Publ. RIMS, Kyoto Univ.
14 (1978), 615-641

Intrinsic Formula for Kuranishi's $\bar{\partial}_b^\varphi$

By

Takao AKAHORI*

Introduction

In 1973, M. Kuranishi constructed a versal family of isolated singularities through the consideration of $\bar{\partial}_b$ equations under a certain assumption. He showed that the deformation of an isolated singularity (V, x), where V is an analytic set in a domain in C^n with an isolated singularity x, corresponds to the deformation of partially complex structures of the real hypersurface $M = V \cap S_\varepsilon^{2n-1}$. But, his method is very complicated and some of his formulas can only be applied under the assumption of the existence of the "ambient space". In this paper the author will show that we can prove the Kuranishi's formulas on abstract partially complex manifold in a much simpler way. And we shall prove Kuranishi's formulas for holomorphic vector bundles on partially complex manifolds. The author wishes to express his hearty gratitude to Professors S. Nakano and M. Kuranishi for their constant encouragement and valuable suggestions during the preparation of this paper.

§ 1. Partially Complex Manifolds and Almost Partially Complex Manifolds

Let M be a differentiable manifold. By a partially complex structure on M, we mean a pair $(M, {}^\circ T'')$ of M and a subbundle ${}^\circ T''$ of $C \otimes TM$, where ${}^\circ T''$ satisfies the following conditions α. 1) and α. 2) :

Communicated by S. Nakano, May 21, 1977.
* Department of Mathematics Ryukyu University Naha, Okirawa, Japan.

$\alpha.1)$ $\circ T'' \cap \circ \bar{T}'' = 0$.

$\alpha.2)$ For any elements X, Y in $\Gamma(\circ T'')$, $[X, Y]$ is in $\Gamma(\circ T'')$.

Here $\Gamma(E)$, for a vector bundle E, denotes the set of differentiable cross sections of E, defined on an unspecified open set of M.

M with $\circ T''$ is also called a partially complex manifold. Let M be a partially complex manifold. Then there exists a following exact sequence of vector bundles.

$$0 \to \circ T'' \oplus \circ T'' \to C \otimes TM \to C \otimes TM / \circ \bar{T}'' \oplus \circ T'' \to 0. \tag{1.1}$$

Differentiably, the above sequence splits and the splitting commutes with the operation of conjugation. So there exists a C^∞ bundle isomorphism map

$$\mu: (\circ \bar{T}'' \oplus \circ T'') \oplus C \otimes TM / \circ \bar{T}'' \oplus \circ T'' \xrightarrow{\sim} C \otimes TM.$$

We shall fix the splitting $C \otimes TM = \circ \bar{T}'' \oplus \circ T'' \oplus F$, where $F = \mu(C \otimes TM / \circ \bar{T}'' \oplus \circ T'')$. F is invariant under the conjugation map of $C \otimes TM$. Especially, we denote the subbundle $\circ \bar{T}'' \oplus F$ of $C \otimes TM$ by $T'N$.

Next we shall define a differential operator $\bar{\partial}_{T'N}$ from $\Gamma(T'N)$ to $\Gamma(T'N \otimes (\circ T'')^*)$. For any element u in $\Gamma(T'N)$ we set

$$\bar{\partial}_{T'N} u(X) = [X, u]_{T'N}, \tag{1.2}$$

where by $[X, u]_{T'N}$, we denote the projection of $[X, u]$ to $T'N$ according to the above splitting $C \otimes TM = \circ T'' \oplus T'N$.

Then we get the following relations $\beta.1)$ and $\beta.2)$

$\beta.1)$ $X(fu) = (Xf)u + f \cdot Xu$,

$\beta.2)$ $[X, Y]u = X(Yu) - Y(Xu)$,

where u in $\Gamma(T'N)$, f being a C^∞ funtion on M, X, Y in $\Gamma(\circ T'')$ and we put $X \cdot u = \bar{\partial}_{T'N} u(X) = [X, u]_{T'N}$.

In fact, from the relation $[X, fu] = (Xf)u + f[X, u]$, we get the relation $[X, fu]_{T'N} = (Xf)u + f[X, u]_{T'N}$. And so we get the relation $\beta.1)$. From the relation $[[X, Y], u] = [X, [Y, u]] - [Y, [X, u]]$, we get the relation $[[X, Y], u]_{T'N} = [X, [Y, u]]_{T'N} - [Y, [X, u]]_{T'N} = [X, [Y, u]_{T'N}]_{T'N} - [Y, [X, u]_{T'N}]_{T'N}$ for any X, Y in $\Gamma(\circ T'')$ and u

in $\Gamma(T'N)$ because of the condition $\alpha.2)$.

From $\beta.1)$ and $\beta.2)$, we can define the operator $\bar{\partial}^{(p)}_{T'N}$ from $\Gamma(T'N\otimes\overset{p}{\wedge}({}^\circ T'')^*)$ to $\Gamma(T'N\otimes\overset{p+1}{\wedge}({}^\circ T'')^*)$ as follows: For any ψ in $\Gamma(T'N\otimes\overset{p}{\wedge}({}^\circ T'')^*)$, we set

$$(1.3)\qquad \bar{\partial}^{(p)}_{T'N}\psi(X_1,\dots,X_{p+1})=\sum_i(-1)^{i+1}X_i\cdot\psi(X_1,\dots,\check{X}_i,\dots,X_{p+1}) +\sum_{i<j}(-1)^{i+j}\psi([X_i,X_j],\dots,\check{X}_i,\dots,\check{X}_j,\dots,\check{X}_{p+1}),$$

where X_i's are in $\Gamma({}^\circ T'')$, and $\check{X}_i$ means to omit X_i. It is easily shown that (1.3) is multilinear in X's with functions as coefficients, we also obtain the following complex.

$$(1.4)\quad 0\longrightarrow\Gamma(T'N)\xrightarrow{\bar{\partial}_{T'N}}\Gamma(T'N\otimes({}^\circ T'')^*)\xrightarrow{\bar{\partial}^{(1)}_{T'N}}\Gamma(T'N\otimes\overset{2}{\wedge}({}^\circ T'')^*)\longrightarrow$$
$$\longrightarrow\Gamma(T'N\otimes\overset{p}{\wedge}({}^\circ T'')^*)\xrightarrow{\bar{\partial}^{(p)}_{T'N}}\Gamma(T'N\otimes\overset{p+1}{\wedge}({}^\circ T'')^*)\longrightarrow$$

Especially, we shall remark that

$$(\bar{\partial}^{(1)}_{T'N}\varphi)(X,Y)=[X,\varphi(Y)]_{T'N}-[Y,\varphi(X)]_{T'N}-\varphi([X,Y]).$$

Now, we shall define an almost partially complex structure at a finite distance from ${}^\circ T''$. (M, E'') is called an almost partially complex structure at a finite distance when the following conditions $\gamma.1)$ and $\gamma.2)$ are satisfied.

$\gamma.1)$ E'' is a subbundle of $C\otimes TM$ and if p denotes the projection of $C\otimes TM={}^\circ T''\oplus T'N$ to ${}^\circ T''$, then $p|E''$ is a C^∞-bundle isomorphisms.

$\gamma.2)$ $E''\cap\bar{E}''=0$.

An almost partially complex sturcture E'' at a finite distance defines an element φ in $\Gamma(X, T'N\otimes({}^\circ T'')^*)$ uniquely, so that $E''=\{X+\varphi(X) : X\in\Gamma({}^\circ T'')\}$. We shall look at this correspondence more closely. φ is divided into two parts: $\varphi=\varphi_1+\varphi_2$ where φ_1 is in $\Gamma(M, {}^\circ T''\otimes({}^\circ T'')^*)$ and φ_2 is in $\Gamma(M, F\otimes({}^\circ T'')^*)$. Using φ_1 and φ_2, we shall rewrite the relation $E''\cap\bar{E}''=0$. We claim that $E''\cap\bar{E}''=\{0\}$ if and only if for any element $Y\neq0$ in ${}^\circ T'''$, $\varphi=\varphi_1+\varphi_2$ satisefies the relations $\varphi_1(\overline{\varphi_1(Y)})\neq\bar{Y}$ or $\varphi_2(\overline{\varphi_1(Y)})\neq\varphi_2(\bar{Y})$.

In fact if there exists an element $Y\neq0$ in ${}^\circ T''$ which satisfies the relations $\varphi_1(\overline{\varphi_1(Y)})=\bar{Y}$ and $\varphi_2(\overline{\varphi_1(Y)})=\overline{\varphi_2(Y)}$, then putting

$X=\overline{\varphi_1(Y)}$ in $^\circ T''$, it satisefies the relation $X+\varphi_1(X)+\varphi_2(X)=\overline{Y+\varphi_1(Y)+\varphi_2(Y)}\neq 0$. Conversely, if there exists two elements X and Y in $^\circ T''$ which satisfy the relation $X+\varphi_1(X)+\varphi_2(X)=\overline{Y+\varphi_1(Y)+\varphi_2(Y)}\neq 0$, then we get the relations $X=\overline{\varphi_1(Y)}$, $\varphi_1(X)=\bar{Y}$ and $\varphi_2(X)=\overline{\varphi_2(Y)}$, by comparing the type of both sides according to the decomposition $C\otimes TM={}^\circ T''\oplus{}^\circ\bar{T}''\oplus F$ and taking $\bar{F}=F$ into account. Therefore we conclude the relation $\varphi_1(\overline{\varphi_1(Y)})=\bar{Y}$ and $\varphi_2(\overline{\varphi_1(Y)})=\overline{\varphi_2(Y)}$. Thus we have proved the following proposition.

Proposition 1.1. *An almost partially complex structure E'' at a finite distance from $^\circ T''$ corresponds to $\varphi=\varphi_1+\varphi_2$ in $\Gamma(M, T'N\otimes({}^\circ T'')^*)$ which satisfies the relation $\varphi_1(\overline{\varphi_1(Y)})\neq\bar{Y}$ or $\varphi_2(\overline{\varphi_1(Y)})\neq\overline{\varphi_2(Y)}$ for any element $Y\neq 0$ in $^\circ T''$ on every point on M. The following formula determines a bijective correspondence.*

$$^\varphi T''=\{X+\varphi(X):\ X\in{}^\circ T''\}$$

§2. The Differential Equation for Integrability

We shall study the necessary and sufficient condition for an almost partially complex structure $^\varphi T''$ to be a partially complex structure. By definition it is necessary and sufficient that for any X', Y' in $\Gamma(^\varphi T'')$, $[X', Y']$ belongs to $\Gamma(^\circ T'')$. We shall rewrite this condition. In the first place, we shall recall that $\Gamma(^\varphi T'')$ is the set of vector fields $X+\varphi(X)$, where X is in $\Gamma(^\circ T'')$. Therefore for any X, Y in $\Gamma(^\circ T'')$, there exists Z in $\Gamma(^\circ T'')$ such that $[X+\varphi(X), Y+\varphi(Y)]=Z+\varphi(Z)$ if and only if $^\varphi T''$ is a partially complex structure. And so we can rewrite this conditon as follows. $^\varphi T''$ is a partially complex structure if and only if for any element X, $Y\in\Gamma(^\circ T'')$, the following relation holds.

$$(2.1)\qquad [X+\varphi(X),\ Y+\varphi(Y)]=[X+\varphi(X),\ Y+\varphi(Y)]_{^\circ T''} +\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{^\circ T''}).$$

Thus we have obtained the following theorem. This is our intrinsic formulation of Kuranishi's integrability condition (Theorem 3.1 in [1]).

Theorem 2.1. *An almost partially complex structure* ${}^{\varphi}T''$ *is a partially complex structure if and only if for any element* X, Y *in* $\Gamma({}^{\circ}T'')$, *it satisfies the relation*

$$[X+\varphi(X),\ Y+\varphi(Y)]=[X+\varphi(X),\ Y+\varphi(Y)]_{{}^{\circ}T''} \\ +\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{{}^{\circ}T''}).$$

We set

$$P(\varphi)(X,\ Y)=[X+\varphi(X),\ Y+\varphi(Y)]-[X+\varphi(X),\ Y+\varphi(Y)]_{{}^{\circ}T''} \\ -\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{{}^{\circ}T''}),$$

then our condition becomes the following differential equation (2.2).

$$P(\varphi)(X,\ Y)=0 \tag{2.2}$$

In the expression (2.2), the terms which are of 0-th order in the components of φ cancel. And the first order terms give $[X, \varphi(Y)]_{T'N}+[\varphi(X),\ Y]_{T'N}-\varphi([X,\ Y])$. And so the first order term of $p(\varphi)$ is equal to $\bar{\partial}^{(1)}_{T'N}(X,\ Y)$. The second order term of $p(\varphi)$ is

$$[\varphi(X),\ \varphi(Y)]_{T'N}-\varphi([X,\ \varphi(Y)]_{{}^{\circ}T''}+[\varphi(X),\ Y]_{{}^{\circ}T''}).$$

We shall write this term as $R_2(\varphi)(X,\ Y)$. This term is obviously skew-symmetric in X and Y and we have $R_2(\varphi)(fX,\ Y)=fR_2(\varphi)(X,Y)$. In fact

$$\begin{aligned} R_2(\varphi)(fX,\ Y) &= [\varphi(fX),\ Y]_{T'N}-\varphi([fX,\ \varphi(Y)]_{{}^{\circ}T''}+[\varphi(fX),\ Y]_{{}^{\circ}T''}) \\ &= -(\varphi(Y)f)\varphi(X)+f[\varphi(X),\ \varphi(Y)]_{T'N} \\ &\quad +\varphi(\varphi(Y)fX)-f\varphi([X, \varphi(Y)]_{{}^{\circ}T''}+[\varphi(X),\ Y]_{{}^{\circ}T''}) \\ &= f([\varphi(X),\ \varphi(Y)]_{T'N}-\varphi([X,\ \varphi(Y)]_{{}^{\circ}T''} \\ &\quad +[\varphi(X),\ Y]_{{}^{\circ}T''})) \\ &= fR_2(\varphi)(X,\ Y). \end{aligned}$$

Therefore $R_2(\varphi)$ is in $\Gamma(M,\ T'N\otimes\overset{2}{\wedge}({}^{\circ}T'')^*)$.

The third order term of $p(\varphi)$ is $-\varphi([\varphi(X),\ \varphi(Y)]_{{}^{\circ}T''})$. We shall write this term as $R_3(\varphi)(X,\ Y)$. This term is obviously skew-symmetric and $R_3(\varphi)(fX,\ Y)=fR_3(\varphi)(X,\ Y)$. In fact

$$\begin{aligned} R_3(\varphi)(fX,\ Y) &= -\varphi([\varphi(fX),\ \varphi(Y)]_{{}^{\circ}T''}) \\ &= -f\varphi([\varphi(X),\ \varphi(Y)]_{{}^{\circ}T''}) \\ &= fR_3(\varphi)(X,\ Y). \end{aligned}$$

Therefore $R_3(\varphi)$ is in $\Gamma(M, T'N\otimes\overset{2}{\wedge}({}^{\circ}T'')^*)$.

Therefore ${}^{\varphi}T''$ is a partially complex structure if and only if φ is the solution of the differential equation

$$p(\varphi)=0,$$

where

$$p(\varphi)=\bar{\partial}^{(1)}_{T'N}\varphi+R_2(\varphi)+R_3(\varphi).$$

§ 3. Kuranishi's $\bar{\partial}^{\varphi}_b$ for Scalar Valued Forms

For $u\in\Gamma(M, \boldsymbol{C})$, we define $\bar{\partial}_b u\in\Gamma(M, ({}^{\circ}T'')^*)$ by

$$\bar{\partial}_b u(X)=Xu,\ \ X\in{}^{\circ}T''.$$

The differential operator $\bar{\partial}_b:\Gamma(M, \boldsymbol{C})\ni u\to\bar{\partial}_b u\in\Gamma(M, ({}^{\circ}T'')^*)$ is called the (tangential) Cauchy-Riemann operator and we can derive some properties of $\bar{\partial}_b$ from the following.

$A.1)$ $X(fu)=Xf\cdot u+f\cdot Xu$,

$A.2)$ $[X, Y]u=X(Yu)-Y(Xu)$,

where $X, Y\in\Gamma(M, {}^{\circ}T'')$, $u, f\in\Gamma(M, \boldsymbol{C})$.

We define differential operators

$$\bar{\partial}^{(q)}_b:\Gamma(M, \overset{q}{\wedge}({}^{\circ}T'')^*)\to\Gamma(M, \overset{q+1}{\wedge}({}^{\circ}T'')^*)$$

by

$$\begin{aligned}(\bar{\partial}^{(q)}_b\varphi)(X_1,\dots, X_{q+1})=&\sum_i(-1)^{i+1}X_i(\varphi(X_1,\dots,\check{X}_i,\dots, X_{q+1}))\\&+\sum_{i<j}(-1)^{i+j}\varphi([X_i, X_j], X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots, X_{q+1})\end{aligned}$$

for all $\varphi\in\Gamma(M, \overset{q}{\wedge}({}^{\circ}T'')^*)$ and $X_1,\dots, X_{q+1}\in\Gamma(M, {}^{\circ}T'')$.

Now, we shall define the operator $\bar{\partial}_{{}^{\varphi}T''}$ for an almost partially complex structure ${}^{\varphi}T''$, so that if ${}^{\varphi}T''$ is a partially complex structure, the operator $\bar{\partial}_{{}^{\varphi}T''}$ coincides with the above definition.

In the first place, we shall define the projection operator from $\boldsymbol{C}\otimes TM$ to ${}^{\varphi}T''$. Recalling ${}^{\circ}T''=\{(Z)_{{}^{\circ}T''}: Z\in\boldsymbol{C}\otimes TM\}$ and ${}^{\varphi}T''=\{(Z)_{{}^{\circ}T''}+\varphi((Z)_{{}^{\circ}T''})\ ;\ Z\in\boldsymbol{C}\otimes TM\}$, we shall put $(Z)_{{}^{\varphi}T''}=(Z)_{{}^{\circ}T''}+\varphi((z)_{{}^{\circ}T''})\in{}^{\varphi}T''$. Now we shall define $\bar{\partial}_{{}^{\varphi}T''}$ from $\Gamma(M,\boldsymbol{C})$ to $\Gamma(M, ({}^{\varphi}T'')^*)$ as follows:

$$\bar{\partial}_{\varphi_{T''}} u(X') = X'u, \ X' \in {}^\varphi T''.$$

Extending this, we define the operator

$$\bar{\partial}^{(p)}_{\varphi_{T''}} : \Gamma(M, \overset{p}{\wedge}({}^\varphi T'')^*) \to \Gamma(M, \overset{p+1}{\wedge}({}^\varphi T'')^*)$$

by

$$\bar{\partial}^{(p)}_{\varphi_{T''}} \phi(X'_1, \ldots, X'_{p+1}) = \sum_i (-1)^{i+1} X'_i \phi(X'_1, \ldots, \check{X}'_i, \ldots, X'_{p+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \phi([X'_i, X'_j]_{\varphi_{T''}}, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X'_{p+1})$$

where $\phi \in \Gamma(M, \overset{p}{\wedge}({}^\varphi T'')^*)$, $X'_1, \ldots, X'_i, \ldots, X'_{p+1} \in \Gamma(M, {}^\varphi T'')$. The above definition is legitimate, since the following condition holds.

B. 1) $X'(fu) = X'f \cdot u + fX'u$

where $X' \in {}^\varphi T''$, $u \in \Gamma(M, \boldsymbol{C})$ and $f \in \Gamma(M, \boldsymbol{C})$.

If ${}^\varphi T''$ is a partially complex structure, it satisfies the relation $[X'_i, X'_j]_{\varphi_{T''}} = [X'_i, X'_j]$. In fact there exists X_i, X_j such that $X'_i = X_i + \varphi(X_i)$ and $X'_j = X_j + \varphi(X_j)$. So

$$\begin{aligned}[X'_i, X'_j]_{\varphi_{T''}} &= [X'_i, X'_j]_{\circ_{T''}} + \varphi([X'_i, X'_j]_{\circ_{T''}}) \\ &= [X_i + \varphi(X_i), X_j + \varphi(X_j)]_{\circ_{T''}} \\ &\quad + \varphi([X_i + \varphi(X_i), X_j + \varphi(X_j)]_{\circ_{T''}}) \\ &= [X'_i, X'_j] \text{ (from Proposition 2. 1)}\end{aligned}$$

So, if ${}^\varphi T''$ is a partially complex structure, the operator $\bar{\partial}^{(p)}_{\varphi_{T''}}$ as defined above becomes

$$\bar{\partial}^{(p)}_{\varphi_{T''}} \phi(X'_1, X'_2, \ldots, X'_{p+1}) = \sum_i (-1)^{i+1} X'_i \phi(X'_1, \ldots, X'_i, \ldots, X'_{p+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \phi([X'_i, X'_j], \ldots, \check{X}'_i, \ldots, \check{X}'_j, \ldots, X'_{p+1}),$$

for any $\phi \in \Gamma(M, \overset{p}{\wedge}({}^\varphi T'')^*)$, and our definition coincides with $\bar{\partial}_b$ taken ${}^\varphi T''$ as the reference structure.

Now, we shall define the operator $\bar{\partial}_b^\varphi$ from $\Gamma(M, \overset{p}{\wedge}({}^\circ T'')^*)$ to $\Gamma(M, \overset{p+1}{\wedge}({}^\circ T'')^*)$ by $\bar{\partial}_b^\varphi = \lambda_\varphi^{(p+1)} \cdot \bar{\partial}^{(p)}_{\varphi_{T''}} \cdot (\lambda_\varphi^{(p)})^{-1}$, where λ_φ is an isomorphism map from ${}^\circ T''$ to ${}^\varphi T''$, defined by

$$\lambda_\varphi : {}^\circ T'' \ni X \to X + \phi(X) \in {}^\varphi T'',$$

and $\lambda_\varphi^{(p)}$ is the isomorphism from $\Gamma(M, \overset{p}{\wedge}({}^\varphi T'')^*)$ to $\Gamma(M, \overset{p}{\wedge}({}^\circ T'')^*)$

induced by λ_φ.

Now, we shall show the following theorem. Compare M. Kuranishi's formulations Propositions 4. 1, 4. 2, 4. 3 and 4. 4 in [1].

Theorem 3. 1. *We shall assume the above conditions. Then, the following assertions* 1), 2) *and* 3) *are equivalent to one another.*

C. 1) ${}^{\varphi}T''$ *is a partially complex structure.*

C. 2) $(\bar\partial^{(p)}_{\varphi T''}, \Gamma(M, \overset{p}{\wedge}({}^{\varphi}T'')^*))$ *is a cochain complex.*

C. 3) $(\bar\partial^{\varphi}_b, \Gamma(M, \overset{p}{\wedge}({}^{\circ}T'')^*)$ *is a cochain complex.*

Proof. From the relation $\bar\partial^{\varphi}_b = \lambda^{(p+1)}_\varphi \cdot \bar\partial^{(p)}_{\varphi T''} \cdot (\lambda^{(p)}_\varphi)^{-1}$, it is clear that the assertion C. 2) is equivalent to C. 3). From the assertion C. 1), we have already proven C. 2).

Therefore it is enough to show that the assertion C. 3) implies the assertion C. 1). The proof is contained in that of the next proposition.

Proposition 3. 2. *For any C^∞ function f, the relation $\bar\partial^{\varphi}_b \bar\partial^{\varphi}_b f = df\ \bar\Lambda\ p(\varphi)$ holds, where $\bar\Lambda$ is the contraction operator.*

Proof. If we prove the relation $\bar\partial^{\varphi}_b \bar\partial^{\varphi}_b f(X, Y) = df\ \bar\Lambda\ p(\varphi)(X, Y)$ for any $X, Y \in {}^{\circ}T''$, we are through.

From the definition, we obtain the relation

$$
\begin{aligned}
(3.2.1)\qquad \bar\partial^{\varphi}_b \theta(X, Y) &= \lambda^{(2)}_\varphi \cdot \bar\partial^{(1)}_{\varphi T''} \cdot (\lambda^{(1)}_\varphi)^{-1} \theta(X, Y),\\
&= \bar\partial^{(1)}_{\varphi T''} \cdot (\lambda^{(1)}_\varphi)^{-1} \theta(X+\varphi(X), Y+\varphi(Y)),\\
&= (X+\varphi(X))(\lambda^{(1)}_\varphi)^{-1}\theta(Y+\varphi(Y))\\
&\quad -(Y+\varphi(Y))(\lambda^{(1)}_\varphi)^{-1}\theta(X+\varphi(X))\\
&\quad -(\lambda^{(1)}_\varphi)^{-1}\theta([X+\varphi(X), Y+\varphi(Y)]_{\varphi T''}),\\
&= (X+\varphi(X))\cdot\theta(Y) - (Y+\varphi(Y))\theta(X)\\
&\quad -\theta([X+\varphi(X), Y+\varphi(Y)]_{\circ T''}).
\end{aligned}
$$

So

$$(3.2.2)\qquad \bar\partial^{\varphi}_b(\bar\partial^{\varphi}_b f)(X, Y) = (X+\varphi(X))\bar\partial^{\varphi}_b f(Y) - (Y+\varphi(Y))\bar\partial^{\varphi}_b f(X)$$

$$-\bar{\partial}_b^{\varphi} f([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''}).$$

And

$$\begin{aligned}(3.2.3)\qquad \bar{\partial}_b^{\varphi} f(Z) &= (\lambda_{\varphi}^{(1)}\cdot \bar{\partial}_{\varphi_{T''}} f)(Z),\\ &= \bar{\partial}_{\varphi_{T''}} f(Z+\varphi(Z)),\\ &= (Z+\varphi(Z))f.\end{aligned}$$

From these formulas, we have

$$\begin{aligned}(3.2.4)\qquad \bar{\partial}_b^{\varphi}\bar{\partial}_b^{\varphi} f(X,\ Y) &= (X+\varphi(X))(Y+\varphi(Y))f-(Y+\varphi(Y))(X\\ &\quad +\varphi(X))f-([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''}\\ &\quad +\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''}))f\\ &= [X+\varphi(X),\ Y+\varphi(Y)]f\\ &\quad -([X+\varphi(X),\ Y+\varphi(Y)])_{\varphi_{T''}} f,\\ &= p(\varphi)(X,\ Y)f,\\ &= df\ \bar{\Lambda}\ p(\varphi)(X,\ Y).\end{aligned}$$

Q. E. D.

Proposition 3.3. *Let f be a C^{∞} function which satisfies the relation $Xf=0$ for any $X\in\Gamma(M,\ {}^{\circ}T'')$. Then we have the relation $\bar{\partial}_b^{\varphi}(df\ \bar{\Lambda}\ \varphi)=df\ \bar{\Lambda}\ p(\varphi)$.*

Proof. We shall show the relation $\bar{\partial}_b^{\varphi}(df\ \bar{\Lambda}\ \varphi)(X,Y)=df\ \bar{\Lambda}\ p(\varphi)(X,Y)$ for any $X,\ Y\in\Gamma(M,\ {}^{\circ}T'')$.

$$\begin{aligned}(3.3.1)\qquad \bar{\partial}_b^{\varphi}(df\ \bar{\Lambda}\varphi)(X,\ Y) &= (X+\varphi(X))(df\ \bar{\Lambda}\,\varphi(Y))\\ &\quad -(Y+\varphi(Y))(df\ \bar{\Lambda}\,\varphi(X))\\ &\quad -df\ \bar{\Lambda}\,\varphi([X+\varphi(X),\ Y+\varphi(Y)_{\circ T''})\\ &= (X+\varphi(X))\varphi(Y)f-(Y+\varphi(Y))\varphi(X)f\\ &\quad -\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''})f,\end{aligned}$$

and from the condition $Xf=0$ for any $X\in(M,\ {}^{\circ}T'')$,

$$\begin{aligned}(3.3.2)\qquad \bar{\partial}_b^{\varphi}(df\ \bar{\Lambda}\varphi)(X,\ Y) &= (X+\varphi(X))\varphi(Y)f-(Y+\varphi(Y))\varphi(X)f\\ &\quad -\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''})f\\ &= (X+\varphi(X))(Y+\varphi(Y))f\\ &\quad -(Y+\varphi(Y))(X+\varphi(X))f\\ &\quad -[X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''}f\\ &\quad -\varphi([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''})f.\end{aligned}$$

So

$$(3.3.3)\qquad \bar{\partial}_b^{\varphi}(df\,\bar{\Lambda}\,\varphi)(X,\ Y)=p(\varphi)f$$
$$=(df\,\bar{\Lambda}\ p(\varphi))(X,\ Y).\qquad \text{Q. E. D.}$$

§ 4. $\bar{\partial}_b^{\varphi}$ for TN Valued Forms

We shall recall Tanaka's theory on holomorphic tangent vector bundle (see for the details, [3]). Let M be a partially complex manifold, with the subbundle S defining its partially complex structure. Put $\hat{T}(M)=C\otimes TM/S$, then we may define the differential operator $\bar{\partial}_{\hat{T}(M)}$ from $\Gamma(M,\ \hat{T}(M))$ to $\Gamma(M,\ \hat{T}(M)\otimes(\bar{S})^*)$ as follows:

$$(4.1)\qquad \bar{\partial}_{\hat{T}(M)}u(X)=\varpi([X,\ Z]),$$

where $u\in\Gamma(M,\ \hat{T}(M))$, $X\in\Gamma(S)$ and ϖ is the canonical projection from $C\otimes TM$ to $\hat{T}(M)$ and $\varpi(Z)=u$.

$\bar{\partial}_{\hat{T}(M)}$ is well-defined by the above formula, because S is integrable. Then, we can prove the following D. 1) a d D. 2).

D. 1) $X(fu)=Xf\cdot u+fXu$,

D. 2) $[X,\ Y]u=X(Y\cdot u)-Y(X\cdot u)$,

where X, Y are in $\Gamma(S)$ and u is in $\Gamma(M,\ \hat{T}(M))$, and we put $X\cdot u=\bar{\partial}_{\hat{T}(M)}u(X)=\varpi([X,\ Z)]$.

Then we define the operator $\bar{\partial}_{\hat{T}(M)}^{(p)}$ from $\Gamma(M,\ \hat{T}(M)\otimes\overset{p}{\wedge}S^*)$ to $\Gamma(M,\ \hat{T}(M)\otimes\overset{p+1}{\wedge}S^*)$ as follows:

For any element ψ in $\Gamma(M,\ \hat{T}(M)\otimes\overset{p}{\wedge}S^*)$, we put

$$\bar{\partial}_{\hat{T}(M)}^{(p)}\psi(X_1,X_2,\ldots,X_{p+1})=\sum_j(-1)^{j+1}X_j\cdot\psi(X_1,\ldots,\check{X}_j,\ldots,X_{p+1})$$
$$+\sum_{i<j}(-1)^{i+j}\psi([X_i,\ X_j],\ X_1,\ldots,\check{X}_i,\ldots,\check{X}_j,\ldots,X_{p+1}).$$

It is well defined because of D. 1) and from D. 2) we can prove the relation $\bar{\partial}_{\hat{T}(M)}^{(p+1)}\cdot\bar{\partial}_{\hat{T}(M)}^{(p)}=0$.

Now, we shall develop Tanaka's theory to the case of an almost partially complex structure ${}^{\varphi}T''$. Of course, it must coincide with the classical one when ${}^{\varphi}T''$ is a partially complex structure.

We shall define the operator $\bar{\partial}_{\hat{T}(M)}^{\varphi}$ from $\Gamma(M,\ \hat{T}(M))$ to $\Gamma(M,\ {}^{\varphi}\hat{T}(M)\otimes({}^{\circ}T'')^*)$, where ${}^{\varphi}\hat{T}(M)=C\otimes TM/{}^{\varphi}T''$ and ${}^{\varphi}T''$ is an almost partially complex structure. Since ${}^{\varphi}T''$ is not always a partially complex structure, (4.1) applied to ${}^{\varphi}\hat{T}(M)$ may not define an operator from $\Gamma(M, {}^{\varphi}\hat{T}(M))$ to $\Gamma(M, {}^{\varphi}\hat{T}(M)\otimes({}^{\circ}T'')^{*})$.

To help this point, first we shall prove the following lemmas.

Lemma 4.1. *Let ϖ be a projection operator from $C\otimes TM$ to $C\otimes TM/{}^{\varphi}T''$. Then, the map $\varpi|_{T'N}$ restricted to $T'N$ is an isomorphism.*

Proof. It is sufficient to prove tqat $\varpi|_{T'N}$ is an injective map. If there are elements x, y in $T'N$ such that the relation $\varpi|_{T'N}(x)=\varpi|_{T'N}(y)$ holds, then $x-y\in{}^{\varphi}T''$, and there is an element $z\in{}^{\circ}T''$ which satisfies the relation $x-y=z+\varphi(z)$. Comparing the components in the decomposition $C\otimes TM={}^{\circ}\bar{T}''\oplus{}^{\circ}T''\oplus F$, we see that $z=0$. Therefore we have $x=y$. Q. E. D.

Lemma 4.2. *Assuming the above condition, we have the relation*

$$(\varpi|_{T'N})^{-1}\circ\varpi(z)=z-(z)_{{}^{\varphi}T''},$$

where $\varpi|_{T'N}: T'N\simeq C\otimes TM/{}^{\varphi}T''$.

Proof. For any element $z\in C\otimes TM$, we can prove that $z-(z)_{{}^{\varphi}T''}$ is in $T'N$. Therefore Lemma 4.2 follows from Lemma 4.1.

Q. E. D.

Lemma 4.3. *Assuming the above condition, the following schema commutes.*

$$\begin{array}{ccc} \bar{\partial}_{T'N}: \Gamma(M, T'N) & \longrightarrow & \Gamma(M, T'N\otimes({}^{\circ}T'')^{*}) \\ \downarrow \varpi & & \downarrow \varpi \\ \bar{\partial}_{\hat{T}(M)}: \Gamma(M, \hat{T}(M)) & \longrightarrow & \Gamma(M, \hat{T}(M)\otimes({}^{\circ}T'')^{*}), \end{array}$$

where we define $\bar{\partial}_{T'N}$ by $\bar{\partial}_{T'N}u(Z)=[Z,u]_{T'N}$ for any element $Z\in\Gamma(M,{}^{\circ}T'')$, and $\varpi: C\otimes TM\to\hat{T}(M)=C\otimes TM/{}^{\circ}T''$ means the canonical projection.

Proof. From the definition of $\bar{\partial}_{T'N}$, it satisfies the relation $\bar{\partial}_{T'N}u(Z)=[Z, u]_{T'N}=[Z, u]-[Z, u]_{{}^{\circ}T''}$. Therefore we obtain the relation

$\varpi(\bar{\partial}_{T'N}u(Z))=\varpi([Z,\ u]-[Z,\ u]_{\varphi T''})=\tilde{\varpi}([Z,\ u])=\bar{\partial}_{T(M)}\varpi(u)(Z).$

Q. E. D.

After these preparations, we shall define the operator $\bar{\partial}_{\varphi\hat{T}(M)}$ from $\Gamma(M,\ \hat{T}(M))$ to $\Gamma(M,\ {}^{\varphi}\hat{T}(M)\otimes({}^{\varphi}T'')^{*})$.

Proposition 4.4. *Let ϖ be the projection form $C\otimes TM$ to ${}^{\varphi}\hat{T}(M)$, and let $\bar{\partial}^{\varphi}_{T'N}$ be the differential operator from $\Gamma(M,\ T'N)$ to $\Gamma(M,\ T'N\otimes({}^{\varphi}T'')^{*})$ defined as follows: For any $u\in\Gamma(M,\ T'N)$, we put $\bar{\partial}^{\varphi}_{T'N}u(Z')=[Z',\ u]-[Z',\ u]_{\varphi T''}$, where Z' is in ${}^{\varphi}T''$. Also let $\bar{\partial}_{\varphi\hat{T}(M)}$ be the differential operator from $\Gamma(M,\ {}^{\varphi}\hat{T}(M))$ to $\Gamma(M,\ {}^{\varphi}\hat{T}(M)\otimes({}^{\varphi}T'')^{*})$ defined as follows: For any $[u]\in\Gamma(M,{}^{\varphi}\hat{T}(M))$, we put $\bar{\partial}_{\varphi\hat{T}(M)}[u](Z')=\varpi([Z',\ \check{u}])$, where Z' is in ${}^{\varphi}T''$ and $\check{u}$ is the unique inverse image of $[u]$ in $\Gamma(T'N)$. Then, the following schema commutes.*

$$\begin{array}{cccc}\bar{\partial}^{\varphi}_{T'N}: & \Gamma(M,\ T'N) & \longrightarrow & \Gamma(M,\ T'N\otimes({}^{\varphi}T'')^{*})\\ & \downarrow\varpi & & \downarrow\varpi\\ \bar{\partial}_{\varphi\hat{T}(M)}: & \Gamma(M,\ {}^{\varphi}\hat{T}(M)) & \longrightarrow & \Gamma(M,\ {}^{\varphi}\hat{T}(M)\otimes({}^{\varphi}T'')^{*}).\end{array}$$

Proof. Before everything, we shall check that the maps $\bar{\partial}^{\varphi}_{T'N}$ and $\bar{\partial}_{\varphi\hat{T}(M)}$ are well-defined. To see this, it is sufficient to prove the relation $\bar{\partial}^{\varphi}_{T'N}u(fZ')=f\bar{\partial}^{\varphi}_{T'N}u(Z')$ for any Z' in $\Gamma(M,\ {}^{\varphi}T'')$ and for any C^{∞} function f on M, and to prove the similar relation

$$\bar{\partial}_{\varphi\hat{T}(M)}u(fZ')=f\bar{\partial}_{\varphi\hat{T}(M)}u(Z').$$

Now we have

$$\begin{aligned}\bar{\partial}^{\varphi}_{T'N}u(fZ')&=[fZ',\ u]-[fZ',\ u]_{\varphi T''}\\&=-u(f)Z'+f[Z',\ u]-(-u(f)Z'+f[Z',\ u])_{\varphi T''}.\end{aligned}$$

The element Z' being in ${}^{\varphi}T''$, we have $Z'=(Z')_{\varphi T''}$ and so

$$\begin{aligned}\bar{\partial}^{\varphi}_{T'N}u(fZ')&=f([Z',\ u]-[Z',\ u]_{\varphi T''})\\&=f\bar{\partial}^{\varphi}_{T'N}u(Z').\end{aligned}$$

Similarly, we have the relation $\bar{\partial}_{\varphi\hat{T}(M)}[u](fZ')=f\bar{\partial}_{\varphi\hat{T}(M)}[u](Z')$. Now Proposition 4.4 is clear from the relation $\varpi(\bar{\partial}^{\varphi}_{T'N}u(Z'))=\varpi([Z',\ u])=$

$\bar{\partial}_{{}^\varphi\hat{T}(M)}[u](Z')$. Q. E. D.

Next we shall define the complex $(\Gamma(M, {}^\varphi\hat{T}(M)\otimes\overset{p}{\wedge}({}^\varphi T'')^*)$, $\bar{\partial}^{(p)}_{{}^\varphi\hat{T}(M)})$. To do this, we have to prove the following proposition.

Proposition 4.5. *We have the relation.*

E. 1) $X'(fu) = X'f\cdot u + fX'u,$

for any X' in ${}^\varphi T''$ and for any u in $\Gamma(M, T'N)$, where we put $X'\cdot u$ *as $\bar{\partial}^\varphi_{T'N}u(X')$. Similarly we have*

E. 2) $X'(f[u]) = X'f[u] + fX'[u],$

for any X' in ${}^\varphi T''$ and for any $[u]$ in $\Gamma(M, {}^\varphi\hat{T}(M))$, where we put $X'[u] = \bar{\partial}_{{}^\varphi\hat{T}(M)}[u](X')$.

Proof. We shall prove E. 1) only, for the proof of E. 2) is similar. From the definition of $\bar{\partial}^\varphi_{T'N}$, we have the following relation:

$$\begin{aligned}\bar{\partial}^\varphi_{T'N}(fu)(X') &= [X', fu] - [X', fu]_{{}^\varphi T''}\\ &= X'f\cdot u + f[X', u] - (X'f\cdot u + f(X', u])_{{}^\varphi T''}\\ &= X'f\cdot u + f([X', u] - [X', u])_{{}^\varphi T''}.\\ \text{(For } (u)_{{}^\varphi T''} &= (u)_{{}^0T''} + \varphi((u)_{{}^0T''}) = 0.)\\ &= X'f\cdot u + f\bar{\partial}^\varphi_{T'N}u(X').\end{aligned}$$

And so

$$X'(fu) = X'f\cdot u + fX'\cdot u.$$

From Proposition 4.5, we have the following proposition.

Proposition 4.6. *The map $\bar{\partial}^{\varphi,(p)}_{T'N}$ is well-defined as a map from $\Gamma(M, T'N\otimes\overset{p}{\wedge}({}^\varphi T'')^*)$ to $\Gamma(M, T'N\otimes\overset{p+1}{\wedge}({}^\varphi T'')^*)$:*

For any ϕ in $\Gamma(M, T'N\otimes\overset{p}{\wedge}({}^\varphi T'')^*)$, we determine $\bar{\partial}^{\varphi,(p)}_{T'N}\phi$ by

$$\bar{\partial}^{\varphi,(p)}_{T'N}\phi(X'_1, \dots, X'_{p+1}) = \sum_j (-1)^{j+1}X'_j\cdot\phi(X'_1, \dots, \check{X}'_j, \dots, X'_{p+1})$$

$$+\sum_{i<j}(-1)^{i+j}\phi([X'_i, X'_j]_{\varphi_{T''}}, \dots, \check{X}_i, \dots, \check{X}_j, \dots, X'_{p+1}),$$

where X'_i *is in* ${}^{\varphi}T''$. *In other words* $\bar{\partial}^{\varphi,(p)}_{T'N}\phi(X'_1, \dots, X'_{p+1})$ *is skew symmetric and multilinear in X's, with functions as coefficients.*

Proof. It is clear that $\bar{\partial}^{\varphi,(p)}_{T'N}\phi(X'_1, \dots, X'_{p+1})$ is skew symmetric. And so it is sufficient to prove the following relation.

$$\bar{\partial}^{\varphi,(p)}_{T'N}\phi(fX'_1, X'_2, \dots, X'_{p+1}) = f\bar{\partial}^{\varphi,(p)}_{T'N}\phi(X'_1, X'_2, \dots, X'_{p+1}). \tag{4.2}$$

We write down the left hand side of (4.2) according to the definition.

$$\begin{aligned}
&\bar{\partial}^{\varphi,(p)}_{T'N}\phi(fX'_1, X'_2, \dots, X'_{p+1}) \\
&= fX'_1\cdot\phi(X'_2, X'_3, \dots, X'_{p+1}) + \sum_{1<j}(-1)^{j+1}X'_j\cdot\phi(fx'_1, \dots, \check{X}_j, \dots, X'_{p+1}) \\
&\quad + \sum_{1<j}(-1)^{1+j}\phi([fX'_1, X'_j]_{\varphi_{T''}}, \check{X}'_1, \dots, \check{X}'_j, \dots, X'_{p+1}) \\
&\quad + \sum_{1<i<j}(-1)^{i+j}\phi([X'_i, X'_j]_{\varphi_{T''}}, fX'_1, \dots, \check{X}'_i, \dots, \check{X}'_j, \dots, X'_{p+1}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
[fX'_1, X'_j]_{\varphi_{T''}} &= (-X'_jf\cdot X'_1 + f[X'_1, X'_j])_{\varphi_{T''}} \\
&= -X'_jf\cdot X'_1 + f[X'_1, X'_j]_{\varphi_{T''}}.
\end{aligned} \tag{4.3}$$

And from Proposition 4.5

$$\begin{aligned}
&X'_j\cdot\phi(fX'_1, \dots, \check{X}'_j, \dots, X'_{p+1}) \\
&= X'_j(f\phi(X'_1, \dots, \check{X}'_j, \dots, X'_{p+1})) \\
&= X'_jf\cdot\phi(X'_1, \dots, \check{X}'_j, \dots, X'_{p+1}) + fX'_j\phi(X'_1, \dots, \check{X}'_j, \dots, X'_{p+1}).
\end{aligned} \tag{4.4}$$

From these formulas, we have Proposition 4.6.

Thus we have the diagrams

$$0 \to \Gamma(M, T'N) \xrightarrow{\bar{\partial}^{\varphi}_{T'N}} (M, T'N \otimes ({}^{\varphi}T'')^*) \xrightarrow{\bar{\partial}^{\varphi,(1)}_{T'N}} \cdots \longrightarrow \Gamma(M, T'N \otimes \overset{p}{\wedge}({}^{\varphi}T'')^*)$$

$$\xrightarrow{\bar{\partial}^{\varphi,(p)}_{T'N}} \Gamma(M, T'N \otimes \overset{p+1}{\wedge}({}^{\varphi}T'')^*) \longrightarrow \cdots,$$

$$0 \to \Gamma(M, {}^{\varphi}\hat{T}(M)) \xrightarrow{\bar{\partial}_{\varphi_{\hat{T}(M)}}} \Gamma(M, {}^{\varphi}\hat{T}(M) \otimes ({}^{\varphi}T'')^*) \xrightarrow{\bar{\partial}_{\varphi_{\hat{T}(M)}}} \cdots \longrightarrow$$

$$\Gamma(M, {}^{\varphi}\hat{T}(M) \otimes \overset{p}{\wedge}({}^{\varphi}T'')^*) \xrightarrow{\bar{\partial}^{(p)}_{\varphi_{\hat{T}(M)}}} \Gamma(M, {}^{\varphi}\hat{T}(M) \otimes \overset{p+1}{\wedge}({}^{\varphi}T'')^*) \to \cdots.$$

Note that they are not necessarily comlpexes.

Now, we shall define Kuranish's $\bar\partial_b^\varphi$.

Definition 4. 7. *We shall define the differential operator* $\bar\partial_b^\varphi$ *from* $\Gamma(M, T'N\otimes\overset{p}{\wedge}({}^\circ T'')^*)$ *to* $\Gamma(M, T'N\otimes\overset{p+1}{\wedge}({}^\circ T'')^*)$ *as follows.*

$$\begin{array}{ccc} \bar\partial_b^\varphi:\ \Gamma(M, T'N\otimes\overset{p}{\wedge}({}^\circ T'')^*) & \longrightarrow & \Gamma(M, T'N\otimes\overset{p+1}{\wedge}({}^\circ T'')^*) \\ \uparrow \lambda_\varphi^{(p)} & & \uparrow \lambda_\varphi^{(p+1)} \\ \bar\partial_{T'N}^{\varphi,(p)}:\ \Gamma(M, T'N\otimes\overset{p}{\wedge}({}^\varphi T'')^*) & \longrightarrow & \Gamma(M, T'N\otimes\overset{p+1}{\wedge}({}^\varphi T'')^*) \end{array}$$

We put $\bar\partial_b^\varphi=\lambda_\varphi^{(p+1)}\cdot\bar\partial_{T'N}^{\varphi,(p)}\cdot(\lambda_\varphi^{(p)})^{-1}$ *where* $\lambda_\varphi^{(p)}$ *is the map induced by*

$$\lambda_\varphi:\ {}^\circ T''\ni X\longrightarrow X+\varphi(X)\in{}^\varphi T''.$$

Theorem 4. 8. *With the above notations the following assertions are equivalent one another.*

1) φ *is integrable.*

2) $(\bar\partial^{(p)}_{{}^\varphi T(M)}, \Gamma(M, {}^\varphi\hat T(M)\otimes\overset{p}{\wedge}({}^\varphi T'')^*))$ *is a cochain complex.*

3) $(\bar\partial_{T'N}^{\varphi,(p)}, \Gamma(M, TN\otimes\overset{p}{\wedge}({}^\varphi T'')^*)$ *is a cochain complex.*

4) $(\bar\partial_b^\varphi, \Gamma(M, T'N\otimes\overset{p}{\wedge}({}^\circ T'')^*))$ *is a cochain complex.*

Proof. It is clear that the assertion 2) is equ valent to the assertion 3) and 4). Moreover we have already proven that the assertion 2) is included in the assertion 3). And so, it is sufficient to prove that we can have the assertion 1) from the assertion 2).

From the relation $\bar\partial_{T'N}^{\varphi,(1)}, \bar\partial_{T'N}^\varphi=0$ contained in the assertion 2), we shall show that φ is a partially complex structure. Our condition means

$$\bar\partial_{T'N}^{\varphi,(1)}\bar\partial_{T'N}^\varphi u(X', Y')=0 \tag{4.5}$$

for any u in $\Gamma(M, T'N)$ and for any X', Y' in ${}^\varphi T''$. From this we have

$$X'\bar\partial_{T'N}^\varphi u(Y')-Y'\cdot\bar\partial_{T'N}^\varphi u(X')-\bar\partial_{T'N}^\varphi u([X', Y']_{{}^\varphi T''})=0. \tag{4.6}$$

And so

$$X'\cdot(Y'\cdot u)-Y'\cdot(X'\cdot u)-[X', Y']_{{}^\varphi T''}u=0, \tag{4.7}$$

for any u in $\Gamma(M, T'N)$ and for any X', Y' in ${}^{\varphi}T''$. Then

$$X'\cdot(Y'\cdot fu)-Y'\cdot(X'\cdot fu)-[X', Y']_{\varphi_{T''}}fu=0 \tag{4.8}$$

for any C^∞ function f on M.

While we have

$$Y'(fu)=(Y'\ f)\cdot u+f\cdot Y'u. \tag{4.9}$$

From Proposition 4.5

$$X'\cdot(Y'\cdot fu)=(X'Y'f)u+(Y'f)\cdot X'u+(X'f)\cdot Y'u+fX'Y'u. \tag{4.10}$$

And

$$[X', Y']_{\varphi_{T''}}fu=[X', Y']_{\varphi_{T''}}f\cdot u+f[X', Y']_{\varphi_{T''}}u. \tag{4.11}$$

From (4.10) and (4.11), the relation (4.8) can be reformed into next relation.

$$\begin{aligned}&((X'\cdot Y'-Y'\cdot X'-[X', Y']_{\varphi_{T''}})f)u\\&+f(X'\cdot Y'-Y'\cdot X'-[X', Y']_{\varphi_{T''}})u=0,\end{aligned} \tag{4.12}$$

for any C^∞ function f and any u in $\Gamma(M, T'N)$. Therefore we have

$$(X'\cdot Y'-Y'\cdot X'-[X', Y']_{\varphi_{T''}})f=0 \text{ for any } C^\infty \text{ function } f. \tag{4.13}$$

And so

$$[X', Y']-[X',Y']_{\varphi_{T''}}=0 \text{ for any } X', Y' \text{ in } \Gamma(M, {}^{\varphi}T''). \tag{4.14}$$

So φ is integrable. Q. E. D.

In the course of Theorem 4.8, we have shown that the operator $X'\cdot Y'-Y'\cdot X'-[X', Y']_{\varphi_{T''}}$, where X', Y' in $\Gamma(M, {}^{\varphi}T'')$, is a first order differential operator from $\Gamma(M, T'N)$ to $\Gamma(M, T'N)$. Moreover, we can prove the following theorem (Proposition 4.2 in [1]).

Theorem 4.9. *For any element* θ *in* $\Gamma(M, T'N\otimes\overset{p}{\wedge}({}^{\varphi}T'')^*)$, *we have the following relation.*

For $p\geqq 1$,

$$\begin{aligned}&\bar{\partial}_{T'N}^{\varphi,(p+1)}\cdot\bar{\partial}_{T'N}^{\varphi,(p)}\theta(X_1', X_2', \dots, X_{p+2}')\\&=\sum_{i<j}(-1)^{i+j+1}(X_i'\cdot X_j'-X_j'\cdot X_i'-[X_i', X_j']_{\varphi_{T''}})\theta(X_1', \dots,\\&\qquad \check{X}_i, \dots, \check{X}_j, \dots X_{p+2}')\end{aligned}$$

$$+\sum_{k<i<j}(-1)^{k+i+j+1}\theta([p(\varphi)(X_i', X_j'), X_k']_{\varphi_{T''}}, \dots, \check{X}_k', \dots, \check{X}_i', \dots, \check{X}_j', \dots, X_{p+2}')$$
$$+\sum_{i<k<j}(-1)^{i+k+j}\theta([p(\varphi)(X_i', X_j'), X_k']_{\varphi_{T''}}, \dots, \check{X}_i', \dots, \check{X}_k', \dots, \check{X}_j', \dots, X_{p+2}')$$
$$+\sum_{i<j<k}(-1)^{i+j+k+1}\theta([p(\varphi)(X_i', X_j'), X_k']_{\varphi_{T''}}, \dots, \check{X}_i', \dots, \check{X}_j', \dots, \check{X}_k', \dots, X_{p+2}'),$$

and for $p=0$

$$\bar{\partial}_{T'N}^{\varphi,(1)}\cdot\bar{\partial}_{T'N}^{\varphi}\theta(X_1', X_2') = (X_1'\cdot X_2' - X_2'\cdot X_1' - [X_1', X_2']_{\varphi_{T''}})\theta.$$

Proof. For ψ in $\Gamma(M, T'N\otimes\overset{p+1}{\wedge}({}^{\varphi}T'')^*)$ we have

$$\bar{\partial}_{T'N}^{\varphi,(p+1)}\psi(X_1', X_2', \dots, X_{p+2}') = \sum_j(-1)^{j+1}X_j'\cdot\psi(X_1', X_2', \dots, \check{X}_j', \dots, X_{p+2}') + \sum_{i<j}(-1)^{i+j}\psi([X_i', X_j']_{\varphi_{T''}}, \dots, \check{X}_i', \dots, \check{X}_j', \dots, X_{p+2}').$$

We shall put $\psi=\bar{\partial}_{T'N}^{\varphi,(p)}\theta$. Then

$$\bar{\partial}_{T'N}^{\varphi,(p)}\theta(X_1', \dots, \check{X}_j', \dots, X_{p+2}') = \sum_{k<j}(-1)^{k+1}X_k'\cdot\theta(X_1', X_2', \dots, \check{X}_k', \dots, \check{X}_j', \dots, X_{p+2}')$$
$$+\sum_{j<k}(-1)^{k+2}X_k'\cdot\theta(X_1', X_2', \dots, \check{X}_j', \dots, \check{X}_k', \dots, X_{p+2}')$$
$$+\sum_{l<m<j}(-1)^{l+m}\theta([X_l', X_m']_{\varphi_{T''}}, \dots, \check{X}_l', \dots, \check{X}_m', \dots, X_j', \dots, X_{p+2}')$$
$$+\sum_{l<j<m}(-1)^{l+m+1}\theta([X_l', X_m']_{\varphi_{T''}}, \dots, \check{X}_l', \dots, \check{X}_j', \dots, \check{X}_m, \dots, X_{p+2}')$$
$$+\sum_{l<m<j}(-1)^{l+m}\theta([X_l', X_m']_{\varphi_{T''}}, \dots, \check{X}_l', \dots, \check{X}_m, \dots, \check{X}_j', \dots, X_{p+2}').$$

Therefore

$$(-1)^{j+1}X_j'\cdot\bar{\partial}_{T'N}^{\varphi,(p)}\theta(X_1', X_2', \dots, \check{X}_j', \dots, X_{p+2}') = \sum_{k<j}(-1)^{k+j}X_j'\cdot X_k'\theta(X_1', X_2', \dots, \check{X}_k', \dots, \check{X}_j', \dots, X_{p+2}')$$
$$+\sum_{j<k}(-1)^{k+j+1}X_j'\cdot X_k'\theta(X_1', X_2', \dots, \check{X}_j', \dots, \check{X}_k', \dots, X_{p+2}')$$
$$+\sum_{l<m<j}(-1)^{l+m+j+1}X_j'\theta([X_l', X_m']_{\varphi_{T''}}, \dots, \check{X}_l', \dots, \check{X}_m', \dots, \check{X}_j', \dots, X_{p+2}')$$

$$+\sum_{l<j<m}(-1)^{l+m+j}X'_j\theta([X'_l, X'_m]_{\varphi_{T''}}, \dots, \check{X}'_l, \dots, \check{X}'_j, \dots, \check{X}'_m, \dots, X'_{p+2})$$
$$+\sum_{l<m<j}(-1)^{l+m+j}X'_j\theta([X'_l, X'_m]_{\varphi_{T''}}, \dots, \check{X}'_l, \dots, \check{X}'_m, \dots, \check{X}'_j, \dots, X'_{p+2}).$$

And

$$(-1)^{i+j}\bar{\partial}^{\varphi,(p)}_{T'N}\theta([X'_i, X'_j]_{\varphi_{T''}}, \dots, \check{X}'_i, \dots, \check{X}'_j, \dots, X'_{p+2})$$
$$=[X'_i, X'_j]_{\varphi_{T''}}\theta[X'_1, \dots, \check{X}'_i, \dots, \check{X}'_j, \dots, X_{p+2})$$
$$+\sum_{k<i<j}(-1)^k X'_k\theta(X'_1, \dots, \check{X}'_k, \dots, \check{X}_i, \dots, \check{X}'_j, \dots, X'_{p+2})$$
$$+\sum_{i<k<j}(-1)^{k+1} X'_k\theta(X'_1, \dots, \check{X}'_i, \dots, \check{X}'_k, \dots, \check{X}'_j, \dots, X'_{p+2})$$
$$+\sum_{i<j<k}(-1)^k X'_k\theta(X'_1, \dots, \check{X}'_i, \dots, \check{X}'_j, \dots, \check{X}'_k, \dots, X'_{p+2})$$
$$+\sum_{l<i<j}(-1)^l\theta([[X'_i, X'_j]_{\varphi_{T''}}, X'_l]_{\varphi_{T''}}, X'_1, \dots, \check{X}'_l, \dots, \check{X}'_j, \dots, \check{X}'_l, \dots, X'_{p+2})$$
$$+\sum_{i<l<j}(-1)^{l+1}\theta([[X'_i, X'_j]_{\varphi_{T''}}X'_l]_{\varphi_{T''}}, X'_1, \dots, \check{X}'_i, \dots, \check{X}'_l, \dots, \check{X}'_j, \dots, X'_{p+2})$$
$$+\sum_{i<j<l}(-1)^l\theta([[X'_i, X'_j]_{\varphi_{T''}}, X'_l]_{\varphi_{T''}}, X'_1, \dots, \check{X}'_i, \dots, \check{X}'_j, \dots, \check{X}'_l, \dots, X'_{p+2}).$$

While

$$(\lambda^{(2)}_\varphi)^{-1}p(\varphi)(X'_i, X'_j)=[X'_i, X'_j]-[X'_i, X'_j]_{\varphi_{T''}}.$$

From these formulas, we have Theorem 4. 9. Q. E. D.

Theorem 4. 10. *With the notation as above, we have*

$$\bar{\partial}^\varphi_b(p(\varphi))=0 \text{ for } \varphi\in\Gamma(M, T'N\otimes(^\circ T'')^*).$$

Proof. The following schema commutes:

$$\begin{array}{llcl}
\bar{\partial}^\varphi_b & : \Gamma(M, T'N\otimes\overset{2}{\wedge}(^\circ T'')^*) & \longrightarrow & \Gamma(M, T'N\otimes\overset{3}{\wedge}(^\circ T'')^*) \\
& \quad\uparrow \lambda^{(2)}_\varphi & & \quad\uparrow \lambda^{(3)}_\varphi \\
\bar{\partial}^{\varphi,(2)}_{T'N} & : \Gamma(M, T'N\otimes\overset{2}{\wedge}(^\varphi T'')^*) & \longrightarrow & \Gamma(M, T'N\otimes\overset{3}{\wedge}(^\varphi T'')^*) \\
& \quad\downarrow \varpi & & \quad\downarrow \varpi \\
\bar{\partial}^{(2)}_{^\varphi\hat{T}(M)} & : \Gamma(M, {}^\varphi\hat{T}(M)\otimes\overset{2}{\wedge}(^\varphi T'')^*) & \longrightarrow & \Gamma(M, {}^\varphi\hat{T}(M)\otimes\overset{3}{\wedge}(^\varphi T'')^*)
\end{array}$$

and $\lambda_\varphi^{(2)}$, $\lambda_\varphi^{(3)}$ and ϖ are isomorphisms. Therefore it is sufficient to show the relation $\bar\partial^{(2)}_{\varphi\hat{T}(M)}\varpi(\lambda_\varphi^{(2)})^{-1}(p(\varphi))=0$ in order to prove $\bar\partial_b^\varphi p(\varphi)=0$.

It is clear that $\varpi(\lambda_\varphi^{(2)})^{-1}(p(\varphi))(X', Y')=\varpi([X', Y'])$, for X', Y' in $\Gamma(W, {}^\varphi T'')$. In fact $p(\varphi)(X, Y)=[X+\varphi(X), Y+\varphi(Y)]-[X+\varphi(X), Y+\varphi(Y)]_{\varphi_{T''}}$, where X, Y in $\Gamma(M, {}^\circ T'')$. And so, we have

$$(\lambda_\varphi^{(2)})^{-1}p(\varphi)(X', Y')=[X', Y']-[X', Y']_{\varphi_{T''}}$$

where X', Y' in $\Gamma(M, {}^\varphi T'')$.

Therefore

$$\varpi\cdot(\lambda_\varphi^{(2)})^{-1}p(\varphi)(X', Y')=\varpi([X', Y']_{\varphi_{T''}}).$$

Now we put $\overline{p(\varphi)}=\varpi\cdot(\lambda_\varphi^{(2)})^{-1}p(\varphi)$ and compute $\bar\partial^{(2)}_{\varphi\hat{T}(M)}p(\varphi)=0$:

$$(4.15)\quad \begin{aligned}&\bar\partial^{(2)}_{\varphi\hat{T}(M)}\overline{p(\varphi)}(X', Y', Z')\\ &=X'\cdot\overline{p(\varphi)}(Y', Z')-Y'\cdot\overline{p(\varphi)}(X', Z')+Z'\cdot\overline{p(\varphi)}(X', Y')\\ &-\overline{p(\varphi)}([X', Y']_{\varphi_{T''}}, Z')+\overline{p(\varphi)}([X', Z']_{\varphi_{T''}}, Y')\\ &\qquad -\overline{p(\varphi)}([Y', Z']_{\varphi_{T''}}, X'),\end{aligned}$$

where $X'\cdot u=\varpi([X', \check{u}])$, $\check{u}$ is the inverse image of $u\in\Gamma({}^\varphi\hat{T}(M))$ in $\Gamma(M, T'N)$. Therefore

$$(4.16)\quad \begin{aligned}X'\cdot\overline{p(\varphi)}(Y', Z')&=X'\cdot\varpi([Y', Z'])\\ &=\varpi([X', [Y', Z']-[Y', Z']_{\varphi_{T''}}])\end{aligned}$$

From (4.16), the right hand side of (4.15) becomes

$$(4.17)\quad \begin{aligned}&\varpi([X', [Y', Z']-[Y', Z']_{\varphi_{T''}}]-[Y', [X', Z']-[X', Z']_{\varphi_{T''}}]\\ &+[Z', [X', Y']-[X', Y']_{\varphi_{T''}}])+\varpi(-[[X', Y']_{\varphi_{T''}}, Z']\\ &+[[X', Z']_{\varphi_{T''}}, Y']-[[Y', Z']_{\varphi_{T''}}, X']).\end{aligned}$$

Therefore (4.17) becomes to

$$(4.18)\quad \varpi([X', [Y', Z']]-[Y', [X', Z']]+[Z', [X', Y']])=0.$$

Q. E. D.

Moreover we have the following theorem. Compare Proposition 4.6 in [1].

Theorem 4.11. *If* φ *and* θ *are in* $\Gamma(M, T'\otimes({}^{\circ}T'')^*)$, *then*

$$p(\varphi+\theta)-p(\varphi)\equiv\bar{\partial}_b^{\varphi}\theta \mod \theta^2.$$

Proof. From the definition of $\bar{\partial}_{T'N}^{(1)}$, we put $\bar{\partial}_{T'N}^{\varphi}u(Z')=[Z', u]-[Z', u]_{\varphi_{T''}}$. And we shall put $Z'\cdot u=[Z', u]-[Z', u]_{\varphi_{T''}}=[Z', u]_{T'N}-\varphi([Z', u]_{{}^{\circ}T''})$.

$$\begin{array}{ccccc}
\Gamma(M, T'N) & \longrightarrow & \Gamma(M, TN\otimes({}^{\circ}T'')^*) & \xrightarrow{\bar{\partial}_b^{\varphi}} & \Gamma(M, T'N\otimes\overset{2}{\wedge}({}^{\circ}T'')^*) \\
\| & & \uparrow \lambda_{\varphi}^{(1)} & & \uparrow \lambda_{\varphi}^{(2)} \\
\Gamma(M, T'N) & \longrightarrow & \Gamma(M, T'N\otimes({}^{\varphi}T'')^*) & \xrightarrow{\bar{\partial}_{T'N}^{\varphi,(1)}} & \Gamma(M, T'N\otimes\overset{2}{\wedge}({}^{\varphi}T'')^*)
\end{array}$$

From the definition of $\bar{\partial}_b^{\varphi}$, $\bar{\partial}_b^{\varphi}=\lambda_{\varphi}^{(2)}\cdot\bar{\partial}_{T'N}^{\varphi,(1)}\cdot(\lambda_{\varphi}^{(1)})^{-1}$.

For θ in $\Gamma(M, T'N\otimes({}^{\circ}T'')^*)$,

$$\begin{aligned}
(4.19)\qquad \bar{\partial}_b^{\varphi}\theta(X, Y) &=\lambda_{\varphi}^{(2)}\cdot\bar{\partial}_{T'N}^{\varphi,(1)}\cdot(\lambda_{\varphi}^{(1)})^{-1}\theta(X, Y) \\
&=\bar{\partial}_{T'N}^{\varphi,(1)}(\lambda_{\varphi}^{(1)})^{-1}\theta(X+\varphi(X), Y+\varphi(Y)) \\
&=(X+\varphi(X))\cdot(\lambda_{\varphi}^{(1)})^{-1}(Y+\varphi(Y)) \\
&\quad -(Y+\varphi(Y))(\lambda_{\varphi}^{(1)})^{-1}\theta(X+\varphi(X)) \\
&\quad -(\lambda_{\varphi}^{(1)})^{-1}\theta([X+\varphi(X), Y+\varphi(Y)]_{\varphi_{T''}}). \\
&=(X+\varphi(X))\theta(Y)-(Y+\varphi(Y))\theta(X) \\
&\quad -\theta([X+\varphi(X), Y+\varphi(Y)]_{{}^{\circ}T''}).
\end{aligned}$$

While

$$\begin{aligned}
(4.20)\quad & p(\varphi+\theta)(X, Y) \\
&=[X+(\varphi+\theta)(X), Y+(\varphi+\theta)(Y)]-[X+(\varphi+\theta)(X), \\
&\qquad\qquad Y+(\varphi+\theta)(Y)]_{\varphi+\theta_{T''}}.
\end{aligned}$$

$$\begin{aligned}
(4.21)\quad & p(\varphi)(X, Y) \\
&=[X+\varphi(X), Y+\varphi(Y)]-[X+\varphi(X), Y+\varphi(Y)]_{\varphi_{T''}}.
\end{aligned}$$

From (4.20) and (4.21)

$$(p(\varphi+\theta)-p(\varphi))(X, Y)$$

$$
\begin{aligned}
&=[X+(\varphi+\theta)(X),\ Y+(\varphi+\theta)(Y)]_{T'N}\\
&\quad-(\varphi+\theta)([X+(\varphi+\theta)(X),\ Y+(\varphi+\theta)(Y)]_{\circ T''})\\
&\quad-[X+\varphi(X),\ Y+\varphi(Y)]_{T'N}+\varphi([X+\varphi(X),\ Y+\varphi(X)]_{\circ T''})\\
&=[\theta(X),\ Y+\varphi(Y)]_{T'N}+[X+\varphi(X),\ \theta(Y)]_{T'N}\\
&\quad-\varphi([X+\varphi(X)\cdot\theta(Y)]_{\circ T''}+[\theta(X),\ Y+\varphi(Y)]_{\circ T''})\\
&\quad-\theta([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''}) \mod \theta^2\\
&=\{[X+\varphi(X),\ \theta(Y)]_{T'N}-\varphi([X+\varphi(X),\ \theta(Y)]_{\circ T''})\}\\
&\quad-\{[Y+\varphi(Y),\ \theta(X)]_{T'N}-\varphi(Y+\varphi(Y),\ \theta(X)]_{\circ T''})\}\\
&\quad-\theta([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''})\\
&=(X+\varphi(X))\cdot\theta(Y)-(Y+\varphi(Y))\cdot\theta(X)\\
&\quad-\theta([X+\varphi(X),\ Y+\varphi(Y)]_{\circ T''})\\
&=\bar{\partial}_b^\varphi\theta(X,\ Y).
\end{aligned}
$$

Q. E. D.

Appendix

Intrinsic formula for Kuranishi's $\bar{\partial}_b^\varphi$ on homorphic vector bundles: Here we start with a given holomorphic vector bundle E on a partially complex manifold M, (in the sence of Tanaka [3]), and consider its deformations. We shall deal with the counterpart of Kuranishi's theory in this setting.

The formulation in this work develops into a briefer treatment of Kuranishi's formulation in [1].

§ A 1. Almost Holomorphic Vector Bundles

Let M be a partially complex manifold with a subbundle S of $C\otimes TM$, as described in the introduction, and let E be a C^∞ vector bundle on M.

We shall define a structure of an almost holomorphic vector bundle on E as follows;

Definition A1. 1. $(E,\ \bar{\partial}_E)$ is an almost holomorphic vector bundle if it satisfies the following relations: $\bar{\partial}_E$ is a differential operator from $\Gamma(E)$ to $\Gamma(E\otimes(\bar{S})^*)$, and satisfies the following relation:

$$\text{(A1. 1)} \qquad X(fu) = Xf\cdot u + fX\cdot u,$$

where u is in $\Gamma(E)$, X is in $\Gamma(S)$ and f is a C^∞ function on M, and we put

$$\text{(A1. 2)} \qquad X\cdot u = \bar{\partial}_E u(X).$$

Proposition A1. 2. *Let* $(E, \bar{\partial}_E)$ *be an almost holomorphic vector bundle. For any* $\phi\in\Gamma(E\otimes\overset{p}{\wedge}(\bar{S})^*)$, *we define* $\bar{\partial}_E^{(p)}\phi$ *by*

$$\begin{aligned}\bar{\partial}_E^{(p)}\phi(X_1,\dots, X_{p+1}) &= \sum_j (-1)^{j+1} X_j\cdot\phi(X_1,\dots,\check{X}_j,\dots, X_{p+1})\\ &+\sum_{i<j}(-1)^{i+j}\phi([X_i, X_j],\dots,\check{X}_i,\dots,\check{X}_j,\dots, X_{p+1})\end{aligned}$$

where X_i *is in* $\Gamma(\bar{S})$ *and we put* $Z\cdot u=\bar{\partial}_E u(Z)$ *for* Z *in* $\Gamma(\bar{S})$ *and* u *in* $\Gamma(E)$. *Then* $\bar{\partial}_E^{(p)}$ *is a map*

$$\bar{\partial}_E^{(p)}: \Gamma(E\otimes\overset{p}{\wedge}(\bar{S})^*)\to\Gamma(E\otimes\overset{p+1}{\wedge}(\bar{S})^*),$$

that is $(\bar{\partial}_E^{(p)}\phi)(X_1,\dots, X_{p+1})$ *is skew symmetric and multilinear in* X's, *with functions as coeffcient.s.*

Proof. It is clear that $\bar{\partial}_E^{(p)}\phi$ is skew symmetric. So it is sufficient to prove the following relation:

$$\text{(A1. 3)} \qquad \bar{\partial}_E^{(p)}\phi(fX_1, X_2,\dots, X_{p+1}) = f\bar{\partial}_E^{(p)}\phi(X_1, X_2,\dots, X_{p+1}),$$

where X_i are in $\Gamma(\bar{S})$ and f is a C^∞ function. To see this we proceed as follows.

$$\begin{aligned}\text{(A1. 4)} \qquad \bar{\partial}_E^{(p)}\phi(fX_1, X_2,\dots, X_{p+1}) &= fX_1\cdot\phi(X_2,\dots, X_{p+1})\\ &+\sum_{j\geq 2}(-1)^{j+1}X_j\cdot\phi(fX_1,\dots,\check{X}_j,\dots, X_{p+1})\\ &+\sum_{1<j}(-1)^{1+j}\phi([fX_1, X_j],\dots,\check{X}_j,\dots, X_{p+1})\\ &+\sum_{1<i<j}(-1)^{i+j}\phi([X_i, X_j], fX_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots, X_{p+1}).\end{aligned}$$

$$\text{(A1. 5)} \qquad [fX_1, X_j] = -(X_j f)X_1 + f[X_1, X_j].$$

$$\text{(A1. 6)} \qquad \phi(fX_1,\dots,\check{X}_j,\dots, X_{p+1}) = f\phi(X_1,\dots,\check{X}_j,\dots, X_{p+1})$$

since $\phi\in\Gamma(E\otimes\overset{p}{\wedge}(\bar{S})^*)$.

$$\begin{aligned}\text{(A1. 7)} \qquad X_j(f\phi(X_1,\dots,\check{X}_j,\dots, X_{p+1})) &= X_j f\cdot\phi(X_1,\dots,\check{X}_j,\dots, X_{p+1})\\ &+fX_j\cdot\phi(X_1,\dots,\check{X}_j,\dots, X_{p+1}).\end{aligned}$$

$(E, \bar{\partial}_E)$ being an almost holomorphic vector bundle, (A1. 3) follows from these formulas.

Proposition A1. 3. *Let $(E, \bar{\partial}_E)$ be an almost holomorphic vector bundle. Then, we can introduce an almost holomorphic structure on* End$E$ *in a natural way.*

Proof. It is sufficient to give the operator $\bar{\partial}_{\mathrm{End}E}$ from $\Gamma(\mathrm{End}E)$ to $\Gamma(\mathrm{End}E \otimes (\bar{S})^*)$. We do this as follows; For any M in $\Gamma(\mathrm{End}E)$, we put

$$(\bar{\partial}_{\mathrm{End}E} M)u = \bar{\partial}_E(Mu) - M\bar{\partial}_E u, \text{ for } u \text{ in } \Gamma(E).$$

This gives the desired map by virtue of the fact that $(E, \bar{\partial}_E)$ is almost holomorphic.

N. Tanaka defines a holomorphic vector bundle $(E, \bar{\partial}_E)$ as follows (N. Tanaka [3]):

① $(E, \bar{\partial}_E)$ is almost holomorphic,

② $[X, Y]u = X(Yu) - Y(Xu)$,

for X, Y in $\Gamma(\bar{S})$, where we put $Z \cdot u = \bar{\partial}_E u(Z)$.

We call the condition ② the integrability condition. If $(E, \bar{\partial}_E)$ is a holomorphic vector bundle, we have the following.

Proposition A1. 4. *Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle. Then. $(\Gamma(E \otimes \overset{p}{\wedge}(\bar{S})^*), \bar{\partial}_E^{(p)})$ is a complex.*

For the proof, see Tanaka [3].

We shall prove the inverse.

Proposition A1. 5. *Let $(E, \bar{\partial}_E)$ be an almost holomorphic vector bundle. If $(\Gamma(E \otimes \overset{p}{\wedge}(\bar{S})^*), \bar{\partial}_E^{(p)})$ is a complex, then $(E, \bar{\partial}_E)$ is a holomorphic vector bundle.*

Proof. From the assumption, we have the following.

$$\bar{\partial}_E^{(1)} \cdot \bar{\partial}_E = 0. \tag{A1. 8}$$

So, for any u in $\Gamma(E)$ and for any X, Y in $\Gamma(\bar{S})$, we have

(A1. 9) $$\bar{\partial}_E^{(1)}\cdot\bar{\partial}_E u(X, Y)=0.$$

(A1. 10) $$X\cdot\bar{\partial}_E u(Y)-Y\cdot\bar{\partial}_E u(X)-\bar{\partial}_E u([X, Y])=0.$$

Therefore

(A1. 11) $$X\cdot Y\cdot u-Y\cdot X\cdot u-[X, Y]u=0.$$

Thus we have proved Proposition A 1. 5.

Proposition A1. 6. *Let $(E, \bar{\partial}_E)$ be an almost holomorphic vector bundle. Then, for any ω in $\Gamma(\mathrm{End}E\otimes(\bar{S})^*)$, $(E, \bar{\partial}_E^\omega)$ is an almost holomorphic vector bundle, where, $\bar{\partial}_E^\omega=\bar{\partial}_E+\omega$. Conversely any almost holomorphic structure on M is of the above form.*

Proof. From the assumption, we have the relation $\bar{\partial}_E^\omega u=\bar{\partial}_E u+\omega(u)$, where ω is in $\Gamma(\mathrm{End}E\otimes(\bar{S})^*)$ and u is in $\Gamma(E)$. Therefore we have the following relation:

(A1. 12) $$\begin{aligned} X(fu)=\bar{\partial}_E^\omega fu(X) &=\bar{\partial}_E fu(X)+\omega(fu)(X)\\ &=f\bar{\partial}_E u(X)+Xf\cdot u+f\omega(u)(X)\\ &=fXu+(Xf)u. \end{aligned}$$

Conversely, let $(E, \bar{\partial}_{E'})$ be another holomorphic structure on E. Then, we put $\omega'=\bar{\partial}_{E'}-\bar{\partial}_E$. Since we have the followings

$$\bar{\partial}'_E fu(X)=f\bar{\partial}'_E u(X)+Xfu$$

and

$$\bar{\partial}_E fu(X)=f\bar{\partial}_E u(X)+Xfu,$$

the operator w is linear with the C^∞ functions on M as coefficients. Therefore ω is in $\Gamma(\mathrm{End}E\otimes(\bar{S})^*)$. Q. E. D.

§ A2. Kuranishi's Formula

In this section, let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle and ω be in $\Gamma(\mathrm{End}E\otimes(\bar{S})^*)$. We shall study the structure of $\bar{\partial}_E^\omega=\bar{\partial}_E+\omega$.

Proposition A2.1. *$(E, \bar\partial_E^\omega)$ is a holomorphic vector pundle if and only if ω satisfies the relation $p(\omega)=\bar\partial^{(1)}_{\mathrm{End}E}\omega+\omega\wedge\omega=0$.*

Compare this proposition with Theorem 2.1 in this paper.

From Proposition A1.6, $(E, \bar\partial_E^\omega)$ is an almost holomorphic vector bundle and we can define $\bar\partial_E^{(p)\cdot\omega}$ from Proposition 1.2. And we have the following proposition.

Proposition A2.2.

$$(\bar\partial_E^{(p+1),\omega}\bar\partial_E^{(p),\omega}\varphi)(X_1, X_2, \dots, X_{p+2})$$
$$=\sum_{i<j}p(\omega)(X_i, X_j)\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2}),$$

where φ is in $\Gamma(E\otimes\overset{p}{\wedge}(\bar S)^*)$. Hence the operator $\bar\partial_E^{(p+1),\omega}\bar\partial_E^{(p),\omega}$ is a differential operator of order 0.

Proof. For any φ in $\Gamma(E\otimes\overset{p}{\wedge}(\bar S)^*)$, we have the relation by simple calculation.

$$\text{(A2.1)}\quad (\bar\partial_E^{(p+1),\omega}\bar\partial_E^{(p),\omega}\varphi)(X_1, \dots, X_{p+2})$$
$$=\sum_{i<j}(X_i\cdot X_j-X_j\cdot X_i-[X_i, X_j])\cdot\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2}),$$

where we put $Z\cdot u=\bar\partial_E^\omega u(Z)$ for u in $\Gamma(E)$ and Z in $\Gamma(\bar S)$. And so

$$\text{(A2.2)}\quad X_j\cdot\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})$$
$$=\bar\partial_E\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})(X_j)+\omega(X_j)\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2}),$$

$$\text{(A2.3)}\quad X_i\cdot X_j\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})$$
$$=\bar\partial_E\{\bar\partial_E\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})(X_j)$$
$$+\omega(X_j)\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})\}(X_i)$$
$$+\omega(X_i)\{\bar\partial_E\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})(X_j)$$
$$+\omega(X_j)\varphi(X_1, \dots, \check X_i, \dots, \check X_j, \dots, X_{p+2})\}$$

while we have the following relation:

$$\text{(A2.4)}\quad (\bar\partial^{(p),\omega}_{\mathrm{End}E}\psi)u=\bar\partial_E^{(p),\omega}(\psi u)+(-1)^{p+1}\psi\bar\partial_E^\omega u$$

for u in $\Gamma(E)$ and ψ in $\Gamma(\mathrm{End}E\otimes\overset{p}{\wedge}(\bar S)^*)$. (This is a direct consequence of the definition of the almost holomorphic structure on End$E$.) In particular for $p=1$, we have

(A2.5) $$(\bar{\partial}^{(1)}_{\mathrm{End}E}\omega)u=\bar{\partial}^{(1)}_{E}(\omega u)+\omega\bar{\partial}_{E}u.$$

Therefore we have the following relation:

(A2.6)
$$\begin{aligned}&(\bar{\partial}^{(p+1),\omega}_{E}\bar{\partial}^{(p),\omega}_{E}\varphi)(X_1,\dots,X_{p+2})\\
&=\sum_{i<j}(X_i\cdot X_j-X_j\cdot X_i-[X_i,X_j])\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2})\\
&=\sum_{i<j}\{\bar{\partial}_E(\omega(X_j)\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2}))(X_i)\\
&\quad-\bar{\partial}_E(\omega(X_j)\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2}))(X_i)\\
&\quad+\omega(X_i)\bar{\partial}_E\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2})(X_j)\\
&\quad-\omega(X_j)\bar{\partial}_E\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots X_{p+2})(X_i)\\
&\quad+\omega(X_i)\cdot\omega(X_j)\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2})\\
&\quad-\omega(X_j)\cdot\omega(X_i)\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2})\\
&=\sum_{i<j}\{(\bar{\partial}^{(1)}_{\mathrm{End}E}\omega+\omega\wedge\omega)(X_i,X_j)\}\varphi(X_1,\dots,\check{X}_i,\dots,\check{X}_j,\dots,X_{p+2}).\end{aligned}$$

Q. E. D.

We shall study $\bar{\partial}^{\omega}_{\mathrm{End}E}$. From the definition,

$$(\bar{\partial}^{\omega}_{\mathrm{End}E}M)u=\bar{\partial}^{\omega}_{E}(Mu)-M\bar{\partial}^{\omega}_{E}u,$$

where u is in $\Gamma(E)$ and M is in $\Gamma(\mathrm{End}E\otimes(\bar{S})^*)$. Hence,

$$\begin{aligned}(\bar{\partial}^{\omega}_{\mathrm{End}E}M)u&=\bar{\partial}^{\omega}_{E}(Mu)-M\bar{\partial}^{\omega}_{E}u\\
&=\bar{\partial}_{E}(Mu)+w(Mu)-M(\bar{\partial}_{E}u+\omega u)\\
&=(\bar{\partial}_{\mathrm{End}E}M)u+\omega\cdot Mu-M\cdot\omega u.\end{aligned}$$

So, we have the relation:

$$\bar{\partial}^{\omega}_{\mathrm{End}E}M=\bar{\partial}^{(p)}_{\mathrm{End}E}M+\omega M-M\omega.$$

Similary, we have the relation:

$$\bar{\partial}^{(p),\omega}_{\mathrm{End}E}M=\bar{\partial}^{(p)}_{\mathrm{End}E}M+\omega\wedge M+(-1)^{p+1}M\wedge\omega,$$

for M in $\Gamma(\mathrm{End}E\otimes\bigwedge^{p}(\bar{S})^*)$.

We can derive the following proposition from these considerations.

Proposition A2.3. $\bar{\partial}^{(2),\omega}_{\mathrm{End}E}p(\omega)=0$.

Proof. Let us calculate $\bar{\partial}^{(2),\omega}_{\mathrm{End}E} p(\omega)$.
From the relation $p(\omega)=\bar{\partial}^{(1)}_{\mathrm{End}E}\omega+\omega\wedge\omega$, we have

$$\bar{\partial}^{(2),\omega}_{\mathrm{End}E}(p(\omega))=\bar{\partial}^{(2)}_{\mathrm{End}E}p(\omega)+\omega\wedge p(\omega)-p(\omega)\wedge\omega,$$

and

$$\begin{aligned}\bar{\partial}^{(2)}_{\mathrm{End}E}(p(\omega)) &= \bar{\partial}^{(2)}_{\mathrm{End}E}(\bar{\partial}^{(1)}_{\mathrm{End}E}\omega+\omega\wedge\omega)\\ &= (\bar{\partial}^{(1)}_{\mathrm{End}E}\omega)\wedge\omega-\omega\wedge(\bar{\partial}^{(1)}_{\mathrm{End}E}\omega)\\ &= p(\omega)\wedge\omega-\omega\wedge p(\omega).\end{aligned}$$

Hence $\bar{\partial}^{(2),\omega}_{\mathrm{End}E}p(\omega)=0$, Q. E. D.

Proposition A2. 4.

$$p(\omega+\theta)-p(\omega)\equiv\bar{\partial}^{(1)\omega}_{\mathrm{End}E}\theta \pmod{\theta^2}$$

for $\omega,\ \theta\in\Gamma(\mathrm{End}E\otimes(\bar{S})^*)$.

Proof.

$$p(\omega+\theta)=\bar{\partial}^{(1)}_{\mathrm{End}E}(\omega+\theta)+(\omega+\theta)\wedge(\omega+\theta),$$
$$p(\omega)=\bar{\partial}^{(1)}_{\mathrm{End}E}\omega+\omega\wedge\omega,$$

while $\bar{\partial}^{(1)\omega}_{\mathrm{End}E}\theta=\bar{\partial}^{(1)}_{\mathrm{End}E}\theta+\omega\wedge\theta+\theta\wedge\omega.$

From these formulas Proposition A 2. 4 follows.

Proposition A2. 5. $\bar{\partial}^{(2),\omega}_{E}(\omega(\bar{\partial}_E f))=p(\omega)(\bar{\partial}_E f)$ *where* f *is* C^∞ *function.*

Proof.

$$\begin{aligned}\bar{\partial}^{(2),\omega}_{E}(\omega(\bar{\partial}_E f)) &= \bar{\partial}^{(2)}_{E}(\omega(\bar{\partial}_E f))+\omega(\omega(\bar{\partial}_E f))\\ &= (\bar{\partial}^{(1)}_{\mathrm{End}E}\omega+\omega\wedge\omega)\bar{\partial}_E f\\ &= p(\omega)(\bar{\partial}_E f).\end{aligned}$$

Q. E. D.

References

[1] Kuranishi, M., Deformation of isolated singularities and $\bar{\partial}_b^{\varphi}$, in preprint.

[2] Kohn, J. J., Boundary of complex manifolds. *Proceeding of the Conference on Complex Analysis*, 1964.

[3] Tanaka, N., "A differential geometric study on strongly pseudo convex manifolds". *Lectures in Mathematics, Department of Mathematics, Kyoto University.*