Publ. RIMS, Kyoto Univ.
15 (1979), 47-52

A Common Fixed Point Theorem for a Sequence of Multivalued Mappings

By

Kenjiro YANAGI*

§ 1. Introduction

Recently, fixed point theorems for multivalued contraction mappings in metric spaces were obtained by many authors ([5], [2], [7], [1], [4], [6], etc). On the other hand, Dube [3] proved a theorem on common fixed points of two multivalued mappings.

In this paper, we shall give a common fixed point theorem for a sequence of multivalued mappings satisfying some conditions in complete metric spaces. In our theorem, we shall obtain an extension of the result of Itoh [4] to the case of a sequence of multivalued mappings.

§ 2. Preliminaries

Let (X, d) be a metric space. For any $x \in X$ and $A \subset X$, we define $D(x, A) = \inf\{d(x, y) : y \in A\}$. Let $CB(X)$ denote the family of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, let $H(A, B)$ denote the distance between A and B in the Hausdorff metric induced by d on $CB(X)$. The following lemmas are direct consequences of definition of Hausdorff metric.

Lemma 1. *If $A, B \in CB(X)$ and $x \in A$, then $D(x, B) \leqq H(A, B)$.*

Lemma 2. *For any $x \in X$, $A, B \in CB(X)$,*

$$|D(x, A) - D(x, B)| \leqq H(A, B).$$

Lemma 3. *Let $A, B \in CB(X)$ and $k \in (1, \infty)$ be given. Then for*

Communicated by S. Matsuura, June 24, 1976. Revised July 1, 1977.
* Department of Mathematics, Yamaguchi University.

$a\in A$, there exists $b\in B$ such that $d(a,b)\leqq kH(A,B)$.

(X,d) is said to be metrically convex if for any $x,y\in X$ with $x\neq y$, there exists an element $z\in X$, $x\neq z\neq y$, such that

$$d(x,z)+d(z,y)=d(x,y).$$

In Assad and Kirk [1], the following is noted.

Lemma 4. *If K is a nonempty closed subset of the complete and metrically convex metric space (X,d), then for any $x\in K$, $y\notin K$, there exists $z\in\partial K$ (the boundary of K) such that*

$$d(x,z)+d(z,y)=d(x,y).$$

§ 3. A Common Fixed Point Theorem

The following is a common fixed point theorem for a sequence of multivalued mappings T_n: $K\to CB(X)$ when K is a nonempty closed subset of a complete metrically convex metric space X.

Theorem. *Let (X,d) be a complete and metrically convex metric space and K be a nonempty closed subset of X. Let T_n $(n=1, 2,\cdots)$ be a sequence of multivalued mappings of K into $CB(X)$. Suppose that there are nonnegative real numbers α, β, γ with $\alpha+(\alpha+3)(\beta+\gamma)<1$ such that*

$$H(T_i(x),T_j(y))\leqq\alpha d(x,y)+\beta\{D(x,T_i(x))+D(y,T_j(y))\}$$
$$+\gamma\{D(x,T_j(y))+D(y,T_i(x))\}$$

for all $x,y\in K$ and for all $i,j=1,2,\cdots$. If $T_n(x)\subset K$ for each $x\in\partial K$ and each $n=1,2,\cdots$, then the sequence T_n $(n=1,2,\cdots)$ has a common fixed point in K.

Proof. Since X is metrically convex, there exists $x_0\in K$ such that $T_1(x_0)\subset K$. If $\alpha=\beta=\gamma=0$, any $z\in T_1(x_0)$ is a common fixed point. In fact, for $n=2,3,\cdots$, $H(T_1(x_0),T_n(z))=0$ and then $z\in T_n(z)$, $n=2,3,\cdots$. Since $H(T_1(z),T_2(z))=0$, we have $z\in T_1(z)$. So we assume that

$\alpha+\beta+\gamma>0$. Since $\alpha+(\alpha+3)(\beta+\gamma)<1$, there exists $k>1$ such that $(1+k\beta+k\gamma)(k\alpha+k\beta+k\gamma)/(1-k\beta-k\gamma)^2<1$. We choose sequences $\{x_n\}$ in K and $\{y_n\}$ in the following way. Let $x_1=y_1\in T_1(x_0)$. By Lemma 3, there exists $y_2\in T_2(x_1)$ such that $d(y_1, y_2)\leqq kH(T_1(x_0), T_2(x_1))$. If $y_2\in K$, let $x_2=y_2$. If $y_2\notin K$, choose an element $x_2\in\partial K$ such that $d(x_1, x_2)+d(x_2, y_2)=d(x_1, y_2)$ by Lemma 4. By introduction we can obtain sequences $\{x_n\}$, $\{y_n\}$ such that for $n=1, 2, \cdots$,

(a) $y_{n+1}\in T_{n+1}(x_n)$,

(b) $d(y_n, y_{n+1})\leqq kH(T_n(x_{n-1}), T_{n+1}(x_n))$,

where

(c) if $y_{n+1}\in K$, then $x_{n+1}=y_{n+1}$, and

(d) if $y_{n+1}\notin K$, then $x_{n+1}\in\partial K$ and

$$d(x_n, x_{n+1})+d(x_{n-1}, y_n)=d(x_n, y_{n+1}).$$

We shall estimate the distance $d(x_n, x_{n+1})$ for $n\geqq 2$. There arise three cases.

(i) The case that $x_n=y_n$ and $x_{n+1}=y_{n+1}$. We have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(y_n, y_{n+1})\\
&\leqq kH(T_n(x_{n-1}), T_{n+1}(x_n))\\
&\leqq k\alpha d(x_{n-1}, x_n)+k\beta\{D(x_{n-1}, T_n(x_{n-1}))+D(x_n, T_{n+1}(x_n))\}\\
&\qquad +k\gamma\{D(x_{n-1}, T_{n+1}(x_n))+D(x_n, T_n(x_{n-1}))\}\\
&\leqq k\alpha d(x_{n-1}, x_n)+k\beta\{d(x_{n-1}, x_n)+d(x_n, x_{n+1})\}\\
&\qquad +k\gamma\{d(x_{n-1}, x_n)+d(x_n, x_{n+1})\}.
\end{aligned}$$

Hence

$$(1-k\beta-k\gamma)d(x_n, x_{n+1})\leqq(k\alpha+k\beta+k\gamma)d(x_{n-1}, x_n),$$

and

$$d(x_n, x_{n+1})\leqq\frac{k\alpha+k\beta+k\gamma}{1-k\beta-k\gamma}d(x_{n-1}, x_n).$$

(ii) The case that $x_n=y_n$ and $x_{n+1}\neq y_{n+1}$. By (d) we obtain that

$$d(x_n, x_{n+1})\leqq d(x_n, y_{n+1})=d(y_n, y_{n+1}).$$

As in the case (i), we have

$$d(y_n, y_{n+1}) \leqq \frac{k\alpha + k\beta + k\gamma}{1 - k\beta - k\gamma} d(x_{n-1}, x_n),$$

thus

$$d(x_n, x_{n+1}) \leqq \frac{k\alpha + k\beta + k\gamma}{1 - k\beta - k\gamma} d(x_{n-1}, x_n).$$

(iii) The case that $x_n \neq y_n$ and $x_{n+1} = y_{n+1}$. In this case $x_{n-1} = y_{n-1}$ holds. We have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqq d(x_n, y_n) + d(y_n, x_{n+1}) \\ &= d(x_n, y_n) + d(y_n, y_{n+1}). \end{aligned}$$

From (b) it follows that

$$\begin{aligned} d(y_n, y_{n+1}) &\leqq kH(T_n(x_{n-1}), T_{n+1}(x_n)) \\ &\leqq k\alpha d(x_{n-1}, x_n) + k\beta\{D(x_{n-1}, T_n(x_{n-1})) + D(x_n, T_{n+1}(x_n))\} \\ &\quad + k\gamma\{D(x_{n-1}, T_{n+1}(x_n)) + D(x_n, T_n(x_{n-1}))\} \\ &\leqq k\alpha d(x_{n-1}, x_n) + k\beta\{d(x_{n-1}, y_n) + d(x_n, x_{n+1})\} \\ &\quad + k\gamma\{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, y_n)\}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\leqq (1 + k\gamma) d(x_n, y_n) + (k\alpha + k\gamma) d(x_{n-1}, x_n) \\ &\quad + k\beta d(x_{n-1}, y_n) + (k\beta + k\gamma) d(x_n, x_{n+1}) \\ &\leqq (1 + k\gamma) d(x_{n-1}, y_n) + k\beta d(x_{n-1}, y_n) + (k\beta + k\gamma) d(x_n, x_{n+1}), \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leqq \frac{1 + k\beta + k\gamma}{1 - k\beta - k\gamma} d(x_{n-1}, y_n).$$

Since $x_{n-1} = y_{n-1}$ and $x_n \neq y_n$, it follows from (ii) that

$$d(x_{n-1}, y_n) \leqq \frac{k\alpha + k\beta + k\gamma}{1 - k\beta - k\gamma} d(x_{n-2}, x_{n-1}).$$

Thus it follows that for every $n \geqq 2$,

$$d(x_n, x_{n+1}) \leqq \frac{(1 + k\beta + k\gamma)(k\alpha + k\beta + k\gamma)}{(1 - k\beta - k\gamma)^2} d(x_{n-2}, x_{n-1}).$$

The case that $x_n \neq y_n$ and $x_{n-1} \neq y_{n+1}$ does not occur. Since

$$\frac{k\alpha+k\beta+k\gamma}{1-k\beta-k\gamma} \leqq \frac{(1+k\beta+k\gamma)(k\alpha+k\beta+k\gamma)}{(1-k\beta-k\gamma)^2},$$

we proved the following; for $n \geqq 2$,

$$d(x_n, x_{n+1}) \leqq \begin{cases} pd(x_{n-1}, x_n), & \text{or} \\ pd(x_{n-2}, x_{n-1}), \end{cases}$$

where $p=(1+k\beta+k\gamma)(k\alpha+k\beta+k\gamma)/(1-k\beta-k\gamma)^2$. Put

$$\delta = p^{-1/2} \max\{d(x_0, x_1), d(x_1, x_2)\},$$

then by induction we can show that

$$d(x_n, x_{n+1}) \leqq p^{n/2}\delta \qquad (n=1, 2, \cdots).$$

It follows that for any $m>n \geqq 1$,

$$d(x_n, x_m) \leqq \delta \sum_{i=n}^{m-1} p^{i/2}.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete and K is closed, $\{x_n\}$ converges to some point $z \in K$. From the definition of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}=y_{n_i}$ $(i=1, 2, \cdots)$. Then for $n=1, 2, \cdots$, we have

$$\begin{aligned} D(x_{n_i}, T_n(z)) &\leqq H(T_{n_i}(x_{n_i-1}), T_n(z)) \\ &\leqq \alpha d(x_{n_i-1}, z) + \beta\{D(x_{n_i-1}, T_{n_i}(x_{n_i-1})) + D(z, T_n(z))\} \\ &\qquad + \gamma\{D(x_{n_i-1}, T_n(z)) + D(z, T_{n_i}(x_{n_i-1}))\} \\ &\leqq \alpha\{d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, z)\} \\ &\qquad + \beta\{d(x_{n_i-1}, x_{n_i}) + d(z, x_{n_i}) + D(x_{n_i}, T_n(z))\} \\ &\qquad + \gamma\{d(x_{n_i-1}, x_{n_i}) + D(x_{n_i}, T_n(z)) + d(z, x_{n_i})\}. \end{aligned}$$

Thus

$$D(x_{n_i}, T_n(z)) \leqq \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\{d(x_{n_i-1}, x_{n_i}) + d(z, x_{n_i})\}.$$

Therefore, $D(x_{n_i}, T_n(z)) \to 0$ as $i \to \infty$. By the inequality $D(z, T_n(z)) \leqq d(x_{n_i}, z) + D(x_{n_i}, T_n(z))$ and the above result, it follows that $D(z, T_n(z))=0$. Since $T_n(z)$ is closed, this implies that $z \in T_n(z)$, $n=1, 2, \cdots$. Q.E.D.

Remark. If $\alpha, \beta, \gamma \geqq 0$, then $\alpha+(\alpha+3)(\beta+\gamma)<1$ if and only if $(1+\beta+\gamma)(\alpha+\beta+\gamma)/(1-\beta-\gamma)^2<1$. Hence putting $T_n=T$ for $n=1, 2, \cdots$ in our theorem, we obtain the result of Itoh [4].

Since every Banach space is metrically convex, we have the following corollary for singlevalued mappings.

Corollary. *Let E be a Banach space and K be a nonempty closed subset of E. Let f_n $(n=1, 2, \cdots)$ a sequence of singlevalued mappings of K into E. Suppose that there are nonnegative real numbers α, β, γ with $\alpha+(\alpha+3)(\beta+\gamma)<1$ such that*

$$\|f_i(x)-f_j(y)\| \leqq \alpha\|x-y\| + \beta\{\|x-f_i(x)\| + \|y-f_j(y)\|\} + \gamma\{\|x-f_j(y)\| + \|y-f_i(x)\|\}$$

for all $x, y \in K$ and for all $i, j=1, 2, \cdots$. If $f_n(\partial K) \subset K$ for each $n=1, 2, \cdots$, then there exists an unique common fixed point $z \in K$.

We conclude this paper by stating an open problem whether our theorem holds when nonnegative real numbers α, β, γ satisfy $\alpha+2\beta+2\gamma<1$ instead of $\alpha+(\alpha+3)(\beta+\gamma)<1$.

Acknowledgement. The author wishes to thank the referee for his critical and helpful comments.

References

[1] Assad, N. A. and Kirk, W. A., Fixed point theorems for set-valued mappings of contractive type, *Pacific J. Math.*, **43** (1972), 553-562.
[2] Dube, L. S. and Singh, S. P., On multivalued contraction mappings, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **14** (1970), 307-310.
[3] Dube, L. S., A theorem on common fixed points of multivalued mappings, *Ann. Soc. Sci. Bruxelles*, **89** (1975), 463-468.
[4] Itoh, S., Multivalued generalized contractions and fixed point theorems, *Research Reports on Information Sciences, Tokyo Institute of Technology*, **A-21** (1975).
[5] Nadler, S. B. Jr., Multivalued contraction mappings, *Pacific J. Math.*, **30** (1969), 457-488.
[6] Ray, B. K., Some fixed point theorems, *Fund. Math.*, **XCII** (1976), 76-90.
[7] Reich, S., Kannan's fixed point theorem, *Bollettino U. M. I.*, **4** (1971), 1-11.