On the Minimal Ordered Morse Functions
on Compact Simply-Connected Manifolds

By

Takao MATUMOTO*

Introduction

Throughout this paper, $(W^n; V, V')$ denotes a compact smooth manifold triad of dimension n in the sense that $\partial W = V \cup V'$ and $V \cap V' = \phi$, and $\mathscr{F}$ the space of C^∞-functions of $(W; V, V')$ into $(I; 0, 1)$ without critical points on the boundary. The codimension 0 stratum $\mathscr{F}^0$ of the natural stratification of $\mathscr{F}$ is the space of excellent functions ($=$ Morse functions with distinct critical values). By definition, $f \in \mathscr{F}$ is called to be ordered and minimal, if the critical values are ordered in the order of their indices and the number of critical points is minimal with respect to the homology group structure of $H_*(W, V)$.

Let f be an excellent function with m critical values $v_1, \ldots, v_m$ and put $u_0 = v_1/2$, $u_m = 1$ and $u_i = (v_i + v_{i+1})/2$ for $1 \leq i \leq m-1$. Then, (W, V) has a filtration $W_f = (W_0, \ldots, W_m)$ defined by $W_i = f^{-1}[0, u_i]$ so that $W_m = W$ and W_i has a presentation of the handle-attaching form $(W_{i-1}, f^{-1}(u_{i-1}); \varphi_i; \lambda_i)$ in the notation of Smale [7]. For another filtration $(W'_0, \ldots, W'_m)$ with $W'_m = W'$, a map $h: W \to W'$ is called to be filtered if $h(W_i) \subset W'_i$ for every i. A filtered map $h: W \to W'$ is a filtered homotopy equivalence if there exists a filtered map $g: W' \to W$ such that $(g \circ h)|W_i: W_i \to W_i$ and $(h \circ g)|W'_i: W'_i \to W'_i$ are homotopy equivalences for every i.

We consider the following condition:

$(*)$ $\pi_1(V) = \pi_1(V') = \pi_1(W, V) = \pi_1(W, V') = 0$, dim $W \geq 6$ and $H_*(W,$

V) *has no torsions.*

The purpose of this paper is to prove

Theorem. *Suppose that* (W, V) *satisfies the condition* (*). *Then, two minimal ordered excellent functions f and f' on* (W, V) *belong to the same arcwise connected component of* $\mathscr{F}^0$ *if and only if the numbers of critical points are equal and there exists a filtered homotopy equivalence* $h: W_f = (W_0, \ldots, W_m) \to W_{f'} = (W_0', \ldots, W_m')$ *such that* $h|V = id$ *and* $h|W_m: W \to W$ *induces the identity on the homology group* $H_*(W, V)$.

Corollary 1. *Suppose the condition* (*). *Then, the space* $\mathscr{O}\mathscr{M}$ *of minimal ordered Morse functions on* (W, V) *is arcwise connected.*

Definition. Let $f \in \mathscr{F}$. An element h of $Diff(W, V; f)$ is a diffeomorphism of W such that $h|V = id$ and h preserves the level of f, i. e.: $f \circ h = f$. A diffeomorphism g of (W, V) is called a *special* diffeomorphism if $g|V = id$ and the induced map $g_*: H_*(W, V) \to H_*(W, V)$ is the identity.

Corollary 2. *Let f be a minimal ordered excellent function on* (W, V). *Suppose the condition* (*). *Then, the natural homomorphism of the group* $\pi_0(Diff(W, V; f))$ *of the isotopy classes of level-preserving diffeomorphisms into the group* $\pi_0(SDiff(W, V))$ *of the isotopy classes of special diffeomorphisms of* (W, V) *is a surjection.*

The theorem above is a generalization of the theorem of connectivity of the space of functions without critical points established by Cerf [2]. Corollary 1 implies that two ordered minimal handle-decomposition on such a manifold become isotopic after some repeat of reorderings and handle-adding operations; Agoston [1] has proved a weaker version of the uniqueness of the minimal handle-decomposition up to isotopy by using only the h-cobordism theorem, and he gave an example which showed that the torsion-freeness was indispensable to his

uniqueness theorem and consequently to Corollary 1. We shall give another stronger counter-example, that is, an example of non-diffeomorphic minimal handle-decompositions in the appendix ($=\S 4$). It is not difficult to find an example which shows the failure of the theorem when $H_*(W, V)$ has some torsion elements. In fact, I acknowledge that Professor F. Laudenbach wrote me such an example. Corollary 2 would be useful in the study of $\pi_0(Diff(W, V))$.

§1. Proof of Theorem

Assume that f_t, $0 \leq t \leq 1$, is a smoothly parametrized family of excellent functions on (W, V) such that $f_0 = f$ and $f_1 = f'$. Then, because the critical points are smoothly parametrized, the usual construction of the gradient-like vector field using a partition of unity gives a smoothly parametrized family of vector fields ξ_t, $0 \leq t \leq 1$, so that every ξ_t is a gradient-like vector field for f_t. Let $W_{i,t}$ denote the i-th filtration associated to f_t and $\partial_+ W_{i,t} = (f_t)^{-1}(u_{i,t})$. Then, because $[0, 1]$ is compact, there exists a sequence $0 = t(0) < t(1) < \ldots < t(\ell) = 1$ such that the flows associated to ξ_t with $t = t(j)$ are transverse with $\partial_+ W_{i,t}$ for every i and $t \in [t(j), t(j+1)]$. Therefore, by sliding along the flows, we get easily an isotopy $(h_j)_s$, $0 \leq s \leq 1$, of diffeomorphisms of (W, V) such that $(h_j)_0 = id$, $(h_j)_s | V = id$ and $(h_j)_1(W_{i,t(j)}) = W_{i,t(j+1)}$. If we compose these ℓ isotopies, we get an isotopy h_t, $0 \leq t \leq 1$, such that $h_0 = id$, $h_t | V = id$ and $h_1(W_{i,0}) = W_{i,1}$. Clearly h_1 is a required filtered homotopy equivalence.

Now we shall prove the sufficiency of the filtered homotopy equivalence condition: Let f and f' be two minimal ordered excellent functions on (W, V). Composing a diffeomorphism of $I = [0, 1]$ onto $[-1, m]$ and adjusting the critical values of excellent functions, we may assume that the critical values of f and f' are $0, 1, 2, 3, \ldots, m-1$. First of all, we note that, for any isotopy h_t of W, $f \circ h_t$ is a family of excellent functions that is, a path in $\mathscr{F}^0$.

If $V = \phi$, it is easy to get an isotopy h_t, $0 \leq t \leq 2$, of W such that $f' \circ h_2$ coincides with f on $f^{-1}[0, 1/3]$: in fact, since $\pi_1(W, V) = 0$, each of the minimal functions f and f' has a unique critical point

of index 0, which is denoted by p_0 or p'_0 respectively. Hence, an isotopy h_t, $0 \leqq t \leqq 1$, can satisfy that $h_0 = id$ and $h_1(p_0) = p'_0$. Then, the uniqueness of the disk neighborhood implies $h_1(f^{-1}[0, 1/3]) = (f')^{-1}[0, 1/3]$ and there is an isotopy h_t, $1 \leqq t \leqq 2$, of W such that $f = f' \circ h_2$ on $f^{-1}[0, 1/3]$. Therefore, we have only to compare f and $f' \circ h_2$ on the manifold $f^{-1}[1/3, m]$ and we may assume $f^{-1}(1/3) \neq \phi$. If $V \neq \phi$, we may assume that f and f' coincide on $f^{-1}[-1, -2/3]$. Remark that f and f' have no critical points of index 1, because $\pi_1(W, V) = 0$.

Therefore, in both cases when $V = \phi$ or $V \neq \phi$, we may assume as a hypothesis of induction that the minimal excellent functions f and f' coincide on $f^{-1}(-\infty, i-2/3]$ and the filtered homotopy equivalence h satisfies $h \mid f^{-1}(-\infty, i-2/3] = id$. Then, h must induce the identify on $H_*(\overline{W}, V_i)$ by the minimality of f and the torsion-freeness of $H_*(W, V)$, $\overline{W}$ and V_i denoting $f^{-1}[i-2/3, m]$ and $f^{-1}(i-2/3)$ respectively. Hence, we have reduced the problem to compare the restricted minimal excellent functions f and f': $(\overline{W}; V_i, V') \to ([i-2/3, m]; i-2/3, m)$ which have no critical points of index less than k with $2 \leqq k \leqq n-3$. Let p be the lowest critical point of index k of f. Choosing a gradient-like vector field ξ for f, we define a left-hand disk $D_L(p)$ whose boundary $S_L(p)$ is contained in V_i, by the union of segments of integral curves of ξ beginning at p; $D_L(p)$ represents an element of $\pi_k(\overline{W}, V_i) = H_k(\overline{W}, V_i)$ [6].

Let w_{i+1} and w'_{i+1} denote the elements of $H_k(\overline{W}, V_i)$ represented by $W_{i+1} \cap \overline{W}$ and $W'_{i+1} \cap \overline{W}$ respectively. Then, we have $(h \mid \overline{W})_* w_{i+1} = w'_{i+1}$; and since $(h \mid \overline{W})_* = id$, $w_{i+1} = w'_{i+1}$. Moreover, because $V_i \cup D_L(p)$ and $V_i \cup D'_L(p')$ are the deformation retracts of $W_{i+1} \cap \overline{W}$ and $W'_{i+1} \cap \overline{W}$ respectively in $(\overline{W}, V_i)$, we see that $D_L(p)$ and $D'_L(p')$ represent the same element of $\pi_k(\overline{W}, V_i)$. Since we restrict ourselves to the case in which $k \leqq n-3$ and $\pi_j(\overline{W}, V_i) = 0$ for $j \leqq k-1$, we can apply the theorem of uniqueness of the handle-attaching due to Hudson [5; Corollary 2.1] in order to get an isotopy h_t, $0 \leqq t \leqq 1$, of $\overline{W}$ starting from the identity h_0 such that $h_1(D_L(p)) = D'_L(p')$; moreover, composing an extension of an isotopy of $D_L(p)$, we may assume that $h_1(p) = p'$.

Note that we have an isotopy g_t, $0 \leq t \leq 1$, of $D'_L(p')$ such that $g_0 = id$ and $f' \circ g_1 \circ h_1 = f$ on a disk neighborhood of p in $D_L(p)$; we may assume that this neighborhood has the form of $f^{-1}[i-\varepsilon, i] \cap D'_L(p)$ for some small positive ε and that the isotopy g_t is extended on $\overline{W}$. It is not difficult to choose an extension g_t so that two smooth collar neighborhoods $(g_1 \circ h_1)(f^{-1}[i-2/3, i-\varepsilon])$ and $(f')^{-1}[i-2/3, i-\varepsilon]$ of V_i in $\overline{W}$ coincide with each other in a neighborhood of $D'_L(p') \cap (f')^{-1}[i-2/3, i-\varepsilon]$. Moreover, by the uniqueness of the smooth collar neighborhood, we have an isotopy g_t, $1 \leq t \leq 2$, of $\overline{W}$ such that $(g_2 \circ h_1)(f^{-1}[i-2/3, i-\varepsilon]) = (f')^{-1}[i-2/3, i-\varepsilon]$, $(g_2 \circ h_1)(D_L(p) \cap f^{-1}[i-\varepsilon, i])) = D'_L(p') \cap (f')^{-1}[i-\varepsilon, i]$ and $f' \circ g_2 \circ h_1 = f$ on $f^{-1}[i-\varepsilon, i] \cap D_L(p)$. Then, because f and $f' \circ g_2 \circ h_1$ have no critical points on $f^{-1}[i-2/3, i-\varepsilon]$, we can apply the theorem of Cerf [2] to deform $f' \circ g_2 \circ h_1$ by a path f'_t in $\mathscr{F}^0$ such that $f'_0 = f' \circ g_2 \circ h_1$, $f'_1 = f$ on $f^{-1}[i-2/3, i-\varepsilon]$ and that for every t, $f'_t = f' \circ g_2 \circ h_1$ on $f^{-1}[i-\varepsilon, \infty)$. Hence, we have only to consider the manifold $f^{-1}[i-\varepsilon, \infty)$. Let $m : [i-\varepsilon, \infty) \to [i-2/3, \infty)$ be a smooth map free from critical points and with $m | [i, \infty) = id$. We rewrite $\overline{W}$, V_i, f, f', $D_L(p)$ and $D'_L(p')$ instead of $f^{-1}[i-\varepsilon, \infty)$, $f^{-1}(i-\varepsilon)$, $m \circ f | f^{-1}[i-\varepsilon, \infty)$, $m \circ f' \circ g_2 \circ f_1 | f^{-1}[i-\varepsilon, \infty)$, $D_L(p) \cap f^{-1}[i-\varepsilon, i]$ and $D'_L(p') \cap (f'_0 \circ g_2 \circ h_1)^{-1}[i-\varepsilon, i]$ respectively. Then, we have $p = p'$, $D'_L(p') = D_L(p)$ and $f = f'$ on $D'_L(p') = D_L(p)$.

Now we may assume that $N(p) = f^{-1}[i-2/3, i+1/3]$ and $N'(p') = (f')^{-1}[i-2/3, i+1/3]$ are the smooth regular neighborhoods of $D_L(p) \cup V$. Taking a small disk-neighborhood D of $p = p'$ in $N(p) \cap N'(p')$, we consider two handles $H(p)$ and $H'(p')$ which are defined by the union of segments contained in $N(p)$ and $N'(p')$ respectively, of the integral curves of $\pm h_1(\xi)$ and $\pm \xi'$ respectively beginning at some points of D. Because $H(p)$ and $H'(p')$ are the smooth regular neighborhoods of $D_L(p) = D'_L(p')$, we have an isotopy h_t, $0 \leq t \leq 1$, of $\overline{W}$ starting from the identity h_0 such that $h_1(H(p)) = H'(p')$ and $h_1 | D_L(p) = id$. It is not difficult to assume $h_1(N(p)) = N'(p')$. On the other hand, there are diffeomorphisms g and g' of $h_1(H(p)) = H'(p')$ onto the standard handle $D^{k, n-k} = \{(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \in D^k \times D^{n-k}; (x_1^2 + \ldots + x_k^2)(y_1^2 + \ldots + y_{n-k}^2) \leq 1\}$, which map each level submanifold onto some level submanifold and restrict to the same diffeomorphism

of $D_L(p) = D'_L(p')$ onto $D^k \times 0$.

Lemma 1. *Let* f_0 *be a standard Morse function on* $D^p \times D^q$, *i. e.*: $f_0 = x_1^2 + \ldots + x_p^2 - y_1^2 - \ldots - y_q^2$. *Then, any auto-diffeomorphism of* $D^{p,q} = \{(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^p \times D^q ; (x_1^2 + \ldots + x_p^2)(y_1^2 + \ldots + y_q^2) \leq 1\}$, *which restricts to the identity on* $D^p \times 0 \cup \partial_- D^{p,q}$, *is isotopic by a path of such auto-diffeomorphisms to a diffeomorphism* h *such that* $f_0 \circ h = f_0$, *where* $\partial_- D^{p,q}$ *denotes* $D^{p,q} \cap \partial D^p \times D^q$.

Proof. Let $\alpha : D^p \to SO(q)$ be a differentiable map into the rotation group satisfying $\alpha | \partial D^p = 1$. Then, the automorphism $d(\alpha) = h$ of $D^{p,q}$ defined by $h(x, y) = (x, \alpha(x)y)$ gives a diffeomorphism of $D^{p,q}$ relative to $D^p \times 0 \cup \partial_- D^{p,q}$. As a consequence, we get a homomorphism $d : \pi_p(SO(q)) \to \pi_0(\mathrm{Diff}(D^{p,q}, D^p \times 0 \cup \partial_- D^{p,q}))$. It is standard in the tubular neighborhood theory to see that d is an isomorphism (cf. [4]). So, any auto-diffeomorphism of $D^{p,q}$ relative to $D^p \times 0 \cup \partial_- D^{p,q}$ has a representative of the form $h(x, y) = (x, \alpha(x)y)$ in the isotopy class, and evidently this diffeomorphism satisfies that $f_0 \circ h = f_0$.

<div align="right">q. e. d.</div>

Since $g' \circ g^{-1}$ is a diffeomorphism of $D^{k,n-k}$ relative to $D^k \times 0 \cup \partial_- D^{k,n-k}$, it is isotopic by a path consisting of a family of diffeomorphisms g_t, $0 \leq t \leq 1$, to a level-preserving diffeomorphism g_1 by Lemma 1. Hence, $(g')^{-1} \circ g_1 \circ g$ is a level-preserving diffeomorphism of $h_1(H(p))$ onto $H'(p')$. By composing some monotone map of $[i - 2/3, i + 1/3]$ onto $[i - 2/3, i + 1/3]$, we may assume that $f' \circ (g')^{-1} \circ g_1 \circ g = f \circ h_1^{-1}$ on $h_1(H(p))$. The isotopy of embeddings $(g')^{-1} \circ g_t \circ g$, $0 \leq t \leq 1$, of $h_1(H(p))$ can be extended to an isotopy of embeddings of $N(p)$ and then to an isotopy h_t, $1 \leq t \leq 2$, of $\overline{W}$. Therefore, $f' \circ h_t$, $0 \leq t \leq 2$, is a path in $\mathscr{F}^0$ such that $f' \circ h_2 = f$ on $H(p)$; moreover, because each of $f' \circ h_2$ and f has no critical points on $N(p) - H(p)$, these two functions can be connected by a path of functions which have no critical points on $N(p) - H(p)$ by the theorem of Cerf [2]. As a consequence, f' can be deformed by a path in $\mathscr{F}^0$ into a f_1' such that $f_1' = f$ on $f^{-1}(-\infty, i + 1/3]$.

If we consider the manifold $f^{-1}[i+1/3, \infty)$, we have diminished the number of the critical points of index k with $k \leq n-3$; Therefore, by the induction on the number of the critical points of index k with $k \leq n-3$, we can obtain a path f'_t in $\mathscr{F}^0$ such that $f'_1 = f$ on $f^{-1}(-\infty, i_0+1/3]$ and $f'_0 = f'$, assuming that i_0 is the heighest critical value of the index k_0 where $n = k_0 + 3$.

If we consider the minimal ordered excellent functions $m-1-f$ and $m-1-f'$, we may assume in the same way that $f'_1 = f$ on $f^{-1}[i_0+2/3, \infty)$, because $n-k_0 = 3 \leq n-3$. Moreover, since f and f'_1 have no critical points on $f^{-1}[i_0+1/3, i_0+2/3]$, they can be connected by a path consisting of the functions without critical points on $f^{-1}[i_0+1/3, i_0+2/3]$ by the theorem of Cerf $\lfloor 2 \rfloor$. Hence, original f and f' are connected by a path in $\mathscr{F}^0$; this completes the proof of Theorem.

Remark. If we review the proof, we have connected f and f' only by the following three types of paths in $\mathscr{F}^0$: type (1) = a path given by adjusting the critical values, type (2) = a path of the form $f \circ h_t$ where h_t is an isotopy of W (in fact, we mentioned only on the isotopy defined on the submanifold of codimension 0 but it is easily extended on W), and type (3) = a path given by the theorem of Cerf; we can easily reduce this case to the path of type (2) by using a smoothly parametrized vector fields. Hence, if f and f' have the same critical values from the beginning, we do not need the path of type (1) and we get an isotopy h_t of W such that $f = f' \circ h_1$. If we modify the isotopy in the collar neighborhood of V, we may assume that this isotopy h_t satisfies $h_t | V = id$.

§ 2. Proof of Corollary 1

Let f and f' be two minimal ordered Morse functions on (W, V). Composing a diffeomorphism of I onto $[-1/2, n+1/2]$ and adjusting the critical values of Morse functions, we may assume that f and f' are minimal *nice* functions, i. e.: the critical values of index k is equal to k.

By the argument in the proof of Theorem, it is sufficient to prove that there exists a path f_t, $0 \leq t \leq 1$, in $\mathcal{O}\mathcal{M}$ such that $f = f_0$ on $f^{-1}(-\infty,\ k+1/3]$ and $f' = f_1$, assuming that f and f' have no critical points of index less than k with $2 \leq k \leq n-3$ and $f(V) = f'(V) = k-2/3$. Now, let $p_1, \ldots, p_j$ be the critical points of index k of f. Choosing a gradient-like vector-field ξ for f, we define left-hand disks $D_L(p_1), \ldots,$ $D_L(p_j)$ whose boundaries are contained in V so that they represent the elements in $\pi_k(W,\ V) = H_k(W,\ V)$.

Lemma 2. *Suppose that $2 \leq k \leq n-2$ and $\pi_k(W, V)$ has no torsions. Then, there is a path f_t', $0 \leq t \leq 1$, in $\mathcal{O}\mathcal{M}$ with $f_1' = f'$ which satisfies the following property: there exists a gradient-like vector field ξ' for f_0' such that for each i with $1 \leq i \leq j$ the left-hand disk $D_L'(p_i')$ associated to f_0' and ξ' represents the same element of $\pi_k(W,\ V)$ as $D_L(p_i)$, where $p_1', \ldots, p_j'$ denote the critical points of index k of f_0'.*

Proof. Since $H_k(W, V)$ has no torsions, the natural map $H_k((f')^{-1}[k-2/3,\ k+1/3],\ V) \to H_k(W,\ V)\ (= \pi_k(W,\ V))$ is an isomorphism. So, applying the basis theorem 6.7 of [6], we obtain a Morse function f_0' on $(f')^{-1}[k-2/3,\ k+1/3]$ and a gradient-like vector field ξ' for f_0' such that the family of $D_L'(p_i')$, $1 \leq i \leq j$, constitutes the corresponding basis in $H_k((f')^{-1}[k-2/3,\ k+1/3],\ V)$. Actually, this f_0' is constructed by a successive use of the following procedure: increasing f' in the neighborhoods of some critical points and then readjusting the Morse function so that there is only one critical value of index k. This implies that f' and f_0' can be connected by a path in $\mathcal{O}\mathcal{M}$.   q. e. d.

Therefore, by using the theorem of Hudson on the uniqueness of the handle-attaching as in the proof of Theorem, we can get an isotopy h_t, $0 \leq t \leq 1$, of $(W,\ V)$ such that $h_0 = id$, $h_t | V = id$ and $f = f_0' \circ h_1$ on $f^{-1}(-\infty,\ k+1/3]$. By the remark preceding Lemma 2, this is enough to get a path f_t, $0 \leq t \leq 1$, in $\mathcal{O}\mathcal{M}$ such that $f_0 = f$ and $f_1 = f'$.

§3. Proof of Corollary 2

Let f be a minimal ordered excellent function on (W, V) and let h be a special diffeomorphism of (W, V). We shall show that there is an isotopy h_t, $0 \leqq t \leqq 1$, of diffeomorphism of (W, V) such that $h_0 = id$, $h_t | V = id$ and $f \circ (h \circ h_1) = f$. We observe first that h^{-1} is a filtered homotopy equivalence of W_f into $W_{f \circ h}$ which induces the identity on $H_*(W, V)$. Then, by our Theorem, $f \circ h$ and f can be connected by a path in $\mathscr{F}^0$, moreover, by the remark at the end of §1, there exists an isotopy h_t, $0 \leqq t \leqq 1$, of the diffeomorphisms of (W, V) such that $h_0 = id$, $h_t | V = id$ and $f = (f \circ h) \circ h_1 = f \circ (h \circ h_1)$.

§4. Appendix : Non-diffeomorphic minimal handle-decompositions

We shall show

Proposition. *There are two different minimal nice functions f and f' on a 5-connected 13-manifold W with $V' = \partial W = S^{12}$ and $H_6(W) = Z/13 \oplus Z/13$ such that there is no diffeomorphism h of W satisfying $h((f')^{-1}(6+1/2)) = f^{-1}(6+1/2)$.*

Agoston's example [1] shows that some diffeomorphic minimal handle-decompositions cannot be isotopic; our example concerns, however, the ones which cannot be mutually diffeomorphic.

Let X_0 be the boundary connected sum $S^6 \times D^7 \natural S^6 \times D^7$ and X_1 be $X_0 - \mathrm{int} D^{13}$. We shall construct two auto-diffeomorphisms h and h' of $\partial X_0 = S^6 \times S^6 \sharp S^6 \times S^6$ so that $W = X_0 \cup_h X_1$ and $W' = X_0 \cup_{h'} X_1$ are mutually diffeomorphic but there is no diffeomorphism $H : W \to W'$ satisfying $H(X_0) = X_0$. Let u_1 and u_2 be the generators of $H_6(S^6 \times S^6 \sharp S^6 \times S^6)$ corresponding the natural embeddings of S^6 onto $S^6 \times 0$ in the first connected sum component and in the second component respectively. Define v_1 and v_2 similarly by using $0 \times S^6$ instead. Then, any auto-diffeomorphism h of ∂X_0 induces a matrix $\mu(h) = M$ which is defined

by $(h_*u_1,\ h_*u_2,\ h_*v_1,\ h_*v_2)^*=M(u_1,\ u_2,\ v_1,\ v_2)^*$, where $*$ stands for the transpose.

Lemma 3. *The anti-homomorphism* $\mu\colon Diff(\partial X_0)\to GL(4,\ \boldsymbol{Z})$ *maps onto the group* $O(4,\ \boldsymbol{Z})$ *of all* 4×4*-matrices of integers* M *which satisfy the condition*:

$$MJM^*=\pm J,\ where\ J=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}.$$

For the proof, we refer the paper of Wall on the diffeomorphisms of 4-manifolds [11]: in fact, reading $N=S^6\times S^6$ and $S=S^6\times S^6$, we have no difficulties to verify this Lemma.

Let $\mu(h)=\begin{pmatrix}R & S\\ P & Q\end{pmatrix}$ and $\mu(h')=\begin{pmatrix}R' & S'\\ P' & Q'\end{pmatrix}$ are the matrices represented by 2×2-matrix blocks.

Lemma 4. *Suppose that* $P'=P$. *Then,* $W=X_0\cup_h X_1$ *is diffeomorphic to* $W'=X_0\cup_{h'}X_1$.

Proof. Since any embedded 6-sphere in X_0 has a trivial normal bundle and the natural map $H_6(X_0)\to H_6(W)$ is onto, the Wall's invariant $\alpha\colon H_6(W)\to\pi_5(SO)$ of the almost closed 5-connected 13-manifold W is zero [12]. So, by Theorem 8 of Wall [loc. cit], we have a handlebody $L\in\mathscr{H}(14,k,7)$ such that $W=\partial L-\mathrm{int}D^{13}$. In the same way, we have another handlebody $L'\in\mathscr{H}(14,k',7)$ such that $W'=\partial L'-\mathrm{int}D^{13}$. Moreover, $P=P'$ implies that $H_6(W)=H_6(W')$ and hence $H_6(\partial L)=H_6(\partial L')$. According to Theorem 6.3 of Smale [10], $\partial L'$ is diffeomorphic to $\partial L'$ and, in particular, W is diffeomorphic to W'.   q. e. d.

Lemma 5. *Suppose that* $P=P'$ *and the elementary divisors of* P *are not all zero. If there exists a diffeomorphism* $H\colon W\to W'$ *satisfying* $H(X_0)=X_0$, *then*

$$\det R \equiv \det R' (\mathrm{mod}\, p_1) \quad \text{and} \quad \det Q \equiv \det Q' (\mathrm{mod}\, p_1),$$

where p_1 denotes the first elementary divisor of P.

The proof of this lemma is similar to that of Theorem 2 of Birman [9]; we have only to replace her J with our J in Lemma 3 and the symmetric matrices with skew-symmetric matrices.

Now, we are in a position to prove Proposition. Since the following two matrices

$$M = \begin{pmatrix} 0 & 4 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ 13 & 0 & 0 & -3 \\ 0 & 13 & 3 & 0 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 13 & 0 & 0 & -6 \\ 0 & 13 & 6 & 0 \end{pmatrix}$$

are the elements of the orthogonal group $O(4, \mathbf{Z})$, we have two auto-diffeomorphisms h and h' of ∂X_0 such that $\mu(h) = M$ and $\mu(h') = M'$ by Lemma 3. Then, $W = X_0 \cup_h X_1$ and $W' = X_0 \cup_{h'} X_1$ are mutually diffeomorphic by Lemma 4; in fact, $H_6(W) = \mathbf{Z}/13 \oplus \mathbf{Z}/13$. We cannot, however, obtain any diffeomorphism $H: W \to W'$ satisfying $H(X_0) = X_0$ according to Lemma 5. Because W has a minimal nice function f with $f^{-1}(-\infty, 6+1/2] = X_0$ and W' has a f' with $(f')^{-1}(-\infty, 6+1/2] = X_0$, this completes the proof of Proposition.

We can remark that the same proof can apply to obtain a 1-connected 5-manifold W with $w_2(W) = 0$ and $H_2(W) = \mathbf{Z}/13 \oplus \mathbf{Z}/13$ which admits two different minimal nice functions f and f' such that there is no diffeomorphism h of W satisfying $h((f')^{-1}(2+1/2)) = f^{-1}(2+1/2)$.

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References added for Appendix

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