Microlocal Properties of Local Elementary Solutions for Cauchy Problems for a Class of Hyperbolic Linear Differential Operators

By

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The purpose of this paper is to present a way of explicit construction of local elementary solutions for Cauchy problems for a class of hyperbolic linear differential operators with multiple characteristics. We also investigate their singularity structure microlocally. The class we deal with in this article is a subclass dealt with by Kashiwara-Kawai [1] and our results can be derived from the very general and precise results given in §6 of Kashiwara-Kawai [1]. (See Miwa [1] for related topics.) However, the argument of Kashiwara -Kawai [1] is highly transcendental in its nature, and, hence, we cannot expect that their results could be a guidepost for the "Structure Theorem" for general systems of micro-differential (=pseudo-differential) equations in the multiple characteristic case, which is originally obtained in Sato-Kawai-Kashiwara [1]⁽*⁾ in the simple-characteristic case⁽**⁾. To the contrary, our results in this article are, though restricted, tied up with the recent results of Nakamura [1] and Hamada-Nakamura [1] whose arguments are constructive in the sense that the coefficients of the expansion used there are determined successively in an explicit manner. Thus, we hope our argument and results in this article will serve as a starting point for the investigation of "Structure Theorem" for general systems with multiple

characteristics, as Kawai [2] did in the simple characteristic case.

A linear differential operator $P(x, D_x) \equiv P\left(x_1, \ldots, x_n, \dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}\right)$ considered in this article is always supposed to satisfy the following conditions (1) and (2). (See Granoff-Ludwig [1] and Ivriĭ [1] for related topics.)

(1) $P(x, D_x)$ is a restriction to an open set $\omega_R \subset R^n$ of a linear differential operator $P(z, D_z) = P\left(z_1, \ldots, z_n, \dfrac{\partial}{\partial z_1}, \ldots, \dfrac{\partial}{\partial z_n}\right)$ with holomorphic coefficients defined in a neighborhood $\omega \supset \omega_R$ of $z \equiv (z_1, \ldots, z_n) = (0, \ldots, 0) \overset{(*)}{\in} C^n$. The principal part of $P(z, D_z)$ shall be denoted by $p_m(z, \zeta) = p_m(z_2, \ldots, z_n, \zeta_1, \ldots, \zeta_n)$.

In the sequel, z' and ζ' stand for $(z_2, \ldots, z_n)$ and $(\zeta_2, \ldots, \zeta_n)$, respectively.

(2) There exist holomorphic functions $\lambda^+(z, \zeta')$, $\lambda^-(z, \zeta')$ and $\lambda_l(z, \zeta')$ $(l=3, \ldots, m)$ which are defined in a complex neighborhood $\tilde{\omega}$ of $\{z=0\} \times S^{n-1}$ and positively homogeneous of degree 1 with respect to ζ' so that they satisfy following conditions (2.a) $\sim$ (2.c). Here S^{n-1} denotes real $(n-1)$-dimensional sphere.

    (2.a) $p_m(z, \zeta) = (\zeta_1 - \lambda^+(z, \zeta'))(\zeta_1 - \lambda^-(z, \zeta')) \overset{m}{\underset{l=3}{\prod}} (\zeta_1 - \lambda_l(z, \zeta'))$.

    (2.b) $\lambda^+(0, \xi')$ (resp. $\lambda^-(0, \xi')$) and $\lambda_l(0, \xi')$ $(l=3, \ldots, m)$ are mutually distinct for any $\xi' \in S^{n-1}$.

    (2.c) The Poisson bracket $\{\zeta_1 - \lambda^+(z, \zeta'), \zeta_1 - \lambda^-(z, \zeta')\}$ of $\zeta_1 - \lambda^+(z, \zeta')$ and $\zeta_1 - \lambda^-(z, \zeta')$ vanishes identically.

Remark 1. Although one may localize our conditions with respect to ζ' by using a neighborhood $\tilde{\omega}_I$ of $\{z=0\} \times I$ for a compact set I of S^{n-1}. We present our conditions in the form described above for simplicity. The I-hyperbolic operators introduced in Kawai [1] can

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(*) We sometimes denote by 0 the origin of C^n, C^{n-1}, R^n or R^{n-1}, if there is no fear of confusion.

be dealt with by this localized expression of the conditions. We leave the detailed discussion to the reader.

Remark 2. Since we are concerned with the local problems in this article, we do not refer to $\tilde{\omega}$ explicitly in the sequel.

When $P(z, D_z)$ satisfies conditions (1) and (2) above, we define the notion of a bicharacteristic of P emanating from a point (z°, ζ°) $\in \{\zeta_1 = \lambda^+(z, \zeta') = \lambda^-(z, \zeta')\}$ as follows:

Definition 1. Assume that the operator P satisfies the conditions above. If (z°, ζ°) satisfies $\zeta_1^\circ = \lambda^+(z^\circ, \zeta^{\circ\prime}) = \lambda^-(z^\circ, \zeta^{\circ\prime})$, then the bicharacteristic associated with P emanating from (z°, ζ°) is by definition the variety with dimension at most two that is obtained by the following procedure:

First consider a bichracteristic $b_{+, (z^\circ, \zeta^\circ)}$ that is associated with $\zeta_1 - \lambda^+(z, \zeta')$ and that emanates from (z°, ζ°). Secondly consider a bicharacteristic $b_{-, (z^1, \zeta^1)}$ that is associated with $\zeta_1 - \lambda^+(z, \zeta')$ and that emanates from $(z^1, \zeta^1) \in b_{-, (z^\circ, \zeta^\circ)}$. Then the set of all points in $b_{-, (z^1, \zeta^1)}$ for some $(z^1, \zeta^1) \in b_{+, (z^\circ, \zeta^\circ)}$ is called a bicharacteristic of P emanating from (z°, ζ°).

In case $\lambda^+(z, \zeta')$ and $\lambda^-(z, \zeta')$ are real for real (z, ζ'), a real bicharacteristic associated with P in the sense of Definition 1 is called time-like if we can find (z^1, ζ^1) so that $(z_1(t_1) - z_1^\circ)(z_1(t_2) - z_1^1) \geqq 0$ holds, where t_1 and t_2 are real paramaters which describe $b_{+, (z^\circ, \zeta^\circ)}$ and $b_{-, (z^1, \zeta^1)}$, respectively.

Remark 1. Of course, one can define the notion of bicharacteristic associated with P by changing the role of λ^+ and λ^-. However, condition (2. c) guarantees that these two procedures yield the same result.

Remark 2. If (z°, ζ°) satisfies $\zeta_1^\circ - \lambda_l(z^\circ, \zeta^{\circ\prime}) = 0$ for some $l(3 \leqq l \leqq m)$, then the definition of the bicharacteristic (=bicharacteristic strip) associated with $\zeta_1 - \lambda_l(z, \zeta')$ emanating from (z°, ζ°) is the same

as usual, i.e. the integral curve of the ordinary differential equations

$$(3)\quad\begin{cases}\dfrac{dz_j}{ds}=\dfrac{\partial p_m}{\partial \zeta_j} & (j=1,\ldots,n)\\[2mm]\dfrac{d\zeta_j}{ds}=-\dfrac{\partial p_m}{\partial z_j} & (j=1,\ldots,n)\end{cases}$$

with initial conditions

$$(4)\quad\begin{cases}z_j(0)=z_j^o & (j=1,\ldots,n)\\ \zeta_j(0)=\zeta_j^o & (j=1,\ldots,n).\end{cases}$$

We may also consider bicharacteristics associated with $\zeta_1-\lambda^+(z,\zeta')$ or $\zeta_1-\lambda^-(z,\zeta')$. They are subsets of the bicharacteristics associated with P in the sense of Definition 1 if

$$\zeta_1^o=\lambda^+(z^o,\zeta^{o\prime})=\lambda^-(z^o,\zeta^{o\prime})$$

holds. In the sequel, we call any of them also bicharacteristics associated with P.

An important result of Hamada-Nakamura [1] claims the following:

Theorem 0. (Hamada-Nakamura) *Consider the following Cauchy problem* $(5)_j (j=0,\ldots,m-1)$:

$$(5)_j\quad\begin{cases}P(z,D_z)E(z,w',\zeta')=0\\ \left(\dfrac{\partial^k}{\partial z_1^k}\right)E_j(z,w',\zeta')\big|_{z_1=0}=\dfrac{1}{(-2\pi\sqrt{-1})^{n-1}}\dfrac{\delta_{jk}}{\langle z'-w',\zeta'\rangle^{n-1}}\\ \hspace{7cm}(k=0,\ldots,m-1)\end{cases}$$

Assume that $P(z,D_z)$ *satisfies conditions* (1) *and* (2). *Then* $(5)_j$ $(j=0,\ldots,m-1,\ resp.)$ *admits a unique solution* $E_j(z,w',\zeta')$ $(j=0,\ldots,m-1,\ resp.)$ *of the following form* (6) *upon the condition that* (z,w',ζ') *is sufficiently close to* $(z,w',\zeta')=(0,0,\xi^{o\prime})$ *where* $\xi^{o\prime}$ *belongs to* S^{n-1}.

$$(6)\quad\sum_{l=3}^{m+1}\left(\frac{\widetilde{F}_l(z,w',\zeta')}{\varphi_l(z,w',\zeta')^{p_l}}+G_l(z,w',\zeta')\log\Phi_l(z,w',\zeta')\right)$$
$$+\int_0^{z_1}\left(\sum_{p=0}^{\infty}\frac{F_p(z,w',\zeta',\tau)}{\Phi(z,w',\zeta',\tau)^p}+G(z,w',\zeta',\tau)\log\Phi(z,w',\zeta',\tau)\right)d\tau$$
$$+H(z,w',\zeta'),$$

where $\varphi_l(z, \ w', \ \zeta') \ (l = 3, \ldots, \ m)$, $\varphi_{m+1}(z, \ w', \ \zeta') \equiv \varphi^+(z, \ w', \ \zeta')$ *and* $\Phi(z, \ w', \ \zeta', \ \tau)$ *are the phase functions of* $P(z, \ D_z)$ *defined in* §2 *of Hamada-Nakamura* [1], *(see also the remark below),* p_l *is a non-negative integer,* $F_l, \ G_l, \ F_p, \ G$ *and* H *are all holomorphic in* $(z, \ w', \zeta')$ *and* $\sum_{p=0}^{\infty} \dfrac{F_p(z, \ w', \ \zeta')}{\Phi(z, \ w', \ \zeta')^p}$ *is holomorphic in* $\{(z, \quad w', \quad \zeta') \ ; \ |z|, \quad |w'|,$ $|\zeta' - \xi^{0'}| \ll 1 \ and \ \Phi(z, \ w', \ \zeta') \neq 0\}$. *Furthermore,* $\tilde{F}_l, \ G_l, \ G$ *and* $\sum_{p=0}^{\infty} \dfrac{F_p}{\Phi^p}$ *are all homogeneous with respect to* ζ'.

Remark. In our case phase functions φ_l and Φ contain parameters w' and ζ', and their definitions should be modified as follows:
First we define $\varphi^+(z, \ w', \ \zeta')$ by

$$(7) \quad \begin{cases} \dfrac{\partial \varphi^{\pm}}{\partial z_1} = \lambda^{\pm}(z, \ \mathrm{grad}_{z'}\varphi^{\pm}), & (7.\text{a}) \\[2mm] \varphi^{\pm}|_{z_1=0} = \langle z' - w', \ \zeta' \rangle & (7.\text{b}) \end{cases}$$

Making use of $\varphi^{\pm}$ thus defined, we choose $\Phi(z, \ \omega', \ \zeta', \ \tau)$ so that it satisfies

$$(8) \quad \begin{cases} \dfrac{\partial \Phi}{\partial z_1} = \lambda^-(z, \ \mathrm{grad}_{z'}\Phi), & (8.\text{a}) \\[2mm] \Phi|_{z_1=\tau} = \varphi^+(\tau, \ z', \ w', \ \zeta') & (8.\text{b}) \end{cases}$$

As shown in Hamada-Nakamura [1], condition (2.c) entails

$$(9) \quad \begin{cases} \dfrac{\partial \Phi}{\partial z_1} + \dfrac{\partial \Phi}{\partial \tau} = \lambda^+(z, \ \mathrm{grad}_{z'}\Phi), & (9.\text{a}) \\[2mm] \Phi|_{\tau=0} = \varphi^-(z, \ w', \ \zeta') & (9.\text{b}) \end{cases}$$

The modification of the definition of φ_l is the same, i.e., φ_l ($l = 3, \ldots, \ m$) is defined by

$$(10) \quad \begin{cases} \dfrac{\partial \varphi_l}{\partial z_1} = \lambda_l(z, \ \mathrm{grad}_{z'}\varphi_l), \\[2mm] \varphi_l|_{z_1=0} = \langle z' - w', \ \zeta' \rangle. \end{cases}$$

Although Theorem 0 is seemingly a little stronger than the results of Hamada-Nakamura [1] in that it contains parameters $(w', \ \zeta')$, one can easily prove it by just the same reasoning as in Hamada-Nakamura [1]. We can verify the homogeneity with respect to ζ'

of $\tilde{F}_l$, G_l, G and $\sum\limits_{p=0}^{\infty} \dfrac{F_p}{\Phi^p}$ by examining the procedure by which these functions are determined.

From now on, we shall always assume the hyperbolicity of $P(x, D_x)$ with respect to the direction $(1, 0, \ldots, 0)$, that is, we shall assume that $\lambda^{\pm}(x, \xi')$ and $\lambda_l(x, \xi')$ $(l=3, \ldots, m)$ are all real for real (x, ξ') with $\xi' \neq 0$. Assuming this reality condition, we prepare some geometric propositions, which will play an important role in the construction and the microlocal investigation of the elementary solutions which we shall construct. Note that phase functions Φ and φ_l are all real-valued for real (z, w', ζ', τ) by virtue of the reality of $\lambda^{\pm}$ and λ_l.

Proposition 1. *For any real $\xi^{o'(*)} \in S^{n-2}$, we can find $\vartheta \in \mathbf{R}^{3(n-1)}$, $\delta > 0$ and a neighborhood ω of the origin of $\mathbf{R}_x^n \times \mathbf{R}_{y'}^{n-1}$ such that*

$$\langle \mathrm{grad}_{(x',y',\xi')}\Phi(x, y', \xi', \tau),\ \vartheta \rangle > 0$$

holds for $(x, y') \in \omega$ and real ξ' and τ with $\left| \dfrac{\xi'}{|\xi'|} - \xi^{o'} \right|$, $|\tau| < \delta$.

Proof. Since $\mathrm{grad}_{(x',y',\xi')}\ \Phi(x, y', \xi', \tau)\,|_{x_1=\tau=0} = (\xi', -\xi',\ (x' - y'))$ holds by the definition of Φ, the proof is obvious.

Corollary. *If $\Phi(x, y', \xi', \tau) = 0$ for real (x, y', ξ', τ) with $-\delta \leq \tau$, $x_1 \leq \delta$, then*

$$\mathrm{Im}\ \Phi(x_1,\ (x', y',\ \xi') + \sqrt{-1}\ \varepsilon\vartheta,\ \tau) > 0$$

holds for any $\varepsilon > 0$, x_1 and τ satisfying $\varepsilon \ll 1$ and $-\delta \leq \tau$, $x_1 \leq \delta$.

Proof. By observing the Taylor expansion of Φ with respect to (x', y', ξ') we find this corollary immediately follows from Proposition 1.

Proposition 2. *Let $\Phi(z, w', \zeta', \tau)$ be a solution of (8). Assume that $(x^o, y^{o'}; u^o, v^{o'})$ is real and satisfies following equations (11)*

(*) The vector with prime such as $\zeta^{o'}$ refers to $(n-1)$-vector such as $(\zeta_2^o, \cdots, \zeta_n^o)$.

for some real $(\xi^{o\prime}, \tau, \alpha, b)$ *with* $a < \tau$, $\xi^{o\prime} \neq 0$ *and* $\alpha > 0$.

$$(11) \quad \begin{cases} \Phi(x^o, y^{o\prime}, \xi^{o\prime}, \tau) = 0, & (11.\text{a}) \\[4pt] x_1^o = \tau, & (11.\text{b}) \\[4pt] (u^o, v^{o\prime}) \equiv (u_1^o, u^{o\prime}, v^{o\prime}) = \\[4pt] \quad = \alpha \operatorname{grad}_{(x, y\prime)} \Phi(x^o, y^{o\prime}, \xi^{o\prime}, \tau)\big|_{\tau = x_1^o} + b(1, 0, 0), & (11.\text{c}) \\[4pt] \alpha \operatorname{grad}_{(\xi\prime, \tau)} \Phi(x^o, y^{o\prime}, \xi^{o\prime}, \tau)\big|_{\tau = x_1^o} + b(0, -1) = (0, 0) & (11.\text{d}) \end{cases}$$

Then $(x^o, y^{o\prime}; u^o, v^{o\prime})$ *lies on a bicharacteristic of* $u_1 - \lambda^+(x, u\prime)$ *emanating from* $(x, y\prime; u, v\prime) \equiv (x_1, x\prime, y\prime; u_1, u\prime, v\prime) = (0, y^{o\prime}, y^{o\prime};$ $\lambda^+(y^{o\prime}, \xi^{o\prime}), \xi^{o\prime}, -\xi^{o\prime})$.

Proof. First we show that (11) entails following equations (12) :

$$(12) \quad \begin{cases} \varphi^+(x^o, y^{o\prime}, \xi^{o\prime}) = 0, & (12.\text{a}) \\[4pt] \operatorname{grad}_{\xi\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime}) = 0, & (12.\text{b}) \\[4pt] u_1^o = \alpha \lambda^+(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})), & (12.\text{c}) \\[4pt] (u^{o\prime}, v^{o\prime}) = \alpha \operatorname{grad}_{(x\prime, y\prime)}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime}) & (12.\text{d}) \end{cases}$$

In fact, by the definition of Φ, we immediately see that (11.a) combined with (11.b) implies (12.a). Furthermore (12.b) and (12.d) are direct consequences of (11.c) and (11.d), respectively. In order to show (12.c) we note that (11.c) and (11.d) combined with (9) entail

$$(13) \quad b = \alpha(\lambda^+(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})) - \lambda^-(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime}))).$$

Hence, combining (13) and (11.c), we find

$$(14) \quad \begin{aligned} u_1^o &= \alpha \lambda^-(x, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})) \\ &\quad + \alpha(\lambda^+(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})) \\ &\quad - \lambda^-(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})) \\ &= \alpha \lambda^+(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})). \end{aligned}$$

Thus we have obtained (12.c)

Now let us consider in S^*M a bicharacteristic ℓ_+ associated with $u_1 - \lambda^+(x, u\prime)$ that emanates from $(x, y\prime, \xi\prime; u_1, u\prime, v\prime, h\prime) = (x^o, y^{o\prime}, \xi^{o\prime};$ $\lambda^+(x^o, \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime})), \operatorname{grad}_{x\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime}), \operatorname{grad}_{y\prime}\varphi^+(x^o, y^{o\prime}, \xi^{o\prime}),$ $0)$. Here M denotes a neighborhood of the origin of R^{3n-2}, where

(x, y', ξ') runs, and S^*M denotes its spherical cotangent bundle. Note that such a bicharacteristic is well-defined in view of (12. c) and (12. d). Then, cleary, (y', ξ', h') is constant along it. On the other hand, (7. a) and (12) entail that h' is equal to $\mathrm{grad}_{\xi'}\,\varphi^+(x, y', \xi')$ along ℓ_+. Therefore

$$(15) \qquad\qquad \mathrm{grad}_{\xi'}\varphi^+(x, y^{o\prime}, \xi^{o\prime}) = 0$$

holds. In particular, at the point p^* where ℓ_+ intersects $\{x_1 = 0\}$, we get

$$(16) \qquad\qquad\qquad x' = y^{o\prime},$$

because $\varphi^+(x, y', \xi') = \langle x' - y', \xi' \rangle$ holds there by virtue of (7. b).

By making use of the same reasoning as above, we find that

$$(17) \qquad\qquad\qquad u_1 = \lambda^+(0, y^{o\prime}, \xi^{o\prime})$$

and

$$(18) \qquad\qquad\qquad v' = -\xi^{o\prime}$$

hold at p^*.

Since the bicharacteristic associated with $u_1 - \lambda^+(x, u')$ considered in $\{(x, y'; u, v')\}$-space is nothing but the projection of ℓ_+ on this space, this completes the proof of Proposition 2.

Proposition 3. *Let $\Phi(z, w', \zeta', \tau)$ be a solution of (8). Assume that $(x^o, y^{o\prime}; u^o, v^{o\prime})$ is real and satisfies following equations (19) for some real $(\xi^{o\prime}, \alpha)$ with $\xi^{o\prime} \neq 0$ and $\alpha > 0$.*

$$(19) \quad \begin{cases} \Phi(x^o, y^{o\prime}, \xi^{o\prime}, \tau)\,|_{\tau=0} = 0, \\ (u^o, v^{o\prime}) = \alpha\,\mathrm{grad}_{(x, y')}\Phi(x^o, y^{o\prime}, \xi^{o\prime}, \tau)\,|_{\tau=0}, \\ \mathrm{grad}_{\xi'}\Phi(x^o, y^{o\prime}, \xi^{o\prime}, \tau)\,|_{\tau=0} = 0. \end{cases}$$

Then $(x^o, y^{o\prime}; u^o, v^{o\prime})$ lies on a bicharacteristic of $u_1 - \lambda^-(x, u')$ emanating from $(x_1, x', y'; u, v) = (0, y^{o\prime}, y^{o\prime}; \lambda^-(0, y^{o\prime}, \xi^{o\prime}), \xi^{o\prime}, -\xi^{o\prime})$.

Since the proof of Proposition 3 can be given in a way analogous to the proof of Proposition 2, we leave the details to the reader.

Proposition 4. *Let* $\Phi(z, w', \zeta', \tau)$ *be a solution of* (8). *Assume that* $(x^o, y^{o'}; u^o, v^{o'})$ *is real and satisfies following equations* (20) *for some real* $(\xi^{o'}, \tau, \alpha)$ *with* $\xi^{o'} \neq 0$, $a \leqq \tau \leqq x_1^o$ *and* $\alpha > 0$.

$$
(20) \quad \begin{cases} \Phi(x^o, y^{o'}, \xi^{o'}, \tau) = 0, \\ (u^o, v^{o'}) = \alpha \ \mathrm{grad}_{(x, y')} \Phi(x, y', \xi', \tau), \\ \mathrm{grad}_{(\xi', \tau)} \Phi(x, y', \xi', \tau) = (0, 0). \end{cases}
$$

Then $(x^o, y^{o'}; u^o, v^{o'})$ *lies on a bicharacteristic of* P *(in the sense of Definition* 1*) emanating from* $(x_1, x', y'; u_1, u', v') = (0, y^{o'}, y^{o'};$ $\lambda^+(0, y^{o'}, \xi^{o'}), \xi^{o'}, -\xi^{o'})$ *with* $\lambda^+(0, y^{o'}, \xi^{o'}) = \lambda^-(0, y^{o'}, \xi^{o'})$.

Proof. As in the proof of Proposition 2, we consider in S^*M a bicharacteristic associated with $u_1 - \lambda^-(x, u')$ emanating from $(x, y', \xi'; u, v', h') = (x^o, y^{o'}, \xi^{o'}; u^o, v^{o'}, 0)$, that is, we consider the following ordinary differential equations:

$$
(21) \quad \begin{cases} \dfrac{dx_1}{dt} = 1, \\[2mm] \dfrac{dx_j}{dt} = -\dfrac{\partial \lambda^-}{\partial u_j} & (j = 2, \ldots, n), \\[2mm] \dfrac{du_j}{dt} = \dfrac{\partial \lambda^-}{\partial x_j} & (j = 1, \ldots, n), \\[2mm] \dfrac{dy_j}{dt} = 0 & (j = 2, \ldots, n), \\[2mm] \dfrac{dv_j}{dt} = 0 & (j = 2, \ldots, n), \\[2mm] \dfrac{d\xi_j}{dt} = 0 & (j = 2, \ldots, n), \\[2mm] \dfrac{dh_j}{dt} = 0 & (j = 2, \ldots, n), \\[2mm] t \leqq 0, \\[2mm] x_j(0) = x_j^o, \\[2mm] u_1(0) = \lambda^-(x^o, \mathrm{grad}_{x'} \Phi(x^o, y^{o'}, \xi^{o'}, \tau)), \\[2mm] u_j(0) = \dfrac{\partial \Phi}{\partial x_j}(x^o, y^{o'}, \xi^{o'}, \tau) & (j = 2, \ldots, n), \\[2mm] y_j(0) = y_j^o & (j = 2, \ldots, n), \end{cases}
$$

$$
\left\{
\begin{aligned}
& v_j(0) = \frac{\partial \Phi}{\partial y_j}(x^o,\ y^{o\prime},\ \xi^{o\prime},\ \tau) && (j=2,\ldots,\ n), \\
& \xi_j(0) = \xi_j^o && (j=2,\ldots,\ n), \\
& h_j(0) = 0 && (j=2,\ldots,\ n).
\end{aligned}
\right.
$$

Then we have

$$
(22) \qquad
\left\{
\begin{aligned}
& x_1(\tau - x_1^o) = \tau, \\
& y'(t) = y^{o\prime}, \\
& \xi'(t) = \xi^{o\prime}.
\end{aligned}
\right.
$$

In view of (20) we now want to show that $(\tau,\ x'(\tau - x_1^o),\ y^{o\prime};\ u(\tau - x_1^o),\ v'(\tau - x_1^o))$ satisfies equations (12) and that

$$
(23) \qquad
\begin{aligned}
& \lambda^+(\tau,\ x'(\tau - x_1^o),\ \mathrm{grad}_{x'}\varphi^+(\tau,\ x'(\tau - x_1^o),\ y^{o\prime},\ \xi^{o\prime})) \\
& \quad = \lambda^-(\tau,\ x'(\tau - x_1^o),\ \mathrm{grad}_{x'}\varphi^+(\tau,\ x'(\tau - x_1^o),\ y^{o\prime},\ \xi^{o\prime}))
\end{aligned}
$$

holds.

In order to show the first statement we first note

$$
\Phi(\tau,\ x'(\tau - x_1^o),\ y^{o\prime},\ \xi^{o\prime},\ \tau) = 0
$$

holds, since Φ is constant along the bicharacteristic of $u_1 - \lambda^-(x,\ u')$ as a solution to the equation

$$
(24) \qquad \frac{\partial \Phi}{\partial x_1} = \lambda^-(x,\ \mathrm{grad}_{x'}\Phi)
$$

Therefore, combining (24) with (8. b) we get (12. a).

We also find

$$
(25) \qquad
\left\{
\begin{aligned}
& u_1(t) = \lambda^-(x(t),\ \mathrm{grad}_{x'}\Phi(x(t),\ y^{o\prime},\ \xi^{o\prime},\ \tau)), \\
& u_j(t) = \frac{\partial \Phi}{\partial x_j}(x(t),\ y^{o\prime},\ \xi^{o\prime},\ \tau) && (j=2,\ldots,\ n), \\
& v_j(t) = \frac{\partial \Phi}{\partial y_j}(x(t),\ y^{o\prime},\ \xi^{o\prime},\ \tau) && (j=2,\ldots,\ n).
\end{aligned}
\right.
$$

Then, setting $t = \tau - x_1^o$, we get the required relations (12. b) through (12. d).

Finally we show (23). Since $\dfrac{\partial \Phi}{\partial \tau}(x^o,\ y^{o\prime},\ \xi^{o\prime},\ \tau) = 0$ by the assumption, (9. a) entails

$$
(26) \qquad \frac{\partial \Phi}{\partial x_1}(x^o,\ y^{o\prime},\ \xi^{o\prime},\ \tau) = \lambda^-(x^o,\ \mathrm{grad}_{x'}\Phi(x^o,\ y^{o\prime},\ \xi^{o\prime},\ \tau))
$$

$$= \lambda^+ (x^\circ, \ \mathrm{grad}_{x'} \Phi(x^\circ, \ y^{\circ\prime}, \ \xi^{\circ\prime}, \ \tau)).$$

On the other hand, condition (2. c) guarantees that $u_1 - \lambda^+(x, u')$ is constant along the integral curve of (21). Therefore we have

(27)
$$\lambda^+ (\tau, \ x'(\tau - x_1^\circ), \ \mathrm{grad}_{x'} \varphi^+ (\tau, \ x'(\tau - x_1^\circ), \ y^{\circ\prime}, \ \xi^{\circ\prime})) =$$
$$= \lambda^- (\tau, \ x'(\tau - x_1^\circ), \ \mathrm{grad}_{x'} \varphi^+ (\tau, \ x'(\tau - x_1^\circ), \ y^{\circ\prime}, \ \xi^{\circ\prime})).$$

Thus, noting the fact that $u_1 - \lambda^-(x, u')$ is constant along the bicharacteristic associated with $u_1 - \lambda^+(x, u')$, we get the required results by repeating the proof of Proposition 2. Q. E. D.

Proposition 5. *Define* $C(a, y^{\circ\prime})$ *by* $\{(x^\circ, y^{\circ\prime}; u^\circ, v^{\circ\prime}) \in \mathbf{R}^{4n-2};$ *there exists real* $\xi^{\circ\prime}$ *such that* $\Phi(x^\circ, y^{\circ\prime}, \xi^{\circ\prime}, a) = 0$, $(u^\circ, v^{\circ\prime}) = \mathrm{grad}_{(x, y')} \Phi(x, y', \xi^{\circ\prime}, a) \mid_{x = x^\circ, y' = y^{\circ\prime}}$, $\mathrm{grad}_{\xi'} \Phi(x^\circ, y^{\circ\prime}, \xi', a) \mid_{\xi' = \xi^{\circ\prime}} = 0$ *and* $a < 0 \leqq x_1^\circ$ *hold*$\}$. *Define* $C(y^{\circ\prime})$ *by* $\bigcap\limits_{\substack{a < 0 \\ |a| < 1}} C(a, y^{\circ\prime})$. *Then each point in* $C(y^{\circ\prime})$ *lies on a bicharacteristic associated with* $u_1 - \lambda^-(x, u')$ *emanating from* $(x_1, x', y'; u_1, u', v') = (0, y^{\circ\prime}, y^{\circ\prime}; \lambda^-(0, y^{\circ\prime}, -v^{\circ\prime}), -v^{\circ\prime}, v^{\circ\prime})$.

Proof. As in the proof of Proposition 4, we find that each point p^* in $C(a, y^{\circ\prime})$ lies on a bicharacteristic associated with $u_1 - \lambda^-(x, u')$ that emanates from $(0, x', y^{\circ\prime}; u, v^{\circ\prime})$ with

$$\begin{cases} \mathrm{grad}_{\xi'} \Phi(0, \ x', \ y^{\circ\prime}, \ \xi', \ a) = 0 \\ (u, \ v^{\circ\prime}) = \mathrm{grad}_{(x', y')} \Phi(0, \ x', \ y', \ \xi', \ a) \mid_{y' = y^{\circ\prime}} \end{cases}$$

holds with real $\xi' \neq 0$. Since

$$\det \left(\frac{\partial^2 \Phi}{\partial \xi_j \partial x_j} (x_1, \ x', \ y', \ \xi', \ a) \mid_{\substack{x_1 = a = 0 \\ x' = y'}} \right)_{2 \leqq j \leqq n} \neq 0$$

holds, we can find a function $X'(y', \ \xi', \ a)$ which is analytic in $(y', \ \xi', \ a)$ so that

$$\begin{cases} x' = X'(y', \ \xi', \ a) \\ y' = X'(y', \ \xi', \ 0) \\ \mathrm{grad}_{\xi'} \Phi(0, \ X'(y', \ \xi', \ a), \ y', \ \xi', \ a) = 0 \end{cases}$$

holds. Therefore p^* lies on a bicharacteristic associated with $u_1 - \lambda^-(x, u')$ emanating from $(x_1, x', y^{\circ\prime}; u, v^{\circ\prime}) = (0, \ X'(y^{\circ\prime}, \ \xi', \ a), \ y^{\circ\prime};$ $\mathrm{grad}_{(x, y')} \Phi(x, y^{\circ\prime}, \xi', a) \mid_{x_1 = 0, x' = X'(y', \xi', a), y' = y^{\circ\prime}})$. Then it clearly follows from

the definition of Φ and the continuity of X' with respect to a that

$$\begin{cases} x' = y^{o\prime} \\ u_1^o = \lambda^-(0, \ y^{o\prime}, \ -v_o') \\ u^{o\prime} = -v^{o\prime} \end{cases}$$

holds. This completes the proof of Proposition 5.

Proposition 6. *Define $K(a, \ y^{o\prime})$ by $\{(x^o, \ y^{o\prime}; \ u^o, \ v^{o\prime}) \in \boldsymbol{R}^{4n-2}$; there exist real $\xi^{o\prime}$ and $\tau (a \le \tau \le x_1^o)$ which satisfies (20) with $\alpha = 1\}$. Define $K(y^{o\prime})$ by $\bigcap\limits_{\substack{a<0 \\ |a| \ll 1}} K(a, \ y^{o\prime})$. Then each point $(x^o, \ y^{o\prime}; \ u^o, \ v^{o\prime})$ in $K(y^{o\prime})$ lies on a time-like bicharacteristic associated with P emanating from $(x_1, \ x', \ y'; \ u_1, \ u', \ v') = (0, \ y^{o\prime}, \ y^{o\prime}; \ \lambda^+(0, \ y^{o\prime}, \ -v^{o\prime}), \ -v^{o\prime}, \ v^{o\prime})$ with $\lambda^+(0, \ y^{o\prime}, \ -v^{o\prime}) = \lambda^-(0, \ y^{o\prime}, \ -v^{o\prime})$.*

Proof. First define $K_1(y^{o\prime})$ and $K_2(a, \ y^{o\prime})$ in the same way as in the definition of $K(a, y^{o\prime})$ by restricting τ to the interval $\{\tau; \ 0 \le \tau \le x_1^o\}$ and $\{\tau; \ a \le \tau \le 0\}$, respectively. Then we immediately see that

$$K(y^{o\prime}) = K_1(y^{o\prime}) \cup \Big(\bigcap\limits_{\substack{a<0 \\ |a| \ll 1}} K_2(a, \ y^{o\prime}) \Big)$$

holds. In view of the way of the proof of Proposition 4 it suffices to show that each point $(x^o, \ y^{o\prime}; \ u^o, \ v^{o\prime})$ in $\bigcap\limits_{\substack{a<0 \\ |a| \ll 1}} K_2(a, \ y^{o\prime})$ enjoys the property stated in the proposition. On the other hand, we can prove by the same reasoning as in the proof of Proposition 5 that $(x^o, \ y^{o\prime}; \ u^o, \ v^{o\prime})$ lies on a bicharacteristic associated with $u_1 - \lambda^-(x, \ u')$ emanating from $(x_1, \ x', \ y'; \ u_1, \ u', \ v') = (0, \ y^{o\prime}, \ y^{o\prime}; \ \lambda^-(0, \ y^{o\prime}, \ -v^{o\prime}), \ -v^{o\prime}, \ v^{o\prime})$ with $\lambda^-(0, \ y^{o\prime}, \ -v^{o\prime}) = \lambda^+(0, \ y^{o\prime}, \ -v^{o\prime})$. This completes the proof of Proposition 6.

Now we embark on the construction of the elementary solution for Cauchy problem for the operator $P(x, \ D_x)$.

Theorem 1. *Let $P(x, \ D_x)$ be a linear differential operator which satisfies conditions (1) and (2). Assume that P is hyperbolic with respect to the direction $(1, 0, \ldots, 0)$ near the origin. Then there*

exists a unique local elementary solution for Cauchy problem for the operator in question, that is, we can find hyperfunctions $E_j(x, y')$ $(j = = 0, \ldots, m-1)$ in a neighborhood of $(x, y') = (0, 0)$ so that it satisfies the following:

$$(28) \quad \begin{cases} P(x, D_x) E_j(x, y') = 0 & (j = 0, \ldots, m-1), \\ \dfrac{\partial^k}{\partial x_1^k} E_j(x, y') \big|_{x_1=0} = \delta_{jk} \delta(x'-y') & (j, k = 0, \ldots, m-1). \end{cases}$$

Furthermore, if $(x^\circ, y^{\circ'}; \sqrt{-1}(u^\circ, v^{\circ'})\infty)$ lies in the singularity spectrum of $E_j(x, y')$, then $v^{\circ'} \neq 0$ and $(x^\circ, y^{\circ'}; u^\circ, v^{\circ'})$ belongs either to a time-like bicharacteristic associated with P (in the sense of Definition 1) which emanates from $(x, y'; u, v') \equiv (x_1, x', y'; u_1, u', v') = = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ with $\lambda^+(0, y^{\circ'}, -v^{\circ'}) = = \lambda^-(0, y^{\circ'}, -v^{\circ'})$ or to a bicharacteristic associated with P which emanates either from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ or from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^-(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ or from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda_l(0, y', -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ $(l = 3, \ldots, m)$.

Proof. First note that the hyperbolicity of P implies that $\lambda^\pm(x, \xi')$ and $\lambda_l(x, \xi')$ $(l = 3, \ldots, m)$ are real valued analytic functions for real (x, ξ') if x is sufficiently close to the origin and $\xi' \neq 0$. This entails that $\varphi_l(x, y', \xi')$ $(l = 3, \ldots, m)$ and $\Phi(x, y', \xi', \tau)$ are real as long as (x, y', ξ') and τ are real as mentioned earlier. In fact, we have only to integrate first order differential equations to obtain φ_l and Φ, hence the reality of $\lambda^\pm$ and λ_l guarantees the reality of φ_l and Φ. Making full use of this reality of φ_l and Φ we shall construct the required elementary solutions for P (as hyperfunctions).

In the sequel, we consider the case $j = m-1$. Other cases can be dealt with by just the same reasoning.

Now we choose a neighborhood ω_1 of the origin of $R_x^n \times R_{y'}^{n-1}$, constants $a < 0$, $\delta > 0$ and open sets I_j, $I'_j \in S_{\xi'}^{n-2} = \{\xi' \in R^{n-1}; |\xi'| = 1\}$ [*] $(j = 1, \ldots, N < \infty)$ so that they satisfy the following:

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[*] If there is no of fear confusion, we sometimes identify a subset I of $S_{\xi'}^{n-2}$ with a positive cone $\bigcup_{c>0} cI \subset R^{n-1}$, without mentioning so.

(29) $I_j' \Subset I_j$

(30) $\overset{N}{\underset{j=1}{\cup}} I_j' = S_{\xi'}^{n-2}$

(31)   There exists $\vartheta_j \in \mathbf{R}^{3(n-1)}$ such that

$\langle \mathrm{grad}_{(x', y', \xi')} \Phi(x_1, x', y', \xi', \tau), \vartheta_j \rangle > 0$

holds if $-\delta < a \leqq \tau \leqq x_1 < \delta$ and if (x, y') belongs to ω_1 and ξ' belongs to I_j.

Since $S_{\xi'}^{n-2}$ is compact, Proposition 1 enables us to find such a finite covering I_j' of $S_{\xi'}^{n-2}$.

First define $F(z, w', \zeta', s)$ by

$$\sum_{p=1}^{\infty} \left(\frac{F_p(z, w', \zeta', s)}{\Phi(z, w', \zeta', s)^p} \right) + G(z, w', \zeta', s) \log \Phi(z, w', \zeta', s),$$

where F_p' and G are those given in Theorem 0 so that

$$U(z, w', \zeta') = \sum_{l=3}^{m+1} \left(\frac{\tilde{F}_l(z, w', \zeta')}{\varphi_l(z, w', \zeta')^{p_l}} + G_l(z, w', \zeta') \log \varphi_l(z, w', \zeta') \right)$$
$$+ \int_0^{x_1} F(z, w', \zeta', s) ds + H(z, w', \zeta')$$

satisfies $(5)_j$ with $j = m - 1$. Clearly $F(z, w', \zeta', s)$ is univalent and holomorphic in the domain $\Omega = \{ \mathrm{Im}\, \Phi(z, w', \zeta', s) > 0 \}$. Furthermore F can be analytically continued across $\partial \Omega$ near $(z^o, w^{o\prime}, \zeta^{o\prime}, s^o) \in \partial \Omega$ if $\Phi(z^o, w^{o\prime}, \zeta^{o\prime}, s^o) \neq 0$. From now on, we shall denote by F this preferred branch of F (on Ω) for simplicity.

We shall now construct a hyperfunction $e_{I_j}(x, y', \xi')$ by making use of $U(z, w', \zeta')$ so that it is defined on $\omega \times I_j$ for a neighborhood ω of the origin of $\mathbf{R}_x^n \times \mathbf{R}_{y'}^{n-1}$ and that it satisfies following condition (32) there:

(32) $\begin{cases} P(x, D_x) e_{I_j}(x, y', \xi') = 0, \\ \dfrac{\partial^k}{\partial x_1^k} e_{I_j}(x, y', \xi') \big|_{x_1 = 0} \\ \qquad = \dfrac{\delta_{m-1, k}}{(-2\pi \sqrt{-1})^{n-1}(\langle x' - y', \xi' \rangle + \sqrt{-1} 0)^{n-1}} \quad (k = 0, \ldots, m-1). \end{cases}$

The reality of Φ combined with the non-vanishing of $\mathrm{grad}_{(z, w', \zeta', s)} \Phi$ on $\{\Phi = 0\}$ entails that F defines a hyperfunciton $f(x, y', \xi', \tau)$ whose singularity spectrum is confined to the set

(33) $\{(x, y', \xi', \tau; \sqrt{-1}(u, v', \eta', \sigma)\infty)\in\sqrt{-1}\ S^*M_1; \Phi(x, y', \xi', \tau) =$
$= 0$ and $(u, v', \eta', \sigma) = c\ \mathrm{grad}_{(x,y',\xi',\tau)}\Phi(x, y', \xi', \tau)\}.$

Here M_1 stands for $\{(x, y', \xi', \tau); (x, y')\in\omega_2,\ \xi'\in I_j,\ a<\tau<\delta\}$,
where ω_2 is a neighborhood of the origin of $R^n_x\times R^{n-1}_{y'}$.

Next define $e_o(x, y', \xi')$ by

(34) $\displaystyle\int Y(\tau-a)\,Y(x_1-\tau)f(x, y', \xi', \tau)\ d\tau.$

We shall now establish the fact that $e_o(x, y', \xi')$ is a well-defined
hyperfunction if $a<x_1<\varepsilon$ and investigate its singularity spectrum.
Here $Y(\tau)$ denotes the Heaviside function.

First we note that the multiplication procedure used to define
the integrand of (34) is legitimate in view of Corollary 2.4.2 of
S-K-K [1] Chapter I §2. 4. In fact, Condition (33) combined with
the fact $\mathrm{grad}_{(x',y',\xi')}\Phi(x, y', \xi', \tau)$ is different from zero if $\tau=x_1$ or
a guarantees it. Once one establishes the fact that the integrand is
well-defined, the well-definedness of integral (34) is clear, because
the support of the integrand is confined to $\{a\leq\tau\leq x_1\}$.

Secondly we determine the maximum possible set for the singularity
spectrum of $e_o(x, y', \xi')$.   Here we apply Corollary 2. 4. 2 and
Theorem 2. 3. 1 of S-K-K [1] Chapter I. Condition (33) combined
with the above quoted Corollary 2. 4. 2 of S-K-K [1] implies that
the singularity spectrum of $Y(\tau-a)\,Y(x_1-\tau)f(x, y', \xi', \tau)$ is contained
in the union of $A_l(l=1,\ldots, 5)$ defined below, if we assume $x_1>a$.

(35) $A_1= \{(x, y', \xi', \tau; \sqrt{-1}(u, v', \eta', \sigma)\infty)\in\sqrt{-1}\ S^*M_1;$
$\Phi(x, y', \xi', \tau) = 0,\ x_1=\tau(\neq a)$ and (u, v', η', σ)
$\equiv(u_1, u', v', \eta', \sigma)=\alpha\ \mathrm{grad}_{(x,y',\xi',\tau)}\Phi(x, y', \xi', \tau)+b(1, 0, 0, 0, -1)$
with $\alpha\geq 0$, b real and $(\alpha, b)\neq(0, 0)\}.$

(36) $A_2= \{(x, y', \xi', \tau; \sqrt{-1}(u, v', \eta', \sigma)\infty)\in\sqrt{-1}\ S^*M_1;$
$\Phi(x, y', \xi', \tau) = 0,\ \tau=a(\neq x_1)$ and (u, v', η', σ)
$=\alpha\ \mathrm{grad}_{(x,y',\xi',\tau)}\Phi(x, y', \xi', \tau)+b(0, 0, 0, 1)$ with $\alpha\geq 0$, b real
and $(\alpha, b)\neq(0, 0)\}.$

(37) $A_3= \{(x, y', \xi', \tau; \sqrt{-1}(u, v', \eta', \sigma)\infty)\in\sqrt{-1}\ S^*M_1;$
$\Phi(x, y', \xi', \tau) = 0$ and (u, v', η', σ)

$$= \alpha \ \mathrm{grad}_{(x,y',\xi',\tau)} \Phi(x, \ y', \ \xi', \ \tau) \ \text{ with } \ \alpha > 0\}.$$

(38) $A_4 = \{(x, \ y', \ \xi', \ \tau \ ; \ \sqrt{-1}(u, \ v', \ \eta', \ \sigma)\infty) \in \sqrt{-1} \ S^*M_1 \ ;$
$x_1 = \tau (\neq a) \ \text{ and } \ (u_1, \ u', \ v', \ \eta', \ \sigma) = b(1, \ 0, \ 0, \ 0, \ -1) \ \text{ with}$
$b(\neq 0) \ \text{ real}\}.$

(39) $A_5 = \{(x, \ y', \ \xi', \ \tau \ ; \ \sqrt{-1}(u, \ v', \ \eta', \ \sigma)\infty) \in \sqrt{-1} \ S^*M_1 \ ; \ \tau = a$
$(\neq x_1) \ \text{ and } \ (u, \ v', \ \eta', \ \sigma) = b(0, \ 0, \ 0, \ -1) \ \text{ with } \ b(\neq 0) \ \text{ real}\}.$

Applying Theorem 2. 3. 1 of S-K-K [1] Chapter I § 2. 3 to the integral (34), we conclude that the singularity spectrum of $e_0(x, y', \xi')$ is confined to the union of the following sets B_l $(l=1, 2, 3)$. There M_2 denotes a sufficiently small neighborhood of $(x, y', \xi') = (0, 0, \xi^{o\prime})$. Note that neither A_4 nor A_5 contributes to the possible singularity set of $e_0(x, \ y', \ \xi')$.

(40) $B_1 = \{(x, \ y', \ \xi', \ ; \ \sqrt{-1}(u, \ v', \ \eta')\infty) \in \sqrt{-1} \ S^*M_2 \ ;$
there exist τ with $a < \tau \leq x_1$ and real α and b $(\alpha > 0)^{(*)}$ such
that $\Phi(x, \ y', \ \xi', \ \tau) = 0, \ x_1 = \tau, \ (u, \ v', \ \eta') \equiv (u_1, \ u', \ v', \ \eta')$
$= \alpha \ \mathrm{grad}_{(x_1, x', y', \xi')} \Phi(x, \ y', \ \xi', \ \tau) + b(1, \ 0, \ 0, \ 0) \ \text{ and that}$
$\alpha \dfrac{\partial}{\partial \tau} \Phi(x, \ y', \ \xi', \ \tau) - b = 0\}.$

(41) $B_2 = \{(x, \ y', \ \xi' \ ; \ \sqrt{-1}(u, \ v', \ \eta')\infty) \in \sqrt{-1} \ S^*M_2 \ ;$
there exist real α and b with $\alpha \gneqq 0^{(**)}$ such that $\Phi(x, y', \xi', a)$
$= 0, \ (u, \ v', \ \eta') = \alpha \ \mathrm{grad}_{(x,y',\xi')} \Phi(x, \ y', \ \xi', \ \tau) \ |_{\tau=a} \ \text{ and that}$
$\alpha \dfrac{\partial}{\partial \tau} \Phi(x, \ y', \ \xi', \ \tau) \ |_{\tau=a} - b = 0\}.$

(42) $B_3 = \{(x, \ y', \ \xi' \ ; \ \sqrt{-1}(u, \ v', \ \eta')\infty) \in \sqrt{-1} \ S^*M_2 \ ;$
there exist $\alpha > 0$ and τ with $a \leq \tau \leq x_1$ such that $\Phi(x, y', \xi', \tau)$
$= 0, \ (u, \ v', \ \eta') = \alpha \ \mathrm{grad}_{(x,y',\xi')} \Phi(x, \ y', \ \xi', \ \tau) \ \text{ and that}$
$\dfrac{\partial}{\partial \tau} \Phi(x, \ y', \ \xi', \ \tau) = 0\}.$

By the same reasoning as above we can conclude that

$$e_1(x, \ y', \ \xi') = \int Y(\tau - a) Y(-\tau) f(x, \ y', \ \xi', \ \tau) d\tau$$

(*)   Note that any points in A_1 with $\alpha = 0$ (hence $b \neq 0$) do not give any contributions to the singularity spectrum of $e_0(x, \ y', \ \xi')$.

(**)   The same remark as in the footnote (*) also applies here.

is a well-defined hyperfunction on $M_2' \equiv \omega_3 \times I_j$ for some neighborhood ω_3 of the origin of $\boldsymbol{R}_x^n \times \boldsymbol{R}_{y'}^{n-1}$ and its singularity spectrum is confined to the union of $B_j' (j=1, 2, 3)$ defined below:

(43) $B_1' = \{(x, y', \xi' ; \sqrt{-1}(u, v', \eta')\infty) \in \sqrt{-1} \ S^*M_2' ;$
        there exists $\alpha > 0$ such that (u, v', η')
        $= \alpha \ \mathrm{grad}_{(x,y',\xi')} \Phi(x, y', \xi', \tau) |_{\tau=0}$ holds with $\Phi(x, y', \xi', 0) = 0\}.$

(44) $B_2' = \{(x, y', \xi' ; \sqrt{-1}(u, v', \eta')\infty) \in \sqrt{-1} \ S^*M_2' ;$
        there exists $\alpha > 0$ such that (u, v', η')
        $= \alpha \ \mathrm{grad}_{(x,y',\xi')} \ \Phi(x, y', \xi', \tau) |_{\tau=a}$ holds with $\Phi(x, y', \xi', a) = 0\}.$

(45) $B_3' = \{(x, y', \xi' ; \sqrt{-1}(u, v', \eta')) \in \sqrt{-1} \ S^*M_2' ;$
        there exist $\alpha > 0$ and τ with $a \leq \tau \leq 0$ such that $\Phi(x, y', \xi', \tau)$
        $= 0, (u, v', \eta') = \alpha \ \mathrm{grad}_{(x,y',\xi')} \Phi(x, y', \xi', \tau)$ and that
        $\dfrac{\partial}{\partial \tau} \Phi(x, y', \xi', \tau) = 0\}.$

Now define (multi-valued) analytic function $\tilde{F}_l(z, w', \zeta') (l=3, \ldots, m+1)$ by

(46) $\dfrac{\tilde{F}_l(z, w', \zeta')}{\varphi_l(z, w', \zeta')^{p_l}} + G_l(z, w', \zeta') \ \log \varphi_l(z, w', \zeta'),$

where $\tilde{F}_l, G_l$ and φ_l are the holomorphic functions used in Theorem 0. Clearly $\tilde{F}_l$ is univalent and holomorphic in the domain $\Omega_l = \{(z, w', \zeta') ; \mathrm{Im} \ \varphi_l(z, w', \zeta') > 0\}$. By the reality of φ_l mentioned earlier, $\tilde{F}_l(z, w', \zeta')$ defines a hyperfunction $f_l(x, y', \xi')$ as its boundary valued attained from the domain Ω_l.

Now we show that

$$e_{I_j}(x, y', \xi') = e_o(x, y', \xi') - e_1(x, y', \xi')$$
$$+ \sum_{l=3}^{m+1} f_l(x, y', \xi') + H(x, y', \xi')$$

satisfies (32). In order to show this we first define a multi-valued analytic function $E(z, w', \zeta')$ as follows:

(47) $E(z, w', \zeta') = \displaystyle\int_a^{z_1} F(z, w', \zeta', s) ds - \int_a^0 F(z, w', \zeta', s) ds$

$+ \displaystyle\sum_{l=3}^{m+1} \tilde{F}_l(z, w', \zeta') + H(z, w', \zeta').$

One can easily verify that

$$f_o(z, w', \zeta') = \int_a^{z_1} F(z, w', \zeta', s)\, ds$$

is univalent and holomorphic in a neighborhood $\tilde{U}$ of

$$\Gamma_j = \{(z, w', \zeta') \; ; \; (\mathrm{Re}\, z, \mathrm{Re}\, w') \in \omega_1, \; \mathrm{Re}\, \zeta' \in I_j, \; |z|, \; |w'| \ll 1,$$
$$\mathrm{Im}(z_1; z', w', \zeta') = \varepsilon(0 \; ; \; \vartheta_j)\, (0 < \varepsilon \ll 1)\}.$$

Furthermore, by the localized version of Bochner's tube theorem (Komatsu [1], S-K-K [1] Chapter I § 3.1 Theorem 3.1.1), we may assume that $\tilde{U}$ contains a conical neighborhood of Γ_j, i. e., $\tilde{U}$ contains

$$\{(z, w', \zeta') \; ; \; |z|, \; |w'|, \; |\mathrm{Im}\, \zeta'| \ll 1, \; (\mathrm{Re}\, z, \mathrm{Re}\, w', \mathrm{Re}\, \zeta')$$
$$\in \omega_1 \times I_j, \left| \frac{\mathrm{Im}\, (z, w', \zeta')}{|\mathrm{Im}(z, w', \zeta')|} - (0, \; \vartheta_j) \right| < c, \text{ where } c \text{ is a constant}$$

depending only on $\mathrm{Re}\, z$, $\mathrm{Re}\, w'$, and $\mathrm{Re}\, \zeta'\}$.

On the other hand, the result on the singularity spectrum of $e_o(x, y', \xi')$ entails that hyperfunction $e_o(x, y', \xi')$ is the boundary value of a holomorphic function $\tilde{f}_o(z, w', \zeta')$ in a conical neighborhood U' of Γ_j. Of course one may assume without loss of generality that $U' = \tilde{U}$. We now want to claim that $f_o(z, w', \zeta') = \tilde{f}_o(z, w', \zeta')$ in $\tilde{U}$. By the definition of $f_o(z, w', \zeta')$,

$$f_o(z, w', \zeta')|_{z_1 = x_1^o} \equiv f_o(x_1^o, z', w', \zeta')$$

$$= \int_a^{x_1^o} (F(z, w', \zeta', s)|_{z_1 = x_1^o})\, ds$$

$$= \int_a^{x_1^o} F(x_1^o, z', w', \zeta', s)\, ds$$

holds if x_1^o is real and sufficiently close to zero. Here $F(x_1^o, z', w', \zeta', s)$ is defined and holomorphic in the domain $\Omega_{x_1^o} \equiv \Omega \cap \{z_1 = x_1^o\} = \{(z', w', \zeta', s) \; ; \mathrm{Im}\, (\Phi(z, w', \zeta', s)|_{z_1 = x_1^o}) > 0\}$. Note that $\mathrm{Im}(\Phi(z, w', \zeta', s)|_{z_1 = x_1^o}) > 0$ even if s is real $(a \le s \le x_1^o)$ as long as $\Phi(x_1^o, \mathrm{Re}\, z', \mathrm{Re}\, w', \mathrm{Re}\, \zeta', s) = 0$ and $\mathrm{Im}(z', w', \xi') = \varepsilon \vartheta_j$, $(0 < \varepsilon \ll 1)$ hold. In view of the definition of the procedure of restriction of hyperfunctions (S-K-K [1] Chapter I § 2.2), the boundary value of $F(x_1^o, z', w', \zeta', s)$ attained from the domain $\Omega_{x_1^o}$ coincides with $f(x, y', \xi', \tau)|_{x_1 = x_1^o}$. Note that the procedure

of restricting $f(x, y', \xi', \tau)$ to the hypersurface $\{x_1 = x_1^o\}$ is a legitimate one by virtue of the previous result (33) on the singularity spectrum of $f(x, y', \xi', \tau)$. Actually, (33) entails more strongly that restricting the hyperfunction $f(x, y', \xi', \tau) Y(\tau - a) Y(x_1 - \tau)$ to $x_1 = x_1^o (\gneqq a)$ is a legitimate procedure and the resulting hyperfunction coincides with $(f(x, y', \xi', \tau)|_{x_1 = x_1^o}) Y(\tau - a) Y(x_1^o - \tau)$. Therefore, the boundary value of a holomorphic function $f_o(x_1^o, z', w', \zeta')$ considered in $\tilde{U}_{x_1^o}$ $= \{z_1 = x_1^o\} \cap \tilde{U}$ attains

$$\int_a^{x_1^o} (f(x, y', \xi', \tau)|_{x_1 = x_1^o}) d\tau = \int Y(\tau - a) Y(x_1^o - \tau) f(x_1^o, x', y', \xi', \tau) d\tau$$

$$= \int (Y(\tau - a) Y(x_1 - \tau) f(x, y', \xi', \tau))|_{x_1 = x_1^o} d\tau = e_o(x, y', \xi')|_{x_1 = x_1^o}.$$

On the other hand, hyperfunction $e_o(x, y', \xi')|_{x_1 = x_1^o}$ is the boundary value of $\tilde{f}_o(z, w', \zeta')|_{z_1 = x_1^o}$ in view of S-K-K [1] Chapter I §2.2.

Therefore $f_o(z, w', \zeta')|_{z_1 = x_1^o} = \tilde{f}_o(z, w', \zeta')|_{z_1 = x_1^o}$. This immediately implies that $f_o(z, w', \zeta') = \tilde{f}_o(z, w', \zeta')$ holds, since x_1^o is an arbitrary real number that is sufficiently close to zero.

By the same reasoning as above we find that

$$\int_a^0 f(z, w', \zeta', \tau) d\tau$$

defines a (univalent) holomorphic function in a conical neighborhood of Γ_j and it attains

$$\int Y(\tau - a) Y(-\tau) f(x, y', \zeta', \tau) d\tau$$

as its boundary value. On the other hand, we have clearly

$$\int_a^{z_1} f(z, w', \zeta', \tau) d\tau - \int_a^0 f(z, w', \zeta', \tau) d\tau$$

$$= \int_0^{z_1} f(z, w', \zeta', \tau) d\tau$$

in a conical neighborhood of Γ_j. Therefore, the boundary value of $E(z, w', \zeta') = U(z, w', \zeta')$ attained from a conical neighborhood of Γ_j coincides with $e_{I_j}(x, y, \xi') \equiv e_o(x, y', \xi') - e_1(x, y', \xi') + \sum_{l=3}^{m+1} f_l(x, y', \xi')$ $+ H(x, y', \xi')$. Since $U(z, w', \xi')$ satisfies (5)$_j$ with $j = m - 1$, we thus find that $e_{I_j}(x, y', \xi')$ satisfies (32) as its boundary value.

Now we appeal to Holmgren's uniqueness theorem (e. g. S-K-K [1] Chapter III § 2. 1 Proposition 2. 1. 3) and conclude that

(48) $$e_{I_j}(x, \, y', \, \xi') = e_{I_{j'}}(x, \, y', \, \xi')$$

holds on $\omega \times (I_j' \cap I_{j'}')$ for a sufficiently small neighborhood ω of the origin of $R_x^n \times R_{y'}^{n-1}$, as $\omega \times (I_j' \cap I_{j'}') \neq \emptyset$. Therefore we can find a hyperfunction $e(x, \, y', \, \xi')$ defined on $\omega \times S_{\xi'}^{n-2}$ so that

(49) $$e(x, \, y', \, \xi') \, |_{\omega \times I_j'} = e_{I_j}(x, \, y', \, \xi')$$

holds. Clearly $e(x, \, y', \, \xi')$ satisfies

(50) $$\begin{cases} P(x, \, D_x) e(x, \, y', \, \xi') = 0, \\[2mm] \dfrac{\partial^k}{\partial x_1^k} e(x, \, y', \, \xi') \, |_{x_1=0} = \\[2mm] \quad = \dfrac{\delta_{j, m-1}}{(-2\pi\sqrt{-1})^{n-1}(\langle x'-y', \, \xi' \rangle + \sqrt{-1} 0)^{n-1}} \quad (j=0, \ldots, \, m-1). \end{cases}$$

Therefore, by making use of the well-known formula (John [1], Gel'fand-Shilov [1])

(51) $$\delta(x') \equiv \delta(x_2, \ldots, \, x_n)$$
$$= \frac{1}{(-2\pi\sqrt{-1})^{n-1}} \int \frac{\omega(\xi')}{(\langle x', \, \xi' \rangle + \sqrt{-1} 0)^{n-1}},$$

we conclude that $\tilde{E}_{m-1}(x, \, y'; a) = \int e(x, \, y', \, \xi') \omega(\xi')$ satisfies (28).

In passing, if we choose $a' > 0$ suitably, we can construct $\tilde{E}_k^-(x, \, y'; a')$ so that it satisfies (28) by replacing $Y(\tau-a) Y(x_1-\tau)$ with $Y(a'-\tau) Y(\tau-x_1)$ in the argument above. Then we appeal to Holmgren's uniqueness theorem to find $E_k(x, \, y')$ so that it satisfies (28) and that it coincides with $\tilde{E}_k(x, \, y'; a)$ in an open neighborhood of $\{x \in \omega; \, x_1 \geq 0\}$ and with $\tilde{E}_k^-(x, \, y'; a')$ in an open neighborhood of $\{x \in \omega; \, x_1 \leq 0\}$. We note that the domain of definition of $E_k(x, y')$ thus defined can be chosen so that it is independent of the choice of a and a'. This fact implies that S. S. $E_k(x, \, y')$ thus constructed is contained in $\bigcap_{a<0}$ S. S. $\tilde{E}_k(x, \, y'; a)$ for $x_1 \geq 0$ and in $\bigcap_{a'>0}$ S. S. $\tilde{E}_k^-(x, y'; a')$ for $x_1 \leq 0$. This will be used in determing the possible singularity set for $E_k(x, \, y')$ below.

Since the uniqueness assertion in the theorem immediately follows

from Holmgren's uniqueness theorem, what remains to be proved is the singularity structure of $E_{m-1}(x, y')$ claimed in the theorem. Since we have already determined the maximum possible singularity set for $e_{I_j}(x, y', \xi')$, hence for $e(x, y', \xi')$, we can conclude by virtue of Theorem 2.3.1 of S-K-K [1] Chapter I § 2.3 that the singularity spectrum of $E_{m-1}(x, y')$ is contained by the union of $C_l(l=0, 1, 2, 3, \ldots, m+1)$ and C_l' $(l=0, 1, 2)$ defined below:

(52) $C_o = \bigcap\limits_{0<|a|\ll 1} \{(x, y'; \sqrt{-1}(u, v')\infty) \in \sqrt{-1}\ S^*\omega ;$
there exist j and τ with $a(\tau-a)<0$, real α and b with $\alpha>0$ and real $\xi' \in I_j'^{(*)}$ such that $\Phi(x, y', \xi', \tau)=0$, $x_1=\tau$, $(u_1, u', v')=$
$= \alpha\ \mathrm{grad}_{(x,y')}\ \Phi(x, y', \xi', \tau)|_{x_1=\tau}+b(1, 0, 0)$ and that
$\alpha\ \mathrm{grad}_{(\xi',\tau)}\Phi(x, y', \xi', \tau)+b(0, -1)=(0, 0)\}.$

(53) $C_1 = \bigcap\limits_{0<|a|\ll 1} \{(x, y'; \sqrt{-1}(u, v')\infty) \in \sqrt{-1}\ S^*\omega ;$
there exist j and real $\xi' \in I_j$ such that $\Phi(x, y', \xi', a)=0$,
$(u, v')=\mathrm{grad}_{(x,y')}\Phi(x, y', \xi', a)$ and $\mathrm{grad}_{\xi'}\ \Phi(x, y', \xi', a)=0$
hold.\}

(54) $C_2 = \bigcap\limits_{0<|a|\ll 1} \{(x, y'; \sqrt{-1}(u, v')\infty) \in \sqrt{-1}\ S^*\omega ;$
there exist j, $\alpha>0$, τ with $(\tau-a)(\tau-x_1)\leqq 0$, $a(x_1-a)<0$ and real $\xi' \in I_j'$ such that $\Phi(x, y', \xi', \tau)=0$ and that
$\mathrm{grad}_{(\xi',\tau)}\Phi(x, y', \xi', \tau)=(0, 0)\}.$

(55) $C_l = \{(x, y'; \sqrt{-1}(u, v')\infty) \in \sqrt{-1}\ S^*\omega ;$
there exist real α and ξ' with $\alpha>0$ such that $\varphi_l(x, y', \xi')$
$=0$, $(u, v')=\alpha\ \mathrm{grad}_{(x,y')}\varphi_l(x, y', \xi')$ and that
$\mathrm{grad}_{\xi'}\varphi_l(x, y', \xi')=0$ $(l=3, \ldots, m+1)\}.$

(56) $C_0' = \{(x, y'; \sqrt{-1}(u, v')\infty) \in \sqrt{-1}\ S^*\omega ;$
there exist $\alpha>0$ and real ξ' such that $\Phi(x, y', \xi', 0)=0$,
$(u, v')=\alpha\ \mathrm{grad}_{(x,y')}\Phi(x, y', \xi', \tau)|_{\tau=0}$ and that
$\mathrm{grad}_{\xi'}\Phi(x, y', \xi', \tau)|_{\tau=0}=0\}.$

(57) $C_1' = C_1$

(58) $C_2' = \bigcap\limits_{0<|a|\ll 1} \{(x, y'; \sqrt{-1}(u, v')\infty) \in \sqrt{-1}\ S^*\omega ;$

(*) Precisely speaking, I_j' has been chosen for $a<0$ and for $a>0$ different notations would better be used. However, a little loose notation used here and in the sequel will not cause any confusion.

there exist j, $\alpha > 0$, $\xi' \in I'_j$ and τ with $a(\tau - a) \leqq 0$ such that $\Phi(x, y', \xi', \tau) = 0$, $(u, v') = \alpha \operatorname{grad}_{(x, y')} \Phi(x, y', \xi', \tau)$ and that $\operatorname{grad}_{(\xi', \tau)} \Phi(x, y', \xi', \tau) = (0, 0)\}$.

Lastly we consider the geometric meaning of these set $C_l(l = 0, \ldots, m+1)$ and $C'_l(l = 0, 1, 2)$.

We first discuss C_l. The conditions on $(x, y'; \sqrt{-1}(u, v')\infty)$ given in (52) is clearly re-written as follows:

$$(59) \quad \begin{cases} \Phi(x, y', \xi', \tau) = 0 & (59.\,\text{a}) \\ x_1 = \tau & (59.\,\text{b}) \\ u_1 = \alpha\left(\dfrac{\partial}{\partial x_1}\Phi(x, y', \xi', \tau) + \dfrac{\partial}{\partial \tau}\Phi(x, y', \xi', \tau)\right) & (59.\,\text{c}) \\ (u', v') = \alpha \operatorname{grad}_{(x', y')}\Phi(x, y', \xi', \tau) & (59.\,\text{d}) \end{cases}$$

holds for some $\alpha > 0$.

Then Proposition 2 imlpies that $(x^\circ, y^{\circ'}; u^\circ, v^{\circ'})$ described by C_o lies on a bicharacteristic associated with P emanating from

$$(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'}).$$

Secondly we discuss C'_o. Then we find by Proposition 3 that a point $(x^\circ, y^{\circ'}; u^\circ, v^{\circ'})$ described by C'_o lies on a bicharacteristic associated with $u_1 - \lambda^-(x, u')$ emanating from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^-(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$.

As for C_1 and C'_1 $(=C_1)$, we apply Proposition 5 to conclude that each point $(x^\circ, y^{\circ'}; u^\circ, v^{\circ'})$ in C_1 lies on a bicharacteristic associated with $u_1 - \lambda^-(x, u')$ emanating from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, -v^{\circ'})$.

As for C_2 and C'_2, Proposition 6 implies that a point $(x^\circ, y^{\circ'}; u^\circ, v^{\circ'})$ described by C_2 or C'_2 lies on a time-like bicharacteristic associated with P emanating from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0; y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ with $\lambda^+(0, y^{\circ'}, -v^{\circ'}) = \lambda^-(0, y^{\circ'}, -v^{\circ'})$.

Lastly we recall the results of Kawai [2] and find that a point $(x^\circ, y^{\circ'}; u^\circ, v^{\circ'})$ described by C_l $(l = 3, \ldots, m+1, \text{resp.})$ lies on a bicharacteristic associated with P emanating from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda_l(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ $(l = 3, \ldots, m+1, \text{resp.})$, if we conventionally define $\lambda_{m+1}(x, u')$ by $\lambda^-(x, u')$.

Summing up the results obtained so far, we conclude that

$E_{m-1}(x, y')$ is a well-defined hyperfunction on ω and, if $(x^\circ, y^{\circ'}$; $\sqrt{-1}(u^\circ, v^{\circ'})\infty)$ is contained in its singualrity spectrum, then $(x^\circ, y^{\circ'}$; $u^\circ, v^{\circ'})$ lies in the union of bicharacteristics emanating from either $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ or $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^-(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ or $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda_l(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ for some $l = 3, \ldots, m$ and a time-like bicharacteristic emanating from $(x_1, x', y'; u_1, u', v') = (0, y^{\circ'}, y^{\circ'}; \lambda^+(0, y^{\circ'}, -v^{\circ'}), -v^{\circ'}, v^{\circ'})$ with $\lambda^+(0, y^{\circ'}, -v^{\circ'}) = \lambda^-(0, y^{\circ'}, -v^{\circ'})$.

This completes the proof of Theorem 1.

Theorem 2. *Let $P(x, D_x)$ be a linear differential operator that satisfies the same conditions as in Theoerm 1. Then there exists an open neighborhood ω_o of the origin of $\mathbf{R}_{x'}^{n-1}$ that satisfies following condition:*

For any m-tuple of hyperfunctions $(\mu_o(x'), \ldots, \mu_{m-1}(x'))$ defined on ω_o, we can find a solution $f(x)$ of the following Cauchy problem (60).

$$(60) \quad \begin{cases} P(x, D_x)f(x) = 0, \\ \dfrac{\partial^k}{\partial x_1^k}f(x)\,|_{x_1=0} = \mu_k(x') \end{cases} \qquad (k = 0, \ldots, m-1).$$

Furthermore, the singularity spectrum of $f(x)$ considered in a sufficiently small neighborhood Ω of the origin of $\mathbf{R}_x^n$ is confined to the union of bicharacteristics associated with P emanating either from $(x_1, x', y'; u_1, u', v') = (0, y', y'; \lambda^+(0, y', \xi'), \xi', -\xi')$ or from $(x_1, x', y'; u_1, u', v') = (0, y', y'; \lambda^-(0, y', \xi'), \xi', -\xi')$ or from $(x_1, x', y'; u_1, u', v') = (0, y', y'; \lambda_l(0, y', \xi'), \xi', -\xi')$ $(l = 3, \ldots, m)$ and time-like bicharacteristic associated with P emanating from $(x_1, x', y'; u_1, u', v') = (0, y', y'; \lambda^+(0, y', \xi'), \xi', -\xi')$ with $\lambda^+(0, y', \xi') = \lambda^-(0, y', \xi')$, where $(y', \sqrt{-1}\,\xi'\infty)$ belongs to the singularity spectrum of $\mu_k(x')$ for some k.

Proof. First choose ω_o sufficiently small so that $\{x_1=0\} \times \omega_o \times \omega_o$ is contained in the domain of the definition of $E_k(x, y')$ constructed in Theorem 1. Since the sheaf $\mathscr{B}$ of hyperfunctions is flabby, we

can find hyperfunctions $\tilde{\mu}_k(x')$ defined on $\boldsymbol{R}^{n-1}$ and supported by $\bar{\omega}_o$ so that it coincides with $\mu_k(x')$ on ω_o. Then, by making use of the regularity property of $E_k(x, y')$ proved in Theorem 1 and Corollary 2. 4. 2 and Theorem 2. 3. 1 of S-K-K [1] Chapter I, we find

$$f(x) = \sum_{k=0}^{m-1} \int E_k(x, y') \tilde{\mu}_k(y') dy'$$

is a well-defined hyperfunction defined on a neighborhood of the origin of $\boldsymbol{R}^n$. It is obvious that $f(x)$ satisfies (60) and that its singularity spectrum is confined to the union of the bicharacteristic associated with P which emanates from $(x_1, x'; u_1, u') = (0, y'; \lambda, \xi')$ with $\lambda = \lambda^+(0, y', \xi')$ or $\lambda^-(0, y', \xi')$ or $\lambda_l(0, y', \xi')$ $(l = 3, \dots, m)$, and a time-like bicharacteristic associated with P emanating from $(x_1, x'; u_1, u') = (0, y', \lambda^+(0, y', \xi'), \xi')$ with $\lambda^+(0, y', \xi') = \lambda^-(0, y', \xi')$, where $(y', \sqrt{-1}\,\xi'\infty)$ belongs to the singularity spectrum of $\tilde{\mu}_k(y')$ for some k. Then, choosing Ω sufficiently small, $f(x)$ enjoys the required singularity structure in Ω. See Kawai [3] Remark 2 in p. 646 for more precise information on the shape of Ω.

<div align="right">Q. E. D.</div>

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