote by (X_t, P_x) , $x \in R^d$ the associated diffusion process.

Let φ_ρ , $\rho > 1$ be the unique solution of the Dirichlet problem :

$$Lu = 0 \quad \text{in } 1 < |x| < \rho$$

and

$$u(x) = \begin{cases} 1 & \text{on } |x| = 1 \\ 0 & \text{on } |x| = \rho. \end{cases}$$

The following fact is a special case of the celebrated Dirichlet principle :

" φ_ρ is the unique solution of the problem minimizing an integral

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$$\int_{1 < |x| < \rho} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.$$

over all $u \in H^{1,2}(\{x : 1 < |x| < \rho\})$ with boundary values 1 and 0 on $|x|=1$ and ρ respectively”.

See Proposition 1.1 for the precise statement.

Hunt [10] made use of the above principle in his investigation on the existence of the 0-order Green function of the process (X_t, P_x) .

We shall further develop this approach and adopt it in a study of the recurrent and transient properties. In Chapter 1 we shall treat the diffusion process associated with a strictly elliptic operator having measurable coefficients and Chapter 2 will be devoted to the study of a class of degenerate diffusion processes. The main parts of the proofs in these chapters will be performed in L^2 -setting.

Our fundamental criterion for recurrence which will be proved in Chapters 1 and 2 is stated as follows :

“The process is recurrent if and only if

$$\lim_{\rho \rightarrow +\infty} \int_{1 < |x| < \rho} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \varphi_\rho}{\partial x_j} dx = 0.”$$

See Theorems 1.1 and 2.1 for the details.

We denote by $A(x)$ the coefficients’ matrix $(a_{ij}(x))_{1 \leq i, j \leq d}$ and define

$$E_1(x) = \frac{(A(x)x, x)}{|x|^2} \text{ for } x \neq 0$$

$$E_2(x) = \frac{|x|^2}{(A^{-1}(x)x, x)} \text{ for } x \neq 0$$

and

$$\bar{E}_1(r) = \int_{S^{d-1}} E_1(r\sigma) d\sigma \text{ for } r > 0$$

where $A^{-1}(x)$ is the inverse of the matrix $A(x)$ and $d\sigma$ is the normalized uniform measure on S^{d-1} .

By means of the above criterion for recurrence, we shall prove

in Chapter 1

Theorem A. *If $\int_1^\infty r^{1-d} \bar{E}_1^{-1}(r) dr = +\infty$, then the process is recurrent.*

Theorem B. *If $\int_1^\infty r^{1-d} E_2^{-1}(r\sigma) dr < +\infty$ on a subset of S^{d-1} with positive uniform measure, then the process is transient.*

In Chapter 2 similar results will be obtained for the diffusion process governed by a hypoelliptic differential operator. See Theorems C and D.

It is to be mentioned that Brown [4] has proved similar results in case of $L = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(a(x) \frac{\partial}{\partial x_i} \right)$ where $a(x)$ is smooth, strictly positive on R^d , but the methods cannot be generalized to either of our cases.

In Chapter 3 we shall give some examples and compare our main results (Theorems A–D) with those which Friedman [7] and Hasminskii [8] have proved by the martingale method. It will then turn out that our test for recurrence contains that of Friedman and Hasminskii in the self adjoint case.

In the last chapter we shall give the exact escape rates to ∞ for the diffusion paths governed by a uniformly elliptic operator; this is an extension of Dvoretzky and Erdős' test [6] for the Brownian motion in space.

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Notations and Definitions

Let Ω be an open domain R^d .

1. $C(\bar{\Omega})$: The class of real continuous functions in $\bar{\Omega}$.
2. $C^1(\bar{\Omega})$: The subclass of $C(\bar{\Omega})$ functions having continuous first partial derivatives on Ω which can be extended continuously to $\bar{\Omega}$.
3. $C_0^\infty(\Omega)$: The class of real, infinitely differentiable functions with compact support in Ω .
4. $H^{1,2}(\Omega)$: The completion of $C^1(\bar{\Omega})$ with respect to the norm

$$\|u\|_{H^{1,2}(\Omega)} = \sqrt{\int_{\Omega} \left\{ u^2 + \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i} \right)^2 \right\} dx}.$$

- 5. $H_{0,c}^{1,2}(\Omega)$: The class of real measurable functions which are of the $H^{1,2}(\Omega')$ class for every bounded open Ω' with closure in Ω .
- 6. $H_0^{1,2}(\Omega)$: The closure of $C_0^\infty(\Omega)$ in $H^{1,2}(\Omega)$.
- 7. Let the matrix $A(x) = (a_{ij}(x))_{1 \leq i, j \leq d}$ be symmetric, positive definite, with real components $a_{ij}(x)$, $1 \leq i, j \leq d$ locally bounded, measurable in R^d . Let the function $k(x)$ be strictly positive, continuous and integrable on Ω . Define a quadratic form by

$$D(u; \Omega) = \int_{\Omega} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.$$

- 8. $H_0^{1,2}(\Omega; kdx)$: The completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^{1,2}(\Omega; kdx)} = \sqrt{\int_{\Omega} u^2 kdx + D(u; \Omega)}.$$

Chapter 1. Criteria for Recurrence and Transience I. Strictly elliptic differential operator

Suppose we are given an elliptic operator defined on R^d as

$$(1.1) \quad L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

where a_{ij} is locally bounded, measurable, symmetric and L is locally uniformly elliptic, i.e., for each compact subset $K \subset R^d$, there exists a constant $\lambda = \lambda(K) > 1$ such that

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2$$

in K and for all $\xi \in R^d$.

Let D be a bounded domain in R^d . A function u on D is called a weak solution of $Lu = f$ for a given $f \in L^2(D)$ if it has weak derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \dots, d$ which belong to $L_{loc}^2(D)$ and

$$(1.2) \quad \int_D \sum_{i,j=1}^d a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} dx = - \int_D v \cdot f dx$$

is fulfilled for all $v \in C_0^\infty(D)$.

Let R_α , $\alpha > 0$ be a family of linear transformations from $L^\infty(D)$ to $C(D)$, the set of all continuous functions on D , satisfying the resolvent equation

$$(1.3) \quad R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0 \text{ for all } \alpha, \beta > 0$$

and the sub Markov property: $0 \leq \alpha R_\alpha f \leq 1$ if $0 \leq f \leq 1$. If $u = R_\alpha f$ satisfies

$$(1.4) \quad (\alpha - L)u = f \text{ for all } f \in L^\infty(D),$$

R_α is called an L -diffusion resolvent. It is easy to see from (1.4) that R_α is a one to one linear operator from $L^\infty(D)$ to $C(D)$. Set $D(A) = \{R_\alpha f : f \in L^\infty(D)\}$, independent of $\alpha > 0$ by (1.3) and define for $u \in D(A)$, $Au = (\alpha - R_\alpha^{-1})u (= Lu)$. A with the domain $D(A)$ is called the generator of the resolvent.

The existence of the unique minimal diffusion process associated with the operator L in an arbitrary bounded domain has been proved by Kanda [12] and Kunita [14]. Their construction is based on the Green function constructed by Littman, Stampacchia and Weinberger [16] and Stampacchia [19] respectively. Note that Kunita treats the more general case $L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j}) + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$.

We shall explain briefly their process. Let D be as above. Littman et al. [16] and Stampacchia [19] have shown that there exists a unique Green operator G_α , $\alpha \geq 0$ such that G_α transforms $L^\infty(D)$ into $C(D)$ and satisfies (1.3) and (1.4). In particular, if ∂D is regular (∂D is of the C^1 -class, for example). $(G_\alpha f)(x)$ converges to 0 as $x \in D$ approaches a boundary point.

Making use of the above Green operator G_α , Kunita and Kanda have constructed a unique minimal diffusion process (X_t^p, ζ^p, P_x^p) , $x \in D$ for which

$$(1.5) \quad G_\alpha f(x) = E_x \left[\int_0^{\zeta^p} e^{-\alpha t} f(X_t^p) dt \right], \text{ for all } f \in L^\infty(D) \text{ and } \alpha \geq 0$$

where ζ^p is the killing time for the process.

It is to be noted that the process has the communication property i. e., for each open subset U of D and each $x \in D$, $P_x[x^p \in U \text{ for some } t \in (0, \zeta^p)] > 0$, since the 0-order Green function is strictly positive. See Littman et al. [16], section 7. It is also easy to see from (1.5) with $\alpha = 0$ and $f \equiv 1$ that the minimal diffusion is transient.

Combining the local existence by Kanda and Kunita with Courrege and Priouret [5], we will now construct a minimal diffusion process

in R^d . Let Σ_ρ be the open sphere with radius ρ and the origin as a center. We then have a class of local diffusions $(X_t^{x_n}, \zeta^{x_n}, P_x^{x_n}), x \in \Sigma_n, n \geq 1$. Proposition 2.4 of Kunita [14] asserts that the stopped process $(X_t^{x_{n+1}}, \sigma_n^{n+1}, P_x^{x_{n+1}}), x \in \Sigma_n$ where $\sigma_n^{n+1} = \inf\{t \geq 0 : |x_t^{x_{n+1}}| \geq n\}$ is equivalent in law to the process $(X_t^{x_n}, \zeta^{x_n}, P_x^{x_n}), x \in \Sigma_n$. Consequently by Theorem 2 of Courrege and Priouret [5], we see that there exists a unique standard diffusion process X_t in R^d which is locally equivalent to that of Kanda and Kunita. Let $\sigma_\rho = \inf\{t \geq 0 : |x_t| \geq \rho\}$. Note that $\sigma_\rho < +\infty$ a. e. since $(X_t^{x_{\rho+1}}, \zeta^{x_{\rho+1}}, P_x^{x_{\rho+1}}), x \in \Sigma_{\rho+1}$ is transient. See the above remark. Now define $\zeta = \lim_{\rho \nearrow +\infty} \sigma_\rho$. Then the process $(X_t, \zeta, P_x), x \in R^d$ is a desired minimal diffusion process in R^d .

What we are concerned with in this chapter is to investigate the recurrent and transient properties of the process (X_t, P_x) constructed above.

Definition 1.1. *The process will be said to be recurrent if $P_x[X_t \in U \text{ for some } t > 0] \equiv 1$ for any open subset U of R^d , otherwise it will be called transient.*

Remark 1.1. From the definition, it is obvious that $\zeta = +\infty$ a. e. if the process is recurrent, and transience is equivalent to

$$(1.6) \quad P_x[\lim_{t \nearrow \zeta} |x_t| = +\infty] \equiv 1.$$

(1.6) will be proved in Corollary 1.2 after some preparation.

Our main results in this chapter are Theorems A and B. In order to prove them, we introduce the stopping time

$$\tau_1 = \inf\{t \geq 0 : |x_t| \leq 1\}$$

and let $\sigma_\rho, \rho > 1$ be as before. Define

$$\begin{aligned} \varphi_\rho(x) &= P_x[\tau_1 < \sigma_\rho], \quad x \in R^d \\ \varphi_\infty(x) &= P_x[\tau_1 < \zeta], \quad x \in R^d. \end{aligned}$$

From the definition of ζ , it is obvious that

$$\lim_{\rho \nearrow +\infty} \varphi_\rho(x) = \varphi_\infty(x), \quad x \in R^d.$$

The following proposition is essentially due to Littman, Stampacchia and Weinberger [16].

Proposition 1.1. *For $1 < \rho < +\infty$, (i) $\varphi_\rho \in H_0^{1,2}(\Sigma_\rho)$ and is a weak solution of $Lu=0$ in $\Sigma_\rho - \bar{\Sigma}_1$, and converges to the boundary values 1 and 0 as $|x| \searrow 1$ and $|x| \nearrow \rho$ respectively.*

(ii) φ_ρ is Hölder continuous in $\Sigma_\rho - \bar{\Sigma}_1$, more precisely, for any compact subset $K \subset \Sigma_{\rho_0} - \bar{\Sigma}_1$, $\rho_0 > 1$, there exist positive constants $C = C(K, \rho_0)$ and $\alpha = \alpha(K, \rho_0)$ such that

$$(1.7) \quad |\varphi_\rho(x) - \varphi_\rho(x')| \leq C \|\varphi_\rho\|_{L^2(\Sigma_{\rho_0} - \bar{\Sigma}_1)} \cdot |x - x'|^\alpha$$

for every $x, x' \in K$, $\rho \in [\rho_0, +\infty)$.

(iii) $\varphi_\rho - \phi_\rho \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$ and φ_ρ minimizes an integral

$$(1.8) \quad \int_{1 < |x| < \rho} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$$

over all $u \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1) + \{\phi_\rho\}$, where ϕ_ρ is any fixed function $C_0^\infty(\Sigma_\rho)$ such that $\phi_\rho(x) \equiv 1$ on $\bar{\Sigma}_1$ and $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1) + \{\phi_\rho\} = \{\phi + \phi_\rho : \phi \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)\}$.

Remark 1. 2. $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1) + \{\phi_\rho\}$ is independent of the choice of ϕ_ρ as long as ϕ_ρ satisfies the above conditions. It is also to be noted that $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1) + \{\phi_\rho\}$ is increasing as $\rho \nearrow +\infty$ after the trivial extension of elements to Σ_ρ^c .

Proof of Proposition 1.1. We first show that there exists a function on Σ_ρ which satisfies all the above properties and is identically equal to 1 on $\bar{\Sigma}_1$.

Combining Theorems 2.1 and 2.3 of Littman et al. [16], we see that there exists a unique function u_ρ on $\Sigma_\rho - \bar{\Sigma}_1$ which is a weak solution of $Lu=0$ in $\Sigma_\rho - \bar{\Sigma}_1$, Hölder continuous i. e. (1.7) holds with u_ρ in place of φ_ρ , and $u_\rho - \varphi_\rho \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$. We will prove the second property of (iii) for u_ρ . Since u_ρ is a weak solution in $\Sigma_\rho - \bar{\Sigma}_1$ and L is uniformly elliptic on $\overline{\Sigma_\rho - \bar{\Sigma}_1}$, a simple limiting procedure

gives that

$$\int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial u_\rho}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = 0 \text{ for all } \phi \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1).$$

Suppose $\varphi \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1) + \{\phi_\rho\}$ and compute

$$\begin{aligned} & \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \\ &= \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \{u_\rho + (\varphi - u_\rho)\}}{\partial x_i} \frac{\partial \{u_\rho + (\varphi - u_\rho)\}}{\partial x_j} dx \\ &= \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial u_\rho}{\partial x_i} \frac{\partial u_\rho}{\partial x_j} + 2 \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial u_\rho}{\partial x_i} \frac{\partial \{\varphi - u_\rho\}}{\partial x_j} dx \\ &+ \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \{\varphi - u_\rho\}}{\partial x_i} \frac{\partial \{\varphi - u_\rho\}}{\partial x_j} dx. \end{aligned}$$

Since $\varphi - u_\rho \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$, the second term is equal to zero as we saw above. Hence we obtain

$$\int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \geq \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial u_\rho}{\partial x_i} \frac{\partial u_\rho}{\partial x_j} dx.$$

This shows that u_ρ minimizes (1.8).

By Corollary 9.1 of Littman et al. [16], all the boundary points of $\Sigma_\rho - \bar{\Sigma}_1$ are regular. This combined with Theorem 3.1 and Definition 3.1 of Littman et al. [16] implies that u_ρ has the boundary values 1 and 0 on $|x|=1$ and ρ respectively.

Now let us extend the function u_ρ by

$$\tilde{u}_\rho(x) = \begin{cases} u_\rho(x) & \text{on } \Sigma_\rho - \bar{\Sigma}_1 \\ 1 & \text{on } \bar{\Sigma}_1. \end{cases}$$

We want to prove that $\tilde{u}_\rho \in H_0^{1,2}(\Sigma_\rho)$. From the fact that $u_\rho - \phi_\rho \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$, we can choose a sequence w_n , $n \geq 1 \in C_0^\infty(\mathbb{R}^d)$ with support in $\Sigma_\rho - \bar{\Sigma}_1$ such that

$$\lim_{n \rightarrow +\infty} \|w_n - (u_\rho - \phi_\rho)\|_{H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)} = 0.$$

Note that a sequence $w_n + \phi_\rho$, $n \geq 1 \in H_0^{1,2}(\Sigma_\rho)$ and is Cauchy in this space, hence there exists a unique element of $H_0^{1,2}(\Sigma_\rho)$ to which

$w_n + \phi_\rho$ converges to $H_0^{1,2}(\Sigma_\rho)$. It is also possible to choose a subsequence of $w_n + \phi_\rho$ so that it converges to \tilde{u}_ρ a. e. on Σ_ρ . Combining all of this, we see that $\tilde{u}_\rho \in H_0^{1,2}(\Sigma_\rho)$. Thus we have constructed a desired function \tilde{u}_ρ .

We now proceed with the proof of Proposition 1.1. It suffices to verify that

$$\varphi_\rho(x) = \tilde{u}_\rho(x) \text{ for all } x \in \Sigma_\rho - \bar{\Sigma}_1.$$

Since \tilde{u}_ρ is a weak solution of $Lu=0$ in $\Sigma_\rho - \bar{\Sigma}_1$, it follows from Theorem 2.1 of Kunita [14] that \tilde{u}_ρ is 0-harmonic i. e.

$$(1.9) \quad E_x[\tilde{u}_\rho(x_{\tau_{V^c}})] = \tilde{u}_\rho(x), \quad x \in V$$

for all sets V such that $\bar{V} \subset \Sigma_\rho - \bar{\Sigma}_1$, where τ_{V^c} is the hitting time for the set V^c . Now letting $V \nearrow \Sigma_\rho - \bar{\Sigma}_1$, we obtain by the continuity of \tilde{u}_ρ and the fact $P_x[\tau_1 = \sigma_\rho = +\infty] = 0$ for every $x \in \Sigma_\rho - \bar{\Sigma}_1$, see the remark on page 446, that

$$(1.10) \quad \varphi_\rho(x) = \tilde{u}_\rho(x) \text{ for all } x \in \Sigma_\rho - \bar{\Sigma}_1.$$

This completes the proof.

Q. E. D.

The next proposition is a major step in our proof.

Proposition 1.2. φ_∞ satisfies (1.7) with $\rho = +\infty$ in the inequality, $\varphi_\infty \in H_{loc}^{1,2}(R^d)$ and

$$\lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) = D(\varphi_\infty : R^d).$$

For the proof of this proposition, we shall make use of the Hilbert space $H_0^{1,2}(R^d : kdx)$, where $k(x)$ is strictly positive, continuous and integrable on R^d . Before proceeding to the proof, we note that if u is an element of $H_0^{1,2}(R^d : kdx)$, then (i) u and $\frac{\partial u}{\partial x_i} \in L_{loc}^2(R^d)$, $i=1, \dots, d$.

$$(ii) \quad \|u\|_{H_0^{1,2}(R^d : kdx)} = \sqrt{\int_{R^d} \{u^2 k + \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\} dx}.$$

Proof of Proposition 1.2. The first statement is immediate from

Proposition 1.1 (ii) and the pointwise convergence of φ_ρ to φ_∞ .

By Proposition 1.1 (iii) and Remark 1.2, we have that

$$D(\varphi_\rho : \Sigma_\rho) = \int_{\Sigma_\rho - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \varphi_\rho}{\partial x_j} dx$$

is non increasing as $\rho \nearrow +\infty$. Therefore the limit of $D(\varphi_\rho : \Sigma_\rho)$ exists and is non negative.

Using the above fact, we first show that

$$(1.11) \quad \lim_{\rho, \rho' \nearrow +\infty} D(\varphi_\rho - \varphi_{\rho'} : R^d) = 0.$$

Without loss of generality, we can assume that $\rho < \rho'$. Then

$$\begin{aligned} D(\varphi_\rho - \varphi_{\rho'} : R^d) &= \int_{\Sigma_{\rho'} - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \{\varphi_\rho - \varphi_{\rho'}\}}{\partial x_i} \frac{\partial \{\varphi_\rho - \varphi_{\rho'}\}}{\partial x_j} dx \\ &= \int_{\Sigma_\rho - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \varphi_\rho}{\partial x_j} dx - 2 \int_{\Sigma_{\rho'} - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \varphi_{\rho'}}{\partial x_j} dx \\ &\quad + \int_{\Sigma_{\rho'} - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_{\rho'}}{\partial x_i} \frac{\partial \varphi_{\rho'}}{\partial x_j} dx. \end{aligned}$$

On the other hand, the property (iii) of Proposition 1.1 implies that

$$\int_{\Sigma_{\rho'} - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_{\rho'}}{\partial x_i} \frac{\partial \{\varphi_{\rho'} - \varphi_\rho\}}{\partial x_j} dx = 0$$

since $\varphi_\rho - \varphi_{\rho'} \in H_0^{1,2}(\Sigma_{\rho'} - \Sigma_1)$. Therefore one has that

$$\begin{aligned} D(\varphi_\rho - \varphi_{\rho'} : R^d) &= \int_{\Sigma_\rho - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \varphi_\rho}{\partial x_j} dx \\ &\quad - \int_{\Sigma_{\rho'} - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_{\rho'}}{\partial x_i} \frac{\partial \varphi_{\rho'}}{\partial x_j} dx \\ &= D(\varphi_\rho : \Sigma_\rho) - D(\varphi_{\rho'} : \Sigma_{\rho'}) \end{aligned}$$

(1.11) now follows from the existence of $\lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho)$.

It is now easy to see that

$$\varphi_\rho \in H_0^{1,2}(R^d : kdx)$$

and

$$\lim_{\rho, \rho' \nearrow +\infty} \|\varphi_\rho - \varphi_{\rho'}\|_{H_0^{1,2}(R^d : kdx)} = 0$$

since $\varphi_\rho(x) \nearrow \varphi_\infty(x) \leq 1$ for all $x \in R^d$ as $\rho \nearrow +\infty$. By the completeness of the space $H_0^{1,2}(R^d : kdx)$, there exists a unique element $\tilde{\varphi} \in H_0^{1,2}(R^d : kdx)$ such that

$$\lim_{\rho \nearrow +\infty} \|\varphi_\rho - \tilde{\varphi}\|_{H_0^{1,2}(R^d : kdx)} = 0.$$

Since φ_ρ converges to φ_∞ pointwise in R^d , $\tilde{\varphi}$ has to be φ_∞ . Thus we have shown that

$$(1.12) \quad \lim_{\rho \nearrow +\infty} \|\varphi_\rho - \varphi_\infty\|_{H_0^{1,2}(R^d : kdx)} = 0$$

which gives $\varphi_\infty \in H_0^{1,2}(R^d : kdx)$ and

$$\lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) = D(\varphi_\infty : R^d). \quad \text{Q. E. D.}$$

Combining Propositions 1.1 and 1.2, we obtain

Corollary 1.1. *φ_∞ is a weak solution of $Lu=0$ in the exterior domain $R^d - \bar{\Sigma}_1$, which has the boundary value 1 on $|x|=1$ and is strictly positive.*

Proof. From Proposition 1.1 (i), we have

$$\int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx = 0 \text{ for } \psi \in C_0^\infty(\Sigma_\rho - \bar{\Sigma}_1).$$

Since we can replace the domain $\Sigma_\rho - \bar{\Sigma}_1$ in the left hand side of the above equality by $R^d - \bar{\Sigma}_1$, we obtain from Proposition 1.2

$$\int_{R^d - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\infty}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx = 0 \text{ for every } \psi \in C_0^\infty(R^d - \bar{\Sigma}_1).$$

This implies that φ_∞ is a weak solution of $Lu=0$ in $R^d - \bar{\Sigma}_1$.

The boundary condition on $|x|=1$ follows from the fact $\varphi_\rho(x) \nearrow \varphi_\infty(x)$ as $\rho \nearrow +\infty$ for all $x \in R^d$, and Proposition 1.1 (i).

Since the killed process (X_t, σ_ρ, P_x) , $x \in \Sigma_\rho$ has the communication property i. e.

$$P_x[x_t \in U \text{ for some } t \in (0, \sigma_\rho)] > 0$$

for every open subset U of Σ_ρ and every $x \in \Sigma_\rho$, it is obvious that

$$P_x[\tau_1 < \sigma_\rho] > 0 \text{ for } x \in \Sigma_\rho - \bar{\Sigma}_1.$$

Since $\varphi_\infty(x) \geq \varphi_\rho(x)$, $x \in R^d$, it follows that $\varphi_\infty(x) > 0$, $x \in R^d$.

Q. E. D.

It should be noted that in the above results the unit sphere centered at the origin can be replaced by an arbitrary sphere in R^d .

From the corollary and the remark above, we have the following

Lemma 1.1. (X_t, P_x) is recurrent if and only if $\varphi_\infty \equiv 1$ on R^d .

Proof. It suffices to show that if $\varphi_\infty \equiv 1$, then

$$(1.13) \quad P_x[x_t \in \bar{U} \text{ for some } t > 0] \equiv 1$$

for any ball U in R^d such that $\partial U \cap \partial \Sigma_1 = \emptyset$.

Let \mathcal{F}_t be the σ -field generated by X_s , $s \leq t$. Define

$$Y_t \equiv \begin{cases} P_{x_t}[\tau_\sigma < \zeta] & \text{if } t < \tau_\sigma \\ 1 & \text{otherwise,} \end{cases}$$

where $\tau_\sigma = \inf\{t \geq 0 : x_t \in \bar{U}\}$. Then it is easy to see that

$$Y_t = P[\tau_\sigma < \zeta | \mathcal{F}_t]$$

and so Y_t is a martingale. From the martingale convergence theorem, one has

$$(1.14) \quad \lim_{t \nearrow +\infty} Y_t = \begin{cases} 1 & \text{if } \tau_\sigma < +\infty \\ 0 & \text{if } \tau_\sigma = +\infty \end{cases} \quad \text{a. e.}$$

On the other hand, from the continuity of $\varphi_\sigma(x) = P_x[\tau_\sigma < \zeta]$,

$$\lim_{t \nearrow +\infty} Y_t \geq \lim_{t \rightarrow +\infty} P_{x_t}[\tau_\sigma < \zeta] \geq \inf_{|x|=1} \varphi_\sigma(x) > 0$$

since each sample path x_t hits the unit sphere infinitely often with probability 1. This implies that τ_σ must be finite with probability 1.

Q. E. D.

Corollary 1.2. If the process is transient, then φ_∞ cannot attain the maximum value 1 in the exterior domain $R^d - \bar{\Sigma}_1$ and (1.6) holds.

Proof. Suppose φ_∞ attains the value 1 at $x_0 \in R^d - \bar{\Sigma}_1$. Let V be an open bounded domain such that $x_0 \in V$ and $\bar{V} \subset R^d - \bar{\Sigma}_1$. Define τ_{V^c} as before. Then it is easy to see from the strong Markov property that

$$\varphi_\infty(x_0) = E_{x_0}[\varphi_\infty(x_{t \wedge \tau_{V^c}})].$$

Combining this equality with the communication property of the process x_t , $t < \tau_{V^c}$ and the continuity of φ_∞ , we obtain

$$\varphi_\infty(x) = 1 \text{ on } \bar{V}.$$

Since V is arbitrary, we have $\varphi_\infty \equiv 1$ on $R^d - \bar{\Sigma}_1$. However this cannot happen by Lemma 1.1. This proves the first assertion.

To prove (1.6), we first note that there exist a point $x_1 \in R^d$ and a compact subset K of R^d such that

$$P_{x_1}[x_t \text{ hits } K \text{ i.o. as } t \nearrow +\infty] > 0$$

if (1.6) is false. We can assume $K = \bar{\Sigma}_1$ without loss of generality.

Define a sequence of stopping times for a fixed $\rho > 1$,

$$\begin{aligned} \tau_1^1 &= \tau_1 \\ \sigma_\rho^1 &= \inf \{t \geq \tau_1 : |x_t| \geq \rho\} \\ \tau_1^n &= \inf \{t \geq \sigma_\rho^{n-1} : |x_t| \leq 1\} \\ \sigma_\rho^n &= \inf \{t \geq \tau_1^n : |x_t| \geq \rho\} \text{ for } n \geq 2. \end{aligned}$$

Note that σ_ρ^n is finite with probability 1 for all n . Observe that

$$\{x_t \text{ hits } K \text{ i.o. as } t \nearrow +\infty\} \subseteq \{\tau_1^n < \zeta\}$$

for every positive integer n . We want to estimate

$$(1.15) \quad P_{x_1}[\tau_1^n < \zeta] = P_{x_1}[\tau_1^1 < \sigma_\rho^1 < \tau_1^2 < \dots < \sigma_\rho^{n-1} < \tau_1^n < \zeta].$$

By repeated use of the strong Markov property, the right hand side is equal to

$$(1.16) \quad E_{x_1}[\tau_1^1 < +\infty, E_{x_{\tau_1^1}}[\tau_1^2 < +\infty, E_{x_{\tau_1^2}}[\dots[\tau_1^{n-1} < +\infty, E_{x_{\tau_1^{n-1}}} [P_{x_{\sigma_\rho^{n-1}}}[\tau_1^n < \zeta] \dots]].$$

Set $c = \sup_{|x|=\rho} \varphi_\infty(x)$, then c is strictly less than 1 by the first assertion. From (1.15) and (1.16), we obtain

$$P_{x_1}[\tau_1^n < \zeta] \leq c^{n-1} P_{x_1}[\tau_1 < \zeta] \searrow 0$$

as $n \nearrow +\infty$.

Hence $P_{x_1}[x_t \text{ hits } K \text{ i. o. as } t \nearrow +\infty] = 0$. Thus we have shown (1.6).

Q. E. D.

We now prove the following fundamental criterion for recurrence.

Theorem 1.1. (X_t, P_x) is recurrent if and only if

$$(1.17) \quad \lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) = 0.$$

Proof. By virtue of Proposition 1.2 and Lemma 1.1, it suffices to show that $\varphi_\infty \equiv 1$ if and only if

$$D(\varphi_\infty : R^d) = 0.$$

The “only if” part is obvious. The proof of the “if” part is given as follows. Suppose $D(\varphi_\infty : R^d) = 0$ i. e.

$$\int_{R^d} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\infty}{\partial x_i} \frac{\partial \varphi_\infty}{\partial x_j} dx = 0.$$

Then by the strict positivity of the matrix $A(x)$, one sees that

$$(1.18) \quad \frac{\partial \varphi_\infty}{\partial x_i} = 0 \text{ a. e., } i = 1, \dots, d.$$

This implies that for a mollifier $\phi_\varepsilon(x)$ defined by

$$\phi_\varepsilon(x) \equiv \varepsilon^{-d} \phi(x/\varepsilon), \quad \varepsilon > 0,$$

where $\phi \in C_0^\infty(R^d)$ is non negative, $\int_{R^d} \phi(x) dx = 1$ and such that $\text{Supp}(\phi) \subseteq \{x \in R^d : |x| \leq 1\}$, one has that the convolution

$$\varphi_\infty * \phi_\varepsilon(x) = a \text{ constant } M_\varepsilon$$

since $\varphi_\infty * \phi_\varepsilon \in C^\infty(R^d)$ and $\frac{\partial \{\varphi_\infty * \phi_\varepsilon\}}{\partial x_i} = \frac{\partial \varphi_\infty}{\partial x_i} * \phi_\varepsilon = 0, \quad i = 1, \dots, d.$

Furthermore, since φ_∞ is bounded and continuous,

$$\lim_{\varepsilon \searrow 0} \varphi_\infty * \phi_\varepsilon(x) = \varphi_\infty(x), \quad x \in R^d.$$

Combining all of this, it follows that φ_∞ must be identical with

the constant 1. This compels the proof.

Q. E. D.

We shall now prove our main theorems by means of Theorem 1. 1.

Proof of Theorem A. First define the following function

$$\phi_\rho(|x|) = \frac{\int_{|x|}^\rho s^{1-d} \bar{E}_1^{-1}(s) ds}{\int_1^\rho s^{1-d} \bar{E}_1^{-1}(s) ds} \quad \text{for } 1 \leq |x| \leq \rho.$$

Then it is clear that $\phi_\rho(|x|) - \phi_\rho(x)$ is absolutely continuous and has bounded partial derivatives on $\Sigma_\rho - \bar{\Sigma}_1$, since L is uniformly elliptic on the set. Therefore the function $\phi_\rho - \phi_\rho$ is an element of $H^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$. See Agmon [1], Theorem 3. 1. Furthermore $\phi_\rho - \phi_\rho$ vanishes on the boundary $\partial(\Sigma_\rho - \bar{\Sigma}_1)$. Hence we see from Theorem 11. 5 of Lions and Magenes [15] page 68 that $\phi_\rho - \phi_\rho \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$.

Now applying Proposition 1. 1 (iii), one obtains

$$\begin{aligned} & \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \phi_\rho(|x|)}{\partial x_i} \frac{\partial \phi_\rho(|x|)}{\partial x_j} dx \\ & \geq \int_{\Sigma_\rho - \bar{\Sigma}_1} \sum_{i,j=1}^d a_{ij} \frac{\partial \varphi_\rho}{\partial x_i} \frac{\partial \varphi_\rho}{\partial x_j} dx. \end{aligned}$$

By a simple computation, the left hand side of the above inequality is equal to

$$\frac{|S^{d-1}|}{\int_1^\rho r^{1-d} \bar{E}_1^{-1}(r) dr} \searrow 0, \quad \text{as } \rho \nearrow +\infty$$

by the assumption. This combined with the above inequality implies that

$$\lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) = 0$$

which completes the proof of Theorem A by Theorem 1. 1.

Q. E. D.

In order to prove Theorem B, we need the following Lemma.

Lemma 1. 2. For every $v \in C_0^\infty(R^d)$ such that $v(x) = 1$ if $|x| \leq 1$

and $= 0$ if $|x| \geq \rho$, it holds that

$$(1.19) \quad D(v : \Sigma_\rho) \geq \int_{S^{d-1}} d\sigma \frac{|S^{d-1}|}{\int_1^\rho r^{1-d} E_2^{-1}(r\sigma) dr}.$$

Proof. By the Schwarz inequality,

$$(1.20) \quad (x, \xi)^2 \leq (A^{-1}(x)x, x) \cdot (A(x)\xi, \xi)$$

for every $(x, \xi) \in R^d \times R^d$.

Using the above inequality, we have

$$\begin{aligned} (1.21) \quad D(v : \Sigma_\rho) &= \int_{\Sigma_\rho - \Sigma_1} \sum_{i,j=1}^d a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx \\ &\geq \int_{\Sigma_\rho - \Sigma_1} \frac{1}{(A^{-1}(x)x, x)} (x, \nabla v)^2 dx \\ &= \int_{\Sigma_\rho - \Sigma_1} E_2(x) \left[\frac{\partial v}{\partial n} \right]^2 dx \\ &= \int_{S^{d-1}} |S^{d-1}| \cdot d\sigma \int_1^\rho r^{d-1} E_2(r\sigma) \left[\frac{\partial v(r\sigma)}{\partial r} \right]^2 dr. \end{aligned}$$

Applying the Schwarz inequality again, we obtain for every $\sigma \in S^{d-1}$;

$$\begin{aligned} (1.22) \quad &\left[\int_1^\rho r^{d-1} E_2(r\sigma) \left[\frac{\partial v(r\sigma)}{\partial r} \right]^2 dr \right] \cdot \left[\int_1^\rho r^{1-d} E_2^{-1}(r\sigma) dr \right] \\ &\geq \left[\int_1^\rho \frac{\partial v(r\sigma)}{\partial r} dr \right]^2 = [v(\rho\sigma) - v(\sigma)]^2 = 1, \end{aligned}$$

from the assumption on $v(x)$. Combining (1.21) and (1.22), we obtain the desired inequality (1.19). Q. E. D.

Proof of Theorem B. By virtue of Proposition 1.1 (iii), one can choose a sequence w_n , $n \geq 1 \in C_0^\infty(\Sigma_\rho - \bar{\Sigma}_1)$ such that w_n converges to $\varphi_\rho - \phi_\rho$ in $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$. Since L is uniformly elliptic on $\bar{\Sigma}_\rho$, the above implies that

$$\lim_{n \rightarrow +\infty} D(w_n + \phi_\rho : \Sigma_\rho) = D(\varphi_\rho : \Sigma_\rho).$$

Applying Lemma 1.2 to the function $w_n + \phi_\rho$, we obtain

$$D(w_n + \phi_\rho : \Sigma_\rho) \geq \int_{S^{d-1}} d\sigma \frac{|S^{d-1}|}{\int_1^\rho r^{1-d} E_2^{-1}(r\sigma) dr}.$$

Letting $n \nearrow +\infty$,

$$\begin{aligned} D(\varphi_\rho : \Sigma_\rho) &\geq \int_{S^{d-1}} d\sigma \frac{|S^{d-1}|}{\int_1^\rho r^{1-d} E_2^{-1}(r\sigma) dr} \\ &\geq \int_{S^{d-1}} d\sigma \frac{|S^{d-1}|}{\int_1^\infty r^{1-d} E_2^{-1}(r\sigma) dr} > 0 \end{aligned}$$

by the assumption. Thus we have shown

$$\lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) > 0.$$

This together with Theorem 1.1 completes the proof of Theorem B.

Q. E. D.

**Chapter 2. Criteria for Recurrence and Transience II.
Hypoelliptic differential operator**

In this chapter we shall study the recurrence and transience problems of the diffusion process associated with a degenerate elliptic operator of self adjoint form.

We assume that the self adjoint operator $L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$ under consideration is of the form

$$(2.1) \quad L = \frac{1}{2} \sum_{i=1}^r X_i^2 + Y, \quad r \leq d$$

on R^d , where

$$(2.2) \quad \begin{aligned} X_i &= \sum_{j=1}^d g_{ji} \frac{\partial}{\partial x_j}, \quad i = 1, \dots, r \\ Y &= \sum_{i=1}^d f_i \frac{\partial}{\partial x_i} \end{aligned}$$

and $g_{ij}, f_i \in C^\infty(R^d)$, $i = 1, \dots, d$, $j = 1, \dots, r$. Our fundamental assumption on the operator L is

$$(*) \quad \text{rank } \mathcal{L}(X_1, \dots, X_r)_x = d, \quad x \in R^d,$$

where $\mathcal{L}(X_1, \dots, X_r)$ is the minimal Lie algebra generated by $X_i, i=1, \dots, r$. See Bony [3], pp. 277–279.

Remark 2.1. A simple computation shows that $Y \in \mathcal{L}(X_1, \dots, X_r)$ under the self adjointness of the operator. Therefore the assumption (*) is equivalent to the weaker one

$$(**) \quad \text{rank } \mathcal{L}(X_1, \dots, X_r, Y)_x = d, \quad x \in R^d.$$

Remark 2.2. Hörmander [9], p. 149, has shown that under the rank condition (**), the second order differential operator (2.1) which is not necessarily self adjoint is hypoelliptic i.e. if f is a.e. equal to a C^∞ function in an open subset D of R^d , any distribution solution u of $Lu=f$ in D is a.e. equal to a C^∞ function in D .

We now construct a minimal diffusion process in the sense of McKean [17] which is associated with the operator L .

From our definition of L , we have

$$a_{ij} = \frac{1}{2} \sum_{k=1}^r g_{ik} g_{jk}, \quad i, j = 1, \dots, d$$

and define

$$G(x) = (g_{ij}(x))_{\substack{1 \leq i \leq d \\ 1 \leq j \leq r}}$$

$$b_i(x) = \sum_{j=1}^d \frac{\partial a_{ji}}{\partial x_j}, \quad i = 1, \dots, d$$

$$b(x) = (b_i(x))_{1 \leq i \leq d}.$$

Let $\beta_t = (\beta_t^i)_{1 \leq i \leq r}$ be an r -dimensional Brownian motion.

Consider the stochastic differential equation in R^d ,

$$(2.3) \quad X_t = X_0 + \int_0^t G(x_s) d\beta_s + \int_0^t b(x_s) ds, \quad x_0 \in R^d.$$

This has a unique continuous solution x_t up to the explosion time ζ . Thus we have obtained the diffusion process $(X_t, \zeta, P_x), x \in R^d$.

Let $P_t(x, dy)$ be the transition probability of the above process. Itô's formula implies that

$$u(t, x_t) - u(0, x_0) = \sum_{i=1}^r \int_0^t X_i u(s, x_s) d\beta_s^i + \int_0^t \left(\frac{\partial}{\partial s} + L \right) u(s, x_s) ds, \quad \text{for } t < \zeta$$

and $u \in C_0^\infty(R^+ \times R^d)$. Since the first term of the right hand side is a martingale with mean 0, we get

$$(2.4) \quad \int_{R^d} P_t(x_0, dy) u(t, y) - u(0, x_0) = \int_0^t \int_{R^d} P_s(x_0, dy) \left(\frac{\partial}{\partial s} + L \right) u(s, y) ds.$$

Consequently the diffusion process constructed above is associated with the operator L . We note that under our fundamental assumption (*), the transition probability $P_t(x, dy)$ has a smooth density i. e. there exists a function $p_t(x, y) \in C^\infty((0, \infty) \times R^d \times R^d)$ such that

$$(2.5) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - L_x \right) p_t(x, y) &= 0 \\ \left(\frac{\partial}{\partial t} - L_y \right) p_t(x, y) &= 0 \end{aligned}$$

where $L_x = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$ and $L_y = \sum_{i,j=1}^d \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial}{\partial y_j} \right)$. In fact, (2.5) is a special case of Theorem 3 of Ichihara and Kunita [11].

Remark 2.3. Under our assumption (*), the main theorem of Stroock and Varadhan [20], combined with Proposition 2.1 of Bony [3], states that the topological support of P_x coincides with the set of all continuous functions taking values in R^d and initial value x , i. e.

$$(2.6) \quad \text{Supp}(P_x) = C_x, \quad x \in R^d,$$

where $C_x = \{ \varphi_t, t \geq 0 : \varphi_0 = x, \varphi_t \text{ is continuous on } [0, e(\varphi)) \text{ and where } e(\varphi) \text{ is the explosion time of } \varphi \}$.

As a direct consequence of this fact, one has the communication property for the process (X_t, ζ, P_x) . More precisely, for each $(t, x) \in (0, \infty) \times R^d$ and each open subset U of R^d , $P_t(x, U) > 0$.

Remark 2.4. The communication property for the process (X_t, ζ, P_x) implies the following fact which is important for our discussion.

Let D be an open domain with compact closure \bar{D} . Define $\tau'_D = \inf \{ t \geq 0 ; x_t \notin \bar{D} \}$. Then we have

$$(2.7) \quad \sup_{x \in D} E_x[\tau_D'] < +\infty.$$

See Stroock and Varadhan [21], Remark 5.2 for the proof.

Denote by $\tilde{E}_2(x)$ the minimal eigenvalue of the matrix $A(x) = (a_{ij}(x))_{1 \leq i, j \leq d}$. We note that

$$\tilde{E}_2(x) \leq E_2(x)$$

if the matrix is not singular at x . Thus the condition in Theorem D below is stronger than the one in Theorem B.

The followings are our main results in this chapter.

Theorem C. *If the condition (*) holds and if*

$$\int_1^\infty r^{1-d} \tilde{E}_1^{-1}(r) dr = +\infty,$$

then the process is recurrent.

Theorem D. *If the condition (*) holds and if*

$$\int_{r_0}^\infty r^{1-d} \tilde{E}_2^{-1}(r\sigma) dr < +\infty$$

for some fixed constant $r_0 > 0$ on a subset of S^{d-1} with positive uniform measure, then the process is transient.

The proofs of the above theorems will follow the same lines as in the preceding chapter, though several steps have to be redone.

We start with the Dirichlet problem. The first boundary value problem for the most general elliptic differential operator has been treated from the martingale point of view by Stroock and Varadhan [21].

In the following we need a somewhat more precise result and we shall derive it first. In order to do it, we introduce the “barrier” function

$$\phi_\rho(x) = 1 - E_x[e^{-\tau_1 \wedge \sigma_\rho}]$$

and define

$$\Gamma_{0,\rho} = \{x \in \partial(\Sigma_\rho - \bar{\Sigma}_1) : \lim_{\substack{y \rightarrow x \\ y \in \Sigma_\rho - \bar{\Sigma}_1}} \phi_\rho(y) = 0\}, \quad 1 < \rho < +\infty$$

$$(2.8) \quad \Gamma_{0,\infty} = S^{d-1} \cap \bigcup_{\rho > 1} \Gamma_{0,\rho}.$$

Proposition 2.1. For each $\rho \in (1, \infty]$, (i) $\varphi_\rho \in C^\infty(\Sigma_\rho - \bar{\Sigma}_1)$ and $L\varphi_\rho = 0$ in $\Sigma_\rho - \bar{\Sigma}_1$

$$(ii) \quad \varphi_\rho(y) \xrightarrow{\phi_\rho} g_\rho(x) \equiv \begin{cases} 1 & \text{if } |x| = 1 \\ 0 & \text{if } |x| = \rho < +\infty \end{cases} \quad \text{as } y \in \Sigma_\rho - \bar{\Sigma}_1 \rightarrow x \in \Gamma_{0,\rho}$$

where $\varphi_\rho(y) \xrightarrow{\phi_\rho} g_\rho(x)$ means that for $\rho < +\infty$, $\varphi_\rho(y_n) \rightarrow g_\rho(x)$ if $y_{n,n \geq 1} \in \Sigma_\rho - \bar{\Sigma}_1$ tends to $x \in \Gamma_{0,\rho}$ and $\lim_{n \rightarrow +\infty} \phi_\rho(y_n) = 0$ and that for $\rho = +\infty$, $\varphi_\infty(y_n) \rightarrow g_\infty(x) \equiv 1$ if $y_{n,n \geq 1} \in R^d - \bar{\Sigma}_1$ tends to $x \in \Gamma_{0,\infty}$ and $\lim_{n \rightarrow +\infty} \phi_\rho(y_n) = 0$ for every $\rho \in (1, \rho_0)$ with a positive constant $\rho_0 = \rho_0(x) > 1$.

Proof. We first discuss the case $1 < \rho < +\infty$. Let D be a domain of R^d . Stroock and Varadhan [21], Chapter 5 and Theorem 7.1 have formulated the first boundary value problem

$$"Lu = f \text{ in } D$$

and

$$u = g \text{ on } \partial D"$$

in two ways, which are called the τ' and τ -first boundary value problem respectively, and shown that under our basic assumption (*), there exists a unique solution of each problem if D is bounded and that both the solutions are the same a. e. in D . They have further proved that the τ' -solution is a weak solution of $Lu = f$ in D . [21] Corollary 8.2. Since φ_ρ is a unique solution of the τ' -first boundary value problem with respect to $D = \Sigma_\rho - \bar{\Sigma}_1$, $f \equiv 0$ and $g = \begin{cases} 1 & \text{on } |x| = 1 \\ 0 & \text{on } |x| = \rho' \end{cases}$, see [21]. Theorem 5.2, this combined with the above facts implies that φ_ρ is also a weak solution of $Lu = 0$ in $\Sigma_\rho - \bar{\Sigma}_1$, i. e.

$$(2.9) \quad \int_{\Sigma_\rho - \bar{\Sigma}_1} \varphi_\rho \cdot L\phi dx = 0 \text{ for all } \phi \in C_0^\infty(\Sigma_\rho - \bar{\Sigma}_1).$$

By virtue of the hypoellipticity of the operator L Remark 2.2, there exists a function $\psi_\rho \in C^\infty(\Sigma_\rho - \bar{\Sigma}_1)$ such that ψ_ρ is a. e. equal to φ_ρ in

$\Sigma_\rho - \bar{\Sigma}_1$ and $L\phi_\rho = 0$.

We want to show that $\varphi_\rho(x) = \phi_\rho(x)$ for all $x \in \Sigma_\rho - \bar{\Sigma}_1$. In order to do this, we note that the transition probability of the stopped process $x_t, t < \tau_1 \wedge \sigma_\rho$ is absolutely continuous with respect to the Lebesgue measure dy because the transition probability $P_t(x, dy)$ of the process (X_t, ζ, P_x) has the smooth density $p_t(x, y)$ with respect to the Lebesgue measure dy and

$$\begin{aligned} P_t^e(x, dy) &= P_x \{x_t \in dy, t < \tau_1 \wedge \sigma_\rho\} \\ &\leq P_x \{x_t \in dy, t < \zeta\} = P_t(x, dy). \end{aligned}$$

Returning to the proof of Proposition 2.1, we first note that $\varphi_\rho(x_{t \wedge \tau_1 \wedge \sigma_\rho})$ is a P_x -martingale, i. e.

$$E_x[\varphi_\rho(x_{t \wedge \tau_1 \wedge \sigma_\rho})] = \varphi_\rho(x), \quad x \in \Sigma_\rho - \bar{\Sigma}_1,$$

holds. Decompose the left hand side as

$$(2.10) \quad \begin{aligned} E_x[\varphi_\rho(x_{t \wedge \tau_1 \wedge \sigma_\rho})] &= E_x[\varphi_\rho(x_t) : t < \tau_1 \wedge \sigma_\rho] \\ &\quad + E_x[\varphi_\rho(x_{\tau_1 \wedge \sigma_\rho}) : t \geq \tau_1 \wedge \sigma_\rho]. \end{aligned}$$

The first term equals $E_x[\phi_\rho(x_t) : t < \tau_1 \wedge \sigma_\rho]$ by the absolute continuity of $P_t^e(x, dy)$. Since $\phi_\rho(x_t)$ converges to $\phi_\rho(x)$ as $t \searrow 0$ and $\lim_{t \searrow 0} P_x[t < \tau_1 \wedge \sigma_\rho] = 1$, one obtains that

$$\lim_{t \searrow 0} E_x[\phi_\rho(x_t) : t < \tau_1 \wedge \sigma_\rho] = \phi_\rho(x).$$

The second term in the right hand side of (2.10) satisfies

$$\begin{aligned} E_x[\varphi_\rho(x_{\tau_1 \wedge \sigma_\rho}) : t \geq \tau_1 \wedge \sigma_\rho] \\ \leq P_x[t \geq \tau_1 \wedge \sigma_\rho] \searrow 0 \end{aligned}$$

as $t \searrow 0$. Consequently it follows that

$$\begin{aligned} &\lim_{t \searrow 0} E_x[\varphi_\rho(x_{t \wedge \tau_1 \wedge \sigma_\rho})] \\ &= \lim_{t \searrow 0} E_x[\varphi_\rho(x_t) : t < \tau_1 \wedge \sigma_\rho] \\ &= \lim_{t \searrow 0} E_x[\phi_\rho(x_t) : t < \tau_1 \wedge \sigma_\rho] = \phi_\rho(x), \end{aligned}$$

which shows that φ_ρ is identical with ϕ_ρ in $\Sigma_\rho - \bar{\Sigma}_1$. The boundary condition (ii) has been already shown by Stroock and Varadhan [21], Theorem 5.2.

For $\rho = +\infty$, [etting $\rho \nearrow +\infty$, one sees that

$$\int_{R^d - \Sigma_1} \varphi_\infty \cdot L\phi dx = 0 \text{ for all } \phi \in C_0^\infty(R^d - \Sigma_1)$$

from (2.9) and the fact that φ_ρ is boundedly convergent to φ_∞ on R^d . This implies that φ_∞ is a weak solution of $Lu=0$ in $R^d - \Sigma_1$. In order to obtain the smoothness of φ_∞ , we need the following Harnack inequality, valid under the condition (*).

Harnack inequality (Bony [2], Theorem 7.2). *If u is a positive solution of $Lu=0$ in an open domain D , then for any compact subset K of D , any point $y_0 \in D$ and any multi index $p=(p_1, \dots, p_d)$ of derivatives, there exists a positive constant $C=C(K, y_0, p)$ independent of u such that*

$$(2.11) \quad \sup_{x \in K} \left| \frac{\partial^{p_1} \dots \partial^{p_d} u(x)}{\partial x_1^{p_1} \dots \partial x_d^{p_d}} \right| \leq C \cdot u(y_0)$$

where $|p| = \sum_{i=1}^d p_i$.

It is easily derived from Bony's Harnack inequality that for any compact subset K of $\Sigma_{\rho_0} - \Sigma_1$ with $\rho_0 > 1$, there exists a positive constant $C=C(K, \rho_0)$ such that

$$|\varphi_\rho(x) - \varphi_\rho(x')| \leq C|x - x'|$$

for every $x, x' \in K$ and $\rho \in [\rho_0, +\infty)$. Since $\varphi_\infty(x) = \lim_{\rho \nearrow +\infty} \varphi_\rho(x)$, $x \in R^d$, we obtain

$$(2.12) \quad |\varphi_\infty(x) - \varphi_\infty(x')| \leq C|x - x'|, \text{ for } x, x' \in K.$$

This implies that φ_∞ is continuous in $R^d - \Sigma_1$. Again using the hypoellipticity of L , we see that φ_∞ is a genuine solution of $Lu=0$.

The boundary condition for φ_∞ follows from the fact that $\varphi_\rho \nearrow \varphi_\infty$ as $\rho \nearrow +\infty$. Q. E. D.

Corollary 2.1. *For any ball $U \subset R^d$, $\varphi_U(x) \equiv P_x[x_t \in \bar{U}$ for some $t > 0] \in C^\infty(R^d - \bar{U})$, $L\varphi_U = 0$ in $R^d - \Sigma_1$ and furthermore φ_U is strictly positive on R^d .*

Proof. The assertions except for the last one follow as after Corollary 1.1. The strict positivity of φ_U follows from Remark 2.3. Q. E. D.

The martingale method used in Lemma 1.1 combined with Corollary 2.1 yields the following

Corollary 2.2. *The process (X_t, p_x) is recurrent if and only if $\varphi_\infty \equiv 1$ on R^d .*

Using a technique similar to that of Corollary 1.2, we can prove

Corollary 2.3. *If the process is transient, then φ_∞ cannot attain the maximum value 1 in the exterior domain $R^d - \bar{\Sigma}_1$ and*

$$(2.13) \quad P_x[\lim_{t \nearrow \zeta} |x_t| = +\infty] \equiv 1 \text{ on } R^d.$$

Finally we summarize the results of [21] on the set of regular points. Let Γ_ρ be the set of all regular points with respect to (X_t, ζ, P_x) and $\Sigma_\rho - \bar{\Sigma}_1$ i. e.

$$\Gamma_\rho = \{x \in \partial(\Sigma_\rho - \bar{\Sigma}_1) : P_x[\tau_{\Sigma_\rho - \bar{\Sigma}_1} = 0] = 1\}$$

where $\tau_{\Sigma_\rho - \bar{\Sigma}_1} = \inf\{t \geq 0; x_t \in (\overline{\Sigma_\rho - \bar{\Sigma}_1})^c\}$, and let

$$\Gamma_{1,\rho} = \{x \in \partial(\Sigma_\rho - \bar{\Sigma}_1) : \sum_{i,j=1}^d a_{ij}(x)x_i x_j > 0\}.$$

Stroock and Varadhan [21], Remark 5.4 and Theorem 6.3 have shown for our case that

$$(2.14) \quad \Gamma_{1,\rho} \subseteq \Gamma_\rho \subseteq \Gamma_{0,\rho} \text{ for } 1 < \rho < +\infty.$$

The following lemma shows that the set Γ_ρ is large enough.

Lemma 2.2. *$\Gamma_{1,\rho}$ is open dense in $\partial(\Sigma_\rho - \bar{\Sigma}_1)$.*

For the proof of this lemma, we introduce the following notation. For a given $Z \in \mathcal{L}(X_1, \dots, X_r)$ where $Z = \sum_{i=1}^d z_i(x) \frac{\partial}{\partial x_i}$, $z_i \in C^\infty(R^d)$,

we denote by $Z(x)$ the vector $(z_i(x))_{1 \leq i \leq d}$.

Proof of Lemma 2.2. Suppose there exists an open subset U of $\{|x| = \bar{\rho}\}$ where $\bar{\rho} = 1$ or $\bar{\rho} (< +\infty)$ such that

$$\sum_{i,j=1}^d a_{ij}(x)x_i x_j = 0 \text{ for all } x \in U.$$

Then it is easy to see that the inner product

$$(x, X_i(x)) = 0 \text{ on } U, \quad i = 1, \dots, r.$$

By virtue of Theorem 2.2 of Bony [3], one has that

$$(x, Z(x)) = 0 \text{ on } U \text{ for all } Z \in \mathcal{L}(x_1, \dots, x_r)$$

which is a contradiction since the Lie algebra $\mathcal{L}(X_1, \dots, X_r)$ has the full rank “ d ”. Therefore $\Gamma_{1,\bar{\rho}}$ must be open dense in $\partial(\Sigma_\rho - \bar{\Sigma}_1)$.

Q. E. D.

Our next task is to extend the variational principle to the degenerate case. In order to do this we make use of the Hilbert space $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1; dx)$.

From the definition of the $(d \times r)$ -matrix $G(x)$ it is clear that $A(x) = \frac{1}{2}(G \cdot G^t)(x)$ where G^t is the transpose of G . Using a standard technique in functional analysis (Yosida [22], page 58), we can obtain the following

Lemma 2.3. *Each $u \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1; dx)$ has the following properties :*

- (i) u and $(G^t \nabla u)_i \in L^2(\Sigma_\rho - \bar{\Sigma}_1)$, $i = 1, \dots, r$ where $\nabla u = \left(\frac{\partial u}{\partial x_i} \right)_{1 \leq i \leq d}$, $\frac{\partial u}{\partial x_i}$ represents the distribution derivative.
- (ii)

$$(2.15) \quad \|u\|_{H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1; dx)} = \sqrt{\int_{\Sigma_\rho - \bar{\Sigma}_1} \{u(x)^2 + \frac{1}{2}(G^t \nabla u, G^t \nabla u)(x)\} dx.}$$

We are now ready to state the main proposition.

Proposition 2.2. *For each $\rho \in (1, +\infty)$, let ϕ_ρ be a C^∞ function*

as in Proposition 1.1 (iii), then

$$(i) \quad \varphi_\rho \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1 : dx) + \{\phi_\rho\}$$

and

$$(ii) \quad \varphi_\rho \text{ minimizes an integral}$$

$$(2.16) \quad \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla u, G^i \nabla u) dx$$

over all $u \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1 : dx) + \{\phi_\rho\}$.

Note that $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1 : dx) + \{\phi_\rho\}$ is independent of the choice of ϕ_ρ and increasing as in the remark after Proposition 1.1.

Our proof is based on a perturbation method. Let X_ε^i , $\varepsilon > 0$ be the minimal diffusion process corresponding to $L^\varepsilon \equiv L + \varepsilon \Delta_d$ where $\Delta_d = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ and τ_ε^i , σ_ρ^ε and φ_ρ^ε be defined as before. Under our basic assumption on the operator L , we have the following lemma which plays an essential role in the proof of Proposition 2.2.

Lemma 2.4 (Stroock and Varadhan [21], Theorems 7.1, 9.2, and Remark 9.2). For each $\rho \in (1, +\infty)$, one has that

$$(2.17) \quad \lim_{\varepsilon \searrow 0} \varphi_\rho^\varepsilon(x) = \varphi_\rho(x) \text{ a. e. in } \Sigma_\rho - \bar{\Sigma}_1.$$

Proof of Proposition 2.2. By virtue of the variational principle for the nondegenerate case, one has that

$$(2.18) \quad \int_{\Sigma_\rho - \Sigma_1} \{ \varepsilon (\nabla \varphi_\rho^\varepsilon, \nabla \varphi_\rho^\varepsilon) + \frac{1}{2} (G^i \nabla \varphi_\rho^\varepsilon, G^i \nabla \varphi_\rho^\varepsilon) \} dx$$

is nonincreasing as $\varepsilon \searrow 0$. Hence its limit exists. Set

$$\tilde{\varphi}_\rho^\varepsilon = \varphi_\rho^\varepsilon - \phi_\rho \text{ and } \tilde{\varphi}_\rho = \varphi_\rho - \phi_\rho.$$

Then it follows from Proposition 1.1 (iii) that

$$\tilde{\varphi}_\rho^\varepsilon \in H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1) \subseteq H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1 : dx).$$

We have, for $\varepsilon > \varepsilon' > 0$

$$\int_{\Sigma_\rho - \Sigma_1} \frac{1}{2} (G^i \nabla (\tilde{\varphi}_\rho^\varepsilon - \tilde{\varphi}_\rho^{\varepsilon'}), G^i \nabla (\tilde{\varphi}_\rho^\varepsilon - \tilde{\varphi}_\rho^{\varepsilon'})) dx$$

$$\begin{aligned}
 &= \int_{\Sigma_\rho - \Sigma_1} \frac{1}{2} (G^i \nabla(\varphi_\rho^i - \varphi_{\rho'}^{i'}), G^i \nabla(\varphi_\rho^i - \varphi_{\rho'}^{i'})) dx \\
 &\leq \int_{\Sigma_\rho - \Sigma_1} \{ \varepsilon' (\nabla(\varphi_\rho^i - \varphi_{\rho'}^{i'}), \nabla(\varphi_\rho^i - \varphi_{\rho'}^{i'})) \\
 &\quad + \frac{1}{2} (G^i \nabla(\varphi_\rho^i - \varphi_{\rho'}^{i'}), G^i \nabla(\varphi_\rho^i - \varphi_{\rho'}^{i'})) \} dx.
 \end{aligned}$$

By the same reasoning as in Proposition 1.2, this equals

$$\begin{aligned}
 &\int_{\Sigma_\rho - \Sigma_1} \{ \varepsilon' (\nabla \varphi_\rho^i, \nabla \varphi_\rho^i) + \frac{1}{2} (G^i \nabla \varphi_\rho^i, G^i \nabla \varphi_\rho^i) \} dx \\
 &- \int_{\Sigma_\rho - \Sigma_1} \{ \varepsilon' (\nabla \varphi_{\rho'}^{i'}, \nabla \varphi_{\rho'}^{i'}) + \frac{1}{2} (G^i \nabla \varphi_{\rho'}^{i'}, G^i \nabla \varphi_{\rho'}^{i'}) \} dx \\
 &\leq \int_{\Sigma_\rho - \Sigma_1} \{ \varepsilon (\nabla \varphi_\rho^i, \nabla \varphi_\rho^i) + \frac{1}{2} (G^i \nabla \varphi_\rho^i, G^i \nabla \varphi_\rho^i) \} dx \\
 &- \int_{\Sigma_\rho - \Sigma_1} \{ \varepsilon' (\nabla \varphi_{\rho'}^{i'}, \nabla \varphi_{\rho'}^{i'}) + \frac{1}{2} (G^i \nabla \varphi_{\rho'}^{i'}, G^i \nabla \varphi_{\rho'}^{i'}) \} dx
 \end{aligned}$$

which tends to zero as $\varepsilon, \varepsilon' \searrow 0$.

Combining this fact with Lemma 2.4, one obtains

$$(2.19) \quad \lim_{\varepsilon, \varepsilon' \searrow 0} \|\tilde{\varphi}_\rho^\varepsilon - \tilde{\varphi}_{\rho'}^{\varepsilon'}\|_{H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx)} = 0.$$

Therefore there exists a unique element $\tilde{\varphi}$ of $H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx)$ to which $\tilde{\varphi}_\rho^\varepsilon$ converges as $\varepsilon \searrow 0$. On the other hand, $\tilde{\varphi}_{\rho'}^{\varepsilon'}$ itself converges to $\tilde{\varphi}_{\rho'}$ a. e. in $\Sigma_{\rho'} - \Sigma_1$ by Lemma 2.4. This implies that $\tilde{\varphi}_{\rho'}$ has to be in $H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx)$ and that $\tilde{\varphi}_{\rho'} = \tilde{\varphi}$ in this space. This completes the proof of the first assertion.

Since φ_ρ is a weak solution of $Lu=0$ in $\Sigma_\rho - \Sigma_1$,

$$\int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi_\rho, G^i \nabla u) dx = 0$$

for every $u \in C_0^\infty(\Sigma_\rho - \Sigma_1)$. Taking the closure in the space $H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx)$, one obtains that

$$(2.20) \quad \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi_\rho, G^i \nabla u) dx = 0$$

for every $u \in H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx)$. Coming to the proof of the minimality,

for $\varphi \in H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx) + \{\phi_\rho\}$

$$\begin{aligned} & \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi, G^i \nabla \varphi) dx \\ &= \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \{\varphi_\rho + \varphi - \varphi_\rho\}, G^i \nabla \{\varphi_\rho + \varphi - \varphi_\rho\}) dx \\ &= \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi_\rho, G^i \nabla \varphi_\rho) dx + 2 \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi_\rho, G^i \nabla (\varphi - \varphi_\rho)) dx \\ & \quad + \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla (\varphi - \varphi_\rho), G^i \nabla (\varphi - \varphi_\rho)) dx. \end{aligned}$$

Since $\varphi - \varphi_\rho \in H_0^{1,2}(\Sigma_\rho - \Sigma_1; dx)$, the second term vanishes (See (2.20)). Hence the following inequality holds.

$$(2.21) \quad \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi, G^i \nabla \varphi) dx \geq \int_{\Sigma_\rho - \Sigma_1} (G^i \nabla \varphi_\rho, G^i \nabla \varphi_\rho) dx$$

which shows that the second assertion is valid. Q. E. D.

Following the same argument as in the preceding chapter, we can obtain the same results for the degenerate case as in Chapter 1. These are summarized in the foollwing.

Theorem 2.1. (i)

$$(2.22) \quad \lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) = D(\varphi_\infty : R^d) < +\infty.$$

(ii) *The process (X_t, P_x) is recurrent if and only if*

$$(2.23) \quad \lim_{\rho \nearrow +\infty} D(\varphi_\rho : \Sigma_\rho) = 0.$$

Proof. For the completion of the proof, it suffices to show that $\varphi_\infty \equiv 1$ on R^d if and only if $D(\varphi_\infty : R^d) = 0$. The “only if” part is obvious. Suppose $D(\varphi_\infty : R^d) = 0$

$$\begin{aligned} D(\varphi_\infty : R^d) &= \int_{R^d - \Sigma_1} \frac{1}{2} (G^i \nabla \varphi_\infty, G^i \nabla \varphi_\infty) dx \\ &= \int_{R^d - \Sigma_1} \frac{1}{2} \sum_{i=1}^r [X_i \varphi_\infty](x) dx. \end{aligned}$$

Since $X_i\varphi_\infty$ is continuous by Proposition 2.1,

$$X_i\varphi_\infty(x) = 0 \text{ in } R^d - \bar{\Sigma}_1, \quad i=1, \dots, r.$$

This leads to

$$(2.24) \quad Z\varphi_\infty(x) \equiv 0 \text{ in } R^d - \bar{\Sigma}_1$$

for every $Z \in \mathcal{L}(X_1, \dots, X_r)$. Since $\text{rank } \mathcal{L}(X_1, \dots, X_r) = d$, one obtains

$$\varphi_\infty \equiv a \text{ constant in } R^d - \bar{\Sigma}_1.$$

Taking the boundary value into consideration (see Proposition 2.1 (ii), (2.14) and Lemma 2.2), we conclude that the above constant is equal to 1. This completes the proof. Q. E. D.

Using Theorem 2.1, we are now able to prove Theorems C and D. The proof of Theorem C is carried out in exactly the same way as in Chapter 1. Note that the space $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1; dx)$ in place of $H_0^{1,2}(\Sigma_\rho - \bar{\Sigma}_1)$ will be used in the proof and that $\bar{E}_1(r)$ is bounded away from zero on bounded intervals by virtue of Lemma 2.2.

In order to prove Theorem D, we need the following lemma similar to Lemma 1.2. Let Ω be the subset of S^{d-1} mentioned in Theorem D. Without loss of generality, we shall take $r_0 = 1$.

Lemma 2.4. *Let a function v be as in Lemma 1.2, then*

$$(2.25) \quad D(v; \Sigma_\rho) \geq \int_a^\rho d\sigma \frac{|S^{d-1}|}{\int_1^\rho r^{1-d} \bar{E}_2^{-1}(r\sigma) dr}$$

for each $\rho \in (1, +\infty)$.

Proof. From the definition, we have

$$D(v; \Sigma_\rho) \geq \int_{\Sigma_\rho - \bar{\Sigma}_1} \bar{E}_2(x) \sum_{i=1}^d \left[\frac{\partial v}{\partial x_i} \right]^2 dx.$$

Using polar coordinates,

$$(2.26) \quad D(v; \Sigma_\rho) \geq \int_{S^{d-1}} |S^{d-1}| d\sigma \int_1^\rho r^{d-1} \bar{E}_2(r\sigma) \left[\frac{\partial v(r\sigma)}{\partial r} \right]^2 dr$$

$$\geq |S^{d-1}| \int_{\mathfrak{a}} d\sigma \int_1^{\rho} r^{d-1} \tilde{E}_2(r\sigma) \left[\frac{\partial v(r\sigma)}{\partial r} \right]^2 dr.$$

Applying the Schwarz inequality for each $\sigma \in \Omega$, we have

$$(2.27) \quad \int_1^{\rho} r^{d-1} \tilde{E}_2(r\sigma) \left[\frac{\partial v(r\sigma)}{\partial r} \right]^2 dr \geq \frac{(v(\rho\sigma) - v(\sigma))^2}{\int_1^{\rho} r^{1-d} \tilde{E}_2^{-1}(r\sigma) dr}.$$

(2.25) now follows from (2.26) and (2.27). Q. E. D.

Proof of Theorem D. Since $\varphi_{\rho} - \phi_{\rho} \in H_0^{1,2}(\Sigma_{\rho} - \bar{\Sigma}_1; dx)$, see Proposition 2.2(i), there exists a sequence $w_n, n \geq 1$ such that $w_n \in C_0^{\infty}(\Sigma_{\rho} - \bar{\Sigma}_1)$ and

$$\lim_{n \nearrow +\infty} \|w_n - (\varphi_{\rho} - \phi_{\rho})\|_{H_0^{1,2}(\Sigma_{\rho} - \bar{\Sigma}_1; dx)} = 0.$$

This gives that

$$(2.28) \quad \lim_{\rho \nearrow +\infty} D(w_n + \phi_{\rho} : \Sigma_{\rho}) = D(\varphi_{\rho} : \Sigma_{\rho}).$$

Applying Lemma 2.4 to $w_n + \phi_{\rho}$, we obtain

$$D(w_n + \phi_{\rho} : \Sigma_{\rho}) \geq \int_{\mathfrak{a}} d\sigma \frac{|S^{d-1}|}{\int_1^{\rho} r^{1-d} \tilde{E}_2^{-1}(r\sigma) dr}.$$

As $n \nearrow +\infty$, (2.28) gives

$$\begin{aligned} D(\varphi_{\rho} : \Sigma_{\rho}) &\geq \int_{\mathfrak{a}} d\sigma \frac{|S^{d-1}|}{\int_1^{\rho} r^{1-d} \tilde{E}_2^{-1}(r\sigma) dr} \\ &\geq \int_{\mathfrak{a}} d\sigma \frac{|S^{d-1}|}{\int_1^{\infty} r^{1-d} \tilde{E}_2^{-1}(r\sigma) dr} > 0, \end{aligned}$$

from the assumption in Theorem D. Thus we have shown that

$$\lim_{\rho \nearrow +\infty} D(\varphi_{\rho} : \Sigma_{\rho}) > 0.$$

This together with Theorem 2.1 completes the proof of Theorem D. Q. E. D.

**Chapter 3. Some Examples and Comparison with
Friedman [7] and Hasminskii [8]**

We first give some examples to illustrate our main results.

Examples 1 (Recurrent case). If $E_1(x) = O\left(\frac{\log|x|}{|x|^{d-2}}\right)$ as $|x| \nearrow +\infty$, then the associated diffusion is recurrent. In particular, if

$$(3.1) \quad \max_{i,j=1,\dots,d} |a_{ij}(x)| = O\left(\frac{\log|x|}{|x|^{d-2}}\right)$$

holds, then the above condition is fulfilled.

The followings are the examples of recurrent diffusion processes corresponding to hypoelliptic operators.

(i) ($d=2$). Let a diffusion (X_t, P_x) be associated with

$$L = \frac{1}{2} \left\{ \frac{\partial^2}{\partial x_1^2} + \frac{x_1^2}{1+x_1^2} \frac{\partial^2}{\partial x_2^2} \right\}.$$

Define first order partial differential operators X_1 and X_2 by

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{x_1}{\sqrt{1+x_1^2}} \frac{\partial}{\partial x_2}.$$

Then it is easy to see that $\text{rank } \mathcal{L}(X_1, X_2)_x = 2, x \in R^2$. The operator L satisfies the inequality (3.1), so the diffusion is recurrent.

(ii) ($d=3$). Let a differential operator on R^3 be defined by

$$L = \sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

where

$$\begin{cases} a_{11}(x) = \frac{1}{2} \frac{\log(2+|x|^2)}{(2+|x|^2)^{1/2}} \\ a_{12}(x) = a_{21}(x) = 0 \\ a_{22}(x) = \frac{1}{2} \frac{x_1^2 \log(2+|x|^2)}{(2+|x|^2)^{5/2}} \\ a_{23}(x) = a_{32}(x) = \frac{1}{2} \frac{x_1^3 \log(2+|x|^2)}{(2+|x|^2)^{5/2}} \\ a_{33}(x) = \frac{1}{2} \frac{x_1^4 \log(2+|x|^2)}{(2+|x|^2)^{5/2}}. \end{cases}$$

Define

$$X_1 = \sqrt{\frac{\log(2 + |x|^2)}{(2 + |x|^2)^{1/2}}} \cdot \frac{\partial}{\partial x_1}$$

$$X_2 = \sqrt{\frac{\log(2 + |x|^2)}{(2 + |x|^2)^{5/2}}} \cdot \left\{ x_1 \frac{\partial}{\partial x_2} + x_1^2 \frac{\partial}{\partial x_3} \right\},$$

and choose a first order partial differential operator Y so that

$$L = \frac{1}{2} \{X_1^2 + X_2^2\} + Y.$$

It is now a simple task to verify that

$$\text{rank } \mathcal{L}(X_1, X_2)_x = 3, \quad x \in R^3.$$

Note that the rank of the coefficients' matrix $(a_{ij}(x))_{i,j=1,2,3}$ is necessarily less than 3, in particular if $x_1=0$, then it is equal to 1.

It is also easy to see that the condition (3.1) is satisfied. Thus the associated diffusion is recurrent.

Examples 2 (Transient case). If there exists a cone V as in Figure 1 such that with positive constants $\alpha > 1$, β and λ_2 ,

$$E_2(x) \text{ (or } \tilde{E}_2(x)) \geq \beta \cdot \frac{(\log |x|)^\alpha}{|x|^{d-2}}$$

for all $x \in V$, $|x| > \lambda_2$, then the associated diffusion is transient. For the special case that

$$L = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(a_i(x) \frac{\partial}{\partial x_i} \right)$$

we have that if

$$(3.2) \quad \min_{i=1, \dots, d} a_i(x) \geq \beta \cdot \frac{(\log |x|)^\alpha}{|x|^{d-2}}$$

for the above x , the diffusion process is transient.

(i) ($d=3$). Let L be an elliptic operator defined by

$$L = \frac{\partial^2}{\partial x_1^2} + x_1^2 \frac{\partial^2}{\partial x_2^2} + x_2^2 \frac{\partial^2}{\partial x_3^2}.$$

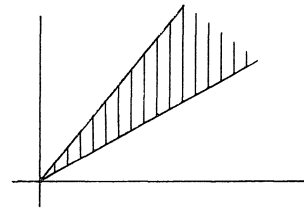


Figure 1

Then the operator L becomes a hypoelliptic one satisfying the fundamental assumption (*) in Chapter 2. It is also easy to verify the assumption in Theorem D. Consequently the process associated with the above operator L is transient.

In the next stage, we want to compare our results with those of Friedman [7] and Hasminskii [8].

We first explain briefly their main results.

Let L be a strictly elliptic differential operator defined by

$$L = \sum_{i,j=1}^d a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

where $a_{ij} \in C^2(R^d)$, symmetric and $b_i \in C^1(R^d)$. Denote the associated diffusion by (X_t, P_x) .

Define

$$S(x) = \frac{1}{|x|} \left\{ \frac{|x|^2 \sum_{i=1}^d \{a_{ii}(x) + x_i b_i(x)\}}{\sum_{i,j=1}^d a_{ij}(x) x_i x_j} - 1 \right\}$$

for $x \neq 0$ and

$$S_+(r) = \max_{|x|=r} S(x), \quad S_-(r) = \min_{|x|=r} S(x)$$

for $r > 0$.

Constructing some semimartingales, they have shown

Theorem 3.1. *If $\int_1^\infty \exp[-\int_1^r S_+(u) du] dr = +\infty$, then the process is recurrent.*

Theorem 3.2. *If $\int_1^\infty \exp[-\int_1^r S_-(u) du] dr < +\infty$, then the process is transient.*

See Friedman [7], Chapter 9, for the proofs of the above results.

In order to compare their theorems with ours, we restrict our consideration to the selfadjoint case. Therefore, from now on, it is assumed that

$$(3.3) \quad b_i(x) = \sum_{j=1}^d \frac{\partial a_{ji}(x)}{\partial x_j}, \quad i=1, \dots, d.$$

The following theorem shows that Theorem A is a generalization of Theorem 3.1 in the selfadjoint case.

Theorem 3.3. *If $\int_1^\infty \exp[-\int_1^r S_+(u) du] dr = +\infty$, then it also holds that*

$$\int_1^\infty r^{1-d} \bar{E}_1^{-1}(r) dr = +\infty.$$

The proof of the above theorem uses the one dimensional diffusion operator defined by

$$\bar{L} = \frac{d}{dr} \left(r^{d-1} \bar{E}_1(r) \frac{d}{dr} \right), \quad r > 0.$$

Lemma 3.1. *Let $v(x) = u(|x|)$ be a smooth function in $|x| > 1$. Then*

$$(3.4) \quad (\bar{L}v)(r) = \left[\int_{|x|=r} (L_x v)(x) ds(x) \right] / |S^{d-1}|,$$

for $r > 1$ where $ds(x)$ is the surface element of $|x| = r$.

Proof. By a simple computation, we obtain

$$\begin{aligned} \int_{|x|=r} (Lv)(x) ds(x) &= \int_{|x|=r} E_1(x) ds(x) \cdot \frac{d^2 u(r)}{dr^2} \\ &+ \int_{|x|=r} \nabla \cdot (A \nabla |x|) ds(x) \cdot \frac{du(r)}{dr}. \end{aligned}$$

The first term is equal to

$$|S^{d-1}| \cdot r^{d-1} \bar{E}_1(r) \frac{d^2 u(r)}{dr^2}.$$

Therefore it suffices to show that]

$$\int_{|x|=r} \nabla \cdot (A \nabla |x|) ds(x) = |S^{d-1}| \cdot \frac{d}{dr} (r^{d-1} \bar{E}_1(r)).$$

Now consider, for $r > 1$

$$\begin{aligned} & |S^{d-1}|r^{d-1}\bar{E}_1(r) - |S^{d-1}|\bar{E}_1(1) \\ &= \int_{|x|=r} (A\nabla|x|, n_+(x))ds(x) + \int_{|x|=1} (A\nabla|x|, n_-(x))ds(x) \end{aligned}$$

where $n_+(x) = -n_-(x) = \nabla|x|$,

$$= \int_{1 < |x| < r} \nabla(A\nabla|x|)dx, \text{ by the divergence theorem.}$$

So by differentiating them with respect to r , one obtains

$$|S^{d-1}|\frac{d}{dr}(r^{d-1}\bar{E}_1(r)) = \int_{|x|=r} \nabla(A\nabla|x|)ds(x). \quad \text{Q. E. D.}$$

Proof of Theorem 3.3. First define the following functions

$$\begin{aligned} \phi_\rho(r) &= \frac{\int_r^\rho u^{1-d}\bar{E}_1^{-1}(u)du}{\int_1^\rho u^{1-d}\bar{E}_1^{-1}(u)du} \\ \Psi_\rho(r) &= \frac{\int_r^\rho \exp[-\int_1^u S_+(t)dt]dt}{\int_1^\rho \exp[-\int_1^u S_+(t)dt]dt} \end{aligned}$$

for $r \in [1, \rho]$.

Then it is clear that

$$(3.5) \quad \begin{cases} (\bar{L}\phi_\rho)(r) = 0 & \text{on } (1, \rho) \\ \phi_\rho(1) = 1, \quad \phi_\rho(\rho) = 0 \\ \Psi_\rho(1) = 1, \quad \Psi_\rho(\rho) = 0. \end{cases}$$

We want to show that

$$(3.6) \quad (\bar{L}\Psi_\rho)(r) \geq 0 \text{ on } (1, \rho).$$

We now compute $Lu_\rho(x)$, where $u_\rho(x) = \Psi_\rho(|x|)$,

$$\begin{aligned} (Lu_\rho)(x) &= E_1(x) \left\{ \frac{d^2\Psi_\rho(|x|)}{d|x|^2} + S(x) \frac{d\Psi_\rho(|x|)}{d|x|} \right\} \\ &\geq E_1(x) \left\{ \frac{d^2\Psi_\rho(|x|)}{d|x|^2} + S_+(|x|) \frac{d\Psi_\rho(|x|)}{d|x|} \right\} \end{aligned}$$

since $\Psi_\rho(|x|)$ is decreasing with respect to $|x|$. From the

definition of Ψ_ρ , we see that

$$\frac{d^2\Psi_\rho(|x|)}{d|x|^2} + S_+(|x|)\frac{d\Psi_\rho(|x|)}{d|x|} = 0.$$

Thus we have shown $(Lu_\rho)(x) \geq 0$ in $1 < |x| < \rho$. Combining this inequality with Lemma 3.1, (3.6) follows immediately. Using (3.5), (3.6) and the maximum principle, we can conclude that

$$(3.7) \quad \Psi_\rho(r) \leq \phi_\rho(r) \quad \text{for } r \in (1, \rho).$$

From the assumption of Theorem 3.3, we have

$$\lim_{\rho \rightarrow +\infty} \Psi_\rho(r) = 1 \quad \text{for every } r > 1$$

and so $\lim_{\rho \rightarrow +\infty} \phi_\rho(r) = 1$ for $r > 1$.

This implies that $\int_1^\infty r^{1-d} \bar{E}_1^{-1}(r) dr = +\infty$. Q. E D.

Finally we shall exhibit a few examples which Friedman and Hasminskii's criterions do not cover.

Examples. (i) ($d=2$). Suppose L is given by

$$L = (3 + \sin x_2) \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

This satisfies the inequality (3.1). Consequently the associated process is recurrent. Now computing $S(x)$ for the operator, one obtains

$$S(x) = \frac{1}{|x|} \frac{x_2^2(3 + \sin x_2) + x_1^2}{x_1^2(3 + \sin x_2) + X_2^2}.$$

Taking the maximum of the function $S(x)$ on $|x|=r$, we have a lower bound,

$$S_+(r) = \max_{|x|=r} S(x) \geq \frac{2}{r}, \quad r > 1.$$

It is now easy to see that

$$\int_1^\infty \exp[-\int_1^r S_+(u) du] dr < +\infty.$$

This shows that Friedman and Hasminskii's test for recurrence does not cover the above case and that the converse of Theorem 3.3 is not valid.

(ii) ($d=2$) Let L be defined by

$$L = a_1(x_2) \frac{\partial^2}{\partial x_1^2} + a_2(x_1) \frac{\partial^2}{\partial x_2^2}$$

where a_1, a_2 are strictly positive and

$$\left. \begin{aligned} a_1(x_2) &\geq \gamma \times (\log |x_2|)^\alpha \\ a_2(x_1) &\geq \delta \times (\log |x_1|)^\beta \end{aligned} \right\} \text{ as } |x_1|, |x_2| \nearrow +\infty$$

with positive constants $\alpha, \beta > 1$ and γ, δ .

It is easy to verify that the inequality (3.2) holds in the cone domain $\{x = (x_1, x_2) ; \frac{1}{2}x_1 < x_2 < \frac{3}{2}x_1\}$. So the associated diffusion is transient. On the other hand, we have

$$S(x) = \frac{1}{|x|} \frac{a_1(x_2)x_2^2 + a_2(x_1)x_1^2}{a_1(x_2)x_1^2 + a_2(x_1)x_2^2}.$$

If $x_1 = x_2$, then $S(x) = \frac{1}{|x|}$. This implies that

$$S_-(r) \leq \frac{1}{r}$$

which shows that the integral

$$\int_1^\infty \exp[-\int_1^r S_-(u) du] dr$$

is divergent. Thus Theorem 3.2 is of no help in proving transience.

Chapter 4. Rate of Escape

Let $L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$ be a uniformly elliptic operator i. e. assume that there exists a positive constant $\lambda > 1$ such that for all $(x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d$

$$(4.1) \quad \lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2.$$

We further assume the smoothness of the coefficients $a_{ij} \in C^\infty(\mathbf{R}^d)$ which ensures the existence of the unique fundamental solution $p_t(x, y)$ of $\partial/\partial t - L$. See McKean [17].

Let (X_t, P_x) , $x \in R^d$ be the associated diffusion in R^d . The following theorem is an extension of the Dvoretzky-Erdős' test for the Brownian motion in space. (See Dvoretzky-Erdős [6].)

Theorem E ($d \geq 3$). *If (4.1) holds and $0 < g(t) \searrow$ as $t \nearrow +\infty$, then*

$$(4.2) \quad P_x[|x_t| \leq \sqrt{t} g(t), \text{ i. o., } t \nearrow +\infty] = 1 \text{ or } 0$$

according as $\int^{+\infty} g(t)^{d-2} dt/t = +\infty$ or $< +\infty$.

Our proof of this theorem is based on the following result which is essentially due to Aronson [2].

Lemma 4.1. *There exist positive constants $\lambda_i, M_i, i=1, 2$ such that*

$$(4.3) \quad \frac{M_1}{t^{d/2}} \exp\left[-\frac{\lambda_1 |x-y|^2}{t}\right] \leq p_t(x, y) \leq \frac{M_2}{t^{d/2}} \exp\left[-\frac{\lambda_2 |x-y|^2}{t}\right]$$

for every $(t, x, y) \in (0, +\infty) \times R^d \times R^d$.

Proof. The upper bound has been given by Aronson [1]. To obtain the global lower bound, we need the following Harnack inequality of Moser [18].

Harnack inequality: *If u is a positive solution of $\partial/\partial t - L$, i. e., $(\partial/\partial t - L)u = 0$ in $(0, T) \times R^d$, then one has*

$$(4.4) \quad \log \frac{u(s, y)}{u(t, x)} \leq C \left\{ \frac{|x-y|^2}{t-s} + \log t/s + 1 \right\}$$

for $0 < s < t < T$, $x, y \in R^d$ where C is a positive constant which depends only on d and λ .

The lower bound: Step i). There exist constants $C_i > 0, i=1, 2$ such that

$$P_x[|x_t - x| \leq C_1 \sqrt{t}] \geq C_2 \text{ for } (t, x) \in R^+ \times R^d.$$

This follows directly from the inequality,

$$E_x[|x_t - x|^2] \leq C_3 t$$

which comes from the upper bound.

Step ii). Set $u(t, y) \equiv p_t(x, y)$ for a fixed $x \in R^d$. Applying Harnack's inequality for $\frac{\partial}{\partial t} - L$ in the domain $\{|y - x|^2 \leq C_1^2 t\}$, we get

$$u(t/2, y) \leq u(t, x) \cdot \exp\{2C \cdot C_1^2 + C \log 2 + C\}.$$

By step i), it is possible to choose a point $y_0(x) \in R^d$ such that

$$|y_0(x) - x| \leq C_1 \sqrt{t}$$

and

$$u(t/2, y_0(x)) \geq M/t^{d/2}$$

where M depends only on C_1, C_2 and d . Therefore one obtains

$$M/t^{d/2} \leq u(t/2, y_0(x)) \leq u(t, x) \exp\{2C \cdot C_1^2 + C \log 2 + C\}.$$

Thus one has

$$(4.5) \quad p_t(x, x) \equiv u(t, x) \geq \tilde{M}/t^{d/2} \text{ for } (t, x) \in (0, \infty) \times R^d$$

with some positive constant \tilde{M} independent of t, x .

Step iii). Applying Harnack's inequality once more, we obtain

$$\frac{u(t/2, x)}{u(t, y)} \leq \exp\left\{\frac{2C|x-y|^2}{t} + C \log 2 + C\right\}$$

(4.5) with t replaced by $t/2$ now yields,

$$(4.6) \quad p_t(x, y) \equiv u(t, y) \geq \frac{M_1}{t^{d/2}} \exp\left[-\frac{\lambda_1|x-y|^2}{t}\right]$$

for every $(t, x, y) \in (0, \infty) \times R^d \times R^d$.

Q. E. D.

From Lemma 4.1, it is easy to see that the 0-order Green function defined by

$$(4.7) \quad G(x, y) = \int_0^\infty p_t(x, y) dt$$

satisfies the inequalities,

$$(4.8) \quad \frac{C_4}{|x-y|^{d-2}} \leq G(x, y) \leq \frac{C_5}{|x-y|^{d-2}}$$

for $x, y \in R^d$, ($d \geq 3$) and some positive constants C_4, C_5 .

Define

$$\begin{aligned} \tau_r^y &= \inf \{t \geq 0 : |x_t - y| \leq r\} \\ \sigma_\rho^y &= \inf \{t \geq 0 : |x_t - y| \geq \rho\} \end{aligned}$$

for $y \in R^d$, $0 < r < \rho < +\infty$. Then

$$u(x_{t \wedge \tau_r^y \wedge \sigma_\rho^y}) \equiv G(x_{t \wedge \tau_r^y \wedge \sigma_\rho^y}, y)$$

is a P_x -martingale for $r < |x - y| < \rho$. Using this, one has the following

Lemma 4.2. For some positive constants C_6, C_7 ,

$$C_6 \left(\frac{r}{|x - y|} \right)^{d-2} \leq P_x[\tau_r^y < +\infty] \leq C_7 \left(\frac{r}{|x - y|} \right)^{d-2} \text{ for } |x - y| > r.$$

Proof. Making use of the martingale defined above and the bounds for $G(x, y)$,

$$\frac{C_4}{|x - y|^{d-2}} \leq u(x) = E_x[u(x_{t \wedge \tau_r^y \wedge \sigma_\rho^y})].$$

As $t \nearrow +\infty$, this tends to

$$\begin{aligned} E_x[u(x_{t \wedge \tau_r^y \wedge \sigma_\rho^y})] &= E_x[u(x_{\tau_r^y}), \tau_r^y < \sigma_\rho^y] + E_x[u(x_{\sigma_\rho^y}), \tau_r^y > \sigma_\rho^y] \\ &\leq \frac{C_5}{r^{d-2}} \cdot P_x[\tau_r^y < \sigma_\rho^y] + \frac{C_5}{\rho^{d-2}} P_x[\tau_r^y > \sigma_\rho^y]. \end{aligned}$$

If we now let $\rho \nearrow +\infty$, we obtain

$$\frac{C_4}{|x - y|^{d-2}} \leq \frac{C_5}{r^{d-2}} P_x[\tau_r^y < +\infty], \text{ for } |x - y| > r.$$

Hence $P_x[\tau_r^y > +\infty] \geq \frac{C_4}{C_5} \left(\frac{r}{|x - y|} \right)^{d-2}$. Similarly we can obtain the upper bound. Q. E. D.

Lemma 4.3. For $T \geq 0, r \geq 0$, define $Q_r(r, T)$ by

$$(4.9) \quad Q_r(r, T) = P_r[|x_t - y| \leq r, \text{ for some } t \geq T],$$

then

$$C_8 \left(\frac{r}{\sqrt{T}}\right)^{d-2} \exp\left[-\frac{\lambda_1 r^2}{T}\right] \leq Q_y(r, T) \leq C_9 \left(\frac{r}{\sqrt{T}}\right)^{d-2}$$

where C_8, C_9 are positive constants.

Proof. Using the Markov property and Lemma 4.2, we obtain

$$\begin{aligned} \int_{|x-y| \geq r} C_6 \left(\frac{r}{|x-y|}\right)^{d-2} p_T(y, x) dx &\leq Q_y(r, T) \\ &\leq \int_{\mathbb{R}^d} C_7 \left(\frac{r}{|x-y|}\right)^{d-2} p_T(y, x) dx \end{aligned}$$

and so

$$\begin{aligned} Q_y(r, T) &\leq \int_{\mathbb{R}^d} C_7 M_2 \left(\frac{r}{\sqrt{T}}\right)^{d-2} \frac{1}{|x-y|^{d-2}} \exp[-\lambda_2 |x-y|^2] dx \\ &= C_9 \left(\frac{r}{\sqrt{T}}\right)^{d-2}. \end{aligned}$$

Similarly we can obtain the lower bound,

$$Q_y(r, T) \geq C_8 \left(\frac{r}{\sqrt{T}}\right)^{d-2} \exp\left[-\frac{\lambda_1 r^2}{T}\right]. \tag{Q. E. D.}$$

Lemma 4.4. Let $K > 1$ be such that

$$C_{10} \equiv C_8 \exp(-\lambda_1) - C_9 \left(\frac{1}{\sqrt{K}}\right)^{d-2} > 0.$$

Define

$$(4.10) \quad P_y(r, T, K) \equiv P_y\{|x, -y| \leq r, \text{ for some } t \in [T, KT]\}.$$

Then for $\sqrt{T} \geq r > 0$

$$(4.11) \quad P_y(r, T, K) \geq C_{10} \left(\frac{r}{\sqrt{T}}\right)^{d-2}.$$

Proof. From the definition of $P_y(r, T, K)$ and $Q_y(r, T)$,

$$P_y(r, T, K) \geq Q_y(r, T) - Q_y(r, KT).$$

Applying Lemma 4.3, we obtain

$$P_y(r, T, K) \geq C_8 \left(\frac{r}{\sqrt{T}}\right)^{d-2} \exp\left(-\frac{\lambda_1 r^2}{T}\right) - C_9 \left(\frac{r}{\sqrt{KT}}\right)^{d-2}$$

$$\geq \left\{ C_8 \exp(-\lambda_1) - C_9 \left(\frac{1}{\sqrt{K}} \right)^{d-2} \right\} \left(\frac{r}{\sqrt{T}} \right)^{d-2}. \quad \text{Q. E. D.}$$

We are now in a position to prove our theorem.

Proof of Theorem E. We first note that it is enough to consider the case,

$$1 \geq g(t) \searrow 0 \quad \text{as } t \nearrow +\infty.$$

Now define

$$E_k = \{ \omega : |x_t(\omega)| \leq \sqrt{t} g(t) \text{ for some } t \in [K^k, K^{k+1}] \}$$

$$\bar{E}_k = \{ \omega : |x_t(\omega)| \leq \sqrt{K^{k+1}} g(K^k) \text{ for some } t \in [K^k, K^{k+1}] \}$$

for every positive integer k . Since g is decreasing, $E_k \subset \bar{E}_k$ for $k \geq 1$. By Lemma 4.3,

$$P_0(\bar{E}_k) \leq C_9 \left(\frac{g(K^k) \sqrt{K^{k+1}}}{\sqrt{K^k}} \right)^{d-2} = C_9 \cdot K^{d-2/2} \cdot \{g(K^k)\}^{d-2}.$$

We now assume the convergence of the integral in our theorem. Then

$$(4.12) \quad \sum_{k=1}^{\infty} P_0(\bar{E}_k) \leq O(1) \sum_{k=1}^{\infty} \{g(K^k)\}^{d-2} = O(1) \int^{+\infty} g(t)^{d-2} \frac{dt}{t} < +\infty$$

and so by the Borel-Cantelli Lemma, the left hand side of (4.2) equals zero.

Next we consider the divergent case. Using a result of Kochen and Stone [13], we shall first show that

$$P_0[|x_t| \leq \sqrt{t} g(t), \text{ i. o.}, t \nearrow +\infty] = 1$$

under the additional assumption

$$(4.13) \quad \frac{1}{(\log t)^2} \leq g(t)^{d-2} \searrow 0 \quad \text{as } t \nearrow +\infty.$$

In order to do this, we introduce the following definition

$$E_{k,y}^\lambda = \{ \omega : |x_t(\omega) - y| \leq \lambda \sqrt{K^k} g(K^{k+1}) \text{ for some } t \in [K^k, K^{k+1}] \}$$

for every $y \in R^d$, positive integer k and positive $\lambda \leq 1$. From Lemma

4. 4, it follows that

$$(4. 14) \quad P_y(E_{k,y}^\lambda) \geq C_{10} \cdot \lambda^{d-2} \{g(K^{k+1})\}^{d-2}$$

and so

$$(4. 15) \quad \sum_{k=1}^\infty P_y(E_{k,y}^\lambda) \geq C_{11} \lambda^{d-2} \int_0^{+\infty} g(t)^{d-2} \frac{dt}{t} = +\infty.$$

Define a stopping time by

$$\sigma_{j,y}^i(\omega) = \begin{cases} \inf \{t \geq 0; |x_t(\omega) - y| \leq \lambda \sqrt{K^i} g(K^{i+1}), K^j \leq t \leq K^{j+1}\} \\ +\infty, \text{ if no such } t \text{ exists.} \end{cases}$$

Using the strong Markov property, for a pair $i > j$,

$$\begin{aligned} P_y \{E_{i,y}^\lambda \cap E_{j,y}^\lambda\} &= \int_{K^j}^{K^{j+1}} P_y \{ |x_t - y| \leq \lambda \sqrt{K^i} g(K^{i+1}) \text{ for some} \\ &\quad t \in [K^i, K^{i+1}] \mid \sigma_{j,y}^i = s\} P_y \{ \sigma_{j,y}^i \in ds\} \\ &\leq \max_{\{|z-y| \leq \lambda \sqrt{K^j} g(K^{j+1})\}} P_z \{ |x_t - z| \leq \lambda g(K^{i+1}) \sqrt{K^i} + \lambda g(K^{j+1}) \sqrt{K^j}, \\ &\quad \text{for some } t \geq K_i - K^{j+1} \} P_y \{ \sigma_{j,y}^i \leq K^{j+1} \}. \end{aligned}$$

By Lemma 4. 3, we have that

$$\begin{aligned} P_z \{ |x_t - z| \leq \lambda \sqrt{K^i} g(K^{i+1}) + \lambda \sqrt{K^j} g(K^{j+1}) \text{ for some } t \geq K^i - K^{j+1} \} \\ \leq C_9 \left\{ \frac{\lambda g(K^{i+1}) \sqrt{K^i} + \lambda g(K^{j+1}) \sqrt{K^j}}{\sqrt{K^i - K^{j+1}}} \right\}^{d-2} \\ = C_9 \cdot \lambda^{d-2} \left\{ \frac{g(K^{i+1}) \sqrt{K^i} + g(K^{j+1}) \sqrt{K^j}}{\sqrt{K^i - K^{j+1}}} \right\}^{d-2}. \end{aligned}$$

On the other hand, (4. 13) implies that

$$\frac{g(K^{i+1}) \sqrt{K^i}}{\sqrt{K^i - K^{j+1}}} \leq D_1 g(K^{i-j\pm 1})$$

and

$$\frac{g(K^{j+1}) \sqrt{K^j}}{\sqrt{K^i - K^{j+1}}} \leq \frac{D_2}{(\log K^{i-j+1})^{2/d-2}} \leq D_2 g(K^{i-j+1})$$

for $i \geq j+2$, where D_1 and D_2 are positive constants which are independent of i, j, λ and g above. This combined with the inequality

(4.14) shows that for $i \geq j+2$

$$P_y \{E_{i,y}^\lambda \cap E_{j,y}^\lambda\} \leq C_{12} P_y \{E_{i-j,y}^\lambda\} \cdot P_y \{E_{j,y}^\lambda\}$$

with a positive constant C_{12} independent of λ, y and g . For $i=j+1$, it is obvious that

$$P_y \{E_{i,y}^\lambda \cap E_{j,y}^\lambda\} \leq P_y (E_{j,y}^\lambda).$$

So from the inequality (4.14) with $k=1$, we have that if $1 \geq \lambda > 1/2$,

$$(4.16) \quad P_y \{E_{i,y}^\lambda \cap E_{j,y}^\lambda\} \leq C_{13} P_y \{E_{i-j,y}^\lambda\} \cdot P_y \{E_{j,y}^\lambda\}$$

for $i > j$ where C_{13} is a positive constant which does not depend upon y and g .

It is now easily seen that

$$(4.17) \quad \lim_{n \nearrow +\infty} \frac{\{\sum_{i=1}^n P_y(E_{i,y}^\lambda)\}^2}{\sum_{j,k=1}^n P_y \{E_{j,y}^\lambda \cap E_{k,y}^\lambda\}} \geq \frac{1}{2C_{13}} > 0$$

for every λ, y and g above. Combining this with a result of Koehen and Stone [13], it follows that

$$(4.18) \quad P_y \{ \overline{\lim}_{n \nearrow +\infty} E_{n,y}^\lambda \} \geq \frac{1}{2C_{13}}.$$

This implies that

$$(4.19) \quad P_y \{ |x_t - y| \leq \lambda \sqrt{t} g(t), i. o., t \nearrow +\infty \} \geq \frac{1}{2C_{13}}$$

for every $\lambda \in (\frac{1}{2}, 1]$, $y \in R^d$ and g satisfying (4.13). We want to show that

$$(4.20) \quad P_y \{ |x_t| \leq \sqrt{t+u} g(t+u), i. o., t \nearrow +\infty \} \geq \frac{1}{2C_{13}}$$

for any fixed positive u . First note that (4.19) gives

$$(4.21) \quad P_y \{ |x_t| \leq \lambda \sqrt{t} g(t), i. o., t \nearrow +\infty \} \geq \frac{1}{2C_{13}}$$

because $\lim_{t \nearrow +\infty} \frac{|y|}{\sqrt{t} g(t)} = 0$. In order to deduce (4.20), we observe that

$$\sqrt{t+u} g(t+u) \geq \sqrt{t} g(t+u)$$

and

$$(2g(t+u))^{d-2} \geq \frac{2^{d-2}}{(\log(t+u))^2} > \frac{1}{(\log t)^2}$$

for all sufficiently large t , where u is a fixed positive number. Hence combining this with (4.21), we obtain the inequality (4.20).

Using (4.20), we shall prove

$$P_x[|x_t| \leq \sqrt{t} g(t), \text{ i. o., } t \nearrow +\infty] = 1, \quad x \in R^d.$$

Define

$$Y_t = P_x\{|x_t(\omega)| \leq \sqrt{t} g(t), \text{ i. o., } t \nearrow +\infty | \mathcal{F}_0^t\}$$

where $\mathcal{F}_0^t = \sigma\{x_s, s \leq t\}$. Then Y_t is an \mathcal{F}_0^t -martingale. Since Y_t is bounded, by the martingale convergence theorem we have

$$\lim_{t \nearrow +\infty} Y_t(\omega) = \begin{cases} 1 & \text{if } |x_t(\omega)| \leq \sqrt{t} g(t), \text{ i. o., } t \nearrow +\infty \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, from the definition of Y_t and the inequality (4.20),

$$Y_t(\omega) = P_{x_t}\{|x_s(\omega)| \leq \sqrt{t+s} g(t+s), \text{ i. o., } s \nearrow +\infty\} \geq \frac{1}{2C_{13}} > 0.$$

This implies that $\lim_{t \nearrow +\infty} Y_t(\omega) = 1$ a. e., i. e.,

$$P_x[|x_t| \leq \sqrt{t} g(t), \text{ i. o., as } t \nearrow +\infty] = 1, \quad x \in R^d.$$

For a general $g(t)$, define

$$g_1(t) = \max\left\{g(t), \sqrt{\frac{1}{(\log t)^2}}\right\}.$$

It is evident that $g_1(t)$ satisfies the condition (4.13).

Therefore, from what we have shown above, it follows that

$$P_x[|x_t| \leq \sqrt{t} g_1(t), \text{ i. o., } t \nearrow +\infty] = 1.$$

On the other hand, from the convergent case,

$$P_x[|x_t| \leq \sqrt{t} \cdot \sqrt{\frac{1}{(\log t)^2}}, \text{ i. o., } t \nearrow +\infty] = 1$$

because of $\int^{+\infty} \frac{dt}{t(\log t)^2} < +\infty$. Hence we finally obtain that

$$P_x[|x_t| \leq \sqrt{t} g(t), i. o., \nearrow + \infty] = 1.$$

Q. E. D.

Bibliography

- [1] Agmon, S., The L^2 -approach to the Dirichlet problem, *Ann. Scuola. Norm. Sup. Pisa*, **13** (1959), 405-448.
- [2] Aronson, D. G., Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.*, **73** (1967), 890-896.
- [3] Bony, J. M., Principe du maximum, unicité du problème de Cauchy et inégalité de Harnack des opérateurs elliptiques dégénérés, *Ann. Inst. Fourier*, **19** (1969), 277-304.
- [4] Brown, L. D., Admissible estimators, recurrent diffusions and insoluble boundary value problems, *Ann. Math. Statist.*, **42** (1971), 855-903.
- [5] Courrége, P. and Priouret, P., Recollements de processus de Markov continus, *C. R. Acad. Sc. Paris*, **259** (1964), 4220-4222.
- [6] Dvoretzky, A. and Erdős, P., Some problems on random walks in space, *Proc. Second Berkeley Sump. Math. Statist. Prob.* (1951), 353-367.
- [7] Friedman, A., *Stochastic differential equations and applications*, vol. I (1975), Academic Press.
- [8] Hasmininskii, R. Z., Ergodic properties of recurrent diffusion processes and stabilization of the problem to the Cauchy problem for parabolic equations, *T. V.* **5** (1960), 196-214.
- [9] Hörmander, L., Hypoelliptic second order differential equations, *Acta Math.*, **119** (1967), 147-171.
- [10] Hunt, G. A., On positive Green functions, *Proc. Nat. Acad. Sci.*, **40** (1954), 816-818.
- [11] Ichihara, K. and Kunita, H., A classification of the second order degenerate elliptic operators and its probabilistic characterization, *Z. Wahr. Verw. Gebiete*, **30** (1974), 235-254.
- [12] Kanda, M., Regular points and Green functions in Markov processes, *Jour. Math. Sci. Japan*, **19** (1967), 246-269.
- [13] Kochen, S. and Stone, C., A note on the Borel-Cantelli Lemma, *Illinois Jour. Math.* **8** (1974), 248-251.
- [14] Kunita, H., General boundary conditions for multidimensional diffusion processes, *Jour. Math. Kyoto Univ.*, **10**, (1970), 273-335.
- [15] Lions, J. L. and Magenes, E., *Problème aux limites non homogènes et applications*, vol. 1, Dunod Paris 1968.
- [16] Littman, W., Stampacchia, G. and Weinberger, H. F., Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola. Norm. Sup. Pisa*, **17** (1963), 43-76.
- [17] McKean, H. P., *Stochastic integrals*, New York Academic Press (1969).
- [18] Moser, J., A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.*, **17** (1964), 101-134.
- [19] Stampacchia, G., Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier*, **15** (1965), 189-258.
- [20] Stroock, D. W. and Varadhan, S. R. S., On the support of diffusion processes with applications to the strong maximum principle, *Proc. Sixth Berkeley Symp. Math. Statist. Prob.*
- [21] Stroock, D. W. and Varadhan, S. R. S., On degenerate elliptic-parabolic operators of second order and their associated diffusions, *Comm. Pure Appl. Math.*, **25** (1972), 651-713.
- [22] Yosida, K., *Functional Analysis*, Springer-Verlag (1966).