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On the General Form of Yamaguti-Nogi-Vaillancourt's Stability Theorem

By

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§ 1. Introduction

It is well known that the Friedrichs scheme is stable in many hyperbolic cases ([2], [5], [10], [15], [17]) and it is quite natural that this simple scheme may be expected to be stable under less restriction.

The theory of pseudo-difference and translation operators has played an important role in the stability theory of difference schemes as in [3], [14], [15], [17]. But the treatments of pseudo-difference operators are rather different from those of pseudo-differential operators, although it seems that both operators work in the same principle. The crucial reason why such different treatments have been needed is as follows: The main properties for a pseudo-differential operator P with symbol $p(x, \xi)$ are derived from the behavior of $p(x, \xi)$ as $|\xi| \to \infty$. On the other hand the properties of a pseudo-difference operator P_h with symbol $p(x, h\xi)$ $(0<h<1)$ are derived from the behavior of $p(x, h\xi)$ as $h \to 0$ (necessarily $|h\xi| \to 0$).

In the present paper we shall study an algebra of a family of pseudo-differential operators and apply this theory directly to the stability theory of the Friedrichs scheme. The class $\{S^m_{\lambda_h}\}$ of pseudo-differential operators is defined by means of a family of basic weight functions $\lambda_h(\xi)$ $(0<h<1)$ as in [7], [8], [12], [13]. For the application to the stability theory we have to define two subclasses $\{\mathring{S}^m_{\lambda_h}\}$ and $\{\tilde{S}^m_{\lambda_h}\}$ of $\{S^m_{\lambda_h}\}$ as the sets of all the symbols $p_h(x, \xi)$ such that $h^{-1}p_h \in \{S^{m+1}_{\lambda_h}\}$ and $h^{-1}\partial^\alpha_\xi p_h \in \{S^{m+1-|\alpha|}_{\lambda_h}\}$ for any $\alpha \neq 0$, respectively. The class $\{\tilde{S}^0_{\lambda_h}\}$ corresponds to the class of usual pseudo-difference operators and the class $\{\mathring{S}^{-1}_{\lambda_h}\}$ does to the class of

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operators of null scheme. Then, setting "the principle of cutting off" a symbol $p_h(x, \xi)$ of our class $\{S^m_{\lambda_h}\}$ by $\chi(\lambda_h(\xi))$ (or $\varphi(\zeta_h(\xi))$) (see Theorem 3.14), we can naturally derive a stability theorem of the Friedrichs scheme for a diagonalizable hyperbolic system by using the well known calculus of pseudo-differential operators.

We should note that the theorem is regarded as the general form of the Yamaguti-Nogi-Vaillancourt stability theorm ([14], [16], [17]) and holds without the restriction on the behavior of symbol $p_h(x, \xi)$ at $x=\infty$.

In Section 2 definitions and preliminaries will be given. In Section 3 algebra of operators of class $\{S^m_{\lambda_h}\}$ and its properties will be given. There and thereafter our theory depends heavily on Calderón-Vaillancourt's theorem. In Section 4 we shall give an improved form*) of the Yamaguti-Nogi-Vaillancourt theorem of Lax-Nirenberg's type as an application of the Friedrichs approximation method (see [4], [11], [14], [17]). But this theorem will not be used for our calculus of the Friedrichs scheme in Section 5, where the algebra of operators of class $\{S^m_{\lambda_h}\}$ will be directly applied to the Friedrichs symbol (5.7) and the general form of the Yamaguti-Nogi-Vaillancourt stability theorem will be derived. The difference scheme may depend on t as well.

The results of this paper are stated in the previous paper [6] with a sketch of proofs.

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§ 2. Definitions and Preliminaries

Let $\alpha=(\alpha_1, \cdots, \alpha_n)$ be a multi-integer of $\alpha_j \geqq 0$. We put $|\alpha|=\alpha_1+\cdots+\alpha_{n'}$, $\alpha!=\alpha_1!\cdots\alpha_n!$ and $\partial^\alpha_\xi=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\partial\xi_n)^{\alpha_n}$.

Definition 2.1. A family $\{\lambda_h(\xi)\}$ $(0<h<1)$ of real valued C^∞-function in R^n_ξ is called a basic weight function, when there exist positive

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*) An essentially improved theorem in the sense that besides the homogeneity of symbol in ξ C^2-smoothness with respect to x and ξ is only assumed, will be published elsewhere.

constants A_0, A_α (independent of h) such that

$$\text{(2.1)} \qquad \text{i)} \quad 1 \leqq \lambda_h(\xi) \leqq A_0\langle\xi\rangle$$

and

$$\text{ii)} \quad |\lambda_h^{(\alpha)}(\xi)| \leqq A_\alpha \lambda_h(\xi)^{1-|\alpha|}$$

for any α, where $\langle\xi\rangle = \{1+|\xi|^2\}^{1/2}$ and $\lambda_h^{(\alpha)}(\xi) = \partial_\xi^\alpha \lambda_h(\xi)$ for α.

Example 1. Let $\zeta_h(\xi) = (h^{-1}\sin h\xi_1, \cdots, h^{-1}\sin h\xi_n)$. Then $\lambda_h(\xi) = \langle\zeta_h(\xi)\rangle$ is a basic weight function. This function satisfies

$$\text{(2.2)} \qquad h \leqq (n+1)^{1/2}\lambda_h(\xi)^{-1}$$

Definition 2.2. $\mathscr{S} = \mathscr{S}(\mathrm{R}^n) = \{f(y) \in C^\infty(\mathrm{R}^n)\,;\ \lim_{|y|\to\infty} |y|^\ell |\partial_y^\alpha f(y)| = 0$ for any ℓ and $\alpha\}$. $\mathscr{S}'$ denotes its dual space which consists of all temperate distributions in the sense of L. Schwartz.

Definition 2.3. **i)** A family of C^∞-symbols $p_h(x, \xi)$ in $\mathrm{R}_x^n \times \mathrm{R}_\xi^n$ $(0<h<1)$ is called of class $\{S_{\lambda_h}^m\}$ $(-\infty<m<\infty)$, where there exist constants $C_{\alpha,\beta}$ (independent of h) such that

$$\text{(2.3)} \qquad |p_{h,(\beta)}^{(\alpha)}(x, \xi)| \leqq C_{\alpha,\beta}\lambda_h(\xi)^{m-|\alpha|}$$

for any α, β, where $p_{h,(\beta)}^{(\alpha)} = \partial_\xi^\alpha D_x^\beta p_h$ $(D_{x_j} = -i\partial_{x_j})$.

ii) The set of all symbols $p_h(x, \xi)$ such that $h^{-1}p_h \in \{S_{\lambda_h}^{m+1}\}$ is denoted by $\{\mathring{S}_{\lambda_h}^m\}$ and the set of all symbols $p_h(x, \xi)$ such that $h^{-1}\partial_\xi^\alpha p_h \in \{S_{\lambda_h}^{m+1-|\alpha|}\}$ for any $\alpha(\neq 0)$ is denoted by $\{\tilde{S}_{\lambda_h}^m\}$.

iii) A family of linear operators $P_h: \mathscr{S} \to \mathscr{S}$ is called a pseudo-differential operators of class $\{S_{\lambda_h}^m\}$ with symbol $p_h(x, \xi)$, where there exists a symbol $p_h(x, \xi)$ of class $\{S_{\lambda_h}^m\}$ such that

$$\text{(2.4)} \qquad P_h u(x) = p_h(X, D_x)u(x) = \int e^{ix\xi}p_h(x, \xi)\hat{u}(\xi)\,d\!\!\bar{}\,\xi$$

for $u \in \mathscr{S}$, where $d\!\!\bar{}\,\xi = (2\pi)^{-n}d\xi$ and $\hat{u}(\xi) = \int e^{-ix\xi}u(x)\,dx$. We denote (2.4) briefly by $P_h = p_h(X, D_x) \in \{S_{\lambda_h}^m\}$, or $\sigma(P_h) = p_h(x, \xi)$.

It is evident that $\{S_{\lambda_h}^{m_2}\} \subset \{S_{\lambda_h}^{m_1}\}$ for $m_2 \leqq m_1$. We set $\{S_{\lambda_h}^{-\infty}\} = \bigcap_m \{S_{\lambda_h}^m\}$, $\{S_{\lambda_h}^\infty\} = \bigcup_m \{S_{\lambda_h}^m\}$.

Definition 2.4. A family of C^∞-symbol $p_h(x, \xi)$ in $R^x_n \times R^n_\xi$ is called of class $\{S^m_{0,\lambda_h}\}$, when there exist constants $C_{\alpha,\beta}$ independent of h such that

$$(2.5) \qquad |p^{(\alpha)}_{h,(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\lambda_h(\xi)^m \qquad \text{for any } \alpha, \beta.$$

The operator P_h corresponding to this symbol is defined by the same way as (2.4). This class will be used only in Section 4.

Example 2. For real m $\lambda_h(\xi)^m \in \{S^m_{\lambda_h}\}$.

In the case $\lambda_h(\xi) = \langle\zeta_h(\xi)\rangle$, we have the following examples which are important for the calculus of difference schemes:

Example 3. $\sin h\xi_j \in \{\mathring{S}^0_{\lambda_h}\}$ and $\cos h\xi_j \in \{\tilde{S}^0_{\lambda_h}\}$.

Example 4. Let $p_h(x, \xi) \in \{S^m_{\lambda_h}\}$. Then $hp_h(x, \xi) \in \{S^{m-1}_{\lambda_h}\}$.

Example 5. Let $p(x, \xi) \in S^m_{\langle\xi\rangle}$, which means that $|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$. Then $p_h(x, \xi) = p(x, \zeta_h(\xi)) \in \{S^m_{\lambda_h}\}$.

Let $P_h = p_h(X, D_x) \in \{S^m_{\lambda_h}\}$ and let's define the formal adjoint P^* by

$$(2.6) \qquad (P_hu, v) = (u, P^*_hv) \qquad \text{for} \quad u, v \in \mathscr{S}.$$

By Theorem 3.1 we get $P^*_h \in \{S^m_{\lambda_h}\}$. Then by the aid of the relation (2.6) for $u \in \mathscr{S}'$ and $v \in \mathscr{S}$, we can extend $P_h: \mathscr{S} \to \mathscr{S}$ to the mapping $P_h: \mathscr{S}' \to \mathscr{S}'$.

Definition 2.5. For real s we define the Sobolev space $\mathscr{H}_{\lambda_h,s}$ by $\mathscr{H}_{\lambda_h,s} = \{u \in \mathscr{S}'; \lambda_h(\xi)^s\hat{u}(\xi) \in L^2(R^n_\xi)$ with norm $\|u\|_{\lambda_h,s} = \|\lambda_h(\xi)^s\hat{u}(\xi)\|_{L^2}$. This is the Hilbert space with inner product $(u, v)_{\lambda_h s} = \int \lambda_h(\xi)^{2s}\hat{u}(\xi) \times \overline{\hat{v}(\xi)}\,d\xi$. When u and v are ℓ-vectors i.e. $u = (u_1 `\cdots; u_\ell)$, $v = {}^t(v_1, \cdots, v_\ell)$, where ${}^t(\cdots)$ is transpose notation, we can define $\mathscr{H}_{\lambda_h,s}$ by the same way with inner product $\sum_{j=1}^{\ell}(u_j, v_j)_{\lambda_h,s}$. $\mathscr{S}$ is dense in $L^2 = \mathscr{H}_{\lambda_h,0}$. When $p_h(x, \xi) = (p_{h,i,j}(x, \xi))$ is a $\ell \times \ell$ matrix function, we say that $p_h \in \{S^m_{\lambda_h}\}$ if all elements $p_{h,i,j}(x, \xi) \in \{S^m_{\lambda_h}\}$. We define P_h by $P_hu = \int e^{ix\xi}p_h(x, \xi) \times \hat{u}(\xi)\,d\xi$, where $u(x) = {}^t(u_1(x), \cdots, u_\ell(x)) \in \mathscr{S}$ and $p_h(x, \xi)\hat{u}(\xi)$

$={}^{t}(\sum_{j=1}^{\ell} p_{h,\ell,j}(x,\xi)\hat{u}_j(\xi),\cdots,\sum_{j=1}^{\ell} p_{h,\ell,j}(x,\xi)\hat{u}_j(\xi))$. In the case the index of Sobolev norm $s=0$, we write briefly $\|u\|_0$, or sometimes $\|u\|$ with no subscript in place of $\|u\|_{\lambda_h,0}$.

§ 3. Algebra of Operators of Class $\{S^m_{\lambda_h}\}$ and Its Properties

Throughout this section we fix a basic weight function $\lambda_h(\xi)$. We assume only that $\lambda_h(\xi)$ satisfies (2.1). In this section we shall employ the methods and results of [8], [9], [12], [13] and for further clarification these papers should be refered to as original references.

Theorem 3.1 (*Fundamental theorem of algebra of* $\{S^m_{\lambda_h}\}$).

i) *Let* $P_h=p_h(X,D_x)\in\{S^m_{\lambda_h}\}$ *and let* P^*_h *be its formal adjoint by* (2.6). *Then* P^*_h *is of class* $\{S^m_{\lambda_h}\}$ *and* $\sigma(P^*_h)=p^*_h(x,\xi)$ *has the asymptotic expansion*

$$(3.1)\qquad p^*_h(x,\xi)\sim\sum_{\alpha}\frac{(-1)^{|\alpha|}}{\alpha!}\overline{p^{(\alpha)}_{h,(\alpha)}(x,\xi)}$$

in the sense $p^*_h(x,\xi)-\sum_{|\alpha|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\overline{p^{(\alpha)}_{h,(\alpha)}(x,\xi)}\in\{S^{m-N}_{\lambda_h}\}$ *for any* N, *where* $\bar{A}$ *denotes the Hermitian adjoint matrix of* A.

ii) *Let* $P_{h,j}=p_{h,j}(X,D_x)\in\{S^{m_j}_{\lambda_h}\}$ $(j=1,2)$ *and set* $P_h=P_{h,1}P_{h,2}$. *Then* P_h *is of class* $\{S^{m_1+m_2}_{\lambda_h}\}$ *and* $\sigma(P_h)=p_h(x,\xi)$ *has the asymptotic expansion*

$$(3.2)\qquad p_h(x,\xi)\sim\sum_{\alpha}\frac{1}{\alpha!}p^{(\alpha)}_{h,1}(x,\xi)p_{h,2,(\alpha)}(x,\xi).$$

We omitt the proof of Theorem 3.1, from which we derive a series of corollaries.

Corollary 3.2. *If* $p_h(x,\xi)$ *is real valued (Hermitian symmetric in the matrix case), from* $P_h\in\{S^m_{\lambda_h}\}$ *we have*

$$(3.3)\qquad P^*_h-P_h\in\{S^{m-1}_{\lambda_h}\}.$$

Corollary 3.3. *If we define the operator* $P_{h,1}\circ P_{h,2}$ *by the symbol* $p_{h,1}(x,\xi)p_{h,2}(x,\xi)$, *from* $P_{h,j}\in\{S^{m_j}_{\lambda_h}\}$ $(j=1,2)$ *we have*

$$(3.4)\qquad P_{h,1}\circ P_{h,2}-P_{h,1}P_{h,2}\in\{S^{m_1+m_2-1}_{\lambda_h}\}.$$

Corollary 3.4. (i) *For* $P_{h,j}\in\{\tilde{S}_{\lambda_h}^{m_j}\}$ $(j=1,2)$ *we have*

$$[P_{h,1}P_{h,2}]=P_{h,1}P_{h,2}-P_{h,2}P_{h,1}\in\{\mathring{S}_{\lambda_h}^{m_1+m_2-1}\} \tag{3.5}$$

under the commutativity condition:

$$p_{h,1}(x,\xi)p_{h,2}(x,\xi)=p_{h,2}(x,\xi)p_{h,1}(x,\xi).$$

(ii) *For* $P_{h,1}=P_{h,1}(D_x)\in\{\tilde{S}_{\lambda_h}^{m_1}\}$ *and* $P_{h,2}\in\{S_{\lambda_h}^{m_2}\}$ *we have*

$$[P_{h,1},P_{h,2}]\in\{\mathring{S}_{\lambda_h}^{m_1+m_2-1}\} \tag{3.6}$$

under the commutativity condition $p_{h,1}(\xi)p_{h,2}(x,\xi)=p_{h,2}(x,\xi)p_{h,1}(\xi)$.

Proof of i). From (3.2)

$$\sigma[P_{h,1},P_{h,2}]\sim\sum_{|\alpha|\geqq1}\frac{1}{\alpha!}(p_{h,1}^{(\alpha)}p_{h,2,(\alpha)}-p_{h,2}^{(\alpha)}p_{h,1(\alpha)})\in\{S_{\lambda_h}^{m_1+m_2-1}\}$$

and

$$h^{-1}\sigma[P_{h,1},P_{h,2}]\sim\sum_{|\alpha|\geqq1}\frac{1}{\alpha!}(h^{-1}p_{h,1}^{(\alpha)}p_{h,2,(\alpha)}-h^{-1}p_{h,2}^{(\alpha)}p_{h,1,(\alpha)})\in\{S_{\lambda_h}^{m_1+m_2}\}.$$

Proof of ii). Noting that $p_{h,1,(\alpha)}(x,\xi)=0$ for $|\alpha|\geqq1$ we have (3.6).

Remark 3.5. Our Corollary 3.4 is so called a commutation theorem ([10]). If $p_{h,1}(\xi)$ is scalar valued, Corollary 3.4 ii) is valid unconditionaly (see Example 3). This fact is used later for the calculus of difference schemes.

Remark 3.6. As for the subclasses $\{\tilde{S}_{\lambda_h}^m\}$, $\{\mathring{S}_{\lambda_h}^m\}$, they form algebras as $\{S_{\lambda_h}^m\}$ in themselves because of the fact that the asymptotic expansion admit term by term differentiation with respect to ξ. Especially we use later the fact that if $P_{h,1}\in\{S_{\lambda_h}^{m_1}\}$ and $P_{h,2}\in\{\mathring{S}_{\lambda_h}^{m_2}\}$, then both $P_{h,1}P_{h,2}$ and $P_{h,1}\circ P_{h,2}\in\{\mathring{S}_{\lambda_h}^{m_1+m_2}\}$.

Theorem 3.7. *For* $P_h\in\{S_{\lambda_h}^m\}$ *we have a constant* C_s *independent of* h *such that*

$$\|P_hu\|_{\lambda_h,s}\leqq C_s\|u\|_{\lambda_h,s+m}\quad\text{for}\quad u\in H_{\lambda_h,s+m}. \tag{3.7}$$

Proof. We begin with the special case of Calderón-Vaillancourt's theorem. ([1])

Lemma 3.8 (*Calderón-Vaillancourt*). *When $p_h(x, \xi) \in \{S^0_{\lambda_h}\}$, it holds that*

$$\|P_h u\|_{\lambda_h,0} \leqq C\|u\|_{\lambda_h,0} \quad \text{for} \quad u \in L^2 = \mathscr{H}_{\lambda_h,0}, \tag{3.8}$$

where C is independent of h.

To derive the estimate (3.7), we consider the operator $\lambda^s_h(D) P_h \lambda_h(D)^{-(s+m)}$ which belongs to $\{S^0_{\lambda_h}\}$ by virtue of Theorem 3.1 ii) and $v(x) = \int e^{ix\xi}\lambda_h(\xi)^{s+m}\hat{u}(\xi)\, d\!\!{}^-\xi$ which belongs to $\mathscr{H}_{\lambda_h,0}$. Then, from the preceding Lemma 3.8 we get

$$\|P_h u\|_{\lambda_h,s} = \|P_h\lambda_h(D)^{-(s+m)}v\|_{\lambda_h,s} = \|\lambda^s_h(D) P_h \lambda_h(D)^{-(s+m)}v\|_{\lambda_h,0}$$
$$\leqq C_s\|v\|_{\lambda_h,0} = C_s\|u\|_{\lambda_h,s+m}. \qquad \text{Q.E.D.}$$

Corollary 3.9. *When $p_h(x, \xi) \in \{S^m_{\lambda_h}\}$, it holds that*

$$|(P_h u, u)| \leqq C\|u\|^2_{\lambda_h,m/2} \quad \textit{for} \quad u \in \mathscr{S}. \tag{3.9}$$

Proof. We put $(P_h u, u) = (\lambda_h(D)^{-m/2}P_h u,\ \lambda_h(D)^{m/2}u) = (P'_h u, \lambda_h(D)^{m/2}u)$, where $P'_h = \lambda_h(D)^{-m/2}P_h \in \{S^{m/2}_{\lambda_h}\}$. Hence by using Theorem 3.7 and Schwarz inequality we get $|(P_h u, u)| \leqq \|P'_h u\|_0 \|\lambda_h(D)^{m/2}u\|_0 \leqq C\|u\|^2_{\lambda_h,m/2}$.

Now we turn to the well known theorem relating to the Friedrichs part and Gårding's inequality. ([4], [8], [12]) Using the same way as in those papers, the following Theorem 3.11 is derived and we only mention its principal statement without proof.

Let $q(\sigma)$ be an even and $C^\infty(R^n)$-function satisfying that $q(\sigma) \geqq 0$, $\operatorname{supp} q(\sigma) \subset \{\sigma : |\sigma| \leqq 1\}$ and $\int q^2(\sigma)\, d\sigma = 1$. We define $F(\xi, \zeta)$ by

$$F(\xi, \zeta) = q((\xi - \zeta)\lambda_h(\xi)^{-1/2})\lambda_h(\xi)^{-n/4} \tag{3.10}$$

and double symbol $p_{F,h}(\xi, x', \xi')$ by

$$p_{F,h}(\xi, x', \xi') = \int F(\xi, \zeta) p_h(x', \zeta) F(\xi', \zeta)\, d\zeta. \tag{3.11}$$

Definition 3.10. The operator $P_{F,h}$ called the Friedrichs part of P_h is defined by

$$\widehat{P_{F,h}u}(\xi) = \int e^{-ix'\xi}\left\{\int e^{ix'\xi'} p_{F,h}(\xi, x', \xi')\,\hat{u}(\xi')\,d\!\!{}^-\xi'\right\}dx' . \tag{3.12}$$

It is well known that if we put $Pu(x) = \iiint e^{i(x\xi - x'\xi + x'\xi')} p(x, \xi, x', \xi')$ $\hat{u}(\xi')\,d\!\!{}^-\xi' dx' d\!\!{}^-\xi$ and $p_L(x, \xi) = \iint e^{-iz\zeta}\langle z\rangle^{-n_0}\langle D_z\rangle^{n_0} p(x, \xi+\zeta, x+z, \xi)$ $\times dz\,d\!\!{}^-\zeta$ $(n_0=2n)$ for double symbol $p(x, \xi, x', \xi')$, $Pu = p_L(X, D_x)u$ holds. $p_L(x, \xi)$ is called the simplified symbol of P. For simplicity the subscript L is omitted here and the simplified symbol of $P_{F,h}$ is denoted by $p_{F,h}(x, \xi)$.

Theorem 3.11. *Let $P_h \in \{S^m_{\lambda_h}\}$. Then we have the following*

$$p_{F,h}(x, \xi) \in \{S^m_{\lambda_h}\}. \tag{3.13}$$

$$p_{F,h}(x, \xi) - p_h(x, \xi) \in \{S^{m-1}_{\lambda_h}\}. \tag{3.14}$$

If $p_h(x, \xi)$ is real valued (Hermitian symmetric),

$$(P_{F,h}u, v) = (u, P_{F,h}v) \qquad \textit{for} \quad u, v \in \mathcal{S}. \tag{3.15}$$

If $p_h(x, \xi) \geqq 0$ (non-negative Hermitian symmetric),

$$(P_{F,h}u, u) \geqq 0 \qquad \textit{for} \quad u \in \mathcal{S}. \tag{3.16}$$

Furthermore, if $p_h(x, \xi) \geqq c_0 \lambda_h(\xi)^m I$ for a constant c_0,

$$(P_{F,h}u, u) \geqq c_0 \|u\|^2_{\lambda_h, m/2} - C\|u\|^2_{\lambda_h, (m-1)/2} \qquad \textit{for} \quad u \in \mathcal{S}. \tag{3.17}$$

Furthermore, if c_0 is positive,

$$(P_{F,h}u, u) \geqq c_1 \|u\|^2_{\lambda_h, m/2}, \tag{3.18}$$

where c_1 can be chosen as positive and independent of h.

For our application to the difference scheme we shall use (3.18), which is derived from (3.16). We shall show it as Lemma 5.7.

Theorem 3.12 (*Lax-Nirenberg*). *Let $P_h (\in \{S^{m_1}_{\lambda_h}\})$ satisfy $p_h(x, \xi) \geqq 0$ and $q_h(\xi)$ $(\in \{\tilde{S}^{m_2}_{\lambda_h}\})$ be a real scalar symbol. Then we have*

(3. 19) $\text{Re}(P_{F,h}q_h^2(D)u, u) \geqq -Ch\|u\|^2_{\lambda_h, m_1/2+m_2}$ *for* $u \in \mathscr{S}$,

where C is independent of h.

Proof. The following identity is easily verified.

$$h^{-1}(P_{F,h}q_h^2(D)u, u) = (P_{F,h}h^{-1/2}q_h(D)u, h^{-1/2}q_h(D)u) + (Q_h q_h u, u),$$

where $Q_h = [P_{F,h}, h^{-1}q_h] \in \{S_{\lambda_h}^{m_1+m_2}\}$ because of (3. 6). Noting that the first term of the right hand is non-negative by virtue of (3. 16) and applying Corollary 3. 9 to the second term, we have (3. 19).

Corollary 3. 13. *If real scalar symbol* $q_h(\xi) \in \{\mathring{S}_{\lambda_h}^{m_2}\}$, *we can replace* $P_{F,h}$ *by* P_h *in* (3. 19); *i.e.*

(3·20) $$\text{Re}(P_h q_h^2(D)u, u) \geqq -Ch\|u\|^2_{\lambda_h, m_1/2+m_2}.$$

Proof. We have only to estimate $(h^{-1}(P_{F,h}-P_h)q_h^2(D)u, u)$. Since $h^{-1}q_h^2(D) \in \{S_{\lambda_h}^{2m_2+1}\}$, it holds that $(P_{F,h}-P_h)h^{-1}q_h^2(D) \in \{S_{\lambda_h}^{m_1+2m_2}\}$. Thus we have $|(h^{-1}(P_{F,h}-P_h)q_h^2(D)u, u)| \leqq C'\|u\|^2_{\lambda_h, m_1/2+m_2}$.

In the following we mention a simple and very useful theorem for the calculus of difference scheme.

Theorem 3. 14 (*The principle of "cutting off"*). *Let* $\chi(t)$ *and* $\varphi(\xi)$ *be* C_0^∞*-function in* R_t^1 *and* R_ξ^n, *respectively. Then we have* $\chi_h(\xi) = \chi(\lambda_h(\xi))$ *and* $\varphi_h(\xi) = \varphi(\zeta_h(\xi)) \in \{S_{\lambda_h}^{-\infty}\}$. *If* $p_h(x, \xi) \in \{S_{\lambda_h}^m\}$, *then we have* $\chi_h p_h$, $\varphi_h p_h \in \{S_{\lambda_h}^{-\infty}\}$ *and if* $p_h(x, \xi) \in \{\mathring{S}_{\lambda_h}^m\}$, *then we have* $\chi_h p_h$, $\varphi_h p_h \in \{\mathring{S}_{\lambda_h}^{-\infty}\}$.

Proof. For $\chi_h(\xi)$ by using (2. 1) ii), we have $|\partial_\xi^\alpha \chi_h(\xi)| \leqq C_{m,\alpha}\lambda_h(\xi)^{m-|\alpha|}$ for any m and α. For $\varphi_h(\xi)$, by using the fact that all the $\partial_\xi^\beta \zeta_h(\xi)$'s are bounded functions, we have $|\partial_\xi^\alpha \varphi_h(\xi)| \leqq C_{m,\alpha}\lambda_h(\xi)^{m-|\alpha|}$ for any m and α. As for $\chi_h p_h$ (or $\varphi_h p_h$), the statement of Theorem 3. 14 is easily seen by virtue of Leibniz formula.

Combining Theorem 3. 14 with Theorem 3. 12, we have the following corollary.

Corollary 3.15. *Let* $\lambda_h(\xi)=\langle\zeta_h(\xi)\rangle$. *If* $p(x,\zeta_h(\xi))\in\{\mathring{S}^m_{\lambda_h}\}$ *and is non-negative Hermitian for* $|\zeta_h(\xi)|\geqq M_0$, *then we have*

$$\mathrm{Re}\,(P_h u, u)\geqq -Kh\|u\|^2_{\lambda_h, m/2}\,. \tag{3.21}$$

Proof. We take a non-negative C_0^∞-function $\varphi(\xi)$ such that $\varphi(\xi)=1$ for $|\xi|\leqq 1$ and consider the symbol $(2h)^{-1}(p_h+\bar{p}_h)+C\Psi_h(\xi)I$, where $\Psi_h(\xi)=\varphi((2M_0)^{-1}\zeta_h(\xi))$. Because of $h^{-1}p_h\in\{S^{m+1}_{\lambda_h}\}$, we can choose sufficiently large C such that $(2h)^{-1}(p_h+\bar{p}_h)+C\Psi_h(\xi)I\geqq 0$ for all x and ξ. If we set $\tilde{p}_h=1/2(p_h+\bar{p}_h)$, $p_h-\tilde{p}_h\in\{S^{-\infty}_{\lambda_h}\}$ by virtue of Theorem 3.14. Then, applying Theorem 3.12 for $\tilde{p}_h+Ch\Psi_h(\xi)I$ $(\in\{S^m_{\lambda_h}\})$ and $q_h(\xi)=1$ $(\in\{\tilde{S}^0_{\lambda_h}\})$ and taking into consideration the fact that the operation $P_h\rightarrow P_{F,h}$ is linear, we get $(\tilde{P}_{F,h}u, u)\geqq -Kh\|u\|^2_{\lambda_h,m/2}-Ch(\Psi_{F,h}u, u)$ and furtheremore $(P_{F,h}u, u)\geqq -Kh\|u\|^2_{\lambda_h,m/2}-Ch(\Psi_{F,h}u, u)$. Again, by applying Theorem 3.11 (3.14) and Theorem 3.14 for Ψ_h, we get

$$(P_{F,h}u, u)\geqq -K'h\|u\|^2_{\lambda_h,m/2}\,. \tag{3.22}$$

On the other hand, from $h^{-1}p_h\in\{S^{m+1}_{\lambda_h}\}$ we have by Theorem 3.11 (3.14) and Corollary 3.9

$$|h^{-1}((P_h-P_{F,h})u, u)|\leqq K''\|u\|^2_{\lambda_h,m/2}\,. \tag{3.23}$$

Combining (3.23) with (3.22), we have (3.21).

§ 4. A Theorem of Lax-Nirenberg's Type

In this section we shall give an alternative proof of Lax-Nirenberg's theorem which was derived by Yamaguti-Nogi and Vaillancourt (see [14] and [17]).

Theorem 4.1. *Let* $k(x,\zeta)$ *be an* $\ell\times\ell$ *matrix and* $C^\infty(\mathrm{R}^n_x\times(\mathrm{R}^n_\zeta-\{0\}))$-*function which is of homogeneous degree* 0 *with respect to* ζ *and satisfies that* $|D^\alpha_x k(x,\zeta)|\leqq C_\alpha$ *for any* α. *Let* $\Lambda(\xi)=(\Lambda_1(\xi),\cdots,\Lambda_n(\xi))$ *be a real* n-*vector valued* $C^2(\mathrm{R}^n_\xi)$-*function which satisfies that* $\Lambda(0)=0$ *and* $\partial^\alpha_\xi\Lambda_j(\xi)$ *are bounded for* $|\alpha|\leqq 2$ *and* $j=1,\cdots,n$. *Assume that* $k(x,\zeta)$ *is non-negative Hermitian, then we have*

$$\mathrm{Re}\,(K_h\Lambda^2_h u, u)\geqq -Ch\|u\|^2 \quad \text{for} \quad u\in L^2(\mathrm{R}^n_x), \tag{4.1}$$

where $\sigma(K_h)=k(x, \Lambda(h\xi))$ *and* $\sigma(\Lambda_h^2)=\sum_{j=1}^{n}\Lambda_j^2(h\xi)$.

Remark. Yamaguti and Nogi proved the theorem in case $k(x, \zeta)$ is independent of x for large $|x|$.

For the proof of Theorem 4.1 we need some lemmas which are shown in [12] and [13].

Lemma 4.2. *Let* $\lambda_h(\xi)$ *be a basic weight function. Then we have*

$$C^{-1}\lambda_h(\xi)\leqq\lambda_h(\xi+\lambda_h(\xi)^{1/2}\sigma)\leqq C\lambda_h(\xi) \tag{4.2}$$

for any $\sigma\in\mathrm{R}_\xi^n$ *satisfying* $|\sigma|\leqq\sigma_0$ (σ_0 *is a positive constant*), *where the constant* C *is independent of* h.

Lemma 4.3. *Let* $\tilde{\lambda}_h(\xi)$ *be a real valued* $\mathrm{C}^1(\mathrm{R}_\xi^n)$*-function such that* $\tilde{\lambda}_h(\xi)\geqq 1$ *and* $\partial_{\xi_j}\tilde{\lambda}_h(\xi)$ $(j=1, \cdots, n)$ *are bounded uniformly with respect to* h. *Then there exists a basic weight function* $\lambda_h(\xi)$ *which satisfies that*

$$c_0\tilde{\lambda}_h(\xi)\leqq\lambda_h(\xi)\leqq c_1\tilde{\lambda}_h(\xi) \tag{4.3}$$

for some positive constants c_0 *and* c_1.

Proofs of these lemmas are omitted. We have only to remark that the proofs can be proceeded as in [12] and [13] independently of h.

We define $\tilde{\lambda}_h(\xi)$ by

$$\tilde{\lambda}_h(\xi)=\{1+h^{-2}\sum_{j=1}^{n}\Lambda_j^2(h\xi)\}^{1/2}. \tag{4.4}$$

Then by the assumption that $\partial_{\xi_i}\Lambda_j(\xi)$ are bounded, we can see that $\tilde{\lambda}_h(\xi)$ satisfies the assumption of Lemma 4.3. Hence we obtain a basic weight function $\lambda_h(\xi)$ which satisfies the inequality (4.3).

From the definition (4.4) and the boundedness of $\Lambda(\xi)$, we can see that $h\tilde{\lambda}_h(\xi)\leqq C_1$. Hence we have

$$h\lambda_h(\xi)\leqq C_2 \quad \text{for some positive } C_2. \tag{4.5}$$

We fix this basic weight function $\lambda_h(\xi)$. Here we recall the class $\{S^m_{0,\lambda_h}\}$ introduced in Definition 2. 4. We note that Theorem 3. 7 holds for the class $\{S^m_{0,\lambda_h}\}$ of pseudo-differential operators, and that Corollary 3. 4 ii) holds when the symbol $p_{h,2}(x, \xi) \in \{S^{m_2}_{0,\lambda_h}\}$ satisfies that $\partial^\alpha_\xi p_{h,2}(x, \xi) \in \{S^{m_2-|\alpha|}_{0,\lambda_h}\}$ for $|\alpha| \leqq 1$, that is

(4. 6) $$h^{-1}\sigma[P_{h,1}, P_{h,2}] \in \{S^{m_1+m_2}_{0,\lambda_h}\}.$$

Correspondingly to Theorem 3. 11 we have the following theorem:

Theorem 4. 4. *Let $p_h(x, \xi) \in \{S^m_{0,\lambda_h}\}$ satisfy that $\partial^\alpha_\xi p_h(x, \xi) \in \{S^{m-|\alpha|}_{0,\lambda_h}\}$ for $|\alpha| \leqq 2$. Then we have*

(4. 7) $$p_{F,h}(x, \xi) \in \{S^m_{0,\lambda_h}\}.$$

(4. 8) $$p_{F,h}(x, \xi) - p_h(x, \xi) \in \{S^{m-1}_{0,\lambda_h}\}.$$

When $p_h(x, \xi)$ is non-negative Hermitian, we have

(4. 9) $$(P_{F,h}u, u) \geqq 0 \qquad \text{for} \quad u \in \mathscr{S}.$$

(For the proof of Theorem 4. 4, see [12].)

Using Theorem 4. 4 we have the following propositions which are similar to Theorem 3. 12 and Corollary 3. 13, respectively.

Proposition 4. 5. *Let P_h $(\in \{S^{m_1}_{0,\lambda_h}\})$ satisfy that the symbol $p_h(x, \xi)$ is non-negative Hermitian such that $\partial^\alpha_\xi p_h(x, \xi) \in \{S^{m-|\alpha|}_{0,\lambda_h}\}$ for $|\alpha| \leqq 2$ and $q_h(\xi)$ $(\in \{\tilde{S}^{m_2}_{0,\lambda_h}\})$ be a real scalar symbol. Then we have*

(4. 10) $$\mathrm{Re}\,(P_{F,h}q_h^2(D)\,u, u) \geqq -Ch\|u\|^2_{\lambda_h, m_1/2+m_2}$$

for $u \in \mathscr{S}$, where C is independent of h.

Proposition 4. 6. *If $q_h(\xi) \in \{\mathring{S}^{m_2}_{\lambda_h}\}$, we can replace $P_{F,h}$ by P_h in (4. 10), that is*

(4. 11) $$\mathrm{Re}\,(P_h q_h^2(D)\,u, u \geqq -Ch\|u\|^2_{\lambda_h, m_1/2+m_2}.$$

Proof of Theorem 4. 1. We choose a C_0^∞-function $\chi(\zeta)$ satisfying that $\chi(\zeta) = 1$ for $|\zeta| \leqq \sqrt{2}\,c_1C$, $\chi(\zeta) = 0$ for large $|\zeta|$ and $0 \leqq \chi(\zeta) \leqq 1$, where c_1 and C are the same as those in the right hands of (4. 3),

(4. 2) respectively.

Consider the following symbol

$$(4.12)\qquad Q_h = k(x, h^{-1}\Lambda(h\xi))\Lambda_h^2(\xi) = (1-\chi_h(\xi))Q_h + \chi_h(\xi)Q_h$$
$$= Q_{1,h} + Q_{2,h}\,,$$

where $\chi_h(\xi) = \chi(\lambda_h(\xi))$. From the definition of χ, it is easily seen that if $|h^{-1}\Lambda(h\xi)| < 1$, $\lambda_h(\xi - \lambda_h(\xi)^{1/2}\sigma) < \sqrt{2}\,c_1 C$ holds for any ξ, σ ($|\sigma| \leqq \sigma_0$) and $0 < h < 1$. Then we have $\chi_h(\xi) = 1$ and $\chi_h(\xi - \lambda_h(\xi)^{1/2}\sigma) = 1$. Hence $Q_{1,h}(x, \xi)$ is C_ξ^2-function and satisfies

$$(4.13)\qquad |Q_{1,h}(x, \xi)| \leqq Ch^2\lambda_h^2(\xi) \leqq C'$$

which is derived from (4. 3) and (4. 5).

Let $\varphi(\sigma)$ be an even non-negative, C_0^∞-function such that $\int \varphi(\sigma)\,d\sigma = 1$ and $T_h(x, \xi)$ be defined by

$$(4.14)\qquad T_h(x, \xi) = \int \varphi(\sigma) Q_{1,h}(x, \xi - \lambda_h(\xi)^{1/2}\sigma)\,d\sigma\,.$$

Then we have

$$(4.15)\qquad T_h(x, \xi) = \int \varphi((\xi - \zeta)/\lambda_h(\xi)^{1/2})\,\lambda_h(\xi)^{-n/2} Q_{1,h}(x, \zeta)\,d\zeta\,.$$

From the assumption $k(x, \zeta) \geqq 0$ (non-negative Hermitian) and the definition of χ we see $T_h(x, \xi) \geqq 0$.

Define P_h by

$$(4.16)\qquad P_h(x, \xi) = h^{-2} T_h(x, \xi)\,\lambda_h(\xi)^{-2}\,.$$

Then we get from (4. 13)

$$(4.17)\qquad |T_h(x, \xi)| \leqq Ch^2\lambda_h(\xi)^2 \quad \text{and} \quad |P_h(x, \xi)| \leqq C\,.$$

Furthermore by the differential calculation we can see

$$(4.18, \text{i})\qquad \partial_{\zeta_j} Q_{1,h}(x, \zeta) = Q_{1,h}^{(1)}(x, \zeta) + \tilde{Q}_{1,h}^{(1)}(x, \zeta)$$

and

$$(4.18, \text{ii})\qquad \partial_\zeta^{(e_i+e_j)} Q_{1,h}(x, \zeta) = Q_{1,h}^{(2)}(x, \zeta) + \tilde{Q}_{1,h}^{(2)}(x, \zeta)\,,$$

where $Q_{1,h}^{(j)}(x, \zeta)$ $(j = 1, 2)$ is the sum of the terms involving the derivatives of $(1 - \chi_h(\zeta))$. Since $\lambda_h(\zeta)$ is bounded where the derivatives of

$(1-\chi_h(\zeta))$ is not zero, $Q_{1,h}^{(j)}(x, \zeta)$ is a form of h^2 times uniformly bounded function. By the fact $|h^{-1}\Lambda(h\zeta)|\leqq C\lambda_h(\zeta)$, $\tilde{Q}_{1,h}^{(j)}(x, \zeta)$ is of the form of $h^2\lambda_h(\zeta)$ times uniformly bounded function. Therefore by using (4. 2) we get from (4. 14)

$$\text{(4. 19, i)} \quad |\partial_{\xi_j}T_h(x, \xi)|\leqq Ch^2\lambda_h(\xi) \quad \text{and} \quad |\partial_{\xi_j}P_h(x, \xi)|\leqq C'\lambda_h(\xi)^{-1}$$

and

$$\text{(4. 19, ii)} \quad |\partial_\xi^{(e_i+e_j)}T_h(x, \xi)|\leqq Ch^2 \quad \text{and} \quad |\partial_\xi^{(e_i+e_j)}P_h(x, \xi)|\leqq C'\lambda_h(\xi)^{-2}.$$

By succesive differentiation of (4. 15) we get

$$\partial_\xi^\alpha T_h=\int\sum_{\beta\leqq\alpha}\partial^\beta\varphi(\sigma)\,[c_{\alpha,\beta}\lambda_h(\xi)^{1/2-|\alpha|}+d_{\alpha,\beta}\lambda_h(\xi)^{-|\alpha|}]Q_{1,h}(x, \sigma)\,d\sigma$$

for $|\alpha|\geqq 3$, where $c_{\alpha,\beta}(\sigma)$ and $d_{\alpha,\beta}(\sigma)$ are bounded functions. Then from (4. 13) we have

$$\text{(4. 20)} \qquad |\partial_\xi^\alpha T_h(x, \xi)|\leqq Ch^2\lambda_h(\xi)^{5/2-|\alpha|}$$

and

$$|\partial_\xi^\alpha P_h(x, \xi)|\leqq C'\lambda_h(\xi)^{1/2-|\alpha|}$$

for $|\alpha|\geqq 3$. By the same calculation as the preceding we can see

$$|D_x^\beta\partial_\xi^\gamma\partial_\xi^\alpha T_h(x, \xi)|\leqq C_{\alpha,\beta,\gamma}h^2\lambda_h(\xi)^{2-|\alpha|}$$

and

$$|D_x^\beta\partial_\xi^\gamma\partial_\xi^\alpha P_h(x, \xi)|\leqq C'_{\alpha,\beta,\gamma}\lambda_h(\xi)^{-|\alpha|} \quad (|\alpha|=0, 1, 2)$$

for any β and γ. Hence $P_h\in\{S_{0,\lambda_h}^0\}$ and $\partial_\xi^\alpha P_h\in\{S_{0,\lambda_h}^{-|\alpha|}\}$ for $|\alpha|\leqq 2$.

On the other hand we define $q_h(\xi)$ by $q_h(\xi)=h\lambda_h(\xi)$ $(\in\{\mathring{S}_{\lambda_h}^0\})$ and apply (4. 11) to P_h and q_h. Then we get

$$\text{(4. 21)} \qquad \mathrm{Re}\,(P_hq_h^2(D)\,u, u)\geqq -Ch\|u\|^2$$

or

$$\text{(4. 22)} \qquad \mathrm{Re}\,(T_hu, u)\geqq -Ch\|u\|^2.$$

To estimate the difference $T_h-Q_{1,h}$ we use the lemma without proof.

Lemma 4. 7 (*P. D. Lax*). *The function* $k(x, \zeta)$ *in Theorem* 4. 1 *can be expanded in a series*

$$k(x,\zeta)=\sum_{\alpha}a_{\alpha}(x)\exp(i(\alpha,\zeta/|\zeta|)=\sum_{\alpha}a_{\alpha}(x)k_{\alpha}(\zeta),\tag{4.23}$$

α varying over all multi-integers so that the series as well as any differentiated series with respect to x and ζ, converges uniformly. Furthermore, $\sum_{\alpha}|\alpha|^{\ell}a_{\alpha}$ are convergent for any ℓ, where $a_{\alpha}=\sup_{x\in R_x^n}|a_{\alpha}(x)|$.

Set $Q_{\alpha,1,h}=(1-\chi_h(\zeta))Q_{\alpha,h}=(1-\chi_h(\zeta))\,a_{\alpha}(x)\exp(i(\alpha,\zeta/|\zeta|)\Lambda_h^2(\zeta)$. Then analogously to (4.18, ii) we get

$$\partial_{\zeta}^{(e_i+e_j)}Q_{\alpha,1,h}(x,\zeta)=Q_{\alpha,1,h}^{(2)}(x,\zeta)+\tilde{Q}_{\alpha,1,h}^{(2)}(x,\zeta),\tag{4.24}$$

where $Q_{\alpha,1,h}^{(2)}$ is the sum of the terms involving the derivatives of $(1-\chi_h(\zeta))$. Therefore we get

$$\partial_{\zeta}^{(e_i+e_j)}Q_{\alpha,1,h}(x,\zeta)=h^2a_{\alpha}(x)\sum_{s=0}^{2}|\alpha|^s b_{\alpha,h,s}(\zeta),\tag{4.25}$$

where $b_{\alpha,h,s}(\zeta)$ $(s=0,1,2)$ are bounded uniformly with respect to α and h.

By applying the Taylor expansion to $Q_{1,h}(x,\zeta)-Q_{1,h}(x,\xi)$, we get

$$\begin{aligned}T_h(x,\xi)-Q_{1,h}(x,\xi)=&-\sum_{j=1}^{n}\lambda_h(\xi)^{1/2}\int\varphi(\sigma)\,\partial_{\xi_j}Q_{1,h}(x,\xi)\,\sigma_j d\sigma\\&+\sum_{\alpha}\sum_{i,j=1}^{n}\lambda_h(\xi)\int\varphi(\sigma)\int_0^1\partial_{\zeta}^{(e_i+e_j)}Q_{\alpha,1,h}(x,\xi-\theta\lambda_h(\xi)^{1/2}\sigma)\\&\times\sigma_i\sigma_j d\theta d\sigma,\end{aligned}\tag{4.26}$$

where the second term of the right hand is obtained by term and term twice differentiation under integration sign. Since $\varphi(\sigma)\sigma_j$ is an odd function the first term of the right hand vanishes. Hence taking into consideration (4.5) and (4.25) we get

$$T_h(x,\xi)-Q_{1,h}(x,\xi)=h\sum_{\alpha}\sum_{s=0}^{2}|\alpha|^s a_{\alpha}(x)\,c_{\alpha,h,s}(\xi),$$

where $c_{\alpha,h,s}(\xi)$ are bounded uniformly with respect to α and h. Therefore by virtue of Lemma 4.7 the operator $h^{-1}(T_h-Q_{1,h})$ is bounded uniformly with respect to h, as it is the uniform limit of L^2-bounded operator whose symbol is $\sum_{\text{finite }\alpha}(\sum_{s=0}^{2}|\alpha|^s a_{\alpha}(x)\,c_{\alpha,h,s}(\xi))$.

Combining (4.22) with the above result we get

(4. 27) $$\mathrm{Re}(Q_{1,h}u, u) \geqq -C_1 h \|u\|^2.$$

As for the estimate of $Q_{2,h}$, we rewrite it in the form

$$Q_{2,h} = \sum_\alpha a_\alpha(x) k_\alpha(\xi) \Lambda_h^2(\xi) \chi_h(\xi),$$

where we can see $|k_\alpha(\xi) \Lambda_h^2(\xi) \chi_h(\xi)| \leqq h^2 \tilde{\lambda}_h^2(\xi) \chi_h(\xi) \leqq c_\beta h^2 \lambda_h(\xi)^{-\beta}$ for any positive β by (4. 3) and the principle of cutting off. Thereofore again from Lemma 4. 7 we get $\|Q_{2,h}\| \leqq C' h^2 \sum_\alpha a_\alpha \leqq C'' h$. Hence $Q_{2,h}$ is a null scheme in the sense of Yamaguti-Nogi and we get (4. 1). Q.E.D.

§ 5. A Stability Theorem for the Friedrichs Scheme

Let $\lambda_h(\xi) = \langle \zeta_h(\xi) \rangle = (1 + h^{-2} \sum_{j=1}^n \sin^2 h\xi_j)^{1/2}$. In this section the scheme may depend on t as well and for clarification of the sense of dependence on t we introduce here a function space $\mathscr{B}_t^1(S^m_{\langle\xi\rangle})$.

Definition 5. 1. Let T be any positive fixed constant. We write $p(t, x, \xi) \in \mathscr{B}_t^1(S^m_{\langle\xi\rangle})$, when $\ell \times \ell$ matrix valued function $p(t, x, \xi)$ defined on $[0, T] \times \mathrm{R}_x^n \times \mathrm{R}_\xi^n$ satisfies the following conditions:

i) $p(t, x, \xi)$, $\dfrac{\partial}{\partial t} p(t, x, \xi) \in S^m_{\langle\xi\rangle}$ for $t \in [0, T]$,

ii) they are uniformly bounded in $S^m_{\langle\xi\rangle}$ with respect to t, i.e. $|\partial_\xi^\alpha \partial_x^\beta p(t, x, \xi)| \leqq C_{\alpha,\beta} \langle\xi\rangle^{m-|\alpha|}$ and
$|\partial_\xi^\alpha \partial_x^\beta \dfrac{\partial}{\partial t} p(t, x, \xi)| \leqq C'_{\alpha,\beta} \langle\xi\rangle^{m-|\alpha|}$

for some constants $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ which are independent of t.

Furthermore if $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ can also be chosen independently of h $(0<h<1)$ for a h-family $p_h(t, x, \xi)$, we write $p_h \in \mathscr{B}_t^1(\{S^m_{\lambda_h}\})$. It is evident that if $p(t, x, \xi) \in \mathscr{B}_t^1(S^m_{\langle\xi\rangle})$, then $p(t, x, \zeta_h(\xi)) \in \mathscr{B}_t^1(\{S^m_{\lambda_h}\})$.

Now we consider the hyperbolic system of the form

(5. 1) $$Lu = D_t u - p(t, x, D_x) = 0 \quad \text{in } [0, T] \times \mathrm{R}_x^n$$

with $u_{|t=0} = u_0 \in L^2(\mathrm{R}_x^n)$ for $u = (u_1, \cdots, u_\ell)$. We assume that $p(t, x, \xi)$ has the form

(5. 2) $$p(t, x, \xi) = p_1(t, x, \xi) + p_0(t, x, \xi)$$

$$(p_j \in \mathscr{B}_t^1(S^j_{\langle\xi\rangle}),\ j=0,1)$$

and that all the eigenvalues $\mu_j(t, x, \xi)$ $(j=1, \cdots, \ell)$ of p_1 are real and satisfy

$$\max_{(t,x)} |\mu_j(t, x, \xi)| \leqq \mu_0 |\xi| \quad (j=1, \cdots, \ell) \tag{5.3}$$

on $[0, T] \times \mathrm{R}_x^m \times \{|\xi| \geqq M_0\}$ for some positive constants μ_0 and M_0. We also assume that $p_1(t, x, \xi)$ is diagonalizable in the sence: there exists $N(t, x, \xi) \in \mathscr{B}_t^1(S^0_{\langle\xi\rangle})$ such that

$$N(t, x, \xi) p_1(t, x, \xi) = \mathscr{D}(t, x, \xi) N(t, x, \xi) \quad \text{on} \quad [0, T] \times \mathrm{R}_x^n \times \{|\xi| \geqq M\} \tag{5.4}$$

and

$$\det |(N(t, x, \xi))| \geqq c_0 \quad \text{on} \quad [0, T] \times \mathrm{R}_x^n \times \{|\xi| \geqq M\} \tag{5.5}$$

for constants $c_0(>0)$, $M(\geqq M_0)$ and $\mathscr{D} = \begin{pmatrix} \mu_1(t, x, \xi) & & 0 \\ & \ddots & \\ 0 & & \mu_\ell(t, x, \xi) \end{pmatrix}$.

Definition 5.2. The Friedrichs scheme for (5.1) is the following:

$$\frac{1}{i} \frac{u(t+k, x) - \widehat{u(t, x)}}{k} = p(t, X, \zeta_h(D_x)) u, \tag{5.6}$$

where $\widehat{u(t, x)} = (2n)^{-1} \sum_{j=1}^{n} (u(t, x+he_j) + u(t, x-he_j))$ and $e_j = (0, \cdots, \underset{\smile}{\overset{j}{1}}, \cdots 0)$.

Here we can consider (5.6) as the operator which works on $u(t, x)$ $(\in L_x^2)$ to $u(t+k, x)$. Since $\tau(=k/h)$ is a fixed real constant, the operator may be denoted by $\boldsymbol{S}_h$ and

$$\sigma(\boldsymbol{S}_h)(t, x, \xi) = q_h(\xi) + i\tau h p_h(t, x, \xi), \tag{5.7}$$

where $q_h(\xi) = n^{-1} \sum_{j=1}^{n} \cos h\xi_j$ and $p_h(t, x, \xi) = p(t, x, \zeta_h(\xi))$. Then we have $q_h \in \mathscr{B}_t^1(\{S^0_{\lambda_h}\})$ and $p_h \in \mathscr{B}_t^1(\{S^1_{\lambda_h}\})$.

Our statement is the following theorem.

Theorem 5.3 (*General form of Yamaguti-Nogi-Vaillancourt's stability theorem*). *For the hyperbolic system of which the principal part p_1 is diagonalizable for large $|\xi|$ (i.e. under the assumptions (5.2), (5.3), (5.4), (5.5)), the Friedrichs scheme with τ ($|\tau| \leqq (\sqrt{n}\mu_0)^{-1}$) is stable in the sense of Lax-Richtmyer.*

Proof of Theorem 5. 3 will involve several propositions which are similar to those in [17] and be done after Lemma 5. 7. In the following calculation we use the notation $O(h)$ that denotes a quantity (real or complex) not larger in absolute value than h times of some positive constants which are independent of h and t. Hereafter $\|\cdot\|$ with no subscript denotes L^2-norm and since the function space $\mathcal{S}$ is dense in $L^2(\mathrm{R}^n)$, we may assume that u belongs to $\mathcal{S}$. First we shall aim at the one-step energy inequality (5. 26) and therefore the terms being $O(h)$ may be neglected for the simplicity of calculation.

We define $\boldsymbol{S}_h^{(1)}$ by

$$(5.8)\qquad \sigma(\boldsymbol{S}_h^{(1)})=q_h(\xi)+i\tau h p_{1,h}(t,x,\xi)\quad (p_{1,h}=p_1(t,x,\zeta_h(\xi))).$$

Since $p_0(t,x,\zeta_h(\xi))\in\mathcal{B}_t^1(\{\boldsymbol{S}_{\lambda_h}^0\})$, we have by Theorem 3. 7

$$(5.9)\qquad \|(\boldsymbol{S}_h-\boldsymbol{S}_h^{(1)})\|\leqq Ch\|u\|=O(h)\|u\|.$$

Then we can neglect the lower term p_0 and p_1 is denoted briefly by p hereafter.

Consider a function $\varphi(\xi)\in C_0(\mathrm{R}_\xi^n)$ satisfying the following conditions:

i) $\varphi(\xi)=1$ for $|\xi|\leqq 4/3$, $\varphi(\xi)=0$ for $|\xi|\geqq 5/3$, and $0<\varphi(\xi)<1$ for $4/3<|\xi|<5/3$.

ii) $\varphi(\xi_1)\geqq\varphi(\xi_2)$ for $|\xi_1|\leqq|\xi_2|$.

We set

$$(5.10)\qquad \varphi_h(\xi)=\varphi(M^{-1}\zeta_h(\xi)),\quad \psi_h(\xi)=\varphi((2M)^{-1}\zeta_h(\xi)),$$

and define $\check{\boldsymbol{S}}_h$ by

$$(5.11)\qquad \sigma(\check{\boldsymbol{S}}_h)(t,x,\xi)=q_h(\xi)+i\tau h\check{p}_h(t,x,\xi),$$

where $\check{p}_h=p_h(t,x,\xi)(1-\psi_h(\xi))$. Then we have

Lemma 5. 4.

$$(5.12)\qquad \|(\boldsymbol{S}_h^{(1)}-\check{\boldsymbol{S}}_h)u\|=O(h)\|u\|.$$

Proof. From $\sigma(\boldsymbol{S}_h^{(1)}-\check{\boldsymbol{S}}_h)=i\tau h p_h(t,x,\xi)\psi_h(\xi)$, where $p_h\psi_h\in\{\boldsymbol{S}_{\lambda_h}^{-\infty}\}$ by the principle of cutting off, we get $h^{-1}(\boldsymbol{S}_h^{(1)}-\check{\boldsymbol{S}}_h)\in\{S_{\lambda_h}^0\}$. Then (5. 12) follows by Theorem 3. 7.

Now set

$$\check{H}_h(t, x, \xi) = N^*(t, x, \zeta_h(\xi))N(t, x, \zeta_h(\xi)) \times (1-\varphi_h(\xi))^2 + \varphi_h(\xi)I, \tag{5.13}$$

where N^* denotes the Hermitian adjoint matrix of N. Then we have

Lemma 5.5. *$\check{H}_h(t, x, \xi)$ is positive Hermitian and satisfies*

$$c_1 I \leqq \check{H}_h(t, x, \xi) \leqq c_2 I \quad (c_1 > 0) \tag{5.14}$$

and $\check{H}_h \in \mathscr{B}_t^1(\{\mathbf{S}_{\lambda_h}^0\})$.

Furhter let H_h denote the Friedrichs part of $\check{H}_h(t, X, D_x)$. Then we have

$$\sigma(H_h) - \sigma(\check{H}_h) \in \mathscr{B}_t^1(\{\mathbf{S}_{\lambda_h}^{-1}\}). \tag{5.15}$$

Proof. For ξ such that $|M^{-1}\zeta_h(\xi)| \leqq 9/6$ and some ξ_0 ($|\xi_0| = 9/6$), $\check{H}_h(t, x, \xi) \geqq \varphi_h(\xi)I \geqq \varphi(\xi_0)I$; and for ξ such that $|M^{-1}\zeta_h(\xi)| \geqq 9/6$, from (5.5) we get $|\det(N(t, x, \zeta_h(\xi)))| \geqq c_0$. Then there exists some positive constant c' such that

$$\begin{aligned}(N^*(t, x, \zeta_h(\xi))N(t, x, \zeta_h(\xi))(1-\varphi_h(\xi))^2 v, v) \\ = (N(t, x, \zeta_h(\xi))(1-\varphi_h(\xi))v, N(t, x, \zeta_h(\xi))(1-\varphi_h(\xi))v) \\ \geqq c'\|(1-\varphi_h(\xi))v\|^2 = c'(1-\varphi_h(\xi))^2\|v\|^2\end{aligned}$$

for any ℓ-vector v. Then, setting $c_1 = \min(\varphi(\xi_0), c'(1-\varphi(\xi_0))^2)$, we get $c_1 I \leqq \check{H}_h(t, x, \xi)$. By the assumption $N(t, x, \xi) \in \mathscr{B}_t^1(S_{\langle\xi\rangle}^0)$ it follows $\check{H}_h \in \mathscr{B}_t^1(\{\mathbf{S}_{\lambda_h}^0\})$, from which we get $\check{H}_h(t, x, \xi) \leqq c_2 I$. Noting the fact that the operation of taking the Friedrichs part of a symbol and the differentiation of the symbol with respect to t are commutative, we get

$$\frac{\partial}{\partial t}\{\sigma(H_h) - \sigma(\check{H}_h)\} = \left(\frac{\partial}{\partial t}\sigma(H_h)\right)_F - \frac{\partial}{\partial t}\sigma(\check{H}_h).$$

Then from (3.14) we get (5.15). Q.E.D.

Lemma 5.6. *$\|u\|_{H_h} = (H_h u, u)^{1/2}$ defines an equivalent norm to $\|u\|$, that is*

$$\alpha\|u\| \leqq \|u\|_{H_h} \leqq \beta\|u\|, \tag{5.16}$$

where α *and* β *are independent of* h *and* t.

Proof. From $\check{H}_h \in \mathscr{B}_t^1(\{S_{\lambda_h}^0\})$ and (3.13) we get $H_h \in \mathscr{B}_t^1(\{S_{\lambda_h}^0\})$. Then $\|u\|_{H_h} \leqq \beta \|u\|$ follows from Theorem 3.7. The other inequality $\alpha \|u\| \leqq \|u\|_{H_h}$ follows from the inequality (5.14) by applying Theorem 3.11, (3.18) for the case $m=0$. For completeness of this proof we shall prove (3.18) as the following Lemma:

Lemma 5.7. *If* $p_h(x, \xi) \geqq c_0 \lambda_h(\xi)^m I$ *for a positive constant* c_0, *then we have* $(P_{F,h}u, u) \geqq c_1 \|u\|_{\lambda_h, m/2}^2$ *for some positive constant* c_1.

Proof. Applying (3.16) to the non-negative Hermitian symbol $p_h(x, \xi) - c_0 \lambda_h(\xi)^m I$, we get $(P_{F,h}u, u) \geqq c_0 (\lambda_{F,h}^m(D, D')u, u)$. From the definition (3.12) and Plancherel's equality we get $(\lambda_{F,h}^m(D, D')u, u) = \int \left\{ \int e^{-ix'\xi} \left\{ \int e^{ix'\xi'} \lambda_{F,h}^m(\xi, \xi') \hat{u}(\xi') d\xi' \right\} dx' \right\} \overline{\hat{u}(\xi)} d\xi$, where $\lambda_{F,h}^m(\xi, \xi') = \int F(\xi, \zeta) \lambda_h(\zeta)^m F(\xi', \zeta) d\zeta$. Via the change of integration order we get

$$(\lambda_{F,h}^m(D, D')u, u) = \int \lambda_h^m(\zeta) \int |u_F(x', \zeta)|^2 dx' d\zeta ,$$

where $u_F(x', \zeta) = \int e^{-ix'\xi'} F(\xi', \zeta) \hat{u}(\xi') d\xi'$ and by using Plancherel's equality again we get

$$\begin{aligned}(\lambda_{F,h}^m(D, D')u, u) &= \int \lambda_h^m(\zeta) \left(\int F(\xi, \zeta)^2 |\hat{u}(\xi)|^2 d\xi \right) d\zeta \\ &= \int \left(\int \lambda_h^m(\xi + \lambda_h(\xi)^{1/2} \sigma) q^2(\sigma) d\sigma |\hat{u}(\xi)|^2 d\xi .\right.\end{aligned}$$

Then, by using the inequality (4.2), we get

$$d_0 \|u\|_{\lambda_h, m/2}^2 \leqq (\lambda_{F,h}^m(D, D')u, u) \leqq d_1 \|u\|_{\lambda_h, m/2}^2$$

for some positive constants d_0, d_1. Noting that c_0 was positive, we get $(P_{F,h}u, u) \geqq c_1 \|u\|_{\lambda_h, m/2}^2$ for $c_1 = c_0^{-1} d_0$, Q.E.D.

Proof of Theorem 5.3. We calculate $\|\check{S}_h u\|_{H_h}^2$ as follows.

$$\begin{aligned}(5.17) \qquad \|\check{S}_h u\|_{H_h}^2 &= (H_h \check{S}_h u, \check{S}_h u) \\ &= (H_h(q_h + i\tau h \check{p}_h)u, \ (q_h + i\tau h \check{p}_h)u)\end{aligned}$$

$$= (q_h^* H_h q_h u,\ u) + i\tau h ([q_h^* H_h \check{p}_h - \check{p}_h^* H_h q_h] u,\ u)$$
$$+ \tau^2 h^2 (\check{p}_h^* H_h \check{p}_h u,\ u) + \boldsymbol{D}_4$$
$$= \boldsymbol{I}_1 + i\tau h \boldsymbol{I}_2 + \tau^2 h^2 \boldsymbol{I}_3 + \boldsymbol{D}_4 ,$$

where the operator defined by the adjoint matrix of A is denoted by A^* which must not be confused with the adjoint operator*), and $\boldsymbol{D}_4$ appears as the term influenced by the difference between the two operators.

As for the estimate of $\boldsymbol{I}_2$, we need the following Propositions 5. 8–5. 10.

Proposition 5. 8. $q_h^* H_h \check{p}_h \equiv q_h^* (H_h \circ \check{p}_h)$, $q_h^* (\check{p}_h^* \circ H_h) \equiv q_h^* (\check{p}_h^* H_h)$, *where* $A \equiv B$ *means* $A - B \in \{S_{\lambda_h}^0\}$ *throughout the Propositions* 5. 8–5. 10.

Proof. Both equalities are verified by considering the difference of operator product and symbol product (Corollary 3. 3).

Proposition 5. 9. $q_h^* (\check{p}_h^* H_h) \equiv (\check{p}_h^* H_h) q_h^*$

Proof. This is verified by the commutation theorem (Corollary 3. 4. (ii)).

Proposition 5. 10. *The modified diagonalization*

$$[N_h^* N_h (1-\varphi_h)^2 + \varphi_h I] p_h (1-\psi_h) = p_h^* (1-\psi_h) [N_h^* N_h (1-\varphi_h)^2 + \varphi_h I] \tag{5. 18}$$

holds and we get

$$q_h^* (H_h \circ \check{p}_h) \equiv q_h^* (\check{p}_h^* \circ H_h) \tag{5. 19}$$

Proof. By the assumption (5. 4) we get $p^* N^* = N^* \mathscr{D}$ and $N^* N p = p^* N^* N$. Substituting $\zeta_h(\xi)$ in place of ξ in the latter identity, we get

$$N_h^* N_h (1-\varphi_h)^2 p_h (1-\psi_h) = p_h^* (1-\psi_h) N_h^* N_h (1-\varphi_h)^2$$

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*) Hereafter we do not use the notation $\bar{A}$ for the Hermitian adjoint matrix of A.

On the other hand we see $\varphi_h(\xi)(1-\psi_h(\xi))=0$ because of the fact that the supports of φ_h and $(1-\psi_h)$ are disjoint. Then we have (5.18). By using Theorem 3.1.ii) and Corollary 3.3 we get (5.19) from (5.18) and (5.15).

Therefore, from Propositions 5.8–5.10 and the self-adjointness of q_h we get $q_h^* H_h \check{p}_h - \check{p}_h^* H_h q_h \in \{S_{\lambda_h}^0\}$. Then by Theorem 3.7 we see $i\tau h I_2 = O(h)\|u\|^2$ and neglect it.

As for the estimate of $h^2 I_3$, we need Propositions 5.11 and 5.12 below.

Proposition 5.11. *$h^2 I_3$ can be deformed as follows:*

(5.20) $$h^2 I_3 = \mathrm{Re}(h^2([\check{N}_h^* \circ \check{D}_h^2 \circ \check{N}_h]u, u)) + O(h)\|u\|^2,$$

where $\check{N}_h = N(x, \zeta_h(\xi))(1-\varphi_h(\xi))$ *and* $\mathcal{D}_h = \mathcal{D}(x, \zeta_h(\xi))(1-\psi_h(\xi))$.

Proof. From matrix calculation we get easily

(5.21) $$\check{p}_h^* \circ \check{H}_h \circ \check{p}_h = \check{N}_h^* \circ \mathcal{D}_h^2 \circ \check{N}_h .$$

On the other hand, by using Theorem 3.1 and Corollary 3.3, we have

$$\check{p}_h^* H_h \check{p}_h = \check{p}_h^* \check{H}_h \check{p}_h + \check{p}_h^* (H_h - \check{H}_h) \check{p}_h$$
$$\equiv \check{p}_h^* \check{H}_h \check{p}_h \equiv \check{p}_h^* (\check{H}_h \circ \check{p}_h) \equiv \check{p}_h^* \circ \check{H}_h \circ \check{p}_h ,$$

where $A \equiv B$ means $A - B \in \{S_{\lambda_h}^1\}$. Multiplying both sides of the above equality by h and noting Example 4 (in Section 2) we get

(5.22) $$h\check{p}_h^* H_h \check{p}_h - h\check{p}_h^* \circ \check{H}_h \circ \check{p} \in \{S_{\lambda_h}^0\}.$$

Hence from (5.21), (5.22) we have (5.20) by using Theorem 3.7.

Proposition 5.12. *The following inequality holds.*

(5.23) $$h^2 I_3 \leqq \mu_0^2 \,\mathrm{Re}(H_h \gamma_h(D) u, u) + O(h)\|u\|^2,$$

where $\gamma_h(\xi) = \sum_{j=1}^{n} \sin^2 h\xi_j$.

Proof. Consider the symbol of $\mu_0^2(\check{N}_h^* \circ \gamma_h(\xi) \circ \check{N}_h) - h^2(\check{N}_h^* \circ \check{\mathcal{D}}_h^2 \circ \check{N}_h)$ which is non-negative Hermitian for $|\zeta_h(\xi)| \geqq M_0$ by the definition of μ_0

and $\in \{\mathring{S}^0_{\lambda_h}\}$. Therefore by applying Corollary 3. 15 we get $\mathrm{Re}([\mu_0^2(\check{N}_h^* \circ \gamma_h(\xi) \circ \check{N}_h) - h^2(\check{N}_h^* \circ \check{\mathscr{D}}_h^2 \circ \check{N}_h]u, u) \geqq -Kh\|u\|^2$. Combining this inequality with (5. 20) we get

$$h^2 \boldsymbol{I}_3 \leqq \mathrm{Re}(\mu_0^2[\check{N}_h^* \circ \gamma_h(\xi) \circ \check{N}_h]u, u) + O(h)\|u\|^2$$
$$= \mathrm{Re}(\mu_0^2[\check{H}_h - \varphi_h(\xi)\boldsymbol{I}] \circ \gamma_h(\xi)u, u)) + O(h)\|u\|^2$$

Then applying the principle of cutting off, we get (5. 23).

As for the estimate of $\boldsymbol{D}_4$, it is seen from the asymptotic expansion (3. 1) and Theorem 3. 1, ii) that $\boldsymbol{D}_4 = O(h)\|u\|^2$. Then we can neglect it.

As for $\boldsymbol{I}_1$, we see that $q_h^* H_h q_h - H_h q_h^2 = (q_h H_h - H_h q_h) q_h \in \{\mathring{S}^{-1}_{\lambda_h}\}$ by virtue of Corollary 3. 4, ii). Hence we have

$$(5.24) \qquad \boldsymbol{I}_1 = (q_h^* H_h q_h u, u) = \mathrm{Re}(H_h q_h^2(D)u, u) + O(h)\|u\|^2.$$

Summarizing (5. 23) and (5. 24) we get

$$(5.25) \qquad \|\check{\boldsymbol{S}}_h u\|_{H_h}^2 - \|u\|_{H_h}^2$$
$$= -\mathrm{Re}(H_h[I - q_h^2(D) - \tau^2\mu_0^2\gamma_h(D)]u, u) + O(h)\|u\|^2,$$

where

$$\sigma[I - q_h^2(D) - \tau^2\mu_0^2\gamma_h(D)] = n^{-2}\{\sum_{j>k}(\cos h\xi_j - \cos h\xi_k)^2$$
$$+ n(1 - \tau^2\mu_0^2 n)\sum_{j=1}^{n}\sin^2 h\xi_j.$$

If $|\tau| \leqq (\sqrt{n}\mu_0)^{-1}$, by applying Theorem 3. 12 to the first term of the right hand of (5. 25), we have

$$\|\check{\boldsymbol{S}}_h u\|_{H_h}^2 - \|u\|_{H_h}^2 \leqq Ch\|u\|^2$$

or equivalently

$$(5.26) \qquad \|\boldsymbol{S}_h u\|_{H_h} \leqq (1 + C'h)\|u\|_{H_h}.$$

In the case that $\boldsymbol{S}_h$ is independent of t, we have

$$\|\boldsymbol{S}_h^j u\|_{H_h} \leqq (1 + C'h)^j \|u\|_{H_h} \leqq C(T)\|u\|_{H_h} \qquad \text{for} \quad 0 \leqq jk \leqq T,$$

which is the desired stability.

In the case that $\boldsymbol{S}_h$ depends on t, we must calculate more carefully.

We rewite (5. 26) in the form

(5. 26′) $\|u((n+1)k\|^2_{H_h(nk)}$ *) $\leqq(1+c''h)\|u(nk)\|^2_{H_h(nk)}$,

setting $t=nk$.

On the other hand we have

$$\begin{aligned}&\|u((n+1)k)\|^2_{H_h((n+1)k)}-\|u((n+1)k)\|^2_{H_h(nk)}\\&=([H_h((n+1)k)-H_h(nk)u((n+1)k),u((n+1)k))\\&=\Big(\int e^{ix\xi}[H_h((n+1)k,x,\xi)-H_h(nk,x,\xi)]\\&\qquad\times\hat{u}((n+1)k,\xi)d\xi,u((n+1)k)\Big)\\&=k\Big(\int e^{ix\xi}G_h(nk,x,\xi)\hat{u}((n+1)k,\xi)d\xi,u((n+1)k)\Big),\end{aligned}$$

where $G_h=\int_0^1\frac{\partial H_h}{\partial t}(nk+\theta k,x,\xi)d\theta$. From Lemma 5. 5 we have $G_h\in\{S^0_{\lambda_h}\}$. Then by using Theorem 3. 7 we can see that the above difference is $O(h)\|u((n+1)k)\|^2$. Further by using the equivalence of $\|\cdot\|_{H_h(t)}$ uniformly with respect to t, we get from (5. 26′)

$$\|u((n+1)k)\|_{H_h((n+1)k)}\leqq(1+c'''h)\|u(nk)\|_{H_h(nk)}.$$

Hence we have

$$\|u(jk)\|_{H_h(jk)}\leqq(1+C'''h)^j\|u(0)\|_{H_h(0)}\leqq C(T)\|u(0)\|_{H_h(0)}$$

Again from the equivalence of $\|\cdot\|_{H_h}$, we get

$$\|u(jk)\|\leqq C(T)\|u(0)\|, \qquad \text{Q.E.D.}$$

Remark. As was mentioned in the remark in [17], our method works as well for the modified Lax-Wendroff scheme

(5. 27) $\sigma(L_h)=I+i\tau hp_h(t,x,\xi)q_h(\xi)-1/2\tau^2h^2p_h^2(t,x,\xi)$,

where $p_h\in\mathscr{B}^1_t(\{S^1_{\lambda_h}\})$. By modifying the above discussion from (5. 17) and thereafter, we can see that the modified Lax-Wendroff scheme with $\tau(|\tau|\leqq2(\sqrt{n}u_0)^{-1})$ is also stable.

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*) $u(t)$ and $H_h(t)$ denote that L^2_x-function $u(t,x)$ and the Friedrichs part H_h at t, respectively.

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