*Publ. RIMS, Kyoto Univ.*
**14** (1978), 655-672

# On The Equivariant Isotopy Classes of Some Equivariant Imbeddings of Spheres

By

Kōjun ABE*

## § 0. Introduction

The purpose of this paper is to study the $G$-isotopy classes of $G$-imbeddings of spheres into spheres, where the spheres are equipped with semi-free linear $G$-actions for a finite group $G$.

Let $V$ be an $m$-dimensional real $G$-module. Throughout this paper we shall assume that $V$ is a product module $V=R^n\oplus V_1$ of a trivial real $G$-module $R^n$ of positive dimension $n$ and an $(m-n)$-dimensional real $G$-module $V_1$ on the $G$-invariant unit sphere $S(V_1)$ of which $G$ acts freely. Let $W$ be a real $G$-module which contains $V$ as a direct summand. Let $S^V$ and $S^W$ denote the one-point compactifications of $V$ and $W$ respectively. Then $S^V$ and $S^W$ are spheres on which $G$ acts linearly. The direct sum of $d$ copies of $V$ will be denoted by $dV$.

**Theorem A.** *Let $G$ be a cyclic group $Z_q$ and let $W=dV\oplus R^k$* for *$k>m+1$. If $d\geqq\max\ \{(n+3)/2,\ (m+2)/(m-n)\}$*, then any *$G$-imbedding of $S^V$ into $S^W$ is $G$-isotopic to the standard imbedding.*

**Theorem B.** *Let $G$ be a cyclic group $Z_q$ for $q>2$ and let $W=dV\oplus R^k$ for $k>m+1$. Suppose that $d\geqq(m+1)/(m-n)$ and $V_1$ is a direct sum of $(m-n)/2$ copies of an irreducible 2-dimensional real $G$-module.*

(1) *If $d=(m+1)/(m-n)$, then there are infinitely many $G$-*

---

Communicated by N. Shimada, July 4, 1977.
* Department of Mathematics, Shinshu University, Matsumoto, 390, Japan.

*imbeddings of $S^V$ into $S^W$ which are not G-isotopic to each other, and*

(2) *if $d>(m+1)/(m-n)$, then any G-imbedding of $S^V$ into $S^W$ is G-isotopic to the standard imbedding.*

The paper is organized as follows. For any $G$-imbedding $f: S^V \to S^W$, we shall show that, by $G$-isotopies, $f|S^n$ can be deformed to be standard in § 1, $f$ can be deformed to be linear on a neighborhood of $S^n$ in § 2 and $f$ can be deformed to be orthogonal on a neighborhood of $S^n$ in § 3. Moreover we shall prove that, if two $G$-imbeddings of $S^V$ into $S^W$ are $G$-isotopic and are orthogonal on a neighborhood of $S^n$, then there exists a $G$-isotopy between them which is orthogonal on a neighborhood of $S^n$ in §3. Then we see that the $G$-isotopy class of $f$ is determined by the homotopy class of the orbit map of $f|(S^V-U)$ relative to the boundary, where $U$ is a neighborhood of $S^n$. In § 4, using the obstruction theory, we shall prove Theorem A and Theorem B.

The author wishes to thank Professors M. Adachi and T. Matumoto for their kind advices and valuable criticism.

## § 1. Imbeddings Can Be Deformed to Be Standard on the Fixed Point Set

In this paper we shall assume that all manifolds and all actions are differentiable of class $C^\infty$. Until Section 3 the results are valid in the case of $G$ a compact Lie group.

In this section we shall prove that any $G$-imbedding of $S^V$ into $S^W$ is $G$-isotopic to a $G$-imbedding which is standard on $S^n$ (see Proposition 1. 3), and if two $G$-imbeddings of $S^V$ into $S^W$, which are standard on $S^n$, are $G$-isotopic, then there exists a $G$-isotopy between them which is standard on $S^n$ (see Proposition 1. 4).

**Definition 1. 1.** Let $M$ be a $G$-submanifold of a $G$-manifold N. Let $I$ denote the unit interval $[0, 1]$ with trivial $G$-action. A smooth map (resp. smooth $G$-map) $f: M\times I\to N$ is said to be an isotopy (resp. $G$-isotopy) if each $f_t: M\to N$ is an imbedding (resp. $G$-imbedding), where $f_t(x)=f(x, t)$, and $f_t$ is independent of $t$ in some neighborhood of 0 and in some neighorhood of 1 (see G. Bredon [1, Chapter VI, § 3]). Two imbeddings (resp. $G$-imbeddings) $f_i: M \to N$ $(i=0, 1)$ are said to be isotopic (resp. $G$-isotopic) if there exists an isotopy (resp. $G$-isotopy) $F: M\times I\to N$ with $F_0=f_0$ and $F_1=f_1$. If $\partial M$ is not empty, we shall consider $M\times I$ as a smooth manifold with corners.

Let $I(S^V, S^W)$ denote the set of all $G$-isotopy classes of $G$-imbeddings $f: S^V\to S^W$. Our purpose is to determine the set of $I(S^V, S^W)$, provided that $W=dV\oplus R^k$ for $k>m+1$.

*Remarks.* **1.** It is easy to see that any $G$-map $f: S^V\to S^W$ is $G$-homotopic to the standard imbedding.

**2.** Using the method of A. Wasserman [7, § 1], we can see that any $G$-imbedding $f: S^V\to S^W$ is $G$-isotopic to the standard imbedding if $d>2m+2$.

The following lemma will be useful.

**Lemma 1.2.** *Let $N$ be a $q$-dimensional manifold on which $G$ acts semi-freely and let $M$ be a $p$-dimensional $G$-submanifold of $N$. Let $K$ denote $I$ or $I\times I$ and let $L$ be a closed subset of $K$ which contains $\partial K$. Let $f: M\times K\to N$ be a continuous $G$-map such that each $f_k: M\to N$ is a $G$-imbedding, where $f_k$ is defined by $f_k(x)=f(x, k)$. If $f$ is a smooth $G$-map on $M\times L'$, where $L'$ is a neighborhood of $L$ in $K$, then there exists a smooth $G$-map $H: M\times K\to N$ such that each $H_k$ is a $G$-imbedding and $H=f$ on $M\times L$, where $H_k: M\to N$ is defined by $H_k(x)= H(x, k)$.*

*Proof.* We shall prove Lemma 1.2 by an equivariant version of J. Munkres' argument [4, Chapter I, § 4]. Let $\{U_i\}$ (resp. $\{V_j\}$) be a family of locally finite countable invariant open sets of $M$ such that $\bigcup_i U_i\subset M^G$ (resp. $\bigcup_j V_j\supset N^G$) and $\bar{U}_i$ (resp. $V_j$) is equivariant diffeomorphic to a $p$-dimensional disc or half disc (resp. $q$-dimensional euclidean space or euclidean half space) with linear $G$-action, where $M^G$ and $N^G$ denote the fixed point set of $M$ and $N$ respectively. We can choose the family $\{U_i\}$ such that, for any $k\in K$ and for any $i$, $f_k(\bar{U}_i)$ is contained in $V_j$ for some $j$ depending on $k$ and $i$. There exists a positive continuous function $\delta_1$ on $M$ as follows. For any continuous map $g : M\times K\to N$, such that $g_k$ is $\delta_1$-approximation to $f_k$ for each $k$, has these properties. Let $\{W_i\}$ be a family of invariant open sets of $M$ with $\bar{W}_i\subset U_i$ and $\bigcup_i W_i\supset M^G$.

Let $C_r(y)$ denote a closed $r$-neighborhood of $y$ in $K$ for a positive number $r$ and $y\in K$. There exists a sufficiently small positive number $r$ such that, for any $z\in C_r(y)$ and for any $y\in K$, $f_z(\bar{U}_1)$ is contained in $V_j$ for some $j$ depending on $y$. Then we can find a finite number of $C_r(y)$, say $C_n=C_r(y_n)$ $(n=1, 2,\dots, l)$, such that $\bigcup_{n=1}^{l} C_n\subset K-L'$. We can assume that $C_n\cap L=\emptyset$ for any $n$. Let $\phi_1 : M\to I$ be an invariant smooth function on $M$ which equals 1 on $\bar{W}_1$ and 0 outside of $U_1$. Let $A_n$ and $B_n$, $n=1, 2,\dots, l$, be open sets of $K$ such that $\bar{A}_n\subset B_n\subset \bar{B}_n\subset$ int $C_n$ and $\bigcup_{n=1}^{l} A_n$ contains $K-L'$. Let $\xi_n : K\to I$, $n=1, 2,\dots, l$, be smooth functions on $K$ which equals 1 on $\bar{A}_n$ and 0 outside of $B_n$.

We shall identify $U_i$ and $V_j$ as euclidean spaces or euclidean half spaces with linear $G$-actions. For any $n$, we can find $j(n)$ such that $f_z(\bar{U}_1)\subset V_{j(n)}$ for any $z\in C_n$. Let $f^{1,n} : M\times C_n\to V_{j(n)}$ be a $G$-map defined by $f^{1,n}_z(x)=\phi_1(x)\cdot f_z(x)$ for $x\in U_1$ and $z\in C_n$, and $f^{1,n}=0$ outside of $U_1\times C_n$. Let $g^{1,n} : M\times K\to V_{j(n)}$ be a $G$-map defined by $g^{1,n}_z(x)=\xi_n(z)\cdot f^{1,n}_z(x)$ for $x\in M$ and $z\in K$. Since $g^{1,n}_z=0$ for $z\notin C_n$, we can extend $g^{1,n}$ trivially on $M\times R$ (resp. $M\times R^2$) if $K=I$ (resp. $I\times I$). Define a smooth $G$-map $h^{1,n} : M\times K\to V_{j(n)}$ by

$$h^{1,n}_z(x)=\int_{C(\varepsilon_n)}\varphi_n(y)\cdot g^{1,n}_{z+y}(x)\,dy \quad \text{for } x\in M \text{ and } z\in K,$$

where $C(\varepsilon_n)$ is a closed $\varepsilon_n$-disc in $K$ and $\varphi_n$ is a smooth function on $R$ or $R^2$ which is positive on int $C(\varepsilon_n)$ and 0 outside of $C(\varepsilon_n)$ and $\int_{C(\varepsilon_n)}\varphi_n(y)\,dy=1$. Choose the positive number $\varepsilon_n$ less than the distance from $B_n$ to the complement of $C_n$. Then $h^{1,n}=0$ outside of $U_1\times C_n$.

Let $F^{1,0}=f$. Assume that $F^{1,n-1} : M\times K\to N$ is defined such that

$F^{1,n-1}$ is smooth on $\overline{W}_1\times(\bar{A}_1\cup\ldots\cup\bar{A}_{n-1})$ and $F^{1,n-1}=F^{1,n-2}$ outside of $U_1\times C_{n-1}$. Moreover we assume that $F_k^{1,n-1}$ is a $\delta_1/2^{l-n+2}$-approximation to $F_k^{1,n-2}$ for each $k\in K$. Then $F_z^{1,n-1}$ is a $\delta_1$-approximation to $f_z$. Let $F^{1,n}: M\times K\to N$ be a $G$-map defined by

$$F_z^{1,n}(x)=F_z^{1,n-1}(x)\ (1-\psi_1(x)\xi_n(z))+h_z^{1,n}(x),\ \text{for } x\in M \text{ and } z\in K.$$

Since $F^{1,n}=h^{1,n}$ on $\overline{W}_1\times\bar{A}_n$, $F^{1,n}$ is smooth on $\overline{W}_1\times(\bar{A}_1\cup\ldots\cup\bar{A}_n)$ (note that, if $F^{1,n-1}$ is smooth on a subset of $M\times K$, $F^{1,n}$ is smooth on the subset of $M\times K$). Since $h^{1,n}=0$ outside of $U_1\times C_n$, $F^{1,n}=F^{1,n-1}$ outside of $U_1\times C_n$. By the argument of J. Munkres [4, Chapter I, § 4], we can choose the positive numbers $\varepsilon_i$ $(i=1,2,\ldots,n)$ so small that $F_k^{1,n}$ is a $\delta_1/2^{l-n+1}$-approximation to $F_k^{n-1}$ for each $k$. Then we can see that $F_z^{1,n}$ is a $\delta_1$-approximation to $f_z$ and $F_z^{1,n+1}$ is defined. By the induction, we have a $G$-map $F^{1,l}: M\times K\to N$ such that $F^{1,l}$ is smooth on $\overline{W}_1\times(\bigcup_{n=1}^{l}\bar{A}_n)$ and $F^{1,l}=F^{1,l-1}$ outside of $U_1\times C_n$. Set $F^1=F^{1,l}$. Since $\bigcup_{n=1}^{l}\bar{A}_n$ contains $K-L'$, $F^1$ is smooth on $\overline{W}_1\times K$. And since $C_n\cap L=\emptyset$, $F^1=f$ on $M\times L$.

There exists a positive continuous function $\delta\leqq\delta_1$ on $M$ such that, for each $k\in K$, any $C^1$-map from $M$ to $N$, which is a $\delta$-approximation in $C^1$-topology to $f_k$, is an imbedding (see J. Munkres [4, Chapter I, Theorem 3.10]). We can choose the positive numbers $\varepsilon_n$, $n=1,2,\ldots,l$, so small that $F_k^1$ is a $\delta/2$-approximation to $f_k$ in $C^1$-topology for each $k\in K$.

By the induction we have $G$-maps $F^i: M\times K\to N$ $(i=2,3,\ldots)$, which is smooth on $(\overline{W}_1\cup\ldots\cup\overline{W}_i)\times K$, such that $F^i=f$ on $M\times L$ and $F^i=F^{i-1}$ outside of $U_i\times K$. Moreover we can choose $F_k^i$ is a $\delta/2^i$-approximation to $F_k^{i-1}$ in $C^1$-topology for each $k\in K$. Define a $G$-map $F: M\times K\to N$ by $F_k(x)=\lim_{i\to\infty}F_k^i(x)$ ; $F_k$ is well defined because $F_k^i=F_k^{i+1}=\ldots$ on some neighborhood of $x$, for sufficiently large $i$. $F: M\times K\to N$ is smooth on $(\bigcup_i\overline{W}_i)\times K$ and $F=f$ on $M\times L$. Moreover $F_k$ is a $\delta$-approximation to $f_k$ in $C^1$-topology, for each $k\in K$.

Let $T$ be a closed invariant neighborhood of $M^G$ in $M$ such that $T$ is contained in $\bigcup_i W_i$. $F_k(M-M^G)$ is contained in $N-N^G$, for each $k$, since $F_k$ is a $G$-imbedding. Let $\bar{F} : (M-M^G)/G\times K\to(N-N^G)/G$ be the orbit map of $F$. Then $\bar{F}$ is a smooth map on a neighborhood of $(T-M^G)/G\times K$ and $\bar{F}=\bar{f}$ on $(M-M^G)/G\times L$, and $\bar{F}_k$ is a $\delta$-approximation to $\bar{f}_k$ for each $k\in K$, where $\bar{f}_k$ is the orbit map of $f_k$. By the relative version of the argument of J. Munkres [4, Chapter I, §4], we have a smooth map $\bar{H}$: $(M-M^G)/G\times K\to(N-N^G)/G$ such that $\bar{H}=\bar{F}$ on $(T-M^G)/G\times K$ and $\bar{H}$ is homotopic to $\bar{F}$ relative to $(T-M^G)/G\times K\cup(M-M^G)/G\times L$. Moreover $\bar{F}_k$ is a $\delta$-approximation to $\bar{f}_k$ in $C^1$-topology, for each $k\in K$. By the covering homotopy property, we have a smooth $G$-map $H : (M-M^G)\times K\to N-N^G$ whose orbit map is $\bar{H}$. Define $H=F$ on $T\times K$. Then $H : M\times K\to N$ is a smooth $G$-map such that $H_k$ is a $\delta$-approximation to $f_k$ in $C^1$-topology, for each $k\in K$, and $H=f$ on $M\times L$. This completes the proof of Lemma 1.2.

Let $f : S^V\to S^W$ be a $G$-imbedding. The fixed point set of $S^V$ and $S^W$ are $S^n$ and $S^{dn+k}$ respectively. Let $f^G : S^n\to S^{dn+k}\subset S^W$ denote an imbedding which is a restriction of $f$ to $S^n$. Let $j : S^V\to S^W$ be the standard imbedding.

**Proposition 1.3.** *Let $f_0 : S^V\to S^W$ be a $G$-imbedding. Then there exists a $G$-isotopy $f : S^V\times I\to S^W$ between $f_0$ and $f_1$ with $f_1^G=j$ on $S^n$.*

*Proof.* Since $dn+k>2n$, we have an isotopy $h : S^{dn+k}\times I\to S^{dn+k}$ such that $h_0=1$ and $h_1\cdot f_0^G=j$. By the isotopy extension theorem, there exists an isotopy $H : S^W\times I\to S^W$ such that $H_0=1$ and $H=h$ on $S^{dn+k}\times I$. Using a result of G. Bredon [1, Chapter VI, Theorem 3.1], we have a $G$-isotopy $K : S^W\times I\to S^W$ such that $K_0=1$ and $K=H$ on $H^G\times I$, where $H^G=\{x\in S^W ; H_t(g\cdot x)=g\cdot H_t(x)$ for any $t\in I$ and $g\in G\}$. Note that $S^{dn+k}\subset H^G$. Let $f : S^V\times I\to S^W$ be a $G$-isotopy between $f_0$ and $f_1$ defined by $f_t=K_t\cdot f_0$. Then $f_1^G=j$ and this completes the proof of Proposition 1.3.

**Proposition 1.4.** *Let $f : S^V\times I\to S^W$ be a $G$-isotopy with $f_i^G=j$ for $i=0, 1$. Then there exists a $G$-isotopy $h : S^V\times I\to S^W$ such that*

$h_i=f_i$ *for* $i=0, 1$ *and* $h_t^G=j$ *for* $0\leqq t\leqq 1$.

*Proof.* Let $f: S^V\times I\to S^W\times I$ be a $G$-imbedding defined by $f(x, t) =(f_t(x), t)$. Let $f^G: S^n\times I\to S^{dn+k}\times I$ be an imbedding which is a restriction of $f$ to $S^n\times I$. Let $E(S^n, S^{dn+k})$ denote the set of all imbeddings of $S^n$ into $S^{dn+k}$ with $C^\infty$-topology. By a result of J. Dax [2, Chapter VI, § 3], $\pi_1(E(S^n, S^{dn+k}))=0$ since $dn+k>2n+2$. Then we have a continuous map $a: I\times I\to E(S^n, S^{dn+k})$ such that, for a sufficiently small $\varepsilon>0$,

$$a(t,s)=\begin{cases} f_t^G \text{ for } (t,s)\in I\times[0, \varepsilon] \\ j \text{ for } (t, s)\in[0, \varepsilon]\times I\cup I\times[1-\varepsilon, 1]\cup[1-\varepsilon, 1]\times \mathrm{I}. \end{cases}$$

Using Lemma 1. 2, we may assume that $\hat{a}: S^n\times I\times I\to S^{dn+k}\times I$ is an isotopy, where $\hat{a}(x, t, s)=(a(t, s)(x), t)$. Then we have an imbedding $\tilde{a}: S^n\times I\times R\to S^{dn+k}\times I\times R$ defined by

$$\tilde{a}(x, t, s)=\begin{cases} \hat{a}(x, t, s),s) \text{ for } 0\leqq s\leqq 1 \\ \hat{a}(x, t, 0),s) \text{ for } s<0 \\ \hat{a}(x, t, 1),s) \text{ for } s>1. \end{cases}$$

$\tilde{a}(S^n\times I\times R)$ is a closed $G$-submanifold of $S^W\times I\times R$, and $\tilde{a}(S^n\times I\times R)$ intersects normally on $\partial(S^n\times I\times R)$ with respect to a product $G$-invariant Riemannian metric on $S^W\times I\times R$. By using the proof of G. Bredon [1, Chapter IV, Theorem 2. 2] with respect to the Riemannian metric, we have an invariant open $\delta$-tubular neighborhood $N$ of $\tilde{a}(S^n\times I\times R)$, where $\delta$ is a $G$-invariant positive real valued function on $\tilde{a}(S^n\times I\times R)$.

The tangent vectors to the curves $\tilde{a}(x\times t\times R)$ define an invariant vector field $\tilde{X}$ on $\tilde{a}(S^n\times I\times R)$ of the form $\tilde{X}(\tilde{a}(x, t, s))=(X(x, s, t), 0, 1)\in T_{\tilde{a}(x,s,t)}(S^W\times I\times R)$, where $T(S^W\times I\times R)$ is the tangent bundle of $S^W\times I\times R$. Identifying $N$ with a $G$-invariant normal bundle to $\tilde{a}(S^n\times I\times R)$ in $S^W\times I\times R$, we denote $p: N\to\tilde{a}(S^n\times I\times R)$ the bundle projection. Let $r: I\to R$ be a $C^\infty$-function such that $r(t)=1$ for $0\leqq t\leqq 1/3$, $0<r(t)<1$ for $1/3<t<2/3$ and $r(t)=0$ for $2/3\leqq t\leqq 1$. Let $Y$ be a $G$-invariant vector field on $S^W\times I\times R$ defined by $Y(v)=r(||v||/\delta(p(v)))\cdot X(p(v))$ for $v\in N$ and $Y=0$ on the outside of $N$, where $||\ ||$ denote the $G$-invariant metric of $S^W\times I\times R$.

Since $\tilde{a}(x, t, s)=(j(x), t, s)$ for $0 \leqq t \leqq \varepsilon$ and $1-\varepsilon \leqq t \leqq 1$, and since $\tilde{a}(x, t, s)=(\hat{a}(x, t, 0), s)$ for $s \leqq 0$ and $\tilde{a}(x, t, s)=(\hat{a}(x, t, 1), s)$ for $s \geqq 1$, $Supp(Y)$ is contained in $S^W \times [\varepsilon, 1-\varepsilon] \times I$ which is compact. We can regard $Y$ as a time-dependent $G$-invariant vector field on $S^W \times I$, and $Y$ generates a $G$-isotopy $F : S^W \times I \times I \to S^W \times I$ (see M. Hirsch [3, Chapter 8, Theorem 1.1]). Since $I$ component of $Y$ is 0, each $F_s : S^W \times I \to S^W \times I$ is level preserving. Let $h : S^V \times I \to S^W$ be a $G$-isotopy defined by $h = p_1 \cdot F_1 \cdot \tilde{f}$, where $p_1 : S^W \times I \to S^W$ is the projection on the first factor. Then $h_t = f_t$ for $t=0$, 1 and $h_t = j$ on $S^n$ for each $t$. This completes the proof of Proposition 1.4.

## § 2. Linearlity on a Neighborhood of the Fixed Point Set

In this section we shall prove that any $G$-imbedding of $S^V$ into $S^W$ is $G$-isotopic to a $G$-imbedding which is linear on a neighborhood of $S^n$ (see Proposition 2.1), and if two $G$-imbeddings of $S^V$ into $S^W$, which are linear on a neighborhood of $S^n$, are $G$-isotopic, then there exists a $G$-isotopy between them which is linear on a neighborhood of $S^n$ (see Proposition 2.3).

Since the fixed point set of $S^V$ is $S^n$ and since $S^V$ is a $G$-submanifold of $S^W$, we can regard $S^n$ as a $G$-submanifold of $S^W$. Let $U$ and $N$ denote invariant open tubular neighborhoods of $S^n$ in $S^V$ and $S^n$ in $S^W$ respectively. We shall identify $U$ and $N$ with invariant normal bundles to $S^n$ in $S^V$ and to $S^n$ in $S^W$ respectively. Let $f : S^V \to S^W$ be a $G$-imbedding with $f^G = j$. We shall assume that $f(U)$ is contained in $N$. Let $f' : U \to N$ be a $G$-bundle monomorphism defined by the differential of $f$.

**Proposition 2.1.** *Let $f : S^V \to S^W$ be a $G$-imbedding with $f^G = j$. Then there exists a $G$-isotopy $h : S^V \times I \to S^W$ such that $h_0 = f$ and $h_1 = f'$ on some invariant neighborhood of $S^n$ in $S^V$.*

In order to prove Proposition 2.1, we start with the following lemma.

**Lemma 2. 2.** *Let $M$ be a G-submanifold of a G-manifold $N$. Let $f: M\times I\rightarrow N$ be a G-isotopy such that $f_t(\partial M)\subset\partial N$ and $f_t(M)$ intersects transversally on $\partial N$ for each $t$. Let $A$ be an invariant subspace of $M$ such that $\bar{A}$ is compact. Then there exists a G-isotopy $F: N\times I\rightarrow N$ such that $F_0=1$ and $F_t\cdot f_0=f_t$ on $A$ for $0\leqq t\leqq 1$.*

*Proof.* Let $\tilde{f}: M\times R\rightarrow N\times R$ be a $G$-imbedding defined by

$$\tilde{f}(x, t)=\begin{cases}(f_t(x), t) & \text{for } 0\leqq t\leqq 1\\ (f_0(x), t) & \text{for } t<0\\ (f_1(x), t) & \text{for } t>1.\end{cases}$$

We can assume that $G$ acts by isometries in some product metric on $N\times R$. Let $\nu$ be an invariant normal bundle of $\tilde{f}(M\times R)$ in $N\times R$ and let $p: \nu\rightarrow\tilde{f}(M\times R)$ be the projection. Then the exponential map is defined on some neighborhood of $\tilde{f}(M\times R)$ in $\nu$ and is an equivariant immersion on a smaller invariant open neighborhood of $\tilde{f}(M\times R)$ (see the proof of G. Bredon [1, Chapter VI, Theorem 2. 2]). Let $B$ be an invariant open neighborhood of $\bar{A}$ such that $\bar{B}$ is compact. Since $\bar{B}$ is compact, the exponential map is a $G$-imbedding on an invariant neighborhood of $\tilde{f}(\bar{B}\times I)$ in $\nu|\tilde{f}(\bar{B}\times I)$. By a method of the proof of G. Bredon [1, Chapter VI, Theorem 2. 2], we have a $G$-imbedding $\varphi: \nu|\tilde{f}(\bar{B}\times I)\rightarrow N\times R$. We shall identify $\nu|\tilde{f}(\bar{B}\times I)$ as the image of $\varphi$.

The tangent vectors to the curves $\tilde{f}(x\times R)$ $(x\in M)$ define an invariant vector field $\tilde{X}$ on $\tilde{f}(M\times R)$ of the form $\tilde{X}(\tilde{f}(x, t))=(X(x, t), 1)\in T_{f(x,t)}(N\times R)$. Note that $Supp(X)$ is contained in $\tilde{f}(M\times I)$. Take an invariant $C^\infty$-partition of unity subordinate to the covering $\{B, M-\bar{A}\}$ of $M$, and let $u$ be the invariant function correspondence to $B$. Let $X'$ be an invariant vector fields on $\tilde{f}(M\times R)$ defined by $X'(\tilde{f}(x, t))=u(x)\cdot X(x, t)$ and $X'=0$ outside of $\tilde{f}(B\times R)$. Then $Supp(X')$ is contained in $\tilde{f}(\bar{B}\times I)$ and $X'=X$ on $\tilde{f}(\bar{A}\times R)$.

Let $r: R\rightarrow[0, 1]$ be a $C^\infty$-function such that $r(t)=1$ for $t\leqq 1$, $0<r(t)<1$ for $1<t<2$ and $r(t)=0$ for $t\geqq 2$. Let $Y$ be an invariant vector field on $N\times I$ defined by $Y(v)=r(||v||)\cdot X'(p(v))$ on $\nu|\tilde{f}(\bar{B}\times I)$ and $Y=0$ outside of $\nu|\tilde{f}(\bar{B}\times I)$, where $||\ ||$ is an invariant Riemannian metric on $\nu$. Then we can regard $Y$ as a time-dependent invariant vector field on $N$. Note that $Supp(Y)$ is contained in $\nu(2)|\bar{B}\times I$ which is compact, where $\nu(2)=\{v\in\nu;\ ||v||\leqq 2\}$. Therefore $Y$ generates a $G$-isotopy $F: N\times I\to N$ such that $F_0=1$ and $F_t\cdot f_0=f_t$ on $A$ for $0\leqq t\leqq 1$. This completes the proof of Lemma 2.2.

*Proof of Proposition* 2.1. Let $g: U\times I\to N\hookrightarrow S^W$ be a homotopy of $G$-imbeddings defined by $g_t(v)=1/(1-t)\cdot f((1-t)v)$ for $0\leqq t<1$ and $v\in U$, and $g_1=f'$. Note that $g_0=f|U$, $\lim_{t\to 1} g_t=f'$ and $g_t$ is a $G$-imbedding for each $t$. By Lemma 1.2 we can assume that $g$ is a $G$-isotopy between $f|U$ and $f'$. By Lemma 2.2 there exists a $G$-isotopy $G: S^W\times I\to S^W$ such that $F_0=1$ and $F_t\cdot g_0=g_t$ on some neighborhood of $S^n$. Let $h: S^V\times I\to S^W$ be a $G$-isotopy defined by $h_t=F_t\cdot f$. Then $h_0=f$ and $h_1=f'$ on some neighborhood of $S^n$. This completes the proof of Proposition 2.1.

By Proposition 1.3 and Proposition 2.1, any element of $I(S^V, S^W)$ is represented by a $G$-imbedding $f: S^V\to S^W$ such that $f^G=j$ and $f=f'$ on an invariant tubular neighborhood of $S^n$.

**Proposition 2.3.** *Let $f: S^V\times I\to S^W$ be a $G$-isotopy such that $f_t^G=j$ $(0\leqq t\leqq 1)$ and $f_i=f_i'$ $(i=0, 1)$ on an invariant tubular neighborhood $U$ of $S^n$. Then there exists a $G$-isotopy $h: S^V\times I\to S^W$ such that $h_i=f_i$ $(i=0, 1)$ and $h_t=h_t'$ on an invariant neighborhood of $S^n$ for $0\leqq t\leqq 1$.*

*Proof.* Let $\tilde{f}: S^V\times I\to S^W\times I$ be a $G$-imbedding defined by $\tilde{f}(x, t)=(f_t(x), t)$. We can assume that $f_t(U)$ is contained in $N$ for each $t$. Let $\tilde{f}': U\times I\to N\times I$ be a $G$-imbedding defined by $\tilde{f}'(v, t)=(f_t'(v), t)$. Let $F: U\times I\times I\to N\times I$ be a $G$-map defined by $F_s(v, t)=(1/(1-s)\cdot f_t((1-s)v), t)$ for $0\leqq s<1$ and $F_1=f'$. Then $F_0=\tilde{f}$ and $\lim_{s\to 1} F_s=\tilde{f}'$ and $F_s$ is a $G$-imbedding for each $s$. Note that, by the definition of $G$-isotopy, there exists a positive number $\varepsilon$ such that $f_t=f_0$ for $0\leqq t\leqq\varepsilon$ and $f_t=f_1$ for $1-\varepsilon\leqq t\leqq 1$. Thus $F_s=f_0'\times 1$ for $0\leqq t\leqq\varepsilon$ and $F_s=f_1'\times 1$ for $1$-$\varepsilon\leqq t\leqq 1$. By Lemma 1.2 we can assume that $F$

is a $G$-isotopy between $\tilde{f}|U\times I$ and $\tilde{f}'$. Let $\tilde{F}: U\times I\times R\rightarrow N\times I\times R\hookrightarrow S^W\times I\times R$ be a $G$-imbedding defined by

$$\tilde{F}(x, t, s)=\begin{cases}(F(x, t, s), s) & \text{for } 0\leqq s\leqq 1\\ (F(x, t, 0), s) & \text{for } s<0\\ (F(x, t, 1), s) & \text{for } s>1.\end{cases}$$

Let $U$ and $U_2$ be invariant open tubular neighborhoods of $S^n$ such that $\bar{U}_1\subset U_2\subset\bar{U}_2\subset U$. Let $\nu$ be an invariant normal bundle of $\tilde{F}(U\times I\times R)$ in $S^W\times I\times R$ and let $p: \nu\rightarrow\tilde{F}(U\times I\times R)$ be the projection. Similarly as the proof of Lemma 2. 2, we have a $G$-imbedding $\varphi: \nu|\tilde{F}(\bar{U}_2\times I\times I)\rightarrow S^W\times I\times R$. We shall identify $\nu|\tilde{F}(\bar{U}_2\times I\times I)$ as the image of $\varphi$.

The tangent vectors to the curves $\tilde{F}(x\times t\times R)$ $(x\times t\in U\times I)$ define an invariant vector field $\tilde{X}$ on $\tilde{F}(U\times I\times R)$ of the form $\tilde{X}(\tilde{F}(x, t, s))=(X(x, t, s), 0, 1)\in T_{\tilde{F}(x,t,s)}(N\times I\times R)$. Note that $Supp(X)$ is contained in $\tilde{F}(U\times[\varepsilon, 1-\varepsilon]\times I)$. Take an invariant partition of unity subordinate to the covering $\{U_2, U-\bar{U}_1\}$ of $U$, and $u$ be the invariant $C^\infty$-function corresponding to $U_2$. Let $X'$ be an invariant vector field on $\tilde{F}(U\times I\times R)$ defined by $X'(\tilde{F}(x, t, s))=u(x)\cdot X(x, t, s)$ for $(x, t, s)\in\bar{U}_2\times I\times R$ and $X'=0$ outside of $\tilde{F}(\bar{U}_2\times I\times R)$. Then $Supp(X')$ is contained in $\tilde{F}(\bar{U}_2\times[\varepsilon, 1-\varepsilon]\times I)$ and $X'=X$ on $\tilde{F}(\bar{U}_1\times I\times R)$.

Let $r: R\rightarrow[0, 1]$ be a $C^\infty$-function such that $r(t)=1$ for $t\leqq 1$, $0<r(t)<1$ for $1<t<2$ and $r(t)=0$ for $t\geqq 2$. Let $Y$ be an invariant vector field on $S^W\times I\times R$ defined by $Y(v)=r(||v||)\cdot X'(p(v))$ for $v\in\nu|\tilde{F}(\bar{U}_2\times I\times I)$ and $Y=0$ outside of $\nu|F(\bar{U}_2\times I\times I)$, where $||\ ||$ is an invariant Riemannian metric on $\nu$. Then we can regard $Y$ as a time-dependent invariant vector field on $S^W\times I$. Note that $Supp(Y)$ is contained in $\nu(2)|\tilde{F}(\bar{U}_1\times[\varepsilon, 1-\varepsilon]\times I)$, where $\nu(2)=\{v\in\nu: ||v||\leqq 2\}$. Then $Y$ generates a $G$-isotopy $H: S^W\times I\times I\rightarrow S^W\times I$ such that $H_0=1$ and $H_s\cdot F_0=F_s$ on $\bar{U}_1\times I$ for $0\leqq s\leqq 1$. Since $I$ component of $Y$ is 0, each $H_s: S^W\times I\rightarrow S^W\times I$ is level preserving equivariant diffeomorphism. Let $h: S^V\times I\rightarrow S^W$ be a $G$-isotopy defined by $h=p_1\cdot H_1\cdot\tilde{f}$, where $p_1: S^W\times I\rightarrow S^W$ is the projection on the first factor. Then $h_i=f_i$ for $i=0, 1$ and $h_t=h'_t$ on $U_1$. This completes the proof of

Proposition 2. 3.

**Definition 2. 4.** Let $f_i : S^V \to S^W$ $(i=0, 1)$ be $G$-imbeddings such that $f_i = f'_i$ on $U$. $f_0$ and $f_1$ are said to be equivalent if there exists a $G$-isotopy $f : S^V \times I \to S^W$ between $f_0$ and $f_1$ such that $f_t = f'_t$ on some neighborhood of $S^n$. Let $I_1(S^V, S^W)$ denote the set of all equivalence classes of these $G$-imbeddings.

**Corollary 2. 5.** *The natural map* $i_1 : I_1(S^V, S^W) \to (S^V, S^W)$ *is bijective.*

*Proof.* By Proposition 1. 3 and Proposition 2. 2, $i_1$ is surjective. By Proposition 1. 4 and Proposition 2. 3, $i_1$ is injective, and Corollary 2. 5 follows.

## § 3. Orthogonality on a Neighborhood of the Fixed Point Set

In this section we shall prove that any $G$-imbedding from $S^V$ into $S^W$ is $G$-isotopic to a $G$-imbedding which is orthogonal on a neighborhood of $S^n$. Moreover we shall prove that, if two $G$-imbeddings $f_0$ and $f_1$, which are orthogonal on $U$, coincide on $U$, then there exists a $G$-isotopy $f$ between $f_0$ and $f_1$ such that $f_t = f_0$ $(0 \leqq t \leqq 1)$ on $U_1$, where $U$ and $U_1$ are invariant neighborhood of $S^n$.

As in § 2, let $U$ and $N$ be invariant normal bundles of $S^n$ in $S^V$ and to $S^n$ in $S^W$ respectively. Note that $U$ and $N$ are isomorphic to product bundles $S^n \times V_1$ and $S^n \times (dV_1 \oplus R^{(d-1)n+k})$ as a $G$-vector bundles over $S^n$ respectively. Let $f : S^V \to S^W$ be a $G$-imbedding with $f^G = j$. Then $f' : U \to N$ induces a continuous map

$$\hat{f} : S^n \to Mon^G(V_1, dV_1 \oplus R^{(d-1)n+k}),$$

where $Mon^G(V_1, dV_1 \oplus R^{(d-1)n+k})$ is the set of all $G$-module monomorphisms from $V_1$ to $dV_1 \oplus R^{(d-1)n+k}$ with usual topology. By Schur's lemma, $Mon^G(V_1, dV_1 \oplus R^{(d-1)n+k})$ is isomorphic to $Mon^G(V_1, dV_1)$.

**Proposition 3. 1.** *Let* $f: S^V \to S^W$ *be a G-imbedding with* $f^G = j$. *Let* $h: S^n \times I \to Mon^G(V_1, dV_1)$ *be a homotopy with* $h_0 = \hat{f}$. *Then there exists a G-isotopy* $F: S^V \times I \to S^W$ *such that* $F_0 = f$ *and* $\hat{F}_1 = h_1$.

*Proof.* Let $p: U \to S^n$ be the bundle projection. Let $F': U \times I \to N$ be a homotopy of $G$-imbeddings defined by $F'_t(u) = h_t(p(u))(u)$ for $u \in U$. Then, by Lemma 1. 2, we can assume that $F'$ is a $G$-isotopy. By Lemma 2. 2, we have a $G$-isotopy $H: S^W \times I \to S^W$ such that $H_0 = 1$ and $H_t \cdot f' = F'_t$ on some invariant neighborhood of $S^n$ for each $t$. Let $F: S^V \times I \to S^W$ be a $G$-isotopy defined by $F_t = H_t \cdot f$. Then $F_0 = f$ and $\hat{F}_1 = h_1$, and this completes the proof of Proposition 3. 1.

Let $O^G(V_1, dV_1)$ denote the set of all $G$-module orthogonal monomorphisms from $V_1$ to $dV_1$. Let $F$ denote the field of real numbers $R$, complex numbers $C$ or quaternionic numbers $H$. Let $U(q, F)$ denote the orthogonal group $O(n)$, the unitary group $U(n)$ or the symplectic group $Sp(n)$ in the case of $F = R$, $C$ or $H$ respectively. Let $Hom^G(V_1, V_1)$ denote the group of $G$-module endmorphisms of $V_1$. Let $V'_{r,s}(F)$ denote the Stiefel manifold (over $F$) of $s$-frames in $F^r$.

**Lemma 3. 2.** *Suppose that* $V_1$ *is isomorphic to* $\bigoplus_i k_i W_i$, *where* $W_i$ *runs over the inequivalent irreducible real G-modules. Then*

$$Mon^G(V_1, dV_1) = \prod_i V'_{dk_i, k_i}(F_i)$$

*and*

$$O^G(V_1, dV_1) = \prod_i U(dk_i, F_i)/U((d-1)k_i, F_i),$$

*where* $F_i = R$, $C$ *and* $H$ *when* dim $Hom^G(W_i, W_i) = 1$, 2 *and* 4 *respectively.*

*Proof.* If $W_i$ is a real restriction of an irreducible complex (resp. quaternionic) $G$-module $W'_i$, then $Hom^G(W_i, W_i)$ is isomorphic to $C$ (resp. $H$) given by the scalar multiplication of $W'_i$. Otherwise $Hom^G(W_i, W_i)$ is isomorphic to $R$ given by the scalar multiplication of $W_i$ (see J. -P. Serre [6, 13. 2]). Therefore $Mon^G(k_i W_i, dk_i W_i)$

and $O^G(k_iW_i, dk_iW_i)$ are identified with $V'_{dk_i,k_i}(F_i)$ and $U(dk_i, F_i)/U((d-1)k_i, F_i)$ respectively. By Schur's lemma $Hom^G(V_1, dV_1)$ is isomorphic to $\bigoplus_i Hom^G(k_iW_i, dk_iW_i)$. Then $Mon^G(V_1, dV_1)$ and $O^G(V_1, dV_1)$ are identified with $\prod_i Mon^G(k_iW_i, dk_iW_i)$ and $\prod_i O^G(k_iW_i, dk_iW_i)$ respectively. This completes the proof of Lemma 3.2.

**Proposition 3.3.** *Let $f: S^V\times I\to S^W$ be a G-isotopy such that $f_t^G=j$, $f_t=f'_t$ on $U$ for each $t$ and $\hat{f}_0=\hat{f}_1$. If $\pi_{n+1}(Mon^G(V_1, dV_1))=0$, then there exists a G-isotopy $h: S^V\times I\to S^W$ such that $h_i=f_i$ for $i=0, 1$ and $\hat{h}_t=\hat{f}_0$ for $0\leqq t\leqq 1$.*

*Proof.* Let $a_f: S^n\times\partial(I\times I)\to Mon^G(V_1, V_1)$ be a continuous map defined by

$$a_f(x, t, s)=\begin{cases}\hat{f}_t(x) & \text{for } s=0 \text{ and } 0\leqq t\leqq 1\\ \hat{f}_0(x) & \text{for } s=1 \text{ and } 0\leqq t\leqq 1\\ \hat{f}_0(x) & \text{for } t=0,\ 1 \text{ and } 0\leqq s\leqq 1.\end{cases}$$

Since $\pi_{n+1}(Mon^G(V_1, dV_1))=0$, the only obstruction to extend $a_f$ to $S^n\times I\times I$ is a well defined cohomology class $o(a_f)\in H^2(S^n\times I\times I, S^n\times\partial(I\times I)$ ; $\pi_1(Mon^G(V_1, dV_1))=\pi_1(Mon^G(V_1, dV_1))$. If $d\geqq 3$, $Mon^G(V_1, dV_1)$ is 2-connected by Lemma 3.2, and $o(a_f)=0$.

Now we will consider the case of $d=1$. In this case $Mon^G(V_1, dV_1)$ is a group $A^G(V_1)$, where $A^G(V_1)$ is the group of all $G$-module automorphisms of $V_1$. Let $b_f: \partial(I\times I)\to Mon^G(V_1, dV_1)=A^G(V_1)$ be a continuous map defined by $b_f(x)=a_f(*, x)$ for $x\in\partial(I\times I)$, where $*$ is a point of $S^n$. Then the above obstruction class $o(a_f)$ is represented by $b_f$. Note that an element of $A^G(V_1)$ can be regarded as an equivariant linear diffeomorphism of $S^W$ in the natural way. Let $g: S^V\times I\to S^W$ be a $G$-isotopy between $f_0$ and $f_1$ defined by $g_t=\hat{f}_0(*)\cdot\hat{f}_t(*)^{-1}\cdot f_t$. Then $b_g(x)=\hat{f}_0(*)$ for any $x\in\partial(I\times I)$, and $o(a_g)=0$. Replacing the $G$-isotopy $f$ between $f_0$ and $f_1$ by $g$, we can assume $o(a_g)=0$.

We now turn to the case $d=2$. If $V_1$ is isomorphic to $\bigoplus_i k_iW_i$, then $Mon^G(V_1, 2V_1)=\prod_i V'_{2k_i,k_i}(F_i)$ by Lemma 3.2. Note that $\pi_1(V'_{2k_i,k_i}(F_i))$ is 0 beside the case $F_i=R$ and $k_i=1$. Let $J$ be the set of index $i$

such that $F_i = R$ and $k_i = 1$. Let $p : \prod_i V'_{2k_i, k_i}(F_i) \to \prod_{i\in J} V_{2,1}(R)$ be the natural projection. Then $p_* : \pi_1(\prod_i V'_{2k_i, k_i}(F_i)) \to \pi_1(\prod_{i\in J} V'_{2,1}(R))$ is isomorphic. Let $r : I \to \prod_{i\in J} V'_{2,1}(R)$ be a continuous map defined by $r(t) = p \cdot \hat{f}_t(*)$. Since $\pi : \prod_{i\in J} GL(2, R) \to \prod_{i\in J} V'_{2,1}$ is a product bundle, there exists a continuous map $\tilde{r} : I \to \prod_{i\in J} GL(2, R)$ such that $\pi \cdot \tilde{r} = r$ and $\tilde{r}(0) = \tilde{r}(1)$. Note that, for each $i \in J$, $GL(2, R)$ can be regarded as the automorphism group $A(2W_i)$ of $G$-module $2W_i$ whose element defines an equivariant linear diffeomorphism of $S^W$. Let $g : S^V \times I \to S^W$ be a $G$-isotopy between $f_0$ and $f_1$ defined by $g_t = \tilde{r}(0) \cdot \tilde{r}(t)^{-1} \cdot f_t$. Since $\pi$ is identified with the natural map $\prod_{i\in J} A^G(2W_i) \to \prod_{i\in J} Mon^G(W_i, 2W_i)$, $p \cdot \hat{g}_t(*) = p \cdot \hat{f}_0(*)$ and $o(g_f) = 0$. Replacing the $G$-isotopy $f$ between $f_0$ and $f_1$ by $g$, we can assume $o(a_f) = 0$.

Therefore we can assume that $a_f$ can be extended to $S^n \times I \times I$. Let $F : U \times I \times I \to N \times I$ be an equivariant map defined by $F(v, t, s) = (a_f(q(v), t, s)(v), t)$, where $q : U \to S^n$ is the bundle projection. Then each $F(\cdot, t, s)$ is a $G$-imbedding, and $F_0(u, t) = (f_t(u), t) = (f'_t(u), t)$ and $F_1(u, t) = (f_0(u), t) = (f'_0(u), t)$ for $(u, t) \in U \times I$. By Lemma 1.2 we can assume that $F$ is a $G$-isotopy. In the same way as the proof of Proposition 2.3, we have a $G$-isotopy $h : S^V \times I \to S^W$ such that $h_i = f_i$ $(i = 0, 1)$ and $h_t = f'_0$ $(0 \leqq t \leqq 1)$ on some invariant neighborhood of $S^n$. Therefore $\hat{h}_t = \hat{f}_0$ for each $t$, and this completes the proof of Proposition 3.3.

*Remark.* I don't know whether Proposition 3.3 is valid without the assumption $\pi_{n+1}(Mon^G(V_1, dV_1)) = 0$.

Now we shall assume $\pi_{n+1}(Mon^G(V_1, dV_1)) = 0$. Choose a continuous map $a_\lambda : S^n \to O^G(V_1, dV_1)$, which represents an element $\lambda$, for each element $\lambda$ of $\pi_n(O^G(V_1, dV_1))$. Let $A = \{a_\lambda ; \lambda \in \pi_n(O^G(V_1, dV_1))\}$

**Definition 3.4.** Let $f_i : S^V \to S^W$, $i = 0, 1$, be $G$-imbeddings, which represent elements of $I_1(S^V, S^W)$, such that $\hat{f}_i$, $i = 0, 1$, are elements of $A$. $f_0$ and $f_1$ are said to be equivalent if there exists a $G$-isotopy $f : S^V \times I \to S^W$ between $f_0$ and $f_1$ such that $\hat{f}_t = \hat{f}_0$ for $0 \leqq t \leqq 1$. Let

$I_2(S^V, S^W)$ denote the set of equivalence classes of these $G$-imbeddings.

**Corollary 3.5.** *If* $\pi_{n+1}(Mon^G(V_1, dV_1))=0$, *the natural map* $i_2 : I_2(S^V, S^W)\to I_1(S^V, S^W)$ *is bijective.*

*Proof.* Let $f : S^V\to S^W$ be a $G$-imbedding which represents an element of $I_1(S^V, S^W)$. By Lemma 3.2 $O^G(V_1, dV_1)$ is a deformation retract of $Mon^G(V_1, dV_1)$. Therefore, by Proposition 3.1, we can assume that $f$ is an element of $A$, and $i_2$ is surjective. By Proposition 3.3, $i_2$ is injective, and this completes the proof of Corollary 3.5.

## §4. Proof of Theorem A and Theorem B

In this section we shall prove that, if $G$ is a finite group and $\pi_{n+1}(Mon^G(V_1, dV_1))=0$, then the $G$-isotopy class of a $G$-imbedding $f : S^V\to S^W$ is determined by the homotopy class of the orbit map of $f|(S^V-U)$ relative to the boundary, where $U$ is an invariant open neighborhood of $S^n$. And, using the obstruction theory, we shall prove Theorem A and Theorem B.

In this section we shall assume that $G$ is a finite group and $\pi_{n+1}(Mon^G(V_1, dV_1))=0$. Let $f_i : S^V\to S^W$, $i=0, 1$, be $G$-imbeddings which represent elements of $I_2(S^V, S^W)$. Let $U$ be an invariant open $\varepsilon$-tubular neighborhood of $S^n$ in $S^V$. We can choose a sufficiently small positive number $\varepsilon$ such that $f_i=f_i'$ on $U$ and $f_i(S^V-U)\subset S^W-T$ for $i=0, 1$. By Corollary 3.5, we have the following :

**Lemma 4.1.** *With the above notations,* $f_0$ *and* $f_1$ *are G-isotopic if and only if there exists a G-isotopy* $f : S^V\times I\to S^W$ *such that* $f_t(S^V-U)$ *is contained in* $S^W-T$ *and* $f_t=f_0$, $0\leqq t\leqq 1$, *on* $U$.

It is clear that free $G$-manifolds $S^V-U$ and $S^W-T$ are equivariant diffeomorphic to $S(V_1)\times D^{n+1}$ and $S(dV_1)\times D^{dn+k+1}$ respectively. Let $L$ and $L'$ denote the orbit spaces $S(V_1)/G$ and $S(dV_1)/G$ respectively. Then the orbit spaces $(S^V-U)/G$ and $(S^W-T)G$ are diffeomorphic to $L\times D^{n+1}$ and $L'\times D^{dn+k+1}$ respectively. Let $\bar{f}_i : L\times D^{n+1}\to L'\times D^{dn+k+1}$,

$i=0, 1$, be imbeddings defined by the orbit maps of $f_i|(S^V-U)$.

**Proposition 4. 2.** *With the above notations, $f_0$ and $f_1$ are G-isotopic if and only if $\bar{f}_0$ and $\bar{f}_1$ are homotopic relative to $L\times S^n$.*

*Proof.* By Lemma 4. 1, if $f_0$ and $f_1$ are $G$-isotopic, then $\bar{f}_0$ and $\bar{f}_1$ are homotopic relative to $L\times S^n$. Conversely if $\bar{f}_0$ and $\bar{f}_1$ are homotopic relative to $L\times S^n$, then $\bar{f}_0$ and $\bar{f}_1$ are isotopic relative to $L\times S^n$ because dim $(L'\times D^{dn+k+1})>2$ dim $(L\times D^{n+1})+1$. Since $G$ is a finite group, $S^V-U\to(S^V-U)/G$ and $S^W-T\to(S^W-T)/G$ are covering spaces. By the covering homotopy property, there exists a $G$-isotopy $h_t: S^V-U\to S^W-T$ $(0\leqq t\leqq 1)$ relative to $\partial(S^V-U)$ such that $h_0=f_0$ and $\bar{h}_t=\bar{f}_t$ for $0\leqq t\leqq 1$. Since $h_1|\partial(S^V-U)=f_1|\partial(S^V-U)$ and $\bar{h}_1=\bar{f}_1$, by the property of the covering space, we have $h_1=f_1$ on $S^V-U$. Therefore $f_0$ and $f_1$ are $G$-isotopic and this completes the proof of Proposition 4. 2.

*Proof of Theorem A.* Suppose that $V_1=\bigoplus_i k_iW_i$, where $W_i$ runs over the inequivalent irreducible real $G$-modules. If $q>2$, $Mon^G(V_1, dV_1)=\prod_i V'_{dk_i,k_i}(C)$ by Lemma 3. 2. Since $V'_{dk_i,k_i}(C)$ is $2(d-1)k_i$-connected, $\pi_{n+1}(Mon^G(V_1, dV_1))=0$ if $d\geqq(n+3)/2$. If $q=2$, $V_1=(m-n)W_1$ and $Mon^G(V_1, dV_1)=V'_{d(m-n),m-n}(R)$, where $W_1$ is the non-trivial 1-dimensional real representation of $Z_2$. Since $V'_{d(m-n),m-n}(R)$ is $(d-1)(m-n)-1$-connected, $\pi_{n+1}(Mon^G(V_1, dV_1))=0$ if $d\geqq(m+2)/(m-n)$. Therefore, combining Corollary 2. 5 and Corollary 3. 5, the set $I(S^V, S^W)$ can be identified with the set $I_2(S^V, S^W)$. Let $f_S: S^V\to S^W$ be the standard imbedding. Let $f: S^V\to S^W$ be a $G$-imbedding which represents an element of $I_2(S^V, S^W)$. Since $\pi_n(Mon^G(V_1, dV_1))=0$, we can assume $\hat{f}=\hat{f}_S$. With the notation of Proposition 4. 2, $f$ and $f_S$ are $G$-isotopic if and only if $\bar{f}$ and $\bar{f}_S$ are homotopic relative to $S^n\times L$.

Note that $L$ (resp. $L'$) is an $m-n-1$ (resp. $d(m-n)-1$)-dimensional lens space or real projective space. Since $d\geqq(m+2)/(m-1)$, $\pi_i(L'\times D^{dn+k+1})=0$ for $2\leqq i\leqq m$. By the obstruction theory of P. Olum [5, Theorem 9. 10 and Theorem 16. 5], $\bar{f}$ and $\bar{f}_S$ are homotopic relative to $L\times S^n$. This completes the proof of Theorem A.

*Proof of Theorem B.* By Lemma 3.2, $Mon^G(V_1, dV_1) = V'_{d(m-n)/2,(m-n)/2}(C)$. Thus $Mon^G(V_1, dV_1)$ is $(d-1)(m-n)$-connected, and if $d \geqq (m+1)/(m-n)$, $\pi_{n+1}(Mon^G(V_1, dV_1)) = 0$. Combining Corollary 2.5 and Corollary 3.5, the set $I(S^V, S^W)$ can be identified with $I_2(S^V, S^W)$.

Let $f: S^V \to S^W$ be a $G$-imbedding which represents an element of $I_2(S^V, S^W)$. Similarly as the proof of Theorem A, we can assume $\bar{f}|L\times S^n = \bar{f}_S|L\times S^n$, and in the case of $d > (m+1)/(m-n)$, $f$ is $G$-isotopic to the standard imbedding $f_S$.

Now consider the case of $d = (m+1)/(m-n)$. Since $H^i(L\times D^{n+1}, L\times S^n; \pi_i(L'\times D^{dn+k+1})) = H^{i-n-1}(L; \pi_i(L')) = 0$ for $i < m$ and $H^{i-1}(L\times D^{n+1}, L\times S^n; \pi_i(L'\times D^{dn+k+1})) = H^{i-n-2}(L; \pi_i(L')) = 0$ for $i < m$, by the obstruction theory, the homotopy classes of maps $\bar{f}: L\times D^{n+1} \to L'\times D^{dn+k+1}$ relative to $L\times S^n$ are in one to one correspondence with the elements of

$$H^m(L\times D^{n+1}, L\times S^n; \pi_m(L'\times D^{dn+k+1})) = \pi_m(L').$$

Since $\dim L' = d(m-n) - 1 = m$ in the case of $d = (m+1)/(m-n)$, by Proposition 4.2, we have $I(S^V, S^W) = Z$. This completes the proof of Theorem B.

*Remark.* Suppose that $V_1 = \bigoplus_i k_i W_i$ and $k_i \geqq 3$ for each $i$ if $\dim Hom^G(W_i, W_i) = 1$, where $W_i$ runs over the inequivalent irreducible real $G$-modules. Then Theorem A is valid when $G$ is a finite group.

## References

[1] Bredon, G., *Introduction to Compact Transformation Groups*, Academic Press, New York and London, 1972.
[2] Dax, J., Étude homotopique des espaces de plongements, *Ann. Sci. Ecole Norm. Sup.*, 5 (1972), 303-377
[3] Hirsch, M, *Differential Topology*, Springer-Verlag, Berlin-Heidelberg New York, 1976.
[4] Munkres, J., *Elementary Differential Topology*, Princeton Univ. Press, 1966.
[5] Olum, P., Obstruction to extensions and homotopies, *Ann. of Math.*, **52** (1950), 1-50.
[6] Serre, J.-P., *Représentations Linéaires des Groupes Finis*, Hermann S. A., Paris, 1971.
[7] Wasserman, A., Equivariant differential topology, *Topology*, **8** (1969), 127-150.