Classifying Subgroups of Solvable Groups

By

J. F. McClendon*

Let G and K be groups (finite or infinite). Two very general questions that might reasonably be asked are (1) does G contain an isomorphic copy of K? (2) can the isomorphic copies of K in G be classified or (in the finite case) counted? The present paper can be viewed as directed toward the second question although the first is touched on lightly. Actually it will be seen that most results concern subgroups that are isomorphic to K in a certain way. Also the answer is in terms of one dimensional cohomology. So it is perhaps accurate to say that the present paper studies relationships between the subgroup structure of a group and its low dimensional cohomology.

One result which can be stated without too much terminology is the following (Corollary 3.7). Let $G=G_0 \supset G_1 \cup .. \supset G_s=1$ be a solvable series with $A_i=G_{i-1}/G_i$. $\{2,..,s\} \supset I$ and $B_i=1$, $i \in I$, $B_i=A_i$, $i \notin I$. Suppose that if $i \in I$ and $G/G_{i-1} \supset W$ and W covers $B_1,.., B_{i-1}$ then $H^j(W, A_i)=0$, $j=1, 2$. Then (1) G has a subgroup K covering $B_1,.., B_s$ and any two such are conjugate. (2) all such$\leftrightarrow \Pi_{i \in I}(A_i/A_i^{K/K_{i-1}})$. This generalizes the classical theorem of P. Hall and the exact relationship to Hall's theorem is discussed in Section 3. A more typical result is Theorem 2.4 whose conclusion reads

$$S((K, \bar{K}) \subset G, E, B_0, M) = \cup H^1(J, \bar{J}, u^*A)/H^1(J, \bar{J}, u^*tB).$$

The left hand side is a certain set of subgroups of G isomorphic to K.

The paper is organized as follows. In Section 1 sets of subgroups

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of an abelian extension are studied which are defined by given maps. In Section 2 it is shown that sometimes the dependence on maps can be dropped. In Section 3 the results of Sections (1) and (2) on abelian extensions are spliced to get results on solvable groups. The results of this section are of a splitting nature. That is, sufficiently strong hypotheses are used so that one can count subgroups simply by counting the number produced at each stage and multiplying the numbers. I hope to study some non-splitting situations at another time. In an appendix some results on relative cohomology and derivations are recalled.

§1. Subgroups of abelian extensions which extend
given maps.

Let K be a group and A a K-module. Recall that a derivation $\alpha : K \to A$ is a function such that $\alpha(xy) = \alpha(x) + x\alpha(y)$. Sometimes A will be written multiplicatively and the condition will be written $\alpha(xy) = \alpha(x)(x \cdot \alpha(y))$. If α and β are derivations then their sum is defined by $(\alpha + \beta)(x) = \alpha(x) + \beta(x)$. It is easily checked that the set $\mathrm{Der}(K, A)$ is an abelian group under this operation. If $\bar{K}$ is a subgroup of K let $\mathrm{Der}(K, \bar{K}, A)$ be the subset of $\mathrm{Der}(K, A)$ satisfying $\alpha(x) = 0$ for $x \in \bar{K}$. It is a subgroup of $\mathrm{Der}(K, A)$. Let $\mathrm{In}(K, A)$ be the group of inner derivations, that is, derivations of the form α_a where $a \in A$, $\alpha_a(x) = a - xa$ and $\mathrm{In}(K, \bar{K}, A)$ the subgroup of inner derivations trivial over $\bar{K}$. Note that if $\alpha \in \mathrm{Der}(K, A)$ then $\alpha(1) = 0$ so $\mathrm{Der}(K, A) = \mathrm{Der}(K, 1, A)$. If B is a normal subgroup of K such that $B \subset \bar{K}$ acting trivially on A then $\mathrm{Der}(K, \bar{K}, A) = \mathrm{Der}(K/B, \bar{K}/B, A)$. In particlar, if $\bar{K}$ is normal and acts trivially on A we have $\mathrm{Der}(K, \bar{K}, A) = \mathrm{Der}(K/\bar{K}, 1, A) = \mathrm{Der}(K/\bar{K}, A)$. In this case relative derivations are more a convenience than a necessity since we can reduce to the absolute case without even changing coefficients.

On the other hand, for any subgroup $\bar{K}$, $\mathrm{Der}(K, \bar{K}, A)$ can be viewed as the one dimensional cohomology group $H^1(K, \bar{K}; A)$ where the latter is defined by Takasu [7]. Namely, we set $I(K, \bar{K}, Z) =$

$\mathrm{Ker}(Z(K) \underset{K}{\otimes} Z \overset{\varepsilon}{\to} Z)$ where $\varepsilon(\sum n_i x_i) = \sum n_i$, and $H^q(K, \bar{K}; A) = $ $\mathrm{Ext}^{q-1}(I(K, \bar{K}, Z), A)$ for $q \geq 1$, and we have an exact sequence

$$0 \to A^K \to A^{\bar{K}} \overset{\delta}{\to} H^1(K, \bar{K}; A) \to H^1(K; A) \to H^1(\bar{K}; A) \overset{\delta}{\to}$$
$$H^2(K, \bar{K}; A) \to \dots$$

where $H^1(K, \bar{K}; A) = \mathrm{Der}(K, \bar{K}, A)$. Here we note that $H^1(K; A) \simeq$ $\mathrm{Der}(K, A)/\mathrm{In}(K, A)$ which differs from $\mathrm{Der}(K, A)$ itself, and this is the point where relative cohomology is necessary to study the set of derivations. See [1] also, but note the different indexing.

In this section we will study the following situation.

$$E: \qquad \begin{array}{ccccccc} 1 & \to & A & \overset{i}{\to} & G & \overset{p}{\to} & D & \to & 1 \\ & & \cup & & \cup & & \cup \\ 1 & \to & B & \to & K & \to & J & \to & 1 \end{array} \qquad B \subset \bar{K} \subset K$$

Here E is an abelian group extension (the sequence is exact and A is abelian). K is a subgroup of G and $B = B(K) = K \cap A$, $J = J(K) = p(K)$. $\bar{K}$ is a subgroup of K containing B and need not be normal in K and $\bar{J} = p\bar{K}$. A gets a left J-module structure by $(xB)a = xax^{-1}$ for $x \in K$. B is a sub J-module. The map $K \to J$ gives both K-module structures. In particular, A has a natural D-module structure.

1.1 Definition. (a) Suppose K and L are subgroups of G. A homomorphism $\phi: K \to L$ is an E-homomorphism if $\phi(K \cap A) \subset L \cap A$. It is an E-isomorphism if it has an inverse which is also an E-homomorphism. It is an E-monomorphism if it is an E-isomorphism onto its image.

(b) $S(K \subset G, E) =$ all subgroups of G which are E-isomorphic to K.

(c) Let $t: \bar{K} \to G$, $u: J \to D$ be given monomorphisms. Define $S(K \subset G, E, t/u)$ to be all L in $S(K \subset G, E)$ such that there is some E-monomorphism $\phi: K \to G$ with $\phi = t$ on $\bar{K}$, $p\phi = up$, and $\phi K = L$.

(d) Let $u: J \to D$. Then u^*A is A with the J-module structure from u. Note that if $p\phi = up$ then in u^*A we have $x \cdot a = \phi(x) a \; \phi(x)^{-1}$: Note also that (J, u^*A) and $(u(J), A)$ are isomorphic in the sense of change of groups.

Monomorphisms $t: (\bar{K}, B) \to (G, A)$, $u: J \to D$ are *compatible* if (1) t induces u on $\bar{J}$ and (2) $t: B \to A$ is a homomorphism of

J-modules $B \to u^*A$. Note that $S(K \subset G,\ E,\ t/u) \neq \phi \Rightarrow t,\ u$ are compatible.

Let $L,\ L' \in S = S(K \subset G,\ E,\ t/u)$ and $\phi : K \to G$, $\phi' : K \to G$ the E-monomorphisms from (c) above. Define $d = d(L,\ L') : K \to G$ by $d = \phi'\phi^{-1}$.

1.2 Lemma. $d(L,\ L') \in \mathrm{Der}(K,\ \bar{K},\ u^*A) = \mathrm{Der}(J,\ \bar{J},\ u^*A)$.

Proof. $p\phi = up = p\phi'$ so $d(K) \subset A$ and $\phi = \phi'$ on $\bar{K}$ so $d(\bar{K}) = 1$. Also $d(xy) = \phi'(xy)\phi(xy)^{-1} = \phi'(x)d(y)\phi(x)^{-1} = \phi'(x)\phi(x)^{-1}(\phi(x)d(y)$ $\phi(x)^{-1}) = d(x)(x \cdot d(y))$ so d is a derivation to u^*A.

1.3 Theorem. *Suppose* $L \in S = S(K \subset G,\ E,\ t/u)$. *Define*
$$\Delta = \Delta_L : S \to \mathrm{Der}(J,\ \bar{J},\ u^*A)/\mathrm{Der}(J,\ \bar{J},\ u^*t(B))$$
by $\Delta(L') = d(L',\ L)$. *Then* Δ *is a bijection.*

Proof. Let M be the set of all E-monomorphisms $\phi : K \to G$ such that $p\phi = up$. Define an action of $\mathrm{Der}(K,\ \bar{K},\ u^*A)$ on M by $(\alpha\phi)(x) = \alpha(x)\phi(x)$. Let $\alpha\phi = \phi'$. Then ϕ' is a homomorphism since $\phi'(xy) = \alpha\phi(xy) = \alpha(x)(x \cdot \alpha(y))\ \phi(x)\phi(y) = \alpha(x)\phi(x)\ \alpha(y)\phi(x)^{-1}\phi(x)$ $\phi(y) = \phi'(x)\ \phi'(y)$. Now $x \in B \Rightarrow \alpha(x) = 1$ so $\phi' = \phi$ on B and so ϕ' monic on B and gives a homomorphism $\bar{\phi} : J \to D$. It is easily checked that $\bar{\phi} = u$ and so $\bar{\phi}$ is monic. Now let $M \supset N =$ those momomorphisms which restrict to t on $\bar{K}$. $\mathrm{Der}(K,\ \bar{K},\ u^*A)$ acts on N since $\phi' = \phi$ on $\bar{K}$. The action is transitive since if $\phi_1,\ \phi_2 \in N$ then $d(\phi_2,\ \phi_1)\phi_1 = \phi_2$ and by the lemma d is in $\mathrm{Der}(K,\ \bar{K},\ u^*A)$. The action is clearly faithful.

There is a surjection $N \to S$, namely, $\phi \to \phi(K) \subset G$. In order to prove the theorem we must show exactly that:
$$\alpha\phi(K) = \alpha'\phi(K) \Leftrightarrow \alpha'\alpha^{-1} \in \mathrm{Der}(K,\ \bar{K},\ u^*t(B))$$
for some (and hence any) $\phi \in N$.

First let $\beta = \alpha'\alpha^{-1} \in \mathrm{Der}(K,\ \bar{K},\ u^*t(B))$. Then $\alpha\phi(x) = \beta(x)^{-1}$ $\alpha'(x)\phi(x) \in t(B)(\alpha'\phi)(K)$. But $\alpha'\phi|B = \phi|B = t|B$ so $t(B) \subset \alpha'\phi(B)$

and hence $\alpha\phi(K) \subset \alpha'\phi(K)$ and vice-versa, proving half the claim.

Now suppose $\alpha\phi(K) = \alpha'\phi(K)$. Then $\alpha'\phi = (\alpha\phi) \circ \nu$ where $\nu : K \to K$ is an E-automorphism of K giving the identity on $\bar{K}$ and J. Now define $\mu(x) = \nu(x)x^{-1}$ and check that $\mu \in \mathrm{Der}(K, \bar{K}, B)$. [Check that $\nu(K)$, $1_K(K) \in S(K \subset G, E, 1_K/1_J K)$ and the lemma gives $\mu = d(\nu, 1) \in \mathrm{Der}(K, \bar{K}, B)$]. Note that $(\alpha\phi) \circ \nu = (\phi \circ \mu)\alpha\phi$. The reason is that $(\alpha\phi)(\nu(x)) = \alpha(\mu(x)x)\phi(\mu(x)x) = \alpha(\mu(x))\phi(\mu(x))\alpha(x)\phi(\mu(x))^{-1}$ $\phi(\mu(x))\phi(x) = \phi(\mu(x))\alpha(x)\phi(x)$ [since $\mu(x) \in B \Rightarrow \alpha\mu(x) = 1$] $= \{((\phi \circ \mu)\alpha)\phi\}(x)$. We see that $\alpha'\phi = (\alpha\phi) \circ \nu = (\phi \circ \mu)\alpha\phi$. Check that $\phi \circ \mu \in \mathrm{Der}(K, \bar{K}, u^*t(B))$ and $\alpha'\alpha'^{-1} = \phi\mu$. This proves the theorem.

Note that it follows from the theorem that $d(L', L)$ is a well defined element of $\mathrm{Der}(J, \bar{J}, u^*A)/\mathrm{Der}(J, \bar{J}, u^*tB)$ and depends only on L and L'. Also $d(L, L') = 0 \Leftrightarrow L = L'$.

1.4 Corollary. *Suppose $\bar{K}$ normal in K and $L \in S = S(K \subset G, E, t/u)$. Then*

$$S \leftrightarrow \mathrm{Der}(J/\bar{J}, u^*A^{\bar{K}})/\mathrm{Der}(J/\bar{J}, u^*tB^{\bar{K}})$$

Derivations can always be viewed as absolute one dimensional cocycles, and they can also be viewed as one dimensional relative cohomology classes and this leads to the following corollary.

1.5 Corollary. 1) $\Delta_L : S \to H^1(J, \bar{J}, u^*A)/H^1(J, \bar{J}, u^*tB) = H^1(K, \bar{K}, u^*A)/H^1(K, \bar{K}, u^*tB)$ *is a bijection.*

2) *If $\bar{K}$ is normal in K then*
$\Delta_L : S \to H^1(J/\bar{J}, 1, u^*A^{\bar{K}})/H^1(J/\bar{J}, 1, u^*tB^{\bar{K}})$ *is a bijection.*

Notice that if we only wanted part 2 we could have used absolute derivations only in Theorem 1.3. However, even in this case relative derivations seem more natural and are required for the cohomology statement anyway-since $(J/\bar{J}, 1)$ can't be changed to $J/\bar{J}$ without changing coefficients. The following simple (simplest) example shows this. Let $G = S_3 =$ symmetric group on 3 symbols and let $A \to G \to D$ be $A \to S_3 \to C_2$ where A is the cyclic subgroup of order 3 and in

general C_n is the cyclic group with n elements. Let $B \to K \to J$ be $1 \to K \to K$ where K is the subgroup $\{1, (12)\}$ of order 2. Let $\bar{K} = B = 1$ and $u : J \to D$ be the isomorphism. Then we have $H^1(K/\bar{K}, 1, u^*A)/H^1(K/\bar{K}, 1, u^*tB) = H^1(C_2, 1, C_3) = C_3$. It is easily seen directly that $S(K \subset G, E, t/u)$ does indeed have three elements. However, $H(K/\bar{K}, u^*A)/H^1(K/\bar{K}, u^*t(1)) = H^1(C_2; C_3) = \{0\}$.

1.6 Corollary. $S \leftrightarrow \operatorname{Ker} \delta : H^1(J, \bar{J}, u^*(A/tB)) \to H^2(J, \bar{J}, u^*tB)$

Proof. $0 \to B \to A \to A/B \to 0$ gives an exact sequence

$$0 \to H^1(J, \bar{J}, B) \to H^1(J, \bar{J}, A) \to H^1(J, \bar{J}, A/B) \to H^2(J, \bar{J}, B) \to \ldots$$

(where A is u^*A etc.) and the corollary follows.

1.7 Corollary. *Assume* $\operatorname{Ker} \delta : H^1(\bar{J}, u^*tB) \to H^2(J, \bar{J}, u^*tB)$ *is 0 and* $H^1(J, u^*tB) \to H^1(J, u^*A)$ *is isomorphic. Then*

$$S \leftrightarrow u^*A^J/[u^*A^J + u^*tB^J]$$

Proof. The hypotheses and the pair $(J, \bar{J})$ give the following commutative diagram with exact rows. Here A is u^*A and B *is* u^*tB.

$$\begin{array}{ccccccc}
0 \to & A^J & \to & A^J & \to H^1(J, \bar{J}, A) & \to H^1(J, A) & \\
& \uparrow & & \uparrow & \uparrow & \uparrow \cong & \\
0 \to & B^J & \to & B^J & \to H^1(J, \bar{J}, B) & \to H^1(J, B) & \to 0
\end{array}$$

The result now follows from a diagram argument.

1.8 Corollary. (1) *Suppose* $\bar{K} = B$ *and* $H^1(J; u^*tB) \to H^1(J; u^*A)$ *isomophic. Then*

$$S \leftrightarrow u^*A/[u^*A^J + u^*tB]$$

(2) *Suppose* $\bar{K} = B = 1$ *and* $H^1(J; u^*A) = 0$. *Then* $S \leftrightarrow u^*A/u^*A^K$.

Proof. Since $\bar{J} = 1$ these results follow from corollary 1.7.

Note that the S_3 example mentioned above illustrates part (2)

since $H^1(C_2 ; u^*C_3) = 0$ and $S \leftrightarrow A/A^K = A = C_3$.

In general a conjugate of an element of $S(K \subset G, E)$ is another element of $S(K \subset G, E)$.

Let $A' = A \cap$ centralizer of t $\bar{K}$. Then $S(K \subset G, E, t/u)$ is closed under conjugation by elements of A'.

1.9 Theorem. *Let $L \in S(K \subset G, E, t/u)$ and $\Delta = \Delta_L$. Then*

$$S \xleftarrow{\hspace{1cm} \Delta \hspace{1cm}} \text{Der}(J, \bar{J}, u^*A)/\text{Der}(J, \bar{J}, u^*tB)$$
$$\cup \hspace{5.5cm} \cup$$
$$A'\text{-conjugates of } L \longleftrightarrow \text{In}(J, \bar{J}, u^*A)/\text{In}(J, \bar{J}, u^*A) \cap \text{Der}(J, \bar{J}, u^*B)$$

Proof. Let $\Gamma = \Delta^{-1}$. It will suffice to show that $\Gamma\langle\alpha\rangle$ is conjugate to L iff $\alpha' \in \text{In}(J, \bar{J}, u^*A)$ for some $\alpha' \in \langle\alpha\rangle$. Let $L = \phi(K)$ where $\phi \in N$ as in the proof of Theorem 1. 3. First suppose $\alpha = \alpha^a \in \text{In}(J, \bar{J}, u^*A)$. Then $(\alpha\phi)(x) = \alpha(x)\phi(x) = a(x^{\cdot}a)^{-1}\phi(x) = a[\phi(x) a\phi(x)^{-1}]^{-1}\phi(x) = a\phi(x)a^{-1}\phi(x)^{-1}\phi(x) = a\phi(x)a^{-1}$, so $(\alpha_a\phi)(K) = c_a(\phi(K))$ and $\Gamma\langle\alpha\rangle$ is an A-conjugate of L. Conversely, $\Gamma\langle\alpha\rangle = L' = aLa^{-1}$ implies $\alpha\phi(K) = (c_a \circ \phi)(K) = (\alpha_a\phi)(K)$ and so $\langle\alpha\rangle = \langle\alpha_a\rangle$.

1.10 Corollary. Assume $H^1(J, \bar{J}, u^*A) = \delta u^*A^J + i_*H^1(J, \bar{J}, u^*tB)$. *Then all elements of $S(K \subset G, E, t/u)$ are conjugate by elements of A'.*

Proof. Here, $u^*A^J = H^\circ(\bar{J}; u^*A) \xrightarrow{\delta} H^1(J, \bar{J}; u^*A)$ is part of the exact sequence of $(J, \bar{J})$. The corollary follows from 1.9 since $\text{In}(J, \bar{J}, A)/[\text{In}(J, \bar{J}, A) \cap \text{Der}(J, \bar{J}, B)] = [\text{In}(J, \bar{J}, A) + \text{Der}(J, \bar{J}, B)]/\text{Der}(J, \bar{J}, B)$ and by the assumptions that the last is $\text{Der}(J, \bar{J}, A)/\text{Der}(J, \bar{J}, B) = S$.

§2. Subgroups of abelian extensions which extend given subgroups

The notation of Section 1 will be used here.

2. 1 Definition. Let $G \supset M \supset B_0$, $D \supset J_0$. Define $S((K, \bar{K}) \subset G, E, B_0/J_0, M)$ to be all L in $S(K \subset G, E)$ such that there is an

E-monomorphism $\phi: K \rightarrow G$ with $\phi B = B_0$, $\phi(K) = L$ $\phi J = J_0$, $\phi \bar{K} = M$. Define $S((K, \bar{K}) \subset G, E, B_0, M)$ by requiring only $\phi B = B_0$ and $\phi(\bar{K}) = M$. If $B = \bar{K}$ drop M and $\bar{K}$ from the notation.

Suppose $t: \bar{K} \rightarrow G$, $\mu: J \rightarrow D$ are given monomorphisms with $t(\bar{K}) = M$, $t(B) = B_0$, $\mu(J) = J_0$. Then we have

$$S = S(K \subset G, E, t/\mu) \subset S' = S((K, \bar{K}) \subset G, E, B_0/J_0, M)$$

2.2 Definition. $(K, \bar{K}, B)$ is a (t/u)-triple if $S = S'$.

We note here the following basic examples of (t/u)-triples.

(1) $J = 1$. Then (B, B, B) is a $(t/1)$-triple.

(2) S non-empty, $B = A$. Then (K, A, A) is a (id/u)-triple.

(3) $B = 1$. Then $(K, 1, 1)$ is a $(1/u)$-triple.

(1) and (2) are clear since in each case S and S' have exactly one element. (3) can be checked directly without much trouble, however, it follows from the theorem below.

Let $\text{Aut}(K, \bar{K}, B)$ be the group of automorphisms of K sending $\bar{K}$ to itself and B to itself (similarly for $\text{Aut}(K, \bar{K})$ etc.). Let $\delta \in \text{Aut}(\bar{K}, B)$ and $\varepsilon \in \text{Aut}(J, \bar{J})$. Say that δ and ε are *compatible* if (1) δ induces ε on $\bar{J}$ and (2) δ is an ε-homomorphism of the J-module B (i.e., $\delta(x \cdot b) = \varepsilon(x) \cdot \delta(b)$ for $x \in J$, $b \in B$). Note that every $\alpha \in \text{Aut}(K, \bar{K}, B)$ gives rise to a compatible pair (δ, ε) where δ is the restriction of α and ε is induced by α. Call $(K, \bar{K}, B)$ a *weak triple* of every compatible pair comes from some $\alpha \in \text{Aut}(K, \bar{K}, B)$.

2.3 Theorem. *If $(K, \bar{K}, B)$ is a weak triple then $(K, \bar{K}, B)$ is a (t/u)-triple for any compatible (t, u).*

Proof. Let $\phi(K) \in S'$ where $\phi: (K, \bar{K}, B) \rightarrow (G, M, B_0)$ is an E-monomorphism. Let $\phi|\bar{K} = s$ and let ϕ induce $v: J \rightarrow J_0$. Define $\varepsilon = v^{-1}u: J \rightarrow J$. Since $u, v: (J, \bar{J}) \rightarrow (J_0, \bar{J}_0)$ we have $\varepsilon \in \text{Aut}(J, \bar{J})$. Define $\delta: \bar{K} \rightarrow \bar{K}$ by $\delta = s^{-1}t$ so $\delta \in \text{Aut}(\bar{K}, B)$. It is easily checked that δ and ε are compatible. Let $\alpha \in \text{Aut}(K, \bar{K}, B)$ with $\alpha|\bar{K} = \delta$

and α inducing ε. Define $\phi' = \phi\alpha : (K, \bar{K}, B) \to (G, M, B_0)$. Then $\phi' | \bar{K} = t$ and ϕ' gives u on J. Hence $\phi(K) = \phi'(K) \in S$ proving the theorem.

Example. Let $\bar{J} \subset J \xrightarrow{\theta} \text{Aut } B$ and $K = B \times_\theta J$, $\bar{K} = B \times_\theta \bar{J}$. If δ and ε are compatible it follows that $\delta(\theta(x)(b)) = \theta(\varepsilon(x))\delta(b)$ for $b \in B = B \times 1$ and $\delta x = \varepsilon x$ for $x \in \bar{J} = 1 \times \bar{J}$. Set $\alpha = \delta \times \varepsilon$. Then α is an Aut $(K, \bar{K}, B)$ and gives (δ, ε). Hence $(K, \bar{K}, B)$ is a weak triple.

2.4 Theorem. *For each* $J_0 \in S_1 = S(J \subset D)$ *choose an isomorphism* $u = u(J_0) : J \to J_0$. *Let* $t : \bar{K} \to M$ *be a fixed isomorphism. Suppose that* $H^2(J, \bar{J}, u^*A) = 0 = H^1(J, \bar{J}, \text{Hom}(B, u^*A))$ *for all choosen* u. *Suppose that* $(K, \bar{K}, B)$ *is a weak triple. Then*

$$S((K, \bar{K}) \subset G, E, B_0, M) = \cup_{J_0 \in S_1} H^1(J, \bar{J}, u^*A) / H^1(J, \bar{J}, u^*tB).$$

Proof. Note first that $S((K, \bar{K}) \subset G, E, B_0, M) = \underset{J_0 \in S'}{\cup} S((K, \bar{K}) \subset G, E, B_0/J_0, M)$ and because $(K, \bar{K}, B)$ is a weak triple this last is $\underset{J_0 \in S'}{\subset} S(K \subset G, E, t/u)$. We will be finished if Corollary 1.5 can be applied; for this we must show that $S(K \subset G, E, t/u)$ is non-empty for $u : J \to J_0$. We have

where $H = (up)^*G$ the pull-back of G along up. Thus a section of $H \to K$ extending t' will give the desired $\phi : K \to G$, making the first diagram commutative. Now define $ob(t') = [\mathscr{E}, t'(K)] \in H^2(K, \bar{K}, A)$. Here $\mathscr{E}$ is the sequence $A \to H \to K$ and $H^2(K, \bar{K}, A)$ is identified with equivalence classes $[\mathscr{E}, L]$ by [7, Theorem 4.1]. I claim that $ob(t') = 0$ iff t' has an extension to a section of $\mathscr{E}$. The reason is that by the proof in [7], $[\mathscr{E}, L] = 0$ iff there is an isomorphism $\Gamma : E \to K \times_\theta A$ (semi-direct product) such that $\Pi\Gamma = p$ and $\Gamma(x) = (p(x), 1) = i_1(x)$ for all $x \in L$. Now if $ob(t') = 0$ then it is easy to see that $\Gamma^{-1}i_1$ is the desired extension. Conversely, if s is such an extension then $\Gamma(e) = (p(e), e(sp(e))^{-1})$ will be the required isomorphism. To

complete the proof it will suffice to show that $H^2(K, \bar{K}, u^*A) = 0$. There is an exact sequence

$$0 \to H^2(J, \bar{J}, M) \to H^2(K, \bar{K}, M) \to H^1(J, \bar{J}, H^1(B, M)) \to \ldots$$

(see [8]) and using $M = u^*A$, $H^1(B, u^*A) = Hom(B, u^*A)$ and our hypotheses give $H^2(K, \bar{K}, A) = 0$.

It may be worthwhile to discuss a relatively concrete example. Suppose $A \to G \to D$ is a given abelian extension. Let S_3 be the symmetric group on three letters. We ask for the copies of S_3 that are in G. First of all let $S_1 = S(S_3 \subset D)$ and $\alpha \in H^2(D, A)$ the classifying element of the given extension. For $J_0 \in S_1$ let $\alpha | J_0 \in H^2(S_3, A)$ be the restriction of α. Let $T = \{J_0 \in S_1 | \alpha | J_0 = 0\}$. If $S_3 \cong L \subset G$ then we must have either $L \cap A = 1$ or $L \cap A \cong C_3$. The L's with $L \cap A = 1$ are the elements of $S(S_3 \subset G, E, 1/J_0)$ where $B \to K \to J$ is $1 \to S_3 \to S_3$. Now $(S_3, 1, 1)$ is a weak triple so that Theorem 2.3 applies. Corollary 1.5 gives $\cup_{J_0 \in T} H^1(S_3, 1, u^*A)$ for these L's, where $u = u(J_0) : S_3 \to D$ is a chosen monomorphism.

Now we consider L's with $L \cap A = C_3$. These are the elements of $S(S_3 \subset G, E, B_0/J_0)$ where $B \to K \to J$ is $C_3 \to S_3 \to C_2$. For each copy of C_2 in D pick a monomorphism $v : C_2 \to D$. For simplicity we make the following assumptions on the C_2-modules v^*A.

Assumption: $H^2(C_2, v^*A) = \mathrm{Hom}(C_3, v^*A)/\mathrm{Hom}_{C_2}((C_3, v^*A) = H^1(C_2, \mathrm{Hom}(C_3, v^*A)) = 0$.

The cohomology of cyclic groups is well known [see, e.g. 2] so these are easily checked in particular cases. From these assumptions we deduce easily that $H^2(C_2, 1, v^*A) = 0 = H^1(C_2, 1, \mathrm{Hom}(C_3, v^*A))$ and Theorem 2.4 can be applied. (S_3, C_3, C_3) is a weak triple because $S_3 = C_3 \times_\phi C_2$. The final result is that

$$S(S_3 \subset G) \leftrightarrow [\bigcup_{J_0 \in T} H^1(S_3, 1, u^*A)] \cup$$
$$\bigcup_{(M, N) \in S_2 \times S_2'} H^1(C_2, 1, v^*A)/H^1(C_2, 1, v^*sC_3)$$

where T is as above, $S_2 = S(C_2 \subset D)$, $S_2' = S(C_3 \subset A)$, $v = v(M) : C_2 \to D$, $s = s(N) : C_3 \to A$ are chosen monomorphisms. If we know that two copies of S_3 in D from T are isomorphic by an automorphism of

D and similarly for any two copies of C_2 in D then this simplifies to

$$S(S_3 \subset G) \leftrightarrow [T \times H^1(S_3, \ 1, \ A] \cup [S_2 \times S_2'$$
$$\times H^1(C_2, \ 1, \ A)/H^1(C_2, \ 1, \ C_3)]$$

In Definition 2.2 we could drop the hypotheses $\bar{K} \supset B$. But then if $S = S((K, \ \bar{K}) \subset G, \ E, \ B_0/J_0, \ M) \ni L$ we have $L \triangleright B_0$, $L \supset M$ so $M^1 = MB_0$ is a subgroup of L and $S = S((K, \ \bar{K}) \subset G, \ E, \ B_0/J_0, \ M)$ with $\bar{K} = \bar{K}B$, $M = MB_0$.

§ 3. Certain subgroups of solvable groups

Let G be a group (finite or infinite). Recall that a solvable series for G is a descending chain

$$G = G_0 \supset G_1 \supset \ldots \supset G_{s-1} \supset G_s = 1$$

with each G_i normal in G and each G_{i-1}/G_i abelian. It will be convenient to write $G_i = F_iG$ and speak of the solvable series F of G. The factors of the series are the groups $G_{i-1}/G_i = A_i(F)$, $i = 1, 2, \ldots, s$.

If K is a subgroup of G then it has induced filtration

$$K = K_0 \supset K_1 \supset \ldots \supset K_{s-1} \supset K_s = 1$$

where $K_i = K \cap G_i$. A_i gets a left K/K_{i-1} module structure by $(xK_{i-1})(yG_i) = (xyx^{-1})G_i$ for $x \in K$ and $y \in G_{i-1}$. Let $B_i = B_i(K) = K_{i-1}/K_i \subset A_i$. The i'th stage is then

$$
\begin{array}{ccccc}
G_{i-1}/G_i = A_i & \longrightarrow & G/G_i & \longrightarrow & G/G_{i-1} \\
\cup & & \cup & & \cup \\
K_{i-1}/K_i = B_i & \longrightarrow & K/K_i & \longrightarrow & K/K_{i-1}
\end{array}
$$

Similarly each quotient group of G has an induced filtration. When confusion is unlikely, all induced filtrations from F will be denoted by F also.

3.1 Definition. (a) Suppose K and L are subgroups of G. A homomorphism $\phi: K \rightarrow L$ is an F-homomorphism if $\phi(K_i) \subset L_i$, $i = 1, \ldots, s$. It is an F-isomorphism if it has an inverse which is also an F-homomorphism. It is an F-monomorphism if it is an F-isomorphism

onto its image.

(b) $S(K \subset G, F) =$ all subgroups of G which are F-isomorphic to K.

(c)   Let $K \supset \bar{K}$. Let $t: \bar{K} \to G$, $u_i: B_i \to A_i$ be given monomorphisms. Define $S(K \subset G, F, \{u_i\}, t)$ to be all $L \in S(K \subset G, F)$ such that there is an F-monomorphism $\phi: K \to G$ with $\phi_i = u_i: K_{i-1}/K_i \to G_{i-1}/G_i$ and $\phi \mid \bar{K} = t$. If $\bar{K} = 1$ drop it from the notation.

(d)   Let $J_i \subset A_i = A_i(F)$, $i = 1, 2, \ldots, s$, and $M \subset G$ be given subgroups. Define $S((K, \bar{K}) \subset G, F, \{J_i\}, M)$ to be all L in $S(K \subset G, F)$ such that there is an F-monomorphism $\phi: K \to G$, $\phi(B_i) = J_i$, $\phi(\bar{K}) = M$. Define $S((K, \bar{K}) \subset G, F, M)$ by requiring only $\phi(\bar{K}) = M$.

(e)   A subset of $S(K \subset G, F)$ is called a conjugate set if all its elements are conjugate. It is called a K-conjugate set if all its elements are conjugate to K.

For the next lemma we use the following notation.

$$S = S((K, K) \subset G, F, \{J_i\}, i = 1, \ldots, s, M)$$
$$S' = S((K/K_{s-1}), p\bar{K}) \subset G/G_{s-1}, F, \{J_i\}, i = 1, \ldots, s-1, pM)$$
$$S''(R) = S((K, \bar{K}) \subset G, E, J_s/R, M)$$

Here E is $1 \to G_{s-1} \to G \xrightarrow{p} G/G_{s-1} \to 1$. Let $\bar{J}_s = M \cap J_s$. Clarly we always have $S \subset \cup \{S''(R)/R \in S'\}$. We are interested in finding sufficient conditions for equality.

3. 2   Lemma. *Let $J_i = B_i$, $M = \bar{K}$ normal in G. B_s and $\bar{B}_s$ normal in G. Suppose S' is a K/K_{s-1}-conjugate set. Then $S''(R) \leftrightarrow S''(K/K_{s-1})$ by conjugation. If $S''(K/K_{s-1})$ is a conjugate set then so is each $S''(R)$ and S is a K-conjugate set. Furthermore $S = \cup S''(R)$.*

Proof. Let $c_x: (K/K_{s-1}, p\bar{K}) \to (R, pM)$ be an isomorphism where c_x is conjugation by $x \in G/G_{s-1}$. Let $x = yG_{s-1}$, and consider $c_y: S''(K/K_{s-1}) \to S''(R)$. Because the compositions of an E-isomorphism and c_y is an E isomorphism and because B_s and $\bar{B}_s$ are normal we see that c_y is well defined. The inverse is $c_{y^{-1}}$ so c_y gives the desired 1-1 correspondence. Now suppose $L \in S''(R)$ then L is a conjugate

of an element of $S''(K/K_{s-1})$ which is in turn a conjugate of K. Thus L is a conjugate of K and hence in S [The reason is that conjugation is an F-monomorphism. Specifically, if $G = G_0 \supset G_1 \supset \ldots \supset G_s = 1$, $G \rhd G_i$, $K_j = K \cap G_j$, $y \in G$, $c_y \colon K \to G$ conjugation by y, then c_y is an F-homomorphism and the equation $(c_y K)_j = c_y(K_j)$ shows that c_{y-1} is also an F-homomorphism which will serve as an inverese to c_y.] Proving $S = \cup S''(R)$ and that S is a conjugate set.

For the following theorem we suppose G, F, K, $\bar{K}$ given and take $J_i = B_i$, $M = \bar{K}$. Here $A_i = G_{i-1}/G_i$, $B_i = K_{i-1}/K_i$, $\bar{B}_i = \bar{K}_{i-1}/\bar{K}_i$. Suppose $\bar{K}$ normal in G and each B_i and $\bar{B}_i$ normal in G/G_i. Assume for $i = 2, \ldots, s$.

Hyp. 1) $S(K/K_i \subset G/G_i, B_i/(K/K_{i-1})) = \{K/K_i\}$ or Hyp. 2) $(K/K_i, (\bar{K}\ K_i/K_i, B_i)$ a weak triple and Ker $\delta \colon H^1(J_i, B_i) \to H^2(J_i, \bar{J}_i, B_i)$ is 0 and $H^1(J_i, B_i) \to H^1(J_i, A_i)$ is isomorphic where $J_i = K/K_{i-1}$, $\bar{J}_i = \bar{K}\ K_{i-1}/K_{i-1}$. Let $I = \{i \mid i = 2, \ldots, s,$ Hypothesis 2 is satisfied and Hypothesis 1 is not.$\}$

3. 3 Theorem. (1) $S = S(K \subset G, F, \{B_i\}, \bar{K})$ *is a conjugate set.*
(2) $S \leftrightarrow \prod_{i \in I} A_i^{J_i}/[A_i^{J_i} + B_i^{J_i}]$.

Proof. The proof is by induction on s. The result is trivially true for $s = 1$ [$S =$ single element, $A_1^{J_1}/A_1^{J_1} + B_1^{J_1} = A/A + B =$ single element] so we assume the result for filtrations of length $s-1$ and prove it for those of length s. We use the notation preceeding Lemma 3. 2. Note that the induction assumption applies to both S' and S''. First suppose Hypothesis 1 is satisfied. Then $S''(K/K_{s-1})$ is a conjugate set and Lemma 3. 2 shows $S = \cup S''(R) = S' \times S''(K/K_{s-1}) \leftrightarrow S'$ and S is a conjugate set. This proves the theorem in the case. Now suppose Hypothesis 2 is satisfied. We have

$$S(K \subset G, E, id/id) \to S''(K/K_{s-1})$$

and by our weak triple assumption and Theorem 2. 3 these sets are equal. The hypotheses of Corollary 1. 10 follow from those here so $S''(K/K_{s-1})$ is a conjugate set. Lemma 3. 2 now shows that S is a

conjugate set and $S \leftrightarrow S' \times S''(K/K_{s-1})$. Corollary 1.7 gives $S''(K/K_{s-1})$ $\leftrightarrow A^J/[A^J + B^J]$ and completes the proof of the theorem.

The theorems until now have assumed $K \subset G$. I now want to include an existence statement. The resulting hypotheses, are, unfortunately, inductive in nature. However it will sometimes be possible to check them all at once rather than at each step.

Suppose $p: E \rightarrow D$ is a group homomorphism and K a subgroup of E and B a subgroup of D. Say that K covers B if $pK = B$. Now suppose $G = G_0 \supset G_1 \supset \ldots \supset G_s = 1$ is a solvable filtration of G with $A_i = G_{i-1}/G_i$ and $A_i \supset B_i$. Say that a subgroup K of G covers B_i provided K_{i-1} covers B_i.

Let B_i be normal in G/G_i. If $G/G_{i-1} \supset W$ and W covers $B_1, \ldots B_{i-1}$ then assume $H^2(W, A_i/B_i) = 0$ for this W and also

either Hypothesis. 1) there can be at most one subgroup G/G_i covering B_i and W

or Hypothesis. 2) if W' covers W and B_i then (W', B_i) is a weak pair and $H^1(W, B_i) \rightarrow H^1(W, A_i)$ is isomorphic.

3.4 Theorem. *In the above sitting*

(1) *There is a subgroup K of G covering $B_1, \ldots, B_s$.*

(2) $S = S(K \subset G, F, \{B_i\})$ *is a conjugate set.*

(3) *Let $I = \{i \mid i = 2, \ldots, s$ such that Hypothesis 2 is satisfied and Hypothesis 1 is not$\}$. Then $S \leftrightarrow \prod_{i \in I} A_i/[A_i^{K/K_{i-1}} + B_i]$.*

Proof. We may assume the result for G/G_{s-1} so we have W_{s-1} covering $B_1, \ldots, B_{s-1}$. Consider the following diagram

$$
\begin{array}{ccccc}
A_s & \longrightarrow & G & \overset{p}{\longrightarrow} & G/G_{s-1} \\
\downarrow & & \downarrow q & & \downarrow \\
A_s/B_s & \longrightarrow & G/B_s & \longrightarrow & G/G_{s-1} \\
& & & & \cup \\
& & & & W_{s-1}
\end{array}
$$

The middle extension splits when pulled back over W_{s-1} and there results a monomorphism of W_{s-1} into G/B_s with image, say, W_s'.

$q^{-1}W'_s$ covers W_{s-1} and B_s. This proves (1) of the theorem. Parts (2) and (3) are now consequences of Theorem 3.3 with $\bar{K}=1$.

Theorem 3.4 and the corollaries that follow can be viewed as generalizations of P. Hall's theoerm [1]. The reader may wish to compare them to some other generalizations from a somewhat different viewpoint [5, 6].

3.5 Corollary. *In 3.4 (2) can be replaced by (2′) any two subgroups covering $B_1, \ldots, B_s$ are conjugate.*

Proof. In the proof of 3.4 we assume (2) and (2′) at the $(s-1)$-stage. Suppose $\bar{W}$ covers $B_1, \ldots, B_s$ and consider the diagram at the beginning of the proof of 3.4. Let $\bar{W}'=q\bar{W}$. $p'\bar{W}'$ covers $B_1, \ldots, B_{s-1}$ so it is in S'. p' is an isomorphism on $\bar{W}'$ and W' so $\bar{W}'$ is F-isomorphic to W' and so conjugate to it (by the theorem). Now the conjugation lifts to show $\bar{W}$ and W are conjugate.

Consider
$$H^2(G/G_{i-1}, A_s) \xrightarrow{q_*} H^2(G/G_{i-1}, A_i/B_i) \xrightarrow{rest} H^2(W, A_i/B_i) \text{ and let}$$
$E_i=(A_i \to G \to G/G_{i-1}) \in H^2(G/G_{i-1}, A_i)$. Then the proof shows the following corollary to be true.

3.6 Corollary. In 3.4 the hypothesis $H^2(W, A_i/B_i)=0$ can be replaced by rest $q_*(E_i)=0$.

Recall the following classical theorem of P. Hall [1]. Let G be a solvable group of order mn with $(m, n)=1$. Call H a Hall subgroup of G if $|H|=m$. Then any two Hall subgroups are conjugate. Let h be the number of Hall subgroups of G. Then h is the product of factors f_i such that (a) $f_i \equiv 1 \pmod{p_i}$ and p_i is a prime factor of m, and (b) f_i is a power of a prime and divides one of the chief factors of G. Now form a chief series for G, $G=G_0 \supset G_1 \supset \ldots \supset G_s=1$. The A_i are elementary abelian. If $|A_i|=q^f$ then $q \mid m$ or $q \mid n$. If $q \mid m$ set $B_i=A_i$ and if $q \mid n$ set $B_i=1$. If W covers $B_1, \ldots, B_s$ then $|W| \mid m$ and is relatively prime to $|A_i/B_i|$ and so $H^2(W, A_i/B_i)=0$.

If $B_i = A_i$ then Hypothesis 1 is satisfied. If $B_i = 1$ then (W', B_i) is a weak triple, $H^1(W, B_i) = 0 = H^1(W, A_i)$ since $|W|$ and $|A_i|$ are relatively prime. Corollary 3.5 gives the conjugacy of the Hall subgroups and that $h = |\prod_{i \in I} A_i / A_i^{K/K_{i-1}}|$ where I is the set of i with $B_i = 1$. Part (b) is clear and part (a) follows from induction and the following elementary counting fact: If A is an elementary q group of order q^n acted on by a p group P then $|A^P| = q^f \Rightarrow q^{n-f} \equiv 1 \pmod{p}$.

The above discussion includes a proof of the following.

3.7 Corollary. *Suppose G is a group and $G = G_0 \supset G_1 \supset \ldots \supset G_s = 1$ a solvable series with $A_i = G_{i-1}/G_i$. Let $\{2, \ldots, s\} \supset I$ and $B_i = 1$, $i \in I$, $B_i = A_i$ $i \notin I$.*

Suppose that if $i \in I$ and $G/G_{i-1} \supset W$ and W covers $B_1, \ldots, B_{i-1}$ then $H^j(W, A_i) = 0$, $j = 1, 2$.

Then (1) *G has a subgroup K covering $B_1, \ldots, B_s$ and any two such are conjugate* (2) *If $S = $ all subgroups covering $B_1, \ldots, B_s$ then $S \leftrightarrow \prod_{i \in I} A_i / A_i^{(K/K_{i-1})}$.*

Appendix

I will recall here a few facts about relative cohomology from [Takasu, 3] for the readers benefit. I will also prove the equivalence of the first cohomology group and the derivation group since this was not done explicitly by Takasu.

Let G be a group, H a subgroup, and A a G-module. $I(G, H, A)$ is the kernel of the natural epimorphism $Z(G) \otimes_H A \to A$, $g \otimes a \to ga$, and $J(G, H, A)$ is the cokernel of the natural monomorphism $A \to \mathrm{Hom}_H(Z(G), A)$, $a \to (g \to ga)$. $H^q(G, H, A)$ is defined to be $\mathrm{Ext}^{q-1}(I(G, H, Z), A)$ for $q \geq 1$. There is a long exact sequence for the pair (G, H),

$$\cdots \to H^q(G, H, A) \to H^q(G, A) \to H^q(H, A) \to H^{q+1}(H, A) \to \cdots$$

and also a long exact sequence for $0 \to A' \to A \to A'' \to 0$ (a short exact sequence of G-modules)

$$\cdots \to H^q(G, H, A') \to H^q(G, H, A) \to$$
$$H^q(G, H, A'') \to H^{q+1}(G, H, A') \to \cdots$$

Relative cohomology can be reduced to absolute cohomology by

$$H^q(G, H, A) = H^{q-1}(G, J(G, H, A)).$$

Let $G \supset H \supset N$ and $G \triangleright N$. There is a spectral sequence which gives in particular

$$H^1(G/N, H/N, A^N) \approx H^1(G, H, A)$$

and an exact sequence

$$0 \to H^2(G/N, H/N, A^N) \to H^2(G, H, A) \to$$
$$H^1(G/N, H/N, H^1(N, A)) \to H^3(G/N, H/N, A^N) \to$$
$$H^3(G, H, A).$$

Now write $I(G, H, Z) = I(G, H)$. The definition tells us that $H^1(G, H, A) = \mathrm{Hom}_G(I(G, H), A)$. Also we have (by definition)

$$0 \to I(G, H) \to Z(G) \otimes_H Z \xrightarrow{\varepsilon} Z \to 0$$
$$0 \to I(G) \to Z(G) \xrightarrow{\varepsilon_0} Z \to 0$$

where $\varepsilon_0(\sum n_i x_i) = \sum n_i$ and $\varepsilon(r \otimes m) = \varepsilon_0(r) m$. Define

$$\mathrm{Hom}'(ZG \otimes_H Z, A) = \{g \in \mathrm{Hom}(ZG \otimes_H Z, A) \mid g(rs \otimes 1)$$
$$= rg(s \otimes 1) + g(r \otimes 1)\varepsilon_0(s))\}$$

and check that it is a subgroup.

Theorem. $\mathrm{Der}(G, H, A) \cong \mathrm{Hom}'(ZG \otimes_H Z, A) \cong \mathrm{Hom}_G(I(G,H), A) = H^1(G, H, A)$

Proof. [cf. MacLane, 4, pp. 106-107]. Let $g \in \mathrm{Der}(G, H, A)$ and define $g' \in \mathrm{Hom}'$ by $g'(x \otimes 1) = g(x)$. It is a well defined homomorphism since $ZG \otimes_H Z$ is free abelian on $G \backslash H$. It is easily checked to be in Hom'. For $u \in \mathrm{Hom}$ define $\hat{u}$ in $\mathrm{Hom}_G(I(G, H), A)$ by $\hat{u} = u \mid I(G, H)$ and check that it is a well defined G-homomorphism. Now define $p : ZG \otimes_H Z \to I(G, H)$ by $p(r \otimes 1) = r \otimes 1 - \varepsilon(r)(1 \otimes 1)$. If $h \in \mathrm{Hom}_G$ then $h \to hp$ is a homomorphism to Hom'. $v \to v \mid G$ is a homomorphism $\mathrm{Hom}' \to \mathrm{Der}$. It is easy to check that these maps

give the desired isomorphisms.

The lemma reveals the relationship between the cohomology of Barr-Rinehart [1, 2] and that of Takasu [8]. It also suggests that Barr-Rinehart made an unfortunate (although defensible) choice of indexing. Recall that Barr-Rinehart defined $\bar{H}_\phi^n(G, A)$ via derivations where $\phi : G \to D$ is a group homomorphism. They noted that for $\phi = \text{identity}$, $\bar{H}_\phi^n(G, A) = H^{n+1}(G, A)$, $n \geqq 1$, where the last is group cohomology. Now let $I_\phi(G, H) = I(G,H) \otimes_{ZG} ZD$ and define $H_\phi^q(G,H,A) = \text{Ext}_G^{q-1}(I_\phi(G, H), A)$ for $q \geqq 1$ [cf. 2, p. 207]. If $\phi = id.$ then this is group cohomology. It is easy to see that $\bar{H}_\phi^n(G, A) = H_\phi^{n+1}(G, 1, A)$, $n \geqq 0$, and in particular $\bar{H}^n(G, A) = H^{n+1}(G, 1, A)$, $n \geqq 0$. The H indexing and relative notation seem best to me since they are consistent with topology and should make the relationship between results in algebraic and topological cohomology transparent.

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