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Classification of $SO(3)$-Actions on Five Manifolds

By

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Introduction

P. Orlik and F. Raymond showed that some invariants classify smooth 3-manifolds with smooth S^1-action, up to equivariant diffeomorphism (preserving the orientation of the orbit space if it is orientable) [6]. And R. W. Richardson JR. studied $SO(3)$-actions on S^5 [7]. Also, K. A. Hudson classified smooth $SO(3)$-actions on connected, simply connected, closed 5-manifolds admitting at least one orbit of dimension three [2].

In this paper, we discuss the equivariant classification of smooth $SO(3)$-actions on closed, connected, oriented, smooth 5-manifolds such that the orbit space is an orientable surface. We call oriented $SO(3)$-manifolds M and N are equivalent if there is an equivariant homeomorphism between M and N which induces an orientation preserving homeomorphism of the orbit spaces M^* to N^*. Since there exist various types as the principal orbit, we classify $SO(3)$-manifolds about each type. It is well known that every subgroups of $SO(3)$ are conjugate to one of the following [4], [5].

$SO(2)$, $O(2)$, Z_n, dihedral group $D_n=\{x, y; x^2=y^n=(xy)^2=1\}$, tetrahedral group $T=\{x, y; x^2=(xy)^3=y^3=1\}$, octahedral group $O=\{x, y; x^2=(xy)^3=y^4=1\}$, and icosahedral group $\mathrm{I}=\{x, y; x^2=(xy)^3=y^5=1\}$.

T, I and O are isomorphic to the alternating groups A_4, A_5 and the symmetric group S_4, respectively. And, as the principal isotropy

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group, we have these groups except $SO(2)$ and $O(2)$.

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§ 0. Preliminaries

Let G be a compact Lie group and M a smooth G-manifold. For $x \in M$, we denote the orbit of x, and the isotropy group at x by $G(x)$ and G_x, respectively. If $H \subset G$, we write $(H) = \{K \subset G ; K$ is conjugate to H by an inner automorphism of $G\}$, and $M_H = \{x \in M ; G_x = H\}$, $M_{(H)} = \{x \in M ; G_x \in (H)\}$, and $F(H, M) = M^H = \{x \in M ; gx = x$ for ${}^{\forall}g \in H\}$.

The maximal orbit type (H) for orbits in M such that $M_{(H)}$ is open dense in M is called the principal orbit type, and H the principal isotropy group. For a principal orbit P and orbit Q, if dim $P >$ dim Q, Q is called a singular orbit. If dim $P =$ dim Q, but the isotropy group K corresponding to Q is not conjugate to H, Q is called an exceptional orbit. And for the orbit space $M^* = M/G$, let $p ; M \longrightarrow M^*$ be a natural projection.

The normal bundle at $x \in G(x)$ has fibre $V_x = TM_x/(TG(x))_x$. For each $g \in G_x$, the differential of g induces a linear map $V_x \longrightarrow V_x$ providing a representation $G_x \longrightarrow GL(V_x)$ called the slice representation. And the following Theorem is given [1].

Slice Theorem. *Some G-invariant open neighbourhood of the zero section of* $G \times_{G_x} V_x$ *is equivariantly diffeomorphic to a G-invariant tubular neighbourhood of the orbit* $G(x)$ *in M by the map* $[g, v] \longrightarrow gv$ *so that the zero section* G/G_x *maps onto the orbit* $G(x)$.

In smooth case, we can choose a suitably small closed disk S_x in V_x called a slice. And it is sufficient to discuss the representation $G_x \longrightarrow O(n)$ ($n =$ dim S_x) because M has a G-invariant metric. The representation of each subgroup of $SO(3)$ is the following.

Slice representation

	G_x	S_x	representation	principal isotropy group
1	e	D^2	trivial	e
2	$SO(3)$	D^5	(a) $\rho+\theta$ θ ; trivial action on R^2 ρ ; standard $SO(3)$-action on R^3 (b) weight two representation ([1] p. 43)	 $SO(2)$ D_2
3	$SO(2)$	D^3	$SO(2) \xrightarrow{\eta_1} O(3)$ $\Cup \qquad \Cup$ $\begin{pmatrix} \cos\theta, & -\sin\theta, & 0 \\ \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{pmatrix} \to \begin{pmatrix} \cos(n\theta), & -\sin(n\theta), & 0 \\ \sin(n\theta), & \cos(n\theta), & 0 \\ 0, & 0, & 1 \end{pmatrix}$	Z_n
4	$O(2)$	D^3	$O(2)=SO(2)\cup bSO(2)$ $\quad b=\begin{pmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{pmatrix}$ (a) $O(2) \xrightarrow{\eta_2} O(3)$ $\eta_2/SO(2)$; trivial $\eta_2(b)=\begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{pmatrix}$ or $\eta_2(b)=\begin{pmatrix} -1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & 1 \end{pmatrix}$ (b) $O(2) \xrightarrow{\eta_3} O(3)$ $\eta_3/SO(2)=\eta_1$ $\eta_3\begin{pmatrix} -1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{pmatrix}=\begin{pmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{pmatrix}$	 $SO(2)$ D_n
5	Z_n	D^2	$Z_n \longrightarrow Z_n/Z_m=Z_q \overset{\xi}{\hookrightarrow} SO(2)$ $(mq=n)$ $\xi(r\theta)=(r, \theta+\nu\xi)$ for $\xi=2\pi/q$ $(q, \nu)=1, 0<\nu<q$ (r, θ) ; polar coordinates of D^2	Z_m
6	D_n	D^2	(a) $D_n \longrightarrow D_n/Z_n \cong Z_2 \overset{\xi}{\hookrightarrow} SO(2)$ (b) for $n=2k$, $D_{2k} \longrightarrow D_{2k}/D_k \cong Z_2 \overset{\xi}{\hookrightarrow} SO(2)$	Z_n D_k
7	A_5	D^2	trivial	A_5
8	A_4	D^2	$A_4 \longrightarrow A_4/V_4=Z_3 \overset{\xi}{\hookrightarrow} SO(2)$ $V_4=\{(i,j)(k,l)$; i, j, k, l are each other different$\}$	D_2
9	S_4	D^2	$S_4 \longrightarrow S_4/A_4 \cong Z_2 \overset{\xi}{\hookrightarrow} SO(2)$	A_4

For a finite group G_x, we considered the representation $G_x \longrightarrow SO(2)$ because $SO(3) \times_{G_x} S_x$ and $SO(3)/G_x$ are orientable.

Next, we outline some basic results we need for the proofs of theorems in this paper.

1. *G acts on a locally compact space M, and assume that all orbits of G are equivalent. Let $x \in M$, and let $N = \{y \in M ; G_y = G_x\}$. Then N is a locally trivial principal fibre bundle with the group $N(G_x)/G_x$ ($N(G_x)$ = the normalizer of G_x). M is an associated fibre bundle with G/G_x as fibre.*

2. We shall often quote the following Tube Theorem [1] V. 4. 2).

Tube Theorem. *Let G be a compact Lie group and let W be a G-space with orbit space $I \times B$, where B is connected, locally connected, paracompact, and of the homotopy type of a* CW-*complex. Suppose that the orbit type on $\{0\} \times B$ is type (G/K) and that on $(0, 1] \times B$ is type (G/H). Then there exists an equivariant map π ; $G/H \longrightarrow G/K$ and, with $S = S(\pi)$, there exists a principal S-bundle $X \longrightarrow B$ (unique up to equivalence) and a G-equivariant homeomorphism $M_\pi \times_S X \cong W$ commuting with the canonical projection to $I \times B$. Moreover, the map $\varphi = \pi \times_S X$; $G/H \times_S X \longrightarrow G/K \times_S X$ gives rise to a G-equivariant homeomorphism f ; $M_\varphi \longrightarrow M \times_S X \cong W$ over $I \times B$.*

$S(\pi)$ in this theorem is given as follows; any G-equivariant map π; $G/H \longrightarrow G/K$ is of the form $R_a^{K,H}$ by $R_a^{K,H}(gH) = ga^{-1}K$ for $a \in G$ satisfying $aHa^{-1} \subset K$ ([1] I. 4. 2). Then we put $S(\pi) = (N(H) \cap a^{-1} N(K)a)/H$.

We shall use this theorem for $H \subset K$ with K/H diffeomorphic to S^1.

§1. Case of the Principal Orbit Type $SO(3)$

Let M be a closed, connected, oriented, smooth 5-dim. manifold with smooth $SO(3)$-action whose principal isotropy group consists of the identity element e. Such manifolds have at most three orbit types, i. e. the principal orbit, exceptional orbit $(SO(3)/Z_\mu)$, and singular orbit $(SO(3)/SO(2))$ with the slice representations 1, 5 and 3 respectively. The orbit space M^* is a 2-dim. surface, and $(M_{(SO(2))})^*$ is the boundary of M^*, and $(M_c)^*$ consists of isolated points in M^*. Here, $M_c=\{x\in M\ ; G_x \text{ is a cyclic group}\}$. From now on, we use M_c in this sense, and orient M^* as follows; Since $SO(3)/H$ is orientable for a finite group H, we orient naturally the tubular neighbourhood $SO(3)\times_H D^2=V$ and $V^*=D^2/H$, and orient M^* by $V^*\subset M^*$. Also, for $B=\{\text{singular orbits}\}\subset M$, the boundary B^* is oriented so that it followed by an inward normal coincides with the orientation of M^*.

For each boundary component B_i^*, $p^{-1}(B_i^*)\longrightarrow B_i^*$ is an $SO(3)/SO(2)$-bundle with the structure group $N(SO(2))/SO(2)\cong Z_2$. Let f (or m) be the number of boundary connected components so that $p^{-1}(B_i^*)\longrightarrow B_i^*$ is a trivial bundle (or non trivial). Then, $M_{SO(2)}$ has $2f+m$ connected components, and B^* has $f+m$ connected components. Let a pair (μ_i, ν_i) be the invariant uniquely determined for each exceptional orbit $SO(3)/Z_{\mu_i}$ ([5], [10]). The purpose of this section is to prove Theorem 1 (where g is the genus of M^*).

Theorem 1. *Let M be a closed, connected, oriented, smooth 5-dim manifold with smooth $SO(3)$-action, and its principal isotropy group e. Then the following orbit invariants*

$$\{g,\ (f,\ m),\ b\in Z_2\ ;\ (\mu_1,\ \nu_1),\dots,\ (\mu_r,\ \nu_r)\}$$

such that (i) $b=0$ *if* $f+m\neq 0$, $b\in Z_2$ *if* $f+m=0$

(ii) $(\mu_i,\ \nu_i)=1,\quad 0<\nu_i<\mu_i$

determine M up to an equivariant homeomorphism (which preserves the orientation of M^*).

From now on, we say that some invariants *determine* M if they determine M up to the above equivalence.

Lemma 1-1. *If* $M_e \cup M_{(SO(2))} = \phi$, *then* $\{g, b \in Z_2\}$ *determine* M.

Proof. A principal $SO(3)$-bundle $M \xrightarrow{p} M^*$ is classified by g and the obstruction class $b \in H^2(M^*; \pi_1(SO(3))) \cong Z_2$. This lemma is immediately proved. q. e. d.

For $x \in M$ with $G_x = Z_{\mu_i}$, G_x-action on the slice $S_x = D^2$ is the slice representation 5, i. e.

$$\xi(r, \theta) = (r, \theta + \nu_i \xi) \qquad \xi = 2\pi/\mu_i \qquad (\mu_i, \nu_i) = 1,\ 0 < \nu_i < \mu_i.$$

Let M have r exceptional orbits, then we have

Lemma 1-2. *If M has no singular orbit, the following orbit invariants determine M*

$$\{g, b \in Z_2; (\mu_1 \nu_1), \dots, (\mu_r, \nu_r)\}$$

such that $(\mu_i, \nu_i) = 1$, $0 < \nu_i < \mu_i$.

Proof. A pair (μ_i, ν_i) specifies a cross section on the boundary of the neighbourhood of the orbit $SO(3)/Z_{\mu_i}$ in the way of Raymond ([5], [6], [10]). We give the brief outline here. $V_i = SO(3) \times_{Z_{\mu_i}} D^2 \supset SO(2) \times_{Z_{\mu_i}} D^2 = U_i$ is a solid torus with $SO(2)$-action equivalent to

$$\theta(r, \gamma, \delta) = (r, \gamma + \nu_i \theta, \delta + \mu_i \theta)$$

(the exceptional orbit $G(x)$ corresponds to $r = 0$). (See Fig. 1-1.)

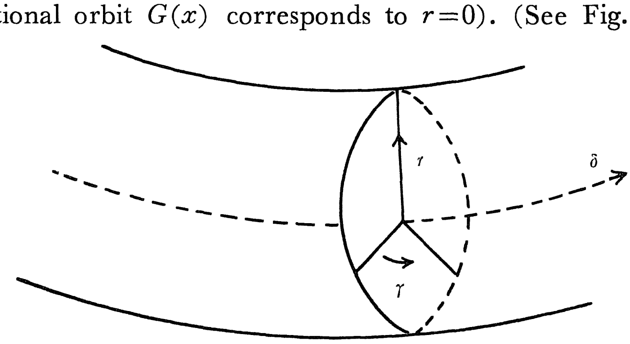


Fig. 1-1

If we give U_i the orientation naturally induced from V_i, then this orients the boundary of the slice m. Let l be an oriented curve on ∂U_i homologous to $SO(2)/Z_{\mu_i}$ in U_i, and so that the ordered pair (m, l) gives the orientation on ∂U_i. For a cross section q' of the bundle $\partial U_i \longrightarrow (\partial U_i)^*$, we orient q' so that the ordered pair (q', h) gives the same orientation as (m, l), where h is an oriented orbit $SO(2)$ on ∂U_i. Then we have

$$m=\mu_i q'+\beta h \quad (\beta>0), \quad l=-\nu_i q'-\rho h, \text{ and } \beta\nu_i\equiv 1 \pmod{\mu_i}$$

and a suitable choice of q' reduces β to $0<\beta<\mu_i$. The pair (μ_i, ν_i) determines a cross section q_i on ∂U_i, uniquely, (therefore on ∂V_i) such that $m=\mu_i q_i+\beta h$, $\beta\nu_i\equiv 1$, $0<\beta<\mu_i$. Thus the pairs $(\mu_1, \nu_1), \dots, (\mu_r, \nu_r)$ specify the cross sections $q_1, q_2, \dots, q_r$ on $\partial V_1, \dots, \partial V_r$. And we have an obstruction class in $H^2(M^*-\mathrm{Int}(\bigcup_{i=1}^{r} V_i^*), \partial(\bigcup_{i=1}^{r} V_i^*)$; $\pi_1(SO(3)))\cong Z_2$, to extend the above cross sections over M^*-Int $(\bigcup_{i=1}^{r} V_i^*)$. Its class is identified with the mod 2 integer b. Thus Lemma 1-2 follows. q. e. d.

Next, we consider the case of $M_{(SO(2))}\neq\phi$.

Lemma 1-3. *If $M_c=\phi$, $M_{(SO(2))}\neq\phi$, then $\{g, (f, m)\}$ determine M.*

Proof. By the Collaring Theorem, $M^*=M_1^*\cup(\bigcup_{i=1}^{f+m} I\times B_i^*)$ with $\{0\}\times B_i^*$ identified with each boundary component B_i^*. Since the equivariant map $SO(3)\longrightarrow SO(3)/SO(2)$ is only a canonical projection π up to equivalence,(1) M is constructed as $E(\rho)\cup(\bigcup_{i=1}^{f+m} M_\pi \times_S Q_i)$ by using the Tube Theorem. Here ρ is a principal $SO(3)$-bundle over M_1^*, and Q_i a principal $S=(N(e)\cap N(SO(2)))/e=O(2)$-bundle over B_i^*. And each attaching map of $E(\rho)$ to $M_\pi\times_S Q_i$ is an $SO(3)$ equivariant map in $\mathrm{Homeo}_{SO(3)}(SO(3)\times S^1)$ ($\mathrm{Homeo}_G M$ for G-space M, denote the group of self equivalences of M over M/G). Also, $\phi_i \in \mathrm{Homeo}_{SO(3)}(SO(3)\times S^1)$ is induced by an injection of S^1 to $SO(3)\times S^1$.

(1) Two equivariant maps f, g; $SO(3)\to SO(3)/SO(2)$ are equivalent if there is an $SO(3)$-equivariant map φ; $SO(3)\to SO(3)$ such that $g\cdot\varphi=f$.

Thus $\mathrm{Homeo}_{SO(3)}(SO(3)\times S^1)\cong\pi_1(SO(3))\cong Z_2$. Since $SO(3)\supset SO(2)$ represents a generator of $\pi_1(SO(3))$, and the bottom of $M_\pi\times_S Q_i$ is $(SO(3)/SO(2))\times_S Q_i$, ϕ_i can be extended into $M_\pi\times_S Q_i$. Thus ϕ_i may be considered as the canonical identification. And we can say M is determined by $\{g, (f, m)\}$ because ρ is a trivial bundle.

q. e. d.

Now we prove Theorem 1.

Proof of Theorem 1. It is sufficient to see the case of $M_c\neq\phi$, $M_{(SO(2))}\neq\phi$. (The other case is given by Lemma 1-1, 1-2 or 1-3.) Let V_i be a suitable small tubular neighbourhood of an exceptional orbit $SO(3)/Z_{\mu_i}$. Then the cross sections $q_1, q_2, \ldots, q_r$ on $\partial V_1, \ldots, \partial V_r$ determined by the pairs $(\mu_1, \nu_1), \ldots, (\mu_r, \nu_r)$, can be extended over $M_2^*=M^*-\mathrm{Int}(\bigcup_{i=1}^{r} V_i^*)-\bigcup_{j=1}^{f+m}[0, 1)\times B_j^*$. We denote this extended cross section by s. Next, we must investigate how to attach $p^{-1}(\{1\}\times B_j^*)$ to $E(\rho)$ where ρ is a principal $SO(3)$-bundle over M_2^*. In Lemma 1-3, we investigated it with respect to a zero cross section of $\rho/(\{1\}\times B_j^*)$. Thus, taking the above section $s/\{1\}\times B_j^*$ in place of the zero cross section, Theorem 1 follows from the proof of Lemma 1-3.

q. e. d.

§ 2. Case of the Principal Orbit Type $SO(3)/A_5$, or $SO(3)/S_4$

Let M be an oriented 5-dim. $SO(3)$-manifold with the principal isotropy group A_5 or S_4. (From now on, we suppose M is closed, connected, smooth and the action is smooth.) Such a manifold M has only principal orbits, and the orbit space M^* is a closed 2-dim. surface. Thus, we have the following theorem immediately because of $N(A_5)=A_5$, $N(S_4)=S_4$.

Theorem 2. *Let M be a closed, connected, oriented, smooth 5-dim. manifold with smooth $SO(3)$-action, and its principal isotropy group A_5 or S_4. Then M is determined only by the genus g of M^* up to* *equivariant homeomorphism which preseves the orientation of M^*.*

§ 3. Case of the Principal Orbit Type $SO(3)/A_4$

A 5-dim. $SO(3)$-manifold M with the principal isotropy group A_4, has at most two orbit types, i. e. the principal orbit and exceptional orbit $SO(3)/S_4$ with the slice representation 9. And the orbit space M^* is a closed 2-dim. surface, and $(M_{(S_4)})^*$ consists of isolated points in M^*.

Let T_g be an oriented closed 2-dim. surface with the genus g, and M_0 be a non trivial Z_2-bundle over T_g.

Lemma 3-1. *Let $M_{(S_4)}=\phi$. Then M is equivariantly homeomorphic to $SO(3)/A_4\times S^2$ if $g=0$, and to $SO(3)/A_4\times T_g$ or $SO(3)/A_4\underset{S_4/A_4}{\times}M_0$ if $g\neq 0$.*

Proof. A bundle $F(A_4, M)\longrightarrow M^*$ is classified by an element of $H^1(M^*;N(A_4)/A_4)=H^1(M^*;Z_2)$.

Case 1. Suppose $g=0$. Clearly, M is equivalent to $SO(3)/A_4\times S^2$.

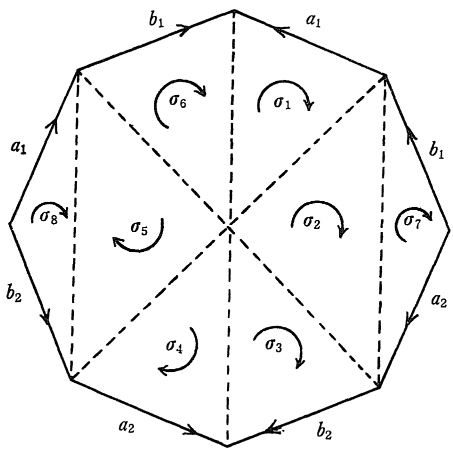
Case 2. Suppose $g=1$. Let ξ, η be Z_2-principal bundles (over M^*) corresponding to $o(\xi)$, $o(\eta)\in H^1(M^*;Z_2)$. If there is an orientation preserving homeomorphism f of M^* such that $f^*(o(\eta))=o(\xi)$, then ξ is equivalent to η in the sense of our classification. And, we can easily construct the homeomorphisms φ_2, φ_3 of M^* inducing automorphisms $(\varphi_2)_*$, $(\varphi_3)_*$ of $H_1(M^*)$ (automorphisms of $\pi_1(M^*)$ are described in [4]).

$$(\varphi_2)_*(a)=ab,\ (\varphi_2)_*(b)=b,\ (\varphi_3)_*(a)=b,\ (\varphi_3)_*(b)=a^{-1}$$

where a, b represent the generators of $H_1(M^*)\cong Z\oplus Z$. Now, we will describe an element of $H^1(M^*;Z_2)$ by a pair (m, n) which maps a and b into mod 2 integer m and n, respectively. Let M_0 be a principal Z_2-bundle (over M^*) corresponding to $(1, 0)$. Then, by operating φ_2, φ_3, $F(A_4, M)$ is equivalent to $SO(3)/A_4\times M^*$ or $SO(3)/A_4\underset{S_4/A_4}{\times}M_0$.

Case 3. Suppose $g=2$. First, we construct homeomorphisms Δ_1, Δ_2 of M^* as Fig. 3-1.

Fig. 3-1

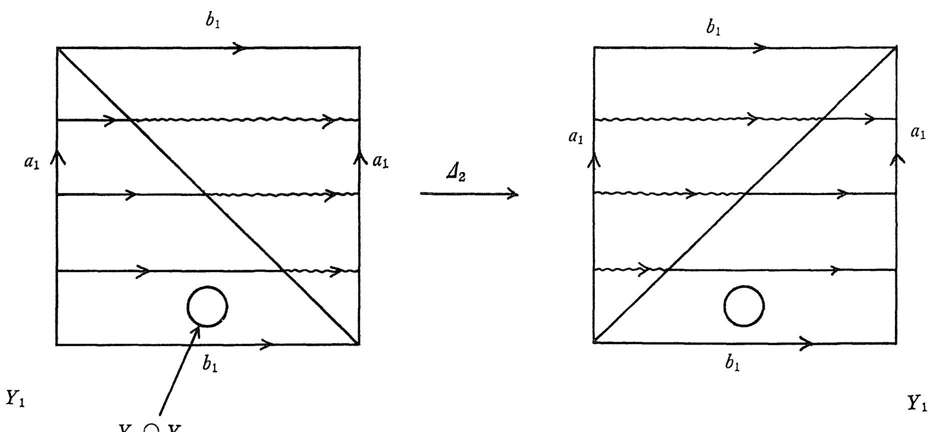


$\Delta_1 : \sigma_i \to \sigma_{i+1} \quad (1 \leq i \leq 5)$

$\sigma_6 \to \sigma_1$

$\sigma_7 \leftrightarrow \sigma_8$

$M^* = T_2$



$T_2 = T_1 \sharp T_1 = Y_1 \cup Y_2 \quad Y_i = T_1 - \mathrm{Int} D^2$
$\Delta_2 / Y_2 =$ identity map

Then, $\varphi_1 = (\Delta_1)^3$, $\varphi_2 = \Delta_2$, $\varphi_3 = \Delta_1^{-1}\Delta_2\Delta_1\Delta_2\Delta_1^{-1}\Delta_2\Delta_1$, $\varphi_4 = \varphi_3\Delta_1$ induce the automorphisms $(\varphi_i)_*(i=1, 2, 3, 4)$ of $H_1(T_g)$;

$(\varphi_1)_*$;	$a_1 \longrightarrow a_2$	$b_1 \longrightarrow b_2$	$a_2 \longrightarrow a_1$	$b_2 \longrightarrow b_1$
$(\varphi_2)_*$;	$a_1 \longrightarrow a_1 b_1$	$b_1 \longrightarrow b_1$	$a_2 \longrightarrow a_2$	$b_2 \longrightarrow b_2$
$(\varphi_3)_*$;	$a_1 \longrightarrow b_1$	$b_1 \longrightarrow a_1^{-1}$	$a_2 \longrightarrow a_2$	$b_2 \longrightarrow b_2$
$(\varphi_4)_*$;	$a_1 \longrightarrow a_2^{-1} a_1^{-1}$	$b_1 \longrightarrow b_1^{-1}$	$a_2 \longrightarrow b_1^{-1} b_2$	$b_2 \longrightarrow a_2$

By suitably operating $\varphi_i (i=1, 2, 3, 4)$ on M^*, each non-trivial bundle

$F(A_4, M) \longrightarrow M^*$ is equivalent to $M_0 \longrightarrow M_0^*$. Here $M_0 \longrightarrow M_0^*$ corresponds to $(1, 0, 0, 0) \in H^1(T_g; Z_2)$. Thus M is equivalent to $SO(3)/A_4 \times T_g$ or $SO(3)/A_4 \underset{S_4/A_4}{\times} M_0$.

Case 4. Suppose $g \geqq 3$. $H^1(M^*; Z_2) \cong \overbrace{Z_2 \oplus \ldots \oplus Z_2}^{2g}$. After applying Case 3 to the last four direct summands, we repeat it to four direct summands between $2g-5$ th and $2g-2$ th. Then an element of $H^1(M^*; Z_2)$ is regarded as $(*, *, \ldots, *, 1, 0, 0, 0, 0, 0)$ or $(*, *, \ldots, *, 0, 0, 0, 0, 0, 0)$ up to equivalence. Repeating this process, we can say M is equivalent to $SO(3)/A_4 \times M^*$, or $SO(3)/A_4 \underset{S_4/A_4}{\times} M_0$ where $M_0 \longrightarrow M_0^*$ corresponds to $(1, 0, \ldots, 0)$.

This completes the proof of Lemma 3-1. q. e. d.

We define $\varepsilon = 0$ if a principal Z_2-bundle over a closed surface is a trivial bundle, and $\varepsilon = 1$ if it is not so.

Let $(M_{(S_4)})^* = \{x_1^*, \ldots, x_r^*\}$ ($M_{(S_4)}$ has r isolated orbits).

Theorem 3. *Let M be a closed, connected, oriented, smooth 5-dim. manifold with smooth $SO(3)$-action, and its principal isotropy group A_4. Then, $\{g, \varepsilon \in \{0, 1\}, r\}$ determines M up to equivariant homeomorphism (which preserves the orientation of M^*) provided* (i) $\varepsilon = 0$ *if $g = 0$, and* (ii) *r is even.*

Proof. For a suitable neighbourhood D_i^* of x_i^* in M^*, $p^{-1}(D_i^*)$ is equivariantly diffeomorphic to $SO(3) \underset{S_4}{\times} D^2$. From S_4-action on D^2, an equivariant sewing between $F(A_4, p^{-1}(\partial D_i^*))$ and $F(A_4, p^{-1}(\partial M_1^*))$ is only the identity map (up to equivalence) where $M_1^* = M^* - \mathrm{Int}(\bigcup_{i=1}^{r} D_i^*)$. Also, M_1^* is regarded as $M_2^* \cup D(r)$. $D(r)$ is given by removing open r disks $\mathrm{Int}(\bigcup_{i=1}^{r} D_i^*)$ from a 2-dim. disk, and M_2^* is an oriented surface with one boundary and the genus g. Then a Z_2-principal bundle over ∂M_2^* is a trivial bundle, and ∂M_2^* is homologous to $\bigcup_{i=1}^{r} \partial D_i^*$ in M_1^*. Thus r is even because $F(A_4, p^{-1}(\partial D_i^*)) \longrightarrow D_i^*$ is a non-trivial bundle. Therefore, if r is given, then the classification of M is reduced to that of principal Z_2-bundles over M_1^*. Moreover, it is reduced to Lemma

3-1 because $F(A_4, p^{-1}(\partial M_1^*)) \longrightarrow \partial M_1^*$ is a trivial bundle. Therefore, $\{g, \varepsilon, r\}$ determine M. q. e. d.

§ 4. Case of the Principal Orbit Type $(SO(3)/D_n)$ $(n \geqq 3)$

Let M be a 5-dim. $SO(3)$-manifold with the principal isotropy group $D_n (n \geqq 3)$. Then M has at most three orbit types, i. e. the principal orbit, exceptional orbit $(SO(3)/D_{2n})$, and singular orbit $(SO(3)/O(2))$ with the slice representations 6-(b) and 4-(b). The orbit space M^* is a 2-dim. surface, $(M_{(O(2))})^*$ becomes the boundary of M^*, and $(M_{(D_{2n})})^*$ consists of isolated points in M^*.

$F(D_n, p^{-1}(\{1\} \times B_i^*)) \longrightarrow \{1\} \times B_i^*$ is a principal $N(D_n)/(D_n \cong Z_2$-bundle when we denote the collar of each boundary component B_i^* by $I \times B_i^*$ with $\{0\} \times B_i^*$ identified with B_i^*. Let f (m) be the number of boundary connected components such that $F(D_n, p^{-1}(\{1\} \times B_i^*)) \longrightarrow \{1\} \times B_i^*$ is a trivial Z_2-bundle (non-trivial Z_2-bundle). Then $f+m$ is the number of connected components of $M_{(O(2))}$. And let M have r exceptional orbits, and ε be the invariant to classify Z_2-bundle $F(D_n, M_{(D_n)}) \longrightarrow (M_{(D_n)})^*$, defined in §3. Then we have the following Theorem 4, and the purpose of this section is to prove this theorem.

Theorem 4. *Let M be a closed, connected, oriented, smooth 5-dim. manifold with smooth $SO(3)$-action, and its principal isotropy group $D_n (n \geqq 3)$. Then the following orbit invariants determine M up to equivariant homeomorphism (which preserves the orientation of M^*)*

$$\{g, \varepsilon \in \{0, 1\}, (f, m), r\}$$

such that (i) $\varepsilon = 0$ *if* $g = 0$, (ii) $m + r$ *is even.*

Lemma 4-1. *If $M_{(O(2))} \neq \phi$, $M_{(D_{2n})} = \phi$, then $\{g, \varepsilon \in \{0, 1\}, (f, m)\}$ determine M up to equivalence. And m must be even.*

Proof. By the Collaring Theorem, $M^* = M_1^* \cup (\bigcup_{i=1}^{f+m} I \times B_i^*)$ with $\{0\} \times B_i^*$ identified with each boundary component B_i^*. Then ρ ; $p^{-1}(M_1^*) \longrightarrow M_1^*$ is an $SO(3)/D_n$-bundle. There is only one simultaneous conjugacy class in $SO(3)$ of pairs (H, K) where $H \subset K$, with H conjugate to D_n, and K to $O(2)$. Thus, the $SO(3)$-equivariant map of $SO(3)/D_n$ to $SO(3)/O(2)$ is only a canonical projection π up to equivalence. (There is a one-one correspondence between the simultaneous conjugacy classes of (H, K) and equivalence classes of equivariant maps $G/H \longrightarrow G/K$ ([1], V. 4. 3).)

Now we put $S=(N(D_n) \cap N(O(2)))/D_n \cong Z_2$, $N=N(D_n)/D_n \cong Z_2$. By the Tube Theorem, M can be constructed as

$$E(\rho) \underset{\phi i}{\cup} \left(\bigcup_{i=1}^{f+m} M_\pi \underset{S}{\times} Q_i\right)$$

where Q_i is a principal S-bundle over B_i^*, and ρ is an $SO(3)/D_n$-bundle over M_1^*. If P_i is the associated principal N-bundle to $\rho/\{1\} \times B_i^*$, then there is a one-one correspondence between classes of S-equivariant maps of Q_i to P_i, and the classes of $SO(3)$-equivariant maps of $SO(3)/D_n \underset{S}{\times} Q_i$ to $SO(3)/D_n \underset{N}{\times} P_i$ ([1], V. 3. 2). Let Q_i be a trivial S-bundle, then the S-equivariant map is either identity map or f_1 given by

$$f_1(1,y)=(b,y) \text{ for } y \in \{1\} \times B_i^*,\ Z_2=\{1,b\}.$$

Thus ϕ_i is either identity map or $\tilde{f}_1$ induced by f_1. But $\tilde{f}_1$ can be extended into $M_\pi \underset{S}{\times} Q_i$ because the bottom of M_π is $SO((3)/O(2)$. Thus ϕ_i may be considered as the identity map (up to equivalence). Similarly, the equivariant map may be considered as the identity map when Q_i, P_i are non-trivial bundles.

From the same argument as §3, it is seen that m must be even, and ρ is determined by ε. Then the lemma is proved. q. e. d.

Lemma 4-2. *If $M_{(O(2))}=\phi$, $M_{(D_{2n})} \neq \phi$, then $\{g, \varepsilon \in \{0, 1\}, r\}$ determine M up to equivalence. In particular, r is even.*

Proof. For $x_i \in M_{D_{2n}}$, by investigating D_{2n}-action on the slice D_i^2 at x_i, M is constructed as $E(\rho) \underset{id}{\cup} \left(\bigcup_{i=1}^{r} SO(3) \underset{D_{2n}}{\times} D_i^2\right)$ where ρ is an $SO(3)/D_n$-bundle over $M^* - \mathrm{Int}\left(\bigcup_{i=1}^{r} D_i^2/D_{2n}\right)$. Since $F(D_n, SO(3) \underset{D_{2n}}{\times} \partial D_i^2) \longrightarrow \partial D_i^2/D_{2n}=S^1$ is a non-trivial Z_2-bundle, r must be even. Thus, Lemma is proved, q. e. d.

In the similar way to §3, it is shown that $r+m$ is even if $M_{SO(2)} \neq \phi$, $M_{(D_{2n})} \neq \phi$. Then, Theorem 4 is immediately given from Lemma 4-1 and 4-2.

§ 5. Case of the Principal Orbit Type $(SO(3)/Z_k)$ $(k \geqq 3)$

Let M be a 5-dim. $SO(3)$-manifold with the principal isotropy group $Z_k (k \geqq 3)$. Then M' has at most four orbit types, i. e. the principal orbit, exceptional orbits $(SO(3)/D_k)$, and $(SO(3)/Z_{kq})$, and singular orbit $(SO(3)/SO(2))$ with the slice representations 6, 5 and 3 respectively. The orbit space M^* is a 2-dim. surface, $(M_{(SO(2))})^*$ is the boundary of M^*, and $(M_{(D_k)})^* \cup (M_c)^*$ consists of isolated points in M^*.

For boundary components $\cup B_i^*$, let f be the number of boundary components so that $F(SO(2), p^{-1}(B_i^*)) \longrightarrow B_i^*$ is a trivial bundle, and m of non-trivial bundle (i. e. $M_{SO(2)}$ has $2f+m$ connected components). And let d be the number of exceptional orbits $(SO(3)/D_k)$, (μ_i, ν_i) be the invariant defined for $SO(3)/Z_{\mu_i k}$ in the same way as §1. And let ε be the invariant defined in §3.

The purpose of this section is to prove the following theorem.

Theorem 5. *Let M be a closed, connected, oriented, 5-dim. smooth manifold with smooth $SO(3)$-action, and its principal isotropy group $Z_k (k \geqq 3)$. Then the following orbit invariants determine M up to an equivariant homeomorphism (which preserves the orientation of M^*)*

$$\{g, \varepsilon \in \{0,1\}, b \in Z, (f,m), d; (\mu_1, \nu_1), \ldots, (\mu_r, \nu_r)\}$$

such that (i) $\varepsilon=0$ *if* $g=0$, (ii) $b=0$ *if* $f+m \neq 0$,
(iii) $(\mu_i, \nu_i)=1$, $0<\nu_i<\mu_i$ (iv) $m+d$ *is even.*

An integer b in this theorem, corresponds to the secondary obstruction class for a principal $O(2)/Z_k$-bundle over M^*. We will make its details clear in the proof of Lemma 5-1.

Lemma 5-1. *If* $M_{(D_k)} \cup M_c \cup M_{(SO(2))} = \phi$, *then* $\{g, \varepsilon \in \{0, 1\}, b \in Z\}$

determine M *up to equivalence. In particular* $\varepsilon=0$ *if* $g=0$.

Proof. ξ; $F(Z_k, M)\longrightarrow M^*$ is a principal $O(2)/Z_k$-bundle, and $H^1(M^*; \pi_0(O(2)/Z_k))\cong\overbrace{Z_2\oplus,\dots,\oplus Z_2}^{2g}$. Thus we may assume ξ corresponds to $\varepsilon=0$ or $\varepsilon=1$. (See §3.) Clearly, if $\varepsilon=0$, then the classification of ξ depends only on g, and an obstruction class $b\in H^2(M^*;\pi_1(O(2)/Z_k))\cong Z$. If $\varepsilon=1$, then M^* is considered as follows;

$$M^*=T_g=M_1^*\cup M_2^*,\ M_1^*=T_1-\mathrm{Int}\,D^2,\ M_2^*=T_{g-1}-\mathrm{Int}\,D^2$$

$$T_g=c_1d_1c_1^{-1}d_1^{-1}\dots c_g^{-1}d_g^{-1},\ T_1=c_1d_1c_1^{-1}d_1^{-1}$$

$\varepsilon=1$ means ξ/c_1 is a non-trivial and $\xi/c_i(i\neq 1)$, ξ/d_j are trivial $O(2)/Z_k$-bundles. Then we can construct a double covering $\tilde{N}$ of $F(Z_k, p^{-1}(M_1^*))$ such that $\tilde{N}\longrightarrow\tilde{N}/(O(2)/Z_k)=\tilde{N}^*$ is a trivial $O(2)/Z_k$-bundle, and $\tilde{N}^*$ is also a double covering of M_1^*. If we specify a cross section $\tilde{s}$; $\tilde{N}^*\longrightarrow\tilde{N}$, then $\tilde{s}$ uniquely determines a cross section s of $\xi/\partial M_1^*$. Thus, ξ is determined by the genus g of M^*, and the obstruction class to extend the specified cross section $s/\partial M_1^*$ over M_2^*, i. e. by $b\in H^2(M_2^*, \partial M_2^*; \pi_1(O(2)/Z_k))\cong Z$. Clearly $\varepsilon=0$ if $g=0$.

q. e. *d.*

Lemma 5-2. *If* $M_{(D_k)}\cup M_c=\phi$, *then* $\{g, \varepsilon\in\{0, 1\}, (f, m)\}$ *determine* M *up to equivalence. In particular,* (i) $\varepsilon=0$ *if* $g=0$, and (ii) m *is even.*

Proof. Let B_i^* $(i=1, 2,\dots, f+m)$ be a boundary component. Then, $M^*=M_1^*\cup(\bigcup_{i=1}^{f+m} I\times B_i^*)$ with $\{0\}\times B_i^*$ identified with B_i^*. We denote an $SO(3)/SO(2)$-bundle $p^{-1}(B_i^*)\longrightarrow B_i^*$ by σ_i, and an $SO(3)/Z_k$-bundle $p^{-1}(M_1^*)\longrightarrow M_1^*$ by ρ. $\rho/\{1\}\times B_i^*$ is a trivial bundle iff σ_i is a trivial bundle. By the Tube Theorem, M can be constructed as $E(\rho)\cup_{id}(\bigcup_i^{f+m} M_{\phi_i})$ where ϕ_i is an equivariant map of $p^{-1}(\{1\}\times B_i^*)$ to $p^{-1}(B_i^*)$. Also we have precisely one (up to equivalence) as ϕ_i. (In fact, we may take a natural projection.) This implies M_{ϕ_i} depends only on (f, m). Thus M depends on ρ and (f, m). Now, we remove $f+m$ open disks from S^2, and denote it by X. Then $M_1^*=T_g\# X=M_2^*\cup M_3^*$ where

$M_2^*=T_g-\text{Int}\,D^2$, $M_3^*=X-\text{Int}\,D^2$, and ρ/M_2^* depends only on ε and g, ρ/M_3^* only on (f, m). Moreover, we may regard $\rho/\partial M_2^*$ is attached to $\rho/\partial M_3^*$ by the identity map. For, ∂X corresponds to $\bigcup_{i=1}^{f+m}\{1\}\times B_i^*$, and M is unchanged up to equivalence even if we exchange an equivariant attaching ϕ_i of M_{ϕ_i}. Hence, ρ depends only on ε and (f, m) and M is determined by ε, g and (f, m). Also, it is seen m is even in the same way as §3. q. e. d.

Let $(M_{(D_k)})^*=\{x_1^*, \dots\dots, x_d^*\}\subset M^*$, i. e. M has d exceptional orbits of type $SO(3)/D_k$.

Lemma 5-3. *If* $M_c\cup M_{(SO(2))}=\phi$, $M_{(D_k)}\neq\phi$, *then* $\{g, \varepsilon\in\{0, 1\}, d, b\in Z\}$ *determine* M *up to equivalence. Moreover* (i) $\varepsilon=0$ *if* $g=0$, *and* (ii) d *is even.*

Proof. For a suitable neighbourhood D_i^* of x_i^*, and a principal $O(2)/Z_k$-bundle ρ over $M_1^*=M^*-\text{Int}\,(\bigcup_{i=1}^{d}D_i^*)$, $F(Z_k, M)$ is constructed as

$$E(\rho)\cup(\bigcup_{i=1}^{d}F(Z_k, p^{-1}(D_i^*))).$$

$E(\rho)$ is attached to $F(Z_k, p^{-1}(D_i^*))$ by $\varphi_i\in\text{Homeo}_{O(2)/Z_k}(O(2)/Z_k\underset{D_k/Z_k}{\times}S^1)$. Since $F(Z_k, p^{-1}(\partial D_i^*))$ is a non-trivial $O(2)/Z_k$-bundle, d is even by the same reason as §3.

Now, if we remove d open disks from S^2, and denote it by Y, then $M_1^*=T_g\,\#\,Y=M_2^*\cup M_3^*$ where $M_2^*=T_g-\text{Int}\,D^2$, $M_3^*=Y-\text{Int}\,D^2$. Let $\tilde{M}_3$ be a double covering of $F(Z_k, p^{-1}(M_3^*))$ such that $\tilde{M}_3\longrightarrow\tilde{M}_3/(O(2)/Z_k)=\tilde{M}_3^*$ is a trivial $O(2)/Z_k$-bundle. Since φ_i is determined by an injection f_i of S^1 to $O(2)/Z_k\underset{D_k/Z_k}{\times}S^1$, we can specify a cross section $\tilde{s}$; $\tilde{M}_3^*\longrightarrow\tilde{M}_3$ by extending the lifting $\{\tilde{f}_i\}$ of $\{f_i\}$. Then $\tilde{s}$ determines a cross section s'; $\partial M_3^*\cap M_2^*\longrightarrow F(Z_k, p^{-1}(\partial M_3^*\cap M_2^*))$ uniquely (i. e. s' depends only on $\{\varphi_i\}$). Taking a specified cross section s over ∂M_2^* defined in §5-1, then we can see that M is determined by ρ/M_2^*, d and how to attach $\rho/\partial M_2^*$ to $\rho/\partial M_3^*$ with respect to the specified cross sections s, s', i. e. by $b\in\text{Homeo}_{O(2)/Z_k}(O(2)/Z_k\times S^1)\cong Z$. Since ρ/M_2^* is classified by $\varepsilon=0$ or 1, the lemma was proved. q. e. d.

If M_c has r connected components, i. e. $(M_c)^*$ has r isolated points $\{x_1^*, \dots, x_r^*\}$, then the following lemma is given in the similar way to §1.

Lemma 5-4. *If $M_{(D_k)} \cup M_{(SO(2))}=\phi$, then the following orbit invariants determine M up to equivalence*;

$$\{g,\ \varepsilon \in \{0,\ 1\},\ b \in Z;\ (\mu_1,\ \nu_1), \dots, (\mu_r,\ \nu_r)\}$$

such that (i) $\varepsilon=0$ *if* $g=0$, (ii) $(\mu_i,\ \nu_i)=1$, $0<\nu_i<\mu_i$.

Now, we prove Theorem 5 from the above lemmas.

Proof of Theorem 5. It is sufficient to see the case of having at least two orbit types except the principal orbit type. First, suppose $M_{(SO(2))}=\phi$. And let $S^2_{(3)}$ be the space given by removing 3 open disks from S^2, $D^2_i (i=1,\ 2)$ a 2-dim. disk, and T'_g a 2-dim. surface with the genus g and one boundary. Then we may regard M^* as

$$S^2_{(3)} \cup T'_g \cup D^2_1 \cup D^2_2$$

by canonically identifying the boundaries of T'_g, D^2_i with three boundary components of $S^2_{(3)}$, respectively. And we may suppose $(M_{(D_k)})^* \subset D^2_1$, $(M_c)^* \subset D^2_2$. Then, $p^{-1}(D^2_1)$ depends only on d, and $p^{-1}(D^2_2)$ only on the invariants $\{(\mu_1,\ \nu_1), \dots, (\mu_r,\ \nu_r)\}$, and $p^{-1}(T'_g)$ on $\{g,\ \varepsilon\}$. Moreover, taking the cross sections on ∂D^2_1, ∂D^2_2, $\partial T'_g$ defined in §5-3, §1 and §5-1, we can see M is determined by the above invariants and the obstruction class to extend this cross sections over $S^2_{(3)}$, i. e. by $b \in H^2(S^2_{(3)},\ \partial S^2_{(3)};\ \pi_1(O(2)/Z_k)) \cong Z$. Also, if we suppose $M_{(SO(2))} \neq \phi$, it is seen b is zero by the argument of §5-2. Thus, M is determined by $\{g,\ \varepsilon,\ b,\ (f,\ m),\ d;\ (\mu_1,\ \nu_1), \dots, (\mu_r,\ \nu_r)\}$ such that $b=0$ if $f+m \neq 0$. Also we can easily seen that $m+d$ is even.

§ 6. Case of the Principal Orbit Type $(SO(3)/Z_2)$

An oriented 5-dim. $SO(3)$-manifold M with the principal isotropy group Z_2, has at most five orbit types, i. e. the principal orbit, exceptional orbits $(SO(3)/D_2)$, $(SO(3)/Z_{2q})$, and singular orbits $(SO(3)/SO(2))$, $(SO(3)/O(2))$ with slice representations 6-(a), 5, 3 and 4-(b), respectively. If M has not a singular orbit type $(SO(3)/O(2))$, then we can apply the argument of §5 to this case in its entirety. Thus, we shall discuss only the case of having orbit $(SO(3)/O(2))$.

It is easily seen that $(M_{(O(2))})^* \cup (M_{(SO(2))})^*$ is the boundary of 2-dim. surface M^*, and $(M_c)^* \cup (M_{(D_2)})^*$ consists of isolated points in M^*. We put $(M_{(O(2))})^* = \cup B_i^*$ where B_i^* is a boundary component, and denote the collar of each boundary component B_i^* by $I \times B_i^*$ with $\{0\} \times B_i^*$ identified with B_i^*. Let k be the number of boundary components so that $F(Z_2, p^{-1}(\{1\} \times B_i^*)) \longrightarrow \{1\} \times B_i^*$ is a trivial bundle, and n of a nontrivial bundle. (Then $M_{(O(2))}$ has $k+n$ connected components.)

Lemma 6-1. *If* $M = M_{(Z_2)} \cup M_{(O(2))}$, *then* $\{g, \varepsilon \in \{0, 1\}, (k, n), b \in Z\}$ *degermine* M *up to equivalence. In particular,* (i) $\varepsilon = 0$ *if* $g = 0$, *and* (ii) n *is even.*

Proof. First, we show there exists only canonical projection as equivariant map of $SO(3)/Z_2$ to $SO(3)/O(2)$, because we want to use the Tube Theorem. For $H \subset K \subset G$, it is known there is a natural one-one correspondence between the equivalence classes of equivariant maps $G/H \longrightarrow G/K$ and orbits of the action of $N(H)/H \times N(K)/K$ on $F(H, G/K)$ where $N(H)/H$ acts on the left and $N(K)/K$ acts on the right ([1], p. 245). Now we take subgroups of $SO(3)$ which consist of the following matrices.

$$H = \left\{ 1, \begin{pmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & -1 \end{pmatrix} \right\} \text{ and } \quad K = \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & \pm 1 \end{array} \right) \in SO(3) \right\}.$$

(K is conjugate to $0(2)$, and H to Z_2.)

Then $F(H, SO(3)/K)/(N(H)/H \times N(K)/K) = \{eK \cup AK \cup A^2K\}$ where

$$A=\begin{pmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ 1, & 0, & 0 \end{pmatrix} \in SO(3), \qquad A^3=1.$$

These orbits correspond to $SO(3)$-equivariant maps,

$$\varphi_i\ ;\ SO(3)/H \longrightarrow SO(3)/K \text{ by } \varphi_i(gH)=gA^iK \quad (i=0,\ 1,\ 2).$$

But, by Bredon ([1] p. 200, Cor 6-3), N_i/H must be homeomorphic to S^1 where N_i is the isotropy group $A^iK(A^i)^{-1}$ of A^iK. This implies that there exists only the canonical projection $\varphi_0(gH)=gK$. We depended on K. A. Hudson [3] for this argument.

We rewrite π for φ_0. Let Q_i be a principal S-bundle over B_i^* for $S=(N(H)\cap N(K))/H\cong Z_2$, and let ρ be an $SO(3)/H$-bundle over $M_1^*=M^*-(\bigcup_{i=1}^{k+n}[0,\ 1)\times B_i^*)$. Then, by the Tube Theorem, M is equivalent to

$$E(\rho)\bigcup_{\psi_i}(\bigcup_{i=1}^{k+n} M_\pi \times_S Q_i).$$

Also, ψ_i is an equivariant homeomorphism of $\rho/\{1\}\times B_i^*$ to the top of $M_\pi\times_S Q_i$, i. e. $SO(3)/K\times_S Q_i$. Since Q_i is a trivial bundle iff P_i is so,

$$\psi_i \in \mathrm{Homeo}_{SO(3)/H}(SO(3)/H\times S^1)=\mathrm{Homeo}_{N(H)/H}(N(H)/H\times S^1)$$
$$\text{or}\quad \psi_i\in \mathrm{Homeo}_{N(H)/H}(N(H)/H\times_S S^1).$$

In §4-1, such a ψ_i was uniquely determined (up to equivalence), but in this case we have various types. In fact,

$$\psi_i^j(eH,\ x)=(g_j(x)H,\ x)=(jg_1(x),\ x) \text{ for } j\in Z$$

is an equivariant homeomorphism in $\mathrm{Homeo}_{SO(3)/H}(SO(3)/H\times S^1)$ where $g_1(x)$ generates $\pi_1(N(H)/H)\cong Z$. And

$$(E(\rho)\bigcup_{\psi_i}(\bigcup_{i=1}^{k-1+n} M_\pi\times_S Q_i))\bigcup_{\psi_k^s}(M_\pi\times_S Q_k)$$

is not equivalent to

$$(E(\rho)\bigcup_{\psi_i}(\bigcup_{i=1}^{k-1+n} M_\pi\times_S Q_i))\bigcup_{\psi_k^t}(M_\pi\times_S Q_k)$$

if $t\neq s$. Thus an obstruction element $b\in Z$ is determined by how to attach $\rho/\{1\}\times B_i^*$ to $SO(3)/H\times_S Q_i$, in the same way as §5-3.

Strictly speaking, if we put

$$M_1^* = M_2^* \cup Y, \qquad M_2^* = T_g - \mathrm{Int}\, D^2, \qquad Y = D^2 - \mathrm{Int}\,(\bigcup_{i=1}^{f+m} D_i^2),$$

then, $b \in Z$ is determined by how to attach $\rho/\partial M_2^*$ to $\rho/\partial Y \cap M_2^*$ with respect to two specifying cross sections, i. e. the cross section on $\partial Y \cap M_2^*$ induced from the above attaching maps $\{\psi_i\}$, and the cross section on ∂M_2^* determined in the similar way to § 5-1. Here ρ/M_2^* depends only on $\{g, \varepsilon\}$. Then Lemma is proved. q. e. d.

Let invariants $\{g, \varepsilon, b, (f, m), d\,;\, (\mu_1, \nu_1), \ldots\ldots, (\mu_r, \nu_r)\}$ be the same as in Theorem 5. Since it is easily checked that $m+n+d$ is even, Lemma 6-1 and Theorem 5 gives the following theorem.

Theorem 6. *Let M be a closed, connected, oriented, smooth 5-dim. manifold with smooth $SO(3)$-action, and its principal isotropy group Z_2. Then the following orbit invariants determine M up to an equivariant homeomorphism (which preserves the orientation of the orbit space M^*)*

$$\{g,\ \varepsilon \in \{0, 1\},\ b \in Z,\ (f, m),\ (k, n),\ d\,;\ (\mu_1, \nu_1), \ldots, (\mu_r, \nu_r)\}$$

such that

(i) $\varepsilon = 0$ *if* $g = 0$, (ii) $m+n+d$ *is even,*

(iii) $b = 0$ *if* $f+m \neq 0$, (iv) $(\mu_i, \nu_i) = 1,\ 0 < \nu_i < \mu_i$.

§ 7. Case of the Principal Orbit Type $(SO(3)/D_2)$

In this section, we treat $SO(3)$-manifold M whose principal orbit type is $(SO(3)/D_2)$. Such a manifold has at most five orbit types, i. e. principal orbit, exceptional orbits $(SO(3)/D_4)$, $(SO(3)/A_4)$, singular orbit $(SO(3)/O(2))$ and fixed point $SO(3)/SO(3)$, with the slice representations 6-(b), 8, 4-(b) and 2-(b) respectively. Then $(M_{(D_4)})^* \cup (M_{(A_4)})^*$ consists of isolated points in a 2-dim. surface M^*, and $(M_{(O(2))})^* \cup (M_{SO(3)})^*$ is the boundary of M^*, and the fixed points set $(M_{SO(3)})^*$ consists of isolated points in ∂M^*. (We shall detail the case of having fixed points in the latter half of this section.)

First, we consider the case M has only principal orbit. Then

$F(D_2, M) \longrightarrow M^*$ is a principal $N(D_2)/D_2 \cong D_3$-bundle. According to the Classification Theorem ([9], 13. 9), the usual bundle equivalence classes of D_3-principal bundles over M^* are in one-one correspondence with the equivalence classes (under inner automorphisms of D_3) of homomorphisms of $\pi_1(M^*)$ into D_3. Let D_3-bundles ξ and η correspond to homomorphisms f and g, respectively. If there is an orientation preserving homeomorphism φ of M^* such that $\varphi^* \circ f = g$, then ξ is equivalent to η in our classification (even if ξ is not equivalent to η as the usual bundle equivalence). So, we shall say f and g with such a homeomorphism are equivalent, too.

Now we put

$$\pi_1(M^*) = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \,;\, [\alpha_1, \beta_1] \ldots . [\alpha_g, \beta_g] = 1\}$$

by the canonical generators. (Here g is the genus of M^*, and $[\alpha_i, \beta_i]$ is the commutator of α_i and β_i.) Then there is a relation;

$$f([\alpha_1, \beta_1][\alpha_2, \beta_2] \ldots [\alpha_g, \beta_g]) = 1 \quad \text{for } f \in \mathrm{Hom}(\pi_1(M^*)\,; D_3)$$

First, we investigate the equivalence classes of Hom $(\pi_1(M^*)\,; Z_p)$ for a prime number p. For a generator x of Z_p we use a symbol $({}^*, {}^*, \ldots, {}^*, x^s, x^t, {}^*, \ldots, {}^*)$ in place of f with $f(\alpha_i) = x^s$, $f(\beta_i) = x^t$. Then, it is easily seen that there are the following relations for φ_i ($i = 1, 2, 3, 4$) constructed in § 3.

$$\text{(i)} \quad \begin{array}{ccccc}
(x^s, x^t) \xrightarrow[\varphi_2^*]{} (x^{s+t}, x^t) \xrightarrow[\varphi_2^*]{} \cdots \longrightarrow & & & & (x^{p-1}, x^t) \\
 & & & & \downarrow \varphi_3^* \\
(1, x) \xrightarrow[\varphi_2^*]{} \cdots\cdots\cdots \longrightarrow (x^{t-1}, x) & & \xrightarrow[\varphi_2^*]{} & & (x^t, x) \\
\varphi_1^* \downarrow \quad & & & & \\
(x, 1) & & & &
\end{array}$$

$$\text{(ii)} \quad \begin{array}{cc}
(x, 1, x, 1) \longrightarrow \cdots\cdots \longrightarrow & (x, 1, x^{-1}, 1) \\
 & \downarrow \varphi_4^* \\
(1, 1, x, 1) \xrightarrow{\varphi_3^*} & (1, 1, 1, x).
\end{array}$$

From these relations (i), (ii), it is seen that the equivalence classes of Hom $(\pi_1(M^*)\,; Z_p)$ are exactly two classes, i. e. $(1, 1, \ldots, 1, 1)$ and $(1, 1, \ldots, 1, 1, x, 1) = f_1$.

Let $\pi\,;\, D_3 \longrightarrow D_3/Z_3 \cong Z_2 = \{1, x\}$ be a natural projection where

$$D_3=\{x, y\ ;\ x^2=y^3=(xy)^2=1\}\supset Z_3=\{1,\ y,\ y^2\}.$$

Since $\varphi_i^*\cdot\pi_*=\pi_*\cdot\varphi_i^*$ $(i=1, 2, 3, 4)$ for π_*; Hom $(\pi_1(M^*)\ ;\ D_3)\longrightarrow$ Hom $(\pi_1(M^*)\ ;\ D_3/Z_3)$, the equivalence classes of Hom $(\pi_1(M^*)\ ;\ D_3)$ are given by computing the classes of $\pi_*^{-1}(1)$ and $\pi_*^{-1}(f_1)$. By the above argument,

$$\pi_*^{-1}(1)=\text{Hom}\ (\pi_1(M^*)\ ;\ Z_3)\subset\text{Hom}\ (\pi_1(M^*)\ ;\ D_3)$$

has only two classes, i. e.

$$(1,\ 1,\dots,\ 1,\ 1)\cdots(1)\ \text{ and }\ (1,\ 1,\dots,\ 1,\ 1,\ 1,\ y)\cdots(2).$$

Also, every elements of $\pi_*^{-1}(f_1)$ take the form of

$$(y^{i_1},\ y^{j_1},\dots,\ y^{i_{g-1}},\ y^{j_{g-1}},\ xy^{i_g},\ y^{j_g})\cdots(3)$$
$$(0\leqq i_k,\ j_k\leqq 2).$$

Applying the above argument $(p=3)$ to the first $2(g-1)$ components of (3), the classes of $\pi_*^{-1}(f_1)$ are in either

$$(1,\ 1,\dots,\ 1,\ 1,\ xy^{i_g},\ y^{j_g})\ \text{ or }\ (1,\ 1,\dots,\ 1,\ y,\ xy^{i_g},\ y^{j_g}).$$

Moreover, considering that $f([\alpha_1, \beta_1]\dots[\alpha_g, \beta_g])=1$ and D_3 is not abelian, we can say that $\pi_*^{-1}(f_1)$ has only classes in the following forms.

$$(1,\ 1,\dots,\ 1,\ 1,\ x,\ 1)\cdots(4)\qquad (1,\ 1,\dots,\ 1,\ 1,\ xy,\ 1)\cdots(5)$$
$$(1,\ 1,\dots,\ 1,\ 1,\ xy^2,\ 1)\cdots(6)\qquad (1,\ 1,\dots,\ 1,\ 1,\ 1,\ y,\ x,\ 1)\cdots(7)$$
$$(1,\ 1,\dots,\ 1,\ 1,\ 1,\ y,\ xy,\ 1)\cdots(8)\qquad (1,\ 1,\dots,\ 1,\ 1,\ 1,\ y,\ yx^2,\ 1)\cdots(9).$$

But (4), (5) and (6) are in the same class under some inner automorphisms. Similarly, (7), (8) and (9) are in one class.

So, we define the number ε to determine M as follows; $\varepsilon=0, 1, 2$ or 3, if $F(D_2, M)\longrightarrow M^*$ corresponds to homomorphism (1), (2), (4) or (7) respectively.

Consequently we have

Lemma 7-1. *Let M have only principal orbits. Then $\{g, \varepsilon\in\{0, 1, 2, 3\}\}$ determine M up to equivalence. If $g=1$, then $\varepsilon\in\{0, 1, 2\}$ and if $g=0$, then $\varepsilon=0$.*

Now, suppose $M_{(D_4)}\neq\phi$, $M_{(A_4)}\cup M_{(O(2))}\cup M_{SO(3)}=\phi$, and put $(M_{(D_4)})^*=\{x_1^*,\dots,x_r^*\}$. By the Slice Theorem, for a suitable neighbourhood D_i^* of x_i^*, $F(D_2, p^{-1}(D_i^*))$ is equivalent to $O/D_2\underset{D_4/D_2}{\times}D^2$. Then it is not difficult to see that equivariant attaching maps between $F(D_2, p^{-1}(\partial(M^*-\mathrm{Int}\ D_i^*)))$ and $F(D_2, p^{-1}(\partial D_i^*))$ can be extended over $F(D_2, p^{-1}(D_i^*))$. Thus M is equivalent to

$$p^{-1}(M_1^*)\underset{id}{\cup}(\bigcup_{i=1}^{r}SO(3)/D_2\underset{D_4/D_2}{\times}D^2)$$

where $M_1^*=M^*-\mathrm{Int}\bigcup_{i=1}^{r}D_i^*$. Here, O is isomorphic to S_4 and D_2 to $V_4\subset A_4$. (V_4 is defined in p. 3.) Thus $O/D_2\cong S_4/V_4\cong S_3$. Moreover, $D_3=\{x, y\ ;\ x^2=(xy)^2=y^3=1\}$ is isomorphic to S_3 by corresponding x to (12) and y to (123) (where (12) and (123) are cycles in S_3). We remark, under this isomorphism, D_4/D_2 is identified with $Z_2=\{1, (13)\}$. Then we can see that the principal $O/D_2\cong D_3$-bundle $F(D_2, p^{-1}(\partial D_i^*))=O/D_2\underset{D_4/D_2}{\times}S^1\longrightarrow S^1=\partial D_i^*$ corresponds to (13), i. e. $xy\in\mathrm{Hom}\ (\pi_1(S^1)\ ;D_3)$. Thus r must be even.

Next, suppose $M_{(A_4)}\neq\phi$, $M_{(D_4)}\cup M_{(O(2))}\cup M_{SO(3)}=\phi$, and put $(M_{(A_4)})^*=\{y_1^*,\dots,y_m^*\}$. Then, $F(D_2, p^{-1}(D_i^*))$ is equivalent to $O/D_2\underset{A_4/D_2}{\times}D^2$ for a suitable neighbourhood D_i^* of y_i^*. And we have two types as $F(D_2, p^{-1}(\partial D_i^*))$, which arise from two different $A_4/D_2\cong Z_3$-actions on the slice D^2 at y_i, (1) and (2).

(1) $\xi_1(r, \theta)=(r, \theta+(2/3)\pi)$, (2) $\xi_2(r, \theta)=(r, \theta+(4/3)\pi)$

where (r, θ) is the polar coordinate of D^2. By the same reason as the case of $M_{(D_4)}\neq\phi$, it is seen M is equivalent to

$$p^{-1}(M_1^*)\underset{id}{\cup}(\bigcup_{i=1}^{d_1+d_2}p^{-1}(D_i^*)).$$

Here d_1 is the number of points in $(M_{(A_4)})^*$ so that the D_3-bundle $F(D_2, p^{-1}(\partial D_i^*))\longrightarrow\partial D_i^*$ corresponds to type (1), and d_2 to type (2). And the bundle of type (1) corresponds to $y\in\mathrm{Hom}(\pi_1(S^1)\ ;D_3)$, and the bundle of type (2) to y^2. ($A_4/V_4\subsetneq S_4/V_4\cong S_3$ and $A_4/V_4=\{1, (123), (132)\}$ where (123) corresponds to y.)

Suppose $M_{(O(2))}\neq\phi$, $M_{(A_4)}\cup M_{(D_4)}\cup M_{SO(3)}=\phi$. We denote each connected component of $(M_{(O(2))})^*$ by B_i^* $(i=1, \ldots, f)$, which is a boundary component of M^*. Applying the argument in §6-1 to this case, the orbit of the action of $N(D_2)/D_2 \times N(O(2))/O(2) = O/D_2$ on $F(D_2, SO(3)/O(2))$ is exactly one. So, there exists only the canonical projection π as the equivariant map of $SO(3)/D_2$ to $SO(3)/O(2)$. Therefore M is equivalent to $p^{-1}(M_1^*) \underset{\psi_i}{\cup} (\bigcup_{i=1}^{f} M_\pi \underset{S}{\times} Q_i)$ where $S=(N(O(2)) \cap N(D_2))/D_2 = D_4/D_2$, and Q_i is a principal S-bundle over B_i^*, and $M_1^* = M^* - \bigcup_{i=1}^{f} ([0, 1) \times B_i^*)$. And by investigating $SO(3)$-equivariant attaching maps $\{\psi_i\}$ of $p^{-1}(\partial M_1^*)$ to $\bigcup_{i=1}^{f} SO(3)/D_2 \underset{S}{\times} Q_i$, we can see M is equivalent to

$$p^{-1}(M_1^*) \underset{id}{\cup} (\bigcup_{i=1}^{f} M_\pi \underset{S}{\times} Q_i)$$

(because the attaching map ψ_i can be extended over $M_\pi \underset{S}{\times} Q_i$).
Here, D_3-bundle $F(D_2, p^{-1}(\{1\} \times B_i^*)) \longrightarrow \{1\} \times B_i^* = S^1$ corresponds to $xy \in \mathrm{Hom}(\pi_1(S^1); D_3)$ if Q_i is a non-trivial $S \cong Z_2$-bundle.

Finally, we shall consider the case of $M_{SO(3)} \neq \phi$, $M_{(A_4)} \cup M_{(D_4)} = \phi$. The results we mention here are the extension of the work of K. A. Hudson [2] (where she treats the case of the orbit space being simply connected), and we make use of her idea.

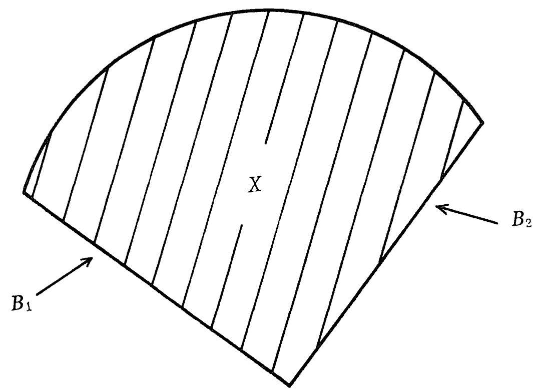
Let $x \in M$ with $G_x = SO(3)$. Then $SO(3)$-action on the slice D^5 at x is given by the weight two representation (Bredon [1], p. 43). $O(2)$ has three different conjugate groups N_0, N_1, N_2 where $N_0 = O(2)$, $N_1 = A^{-1}N_0A$, $N_2 = A^{-1}N_1A$, for A in §6-1. Using this notation, $D^5/SO(3)$ can be illustrated below (Fig. 7-1), and B_i $(i=1, 2)$ is in the boundary of M^*. Thus the neighbourhood in M^* of a boundary component having two orbit types, $(SO(3)/O(2))$ and fixed points can be illustrated as Fig. 7-2. According to Richardson ([8], 5-2), $p^{-1}(S_i)$ is homeomorphic to $S^4 = \partial D^5$. And it is clear that there is no boundary component with exactly one fixed point (by reason of $j_i \neq j_{i+1}$).

Now, we put

$$A = S_1 \cup \ldots \cup S_n \cup L_1 \cup L_2 \cup \ldots \cup L_n$$

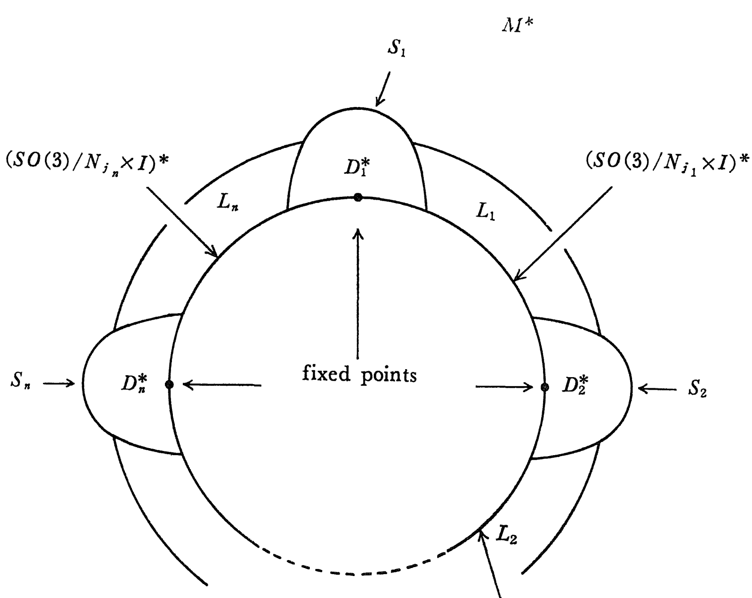
$$N^* = \overline{M^* - (A \cup \bigcup_{i=1}^{n} D_i^*)} \qquad \text{(see Fig. 7-2).}$$

Fig. 7-1



$G_x = N_i$ if $x^* \in B_1$
$G_x = N_j$ if $x^* \in B_2$ $(i, j \in \{0,1,2\}$ and $i \neq j)$
$G_x = SO(3)$ if $x^* \in B_1 \cap B_2$
$G_x = D_2$ if $x^* \in X - (B_1 \cup B_2)$

Fig. 7-2



$j_i \in \{0, 1, 2\}$ $j_i \neq j_{i+1}$, $j_1 \neq j_n$

Then, principal D_3-bundle $F(D_2, p^{-1}(B)) \longrightarrow B = A \cap N^*$ is a trivial D_3-bundle or a bundle corresponding to $xy \in \mathrm{Hom}(\pi_1(S^1); D_3)$. For, this bundle is equivalent to $I \times D_3/\sim$, with $\{0\} \times D_3$ identified with $\{1\} \times D_3$ by a D_3-equivariant map ϕ which induces an equivariant map of $M_{\pi_{j_n}}(\pi_{j_n}; SO(3)/D_2 \longrightarrow SO(3)/N_{j_n}$ projection). Since we can assume $N_{j_n} = N_0$, ϕ must be

$$\phi(gD_2) = gaD_2, \quad a \in (N(D_2) \cap O(2))/D_2 = D_4/D_2 = \{1, xy\}$$

In both cases, the equivariant attaching map of $p^{-1}(B)$ over B is only identity map (up to equivalence).

We denote the above B by B_i $(i=1, \dots, f)$ for each component. Here B_j means $\{1\} \times B_j^*$ in § 6 when there is no fixed point on this boundary component. And we define $\delta(i)=0$ if $p^{-1}(B_i) \longrightarrow B_i$ is a trivial bundle, and $\delta(i)=1$ if it corresponds to $xy \in \mathrm{Hom}(\pi_1(B_i); D_3)$. ($B_i$ is a component of $\{1\} \times \partial M^*$ for the collar $I \times \partial M^*$ of $M^* = \{0\} \times \partial M^*$.)

For the open neighbourhoods $\bigcup_{i=1}^{r} \mathrm{Int}\, D_i^*$, $\bigcup_{j=1}^{d_1+d_2} \mathrm{Int}\, D_j^*$, $\bigcup_{k=1}^{f} [0, 1) \times B_k^*$ in M^* of $(M_{(D_4)})^*$, $(M_{(A_4)})^*$ and $(M_{SO(3)})^* \cup (M_{(O(2))})^* = \bigcup_{k=1}^{f} B_k^*$, let $(M')^*$ be a subspace which is given by removing these open neighbourhoods from M^*. Then we put $\varepsilon=4$, 5, 6 or 7 if $\varphi(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_g, \beta_g) = (1, 1, \dots, 1, x, y)$, $(1, 1, \dots, 1, 1, y, x, y)$, $(1, 1, \dots, 1, x, y^2)$ or $(1, 1, \dots, 1, y, x, y^2)$ for $\varphi \in \mathrm{Hom}(\pi_1((M')^*); D_3)$. Given r, (d_1, d_2), $\delta(1) + \dots + \delta(f)$, we can see each equivalence class of homomorphisms of $\pi_1((M')^*)$ to D_3, is represented by one of the above four types. That is, $\{r, (d_1, d_2), \delta(1) + \dots + \delta(f), \varepsilon\}$ determine $M_{(D_2)}$ up to equivalence. Moreover, $\delta(1) + \dots + \delta(f) + r$ must be even because of

$$\underbrace{(xy) \dots (xy)}_{r+\delta(1)+\dots+\delta(f)} \cdot y^{d_1+2d_2} = \varphi([\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]) = y^k \quad (k=0, 1, 2).$$

Let B_i^* be a boundary component with n fixed points $\{p_1, \dots, p_n\}$ $(n \neq 1)$ which are arranged in this order. And let $C_{k(i)}$ be the closed are on B_i^* joining p_k and p_{k+1}. (if $k=n$, then $p_{n+1}=1$) As Fig. 7-2, $C_{k(i)}$ is the orbit space of $SO(3)/N_{j_{k(i)}} \times I/\sim$ given by collapsing $SO(3)/N_{j_{k(i)}} = (\{0\} \cup \{1\})$ to $SO(3)/SO(3) \times (\{0\} \cup \{1\})$.

Here, $N_{j_{k(i)}}$ is conjugate to $O(2)$, and $j_{k(i)}$ values in $\{0, 1, 2\}$. Then, by corresponding each $C_{k(i)}$ to $j_{k(i)}$, B_i^* gives an ordered n-tuple $(j_{1(i)}, \ldots, j_{k(i)})$ such that $j_{k(i)} \neq j_{k+1(i)}$, $j_{1(i)} \neq j_{n(i)}$ (because $SO(3)$ acts on the slice at each fixed point by the weight two representation).

Then we obtain the following theorem.

Theorem 7. *Let M be a closed, connected, oriented, smooth 5-dim. manifold with smooth $SO(3)$-action, and its principal isotropy group D_2, then the following orbit invariants determine M up to equivariant homeomorphism (which preserves the orientation of M^*)*

$$\{g, \varepsilon, r, (d_1, d_2), (\delta(i); (j_{1(i)}, \ldots, j_{n(i)})), \quad i=1, \ldots, f\}$$

such that

(i) $\varepsilon \in \{0, 1, 2, 3, 4, 5, 6, 7\}$,

(ii) $\delta(i) \in \{0, 1\}$,

(iii) $j_{k(i)} \in \{0, 1, 2\}$, $n \neq 1$,

(iv) $\delta(1)+\delta(2)+\ldots+\delta(f)+r$ is even,

(v)

$\varepsilon=0$	*if*	$g=0$, $d_1+2d_2 \equiv 0$	(mod 3)
$\varepsilon \in \{0, 1, 2\}$	*if*	$g=1$, $d_1+2d_2 \equiv 0$	(mod 3)
$\varepsilon \in \{0, 1, 2, 3\}$	*if*	$g \geqq 2$, $d_1+2d_2 \equiv 0$	(mod 3)
$\varepsilon=4$	*if*	$g=1$, $d_1+2d_2 \equiv 1$	(mod 3)
$\varepsilon \in \{4, 5\}$	*if*	$g \geqq 2$, $d_1+2d_2 \equiv 1$	(mod 3)
$\varepsilon=6$	*if*	$g=1$, $d_1+2d_2 \equiv 2$	(mod 3)
$\varepsilon \in \{6, 7\}$	*if*	$g \geqq 2$, $d_1+2d_2 \equiv 2$	(mod 3)

($g=0$, $d_1+2d_2 \equiv 1$, *or* $g=0$, $d_1+2d_2 \equiv 2$ *do not appear*).

Proof. First, if $d_1+2d_2 \equiv 0$ (mod 3). Then the classification of $p^{-1}((M')^*) \longrightarrow (M')^*$ is reduced to Lemma 7-1, i. e. $\varepsilon \in \{0, 1, 2\}$ and g determine $p^{-1}((M')^*)$.

Next, if $d_1+2d_2 \equiv 1$ (mod 3), then we have to compute the equivalence classes of $\mathrm{Hom}(\pi_1(M')^*); D_3)$ which satisfy the condition

$$\varphi([\alpha_1, \beta_1][\alpha_2, \beta_2] \ldots [\alpha_g, \beta_g]) = y.$$

The similar argument to Lemma 7-1, concludes that the equivalence classes are $(1, 1, \ldots, 1, 1, x, y)$ and $(1, 1, \ldots, 1, 1, y, x, y)$.

Similarly, if $d_1+2d_2\equiv 2 \pmod 3$, then

$$\varphi([\alpha_1, \beta_1][\alpha_2, \beta_2]\ldots\ldots[\alpha_g, \beta_g])=y^2,$$

and the classes are only $(1, 1, \ldots, 1, x, y^2)$ and $(1, 1, \ldots, 1, y, x, y^2)$.

Thus ε satisfying the condition (v) determines p^{-1} $((M')^*)$ up to equivalence. Since $M_{(D_2)}$ determine M if r, (d_1, d_2), $(\delta(i)$; $(j_{1(i)}, \ldots, j_{n(i)})$ are given, these invariants and g, ε classify M up to equivalence.

q. e. d.

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