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Holonomic Quantum Fields. II
—The Riemann-Hilbert Problem—

By

Mikio SATO,* Tetsuji MIWA* and Michio JIMBO*

Chapter 2. Application to the Riemann-Hilbert Problem

Introduction

This paper is a continuation of our previous note [1], hereafter referred to as I, and constitutes the second chapter of the series. As stated in I, our aim in this series is to reveal the intimate relation between (i) deformation theory for systems of linear (ordinary and partial) differential equations, and (ii) field operators belonging to the Clifford group. In the present article we study the Riemann-Hilbert problem on $\boldsymbol{P}^1_{\boldsymbol{C}}$. Because the exposition may be viewed as a prototype of our theory, we have included it here, thereby changing the organization of the series from the one announced in I.

The Riemann-Hilbert problem has a rather long history. Let

$$(2.0.1) \qquad \frac{dy}{dx} = A(x)\,y\,, \quad y = {}^t(y_1, \cdots, y_m)$$

be a system of linear ordinary differential equations with a rational coefficient matrix $A(x)$. Denote by $\{a_1, \cdots, a_n\}$ the set of poles of $A(x)$, and let $Y(x)$ be a fundamental solution matrix of (2.0.1). In general $Y(x)$ is a multi-valued function having $a_1, \cdots, a_n$ and possibly $a_\infty = \infty$ as its branch points, and when x makes a negative(*) circuit around a_ν, it undergoes a transformation

$$(2.0.2) \qquad Y(x) \mapsto Y(x)\,M_\nu \quad (\nu = 1, \cdots, n, \infty)\,.$$

Here $M_\nu \in GL(m, \boldsymbol{C})$ are constant matrices subject to the relation

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* Research Institute for Mathematical Sciences, Kyoto University.

(*) Throughout this article we adopt this somewhat unusual convention.

(2. 0. 3) $$M_1M_2\cdots M_nM_\infty=1\,.$$

In 1857 Riemann [4][(*)] posed the question whether there exists, for given $a_1, \cdots, a_n \in \boldsymbol{P}^1_{\boldsymbol{C}}$ and $M_1, \cdots, M_n \in GL(m, \boldsymbol{C})$, a differential equation (2. 0. 1) which has a solution matrix $Y(x)$ having precisely the monodromic property (2. 0. 2). He imposed a further condition that $Y(x)$ be at most regularly singular at the branch points a_ν $(\nu=1, \cdots, n, \infty)$; namely that its singularities there be of the form

(2. 0. 4) $$Y(x)=\Phi_\nu(x)\cdot(x-a_\nu)^{-L_\nu} \qquad (\nu=1, \cdots, n, \infty)^{(**)}$$

where $\Phi_\nu(x)$ is an invertible meromorphic matrix at $x=a_\nu$, and L_ν is a constant matrix such that $e^{2\pi i L_\nu}=M_\nu$. Since in 1900 Hilbert [5] included the above problem to his mathematical problems, it has often been called the Riemann-Hilbert problem.

Considerable efforts has been made by a number of people [6], [7], [8], [10], [11], [12], [13], [14], toward the solution of the Riemann-Hilbert problem. Among them we note the names of (1) J. Plemelj [10], who presented an existence proof of a solution on $\boldsymbol{P}^1_{\boldsymbol{C}}$ for arbitrary m and n, (2) G. D. Birkhoff [11], who solved independently the original problem and its various generalizations previously proposed by himself, and (3) H. Röhrl [14], who extended the result of Plemelj to an arbitrary Riemann surface.

A solution $Y(x)$ to the Riemann-Hilbert problem (and hence the coefficient $A(x)$ of the differential equation (2. 0. 1)) depends on the initially specified branch points a_ν and the monodromy matrices M_ν, sometimes referred to as the Riemann data. L. Schlesinger [9] discussed this point as a deformation theory of differential equation (2. 0. 1). Assuming that $A(x)$ has the form $\sum_{\nu=1}^{n}\frac{A_\nu}{x-a_\nu}$, he asked for the condition for (2. 0. 1) to have *constant* monodromy under the variation of the position of branch points, and obtained his celebrated equations (see § 2. 3, Proposition 2. 3. 12)

(2. 0. 5) $$dA_\mu=-\sum_{\nu(\neq\mu)}[A_\mu, A_\nu]\,d(a_\mu-a_\nu)/(a_\mu-a_\nu) \qquad (\mu=1, \cdots, n).$$

The methods so far employed to solve the Riemann-Hilbert problem

(*) Riemann himself treated an n-th order equation for one unknown function.
(**) For $\nu=\infty$ we replace $x-a_\nu$ by $1/x$.

are roughly classified as follows:

(1) the continuity method (Schlesinger [8])

(2) reduction to integral equations (Hilbert [6], Plemelj [10], Birkhoff [11], Muskhelishvili [13])

(3) series expansions involving hyperlogarithms (Lappo-Danilevski [12])

(4) the method using fibre bundles (Röhrl [14]).

In the present paper we present still another and an entirely different one, namely

(5) the method of quantum field theory.

The idea lies in the following point. Let $\psi^{(i)}(x)$, $\psi^{*(i)}(x)$ $(i=1, \cdots, m)$ denote free fermion operators on $\boldsymbol{P}^1_{\boldsymbol{R}}$ (see § 2. 1). Let φ be a field operator satisfying the commutation relation of the form

$$\varphi\cdot\psi^{(j)}(x)=\sum_{i=1}^{m}\psi^{(i)}(x)\cdot\varphi\cdot m_{ij}(x),\tag{2.0.6}$$

$$\varphi\cdot\psi^{*(j)}(x)=\sum_{i=1}^{m}\psi^{*(i)}(x)\cdot\varphi\cdot m^{*}_{ij}(x)$$

where the matrices $(m_{ij}(x))=M(x)$, $(m^{*}_{ij}(x))={}^{t}M(x)^{-1}$ are related to the monodromy M_{ν}. Then the vacuum expectation value

$$Y(x_0;x)=-2\pi i(x_0-x)\left(\langle\psi^{*(i)}(x_0)\psi^{(j)}(x)\varphi\rangle/\langle\varphi\rangle\right)_{i,j=1,\cdots,m}\tag{2.0.7}$$

provides a solution to the Riemann-Hilbert problem normalized as $Y=1$ at $x=x_0$. The relation (2. 0. 6) indicates that φ induces a "rotation" in the space of free fermion operators; indeed we shall construct a class of field operators $\varphi(a;L)$ "belonging to the Clifford group", and show that their product

$$\varphi=\frac{\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)}{\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle}\tag{2.0.8}$$

has the required properties.

The advantage of our approach is that the monodromic structure is quite apparent in the concise expression (2. 0. 7)-(2. 0. 8) of the solution, where the "deformation parameters" a_{ν} and the exponent matrices L_{ν} are explicitly incorporated.

We should note here that the theory of Clifford groups expounded in I is not directly applicable, since we are dealing with infinite dimensional orthogonal spaces. One might think of constructing its infinite dimensional version by defining suitably the notions of $A(W)$, $G(W)$, Nr, etc. However it seems a rather lengthy, if not impossible, way to recover the fundamental results obtained in I to an extent sufficient for application. Since our interest lies not in developing the general theory but in concrete results, we prefer to give direct proofs to individual formulas which we need in our construction. In the course the finite dimensional "theory of rotation" turns out to be a useful *guiding principle.*

This paper is organized as follows.

§ 2. 1 is a preparatory paragraph for generalities on free fermion operators in one dimensional space.

In § 2. 2 the Riemann-Hilbert problem is formulated in terms of operator theory. We show that the following are equivalent: (i) to find a multi-valued analytic function with a prescribed monodromic property, (ii) to construct a field operator which induces a specified rotation of the type (2. 0. 6). Making use of a solution to (i) in the case of only two branch points a and ∞, we construct the field operator $\varphi(a, L)$ mentioned above. By virtue of the product formula (I. § 1. 4) we then obtain an infinite series expansion of the matrix $Y(x_0; x)$ in (2. 0. 7).

The arguments in these paragraphs are instructive but rather formal ones. In the latter half of the paper we shall make precise the formulas thus derived in a direct and mathematically rigorous way.

We begin § 2. 3 by supplying a convergence proof of the above infinite series. Assuming $|L_\nu|$ $(\nu=1, \cdots, n)$ to be sufficiently small, we show that this series converges for complex $x_0, x, a_1, \cdots, a_n$ to give a solution to the Riemann-Hilbert problem. Then we discuss some properties of $Y(x_0; x)$, including the linear total differential equation it satisfies in the variables $(x_0, x, a_1, \cdots, a_n)$, and its behavior under coalescence of branch points. We note that in the latter process formation of irregular singularities does not take place if the exponents L_ν are kept fixed. Indeed, by such a limit, $Y(x_0; x)$ is shown to become a solution to a Riemann-Hilbert problem, whose Riemann data are obtained by "fusing" the initial ones (see p. 254). Applying these results we give the commutation relation among $\varphi(a; L)$'s, and calculate the operator (2. 0. 8) in the limit where some of a_ν's coincide.

In the final § 2.4 we give a formula expressing the τ-function $\langle\varphi(a_1; L_1)\cdots\varphi(a_n; L_n)\rangle$ in terms of a solution $\{A_1, \cdots, A_n\}$ of the Schlesinger equations (2.0.5). We then study the behavior of A_ν's in the limit where some of a_ν's behave like $a_\nu = tb_\nu$, $t\to 0$ or $t\to\infty$. We shall calculate their asymptotic expansions in powers of t. We also derive the total differential equations satisfied by $Y(x_0; x)$ and by A_ν's in these limits, and calculate the corresponding limits of the τ-function.

Main results of this paper has been announced in the series of papers [2], specifically in VI.

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§ 2.1. 1 Dimensional Space Theory

Let $W^{\boldsymbol{R}}$ denote the space of real-valued square-integrable functions on 1 dimensional space which we denote by $X \underset{\text{def}}{=} \{x | x\in \boldsymbol{R}^1\}$. The inner product in $W^{\boldsymbol{R}}$ is defined by

$$(2.1.1)^{(*)} \qquad \langle w, w'\rangle = \int dx w(x) w'(x), \quad w, w' \in W^{\boldsymbol{R}}.$$

It is uniquely extended to the non-degenerate symmetric inner product in the complexification $W = W^{\boldsymbol{R}} \underset{\boldsymbol{R}}{\otimes} \boldsymbol{C}$, so that W is an orthogonal vector space.

Set $M = \{u | u\in GL(1, \boldsymbol{R})\}$, $M_\pm = \{u\in M | u \gtrless 0\}$ and set also $\underline{du} = \frac{du}{2\pi|u|}$. In accordance with $1+1$ space-time theory we define the Fourier transformation as follows:

$$(2.1.2) \qquad w(x) = \int \underline{du} \sqrt{0+iu}\, e^{ixu} w(u),$$

$$w(u) = \int dx \sqrt{0-iu}\, e^{-ixu} w(x)$$

(*) $\int\cdots\int dx_1\cdots dx_m$ (resp. $\int\cdots\int \underline{du_1}\cdots\underline{du_m}$) means $\int_{-\infty}^{+\infty} dx_1\cdots\int_{-\infty}^{+\infty} dx_m$ (resp. $\int_{-\infty}^{+\infty}\underline{du_1}\cdots\int_{-\infty}^{+\infty}\underline{du_m}$) unless otherwise stated.

where

$$\sqrt{0\pm iu}=\begin{cases} e^{\pm\pi i/4}|u|^{1/2} & u\in M_+ ,\\ e^{\mp\pi i/4}|u|^{1/2} & u\in M_- .\end{cases}$$

Remark. For the sake of notational simplicity we use the same symbol $w(\cdot)$ in both x and u representations of $w\in W$.

Then we have

$$\langle w, w'\rangle = \int \underline{du}\, w(u)\, w'(-u). \tag{2.1.3}$$

In this and following chapters we shall construct and analyze a class of field operators "belonging to the Clifford group $G(W)$" over W. In principle one may proceed as follows:

(i) Specify a rotation T in the above W.

(ii) Apply formulas (1.5.7), (1.5.8) in I and find $\varphi\in G(W)$ such that $T_\varphi = T$.

However the second step is ambiguous, for our orthogonal space W is infinite dimensional. As mentioned in the introduction, we do not pursue here the course of defining $A(W)$, $G(W)$, etc. Instead we shall lay aside the above W for the moment and start with the following special functions indexed by X or M, which are supposed to play a role of ideal basis of W.

Let $\psi(x_0)$ $(x_0\in X)$ denote $\delta(x-x_0)\in\mathcal{B}(X)$. We identify it with its Fourier transform $\sqrt{0-iu}\,e^{-ix_0u}\in\mathcal{B}(M)$. Likewise $\psi(u_0)$ $(u_0\in X)$ denotes $\sqrt{0-iu_0}e^{-ixu_0}\in\mathcal{B}(X)$ and it is identified with its Fourier transform $2\pi|u|\delta(u+u_0)\in\mathcal{B}(M)$. Namely we have the following scheme.

		x-representation	u-representation
(2.1.4)	$\psi(x_0)\leftrightarrow$	$\delta(x-x_0)$	$\sqrt{0-iu}\,e^{-ix_0u}$
(2.1.5)	$\psi(u_0)\leftrightarrow$	$\sqrt{0-\mathrm{i}u_0}\,e^{-ixu_0}$	$2\pi\lvert u\rvert\delta(u+u_0)$

$$\psi(x) = \int \underline{du}\sqrt{0+iu}\, e^{ixu}\psi(u) \tag{2.1.6}$$

$$\psi(u) = \int dx\sqrt{0-iu}\, e^{-ixu}\psi(x)$$

We denote by $J(x,x')$ (resp. $J(u,u')$) the table of inner product for $\psi(x)$'s (resp. $\psi(u)$'s). Namely they are hyperfunction kernels given by

(2. 1. 7)
$$J(x,x') \underset{\text{def}}{=} \langle\psi(x),\psi(x')\rangle=\delta(x-x').$$
$$J(u,u') \underset{\text{def}}{=} \langle\psi(u),\psi(u')\rangle=2\pi|u|\delta(u+u').$$

As in the finite dimensional case, an expectation value is a bilinear form $\langle\ \ \rangle$: $(w,w')\mapsto\langle ww'\rangle$ such that

(2. 1. 8)
$$\langle ww'\rangle+\langle w'w\rangle=\langle w,w'\rangle.$$

Our choice of the expectation value is

(2. 1. 9)
$$K(x,x') \underset{\text{def}}{=} \langle\psi(x)\psi(x')\rangle=\frac{1}{2\pi}\frac{i}{x-x'+i0},$$
$$K(u,u') \underset{\text{def}}{=} \langle\psi(u)\psi(u')\rangle=2\pi u_+\delta(u+u').^{(*)}$$

We shall also make use of $\psi(x)$ (resp. $\psi(u)$) "with several components" indexed by $X\times\{1,\cdots,m\}$ (resp. $M\times\{1,\cdots,m\}$). Namely for $i=1,\cdots,m$ let $\psi^{(i)}(x)$ (resp. $\psi^{(i)}(u)$) be a copy of $\psi(x)$ (resp. $\psi(u)$). The inner product and the expectation value among them are specified by an $m\times m$ non-degenerate symmetric matrix $\Lambda=(\lambda_{ij})$ as follows:

(2. 1. 10)
$$\langle\psi^{(i)}(x),\psi^{(j)}(x')\rangle_J=\lambda_{ij}J(x,x')$$
$$\langle\psi^{(i)}(x)\psi^{(j)}(x')\rangle_K=\lambda_{ij}K(x,x').$$

Accordingly J, K are now $m\times m$ matrices of hyperfunction kernels. In the sequel we shall mainly deal with the case $\Lambda=1_m$, and also $\Lambda=\begin{pmatrix} & 1_{\frac{m}{2}} \\ 1_{\frac{m}{2}} & \end{pmatrix}$ with even m. In the latter case we set $\psi^{*(i)}(x)=\psi^{(i+m/2)}(x)$ $(i=1,\cdots,m/2)$.

Remark. $\psi^{(i)}(x)$ and $\psi^{(i)}(u)$ are regarded as ideal basis of $W\otimes\boldsymbol{C}^m$. In general, let W_1, W_2 be orthogonal vector spaces equipped with the inner product $\langle\ ,\ \rangle_{W_1}$, $\langle\ ,\ \rangle_{W_2}$. Their tensor product $W=W_1\otimes W_2$ is naturally endowed with an orthogonal structure by setting $\langle w_1\otimes w_2, w_1'\otimes w_2'\rangle_W=\langle w_1,w_1'\rangle_{W_1}\cdot\langle w_2,w_2'\rangle_{W_2}$ $(w_1,w_1'\in W_1,\ w_2,w_2'\in W_2)$. We

(*) $u_\pm=\theta(\pm u)\cdot u$, $\theta(u)=1$ $(u>0)$, $=0$ $(u<0)$.

denote by ι (resp. ι_ν) the element of $\mathrm{Hom}_{\boldsymbol{C}}(W, W^*)$ (resp. $\mathrm{Hom}_{\boldsymbol{C}}(W_\nu, W_\nu^*)$) which defines the inner product $\langle\ ,\ \rangle_W$ (resp. $\langle\ ,\ \rangle_{W_\nu}$), i.e. $\iota(w)(w') = \langle w, w'\rangle_W$, $\iota_\nu(w_\nu)(w'_\nu) = \langle w_\nu, w'_\nu\rangle_{W_\nu}$ $(\nu=1,2)$. Also a κ-norm on $A(W_1)$ induces one on $A(W)$; namely let $\kappa_1 \in \mathrm{Hom}_{\boldsymbol{C}}(W_1, W_1^*)$ be an element such that $\kappa_1 + {}^t\kappa_1 = \iota_1$. Then $\kappa = \kappa_1 \otimes \iota_2 \in \mathrm{Hom}_{\boldsymbol{C}}(W, W^*)$ clearly satisfies $\kappa + {}^t\kappa = \iota$ (see § 1. 5).

Let $W_2 = \boldsymbol{C}^m$ and choose a basis $e_1, \cdots, e_m$ such that $\langle e_i, e_j\rangle_{W_2} = \lambda_{ij}$. By setting $\psi^{(i)}(x) = \psi(x) \otimes e_i$ we are led to formulas (2. 1. 10).

Take an infinite set of hyperfunctions $(\rho_m(x_1, \cdots, x_m))_{m\in\boldsymbol{N}}$ where $\rho_m(x_1, \cdots, x_m)$ belongs to $\mathscr{B}(X^m)$. We consider an equivalence relation; $(\rho_m(x_1, \cdots, x_m))_{m\in\boldsymbol{N}} \sim (\rho'_m(x_1, \cdots, x_m))_{m\in\boldsymbol{N}}$ if and only if $\sum_{\sigma\in\mathfrak{S}_m} (\mathrm{sgn}\,\sigma) \times \rho_m(x_{\sigma(1)}, \cdots, x_{\sigma(m)}) = \sum_{\sigma\in\mathfrak{S}_m} (\mathrm{sgn}\,\sigma)\rho'_m(x_{\sigma(1)}, \cdots, x_{\sigma(m)})$ for all $m \in \boldsymbol{N}$, $\mathfrak{S}_m$ denoting the symmetric group of degree m. We call an equivalence class a norm and denote it by

$$(2.1.11)\qquad \sum_{m=0}^{\infty} \frac{1}{m!} \int\cdots\int dx_1 \cdots dx_m \rho_m(x_1, \cdots, x_m)\psi(x_m)\cdots\psi(x_1)$$

symbolically. We denote by $A(W)$ the set of norms, which is endowed naturally with the structure of a vector space. The product of two norms

$$\sum_{m=0}^{\infty} \frac{1}{m!} \int\cdots\int dx_1 \cdots dx_m \rho_m^{(\mu)}(x_1, \cdots, x_m)\psi(x_m)\cdots\psi(x_1)\ (\mu=1,2)$$

is defined to be

$$\sum_{m=0}^{\infty} \frac{1}{m!} \int\cdots\int dx_1 \cdots dx_m \sum_{k=0}^{m} \rho^{(1)}(x_1, \cdots, x_k)\rho^{(2)}(x_{k+1}, \cdots, x_m)\psi(x_m)\cdots\psi(x_1).$$

We also use the u-representation of a norm

$$(2.1.12)\qquad \sum_{m=0}^{\infty} \frac{1}{m!} \int\cdots\int \underline{du_1} \cdots \underline{du_m} \rho_m(u_1, \cdots, u_m)\psi(u_m)\cdots\psi(u_1).$$

(2. 1. 11) and (2. 1. 12) represent the same norm if and only if they are transformed to each other by (2. 1. 6). In general the transformation into u-(resp. x-) representation from x-(resp. u-) representation may happen to be ill-defined. Hence strictly speaking the above two definitions do not coincide. But since we are interested not in the whole set $A(W)$ of

norms but in individual elements, the reader should not worry about this point.

The "operator algebra" $A(W)$ is the same thing as $\Lambda(W)$ as a vector space, but they differ in product rule as explained below. To distinguish an element of $A(W)$ from a norm we refer to the former as an operator. For $a\in A(W)$ we denote by $\mathrm{Nr}(a)$ the corresponding norm in $\Lambda(W)$; conversely for $a\in\Lambda(W)$ $:a:$ will represent the corresponding operator (in physicist's terminology : : is called the normal ordering).

The product in $A(W)$ is not always well-defined. It is well-defined when the following formal definition makes sense.

First we note the following formula which generalizes (1. 5. 2).

$$\mathrm{Nr}(:w_1\cdots w_k:\,:w_1'\cdots w_l':) \tag{2.1.13}$$

$$=\sum \mathrm{sgn}\begin{pmatrix}1\cdots\cdots\cdots\cdots\cdots\cdots k\\ \mu_1'\cdots\mu_{k-m}'\ \ \mu_m\cdots\mu_1\end{pmatrix}\mathrm{sgn}\begin{pmatrix}1\cdots\cdots\cdots\cdots\cdots l\\ \nu_1\cdots\nu_m\ \ \nu_1'\cdots\nu_{l-m}'\end{pmatrix}$$

$$\times\langle w_{\mu_1}w'_{\nu_{\sigma(1)}}\rangle\cdots\langle w_{\mu_m}w'_{\nu_{\sigma(m)}}\rangle w_{\mu_1'}\cdots w_{\mu_{k-m}'}w'_{\nu_1'}\cdots w'_{\nu_{l-m}'}.$$

Here the sum is taken over all the partitions $\{1,\cdots,k\}=\{\mu_1,\cdots,\mu_m\}\cup\{\mu_1',\cdots,\mu_{k-m}'\}$ $(\mu_1<\cdots<\mu_m,\ \mu_1'<\cdots<\mu_{k-m}')$, $\{1,\cdots,l\}=\{\nu_1,\cdots,\nu_m\}\cup\{\nu_1',\cdots,\nu_{l-m}'\}$ $(\nu_1<\cdots<\nu_m,\ \nu_1'<\cdots<\nu_{l-m}')$ and $\sigma\in\mathfrak{S}_m$. We define products of operators by termwise application of (2. 1. 13). For example

$$\mathrm{Nr}(\psi(x)\psi(x'))=\langle\psi(x)\psi(x')\rangle+\psi(x)\psi(x')$$

$$\mathrm{Nr}(\psi(x):\psi(x')\psi(x''):)=\langle\psi(x)\psi(x')\rangle\psi(x'')$$

$$-\langle\psi(x)\psi(x'')\rangle\psi(x')+\psi(x)\psi(x')\psi(x'').$$

Remark. Originally $\psi(x)$ means the delta function supported on x as an ideal element of W. Now it means sometimes a norm and sometimes an operator. In the above, $\psi(x)$ and $\psi(x')$ in $\mathrm{Nr}(\psi(x)\psi(x'))$ or $\langle\psi(x)\psi(x')\rangle$ are operators, while those of $\psi(x)\psi(x')$ are norms. If $\psi(x)$ and $\psi(x')$ are considered as operators (resp. norms) they satisfy

$$[\psi(x),\psi(x')]_+=\delta(x-x'),$$

$$(\text{resp. } [\psi(x),\psi(x')]_+=0).$$

In general, let $\varphi^{(j)}$ $(j=1,2)$ be operators given by

$$\mathrm{Nr}(\varphi^{(j)})=\sum_{m=0}^{\infty}\frac{1}{m!}\int\cdots\int dx_1\cdots dx_m\rho_m^{(j)}(x_1,\cdots,x_m)\psi(x_m)\cdots\psi(x_1).$$

Then we have

$$\mathrm{Nr}(\varphi^{(1)}\varphi^{(2)})=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{m_1!}\frac{1}{m_2!}\int\cdots\int dx_1\cdots dx_{m_1}\int\cdots\int dx_1'\cdots dx_{m_2}'$$

$$\times\rho_{m_1}^{(1)}(x_1,\cdots,x_{m_1})\rho_{m_2}^{(2)}(x_1',\cdots,x_{m_2}')\mathrm{Nr}(:\psi(x_{m_1})\cdots\psi(x_1)::\psi(x_{m_2}')\cdots\psi(x_1'):)$$

where $\mathrm{Nr}(:\psi(x_{m_1})\cdots\psi(x_1)::\psi(x_{m_2}')\cdots\psi(x_1'):)$ is given by (2.1.13).

The above formal definition of products has ambiguities caused by several operations on hyperfunctions.

As for substitution, integration and product, intrinsic definitions are given in [16] (see also [17]), but the conditions for their well-definedness may fail sometimes. Moreover infinite sums of hyperfunctions are nonsensical in general. Yet these difficulties are not overwhelming as long as we are interested in handling explicit formulas and not the general theory (see § 2.3 below).

This is in particular the case if we restrict ourselves to the following class of operators. An operator g is said to be *in class G* if its norm has the form

(2.1.14) $$\mathrm{Nr}(g)=cw_1\cdots w_k\exp(\rho/2)^{(*)}$$

$$c\in\boldsymbol{C},\ w_j=\int dx\,c_j(x)\psi(x)\quad(j=1,\cdots,k)$$

$$\rho=\iint dxdx'R(x,x')\psi(x)\psi(x')$$

where $c_j\in\mathscr{B}(X)$ and $R\in\mathscr{B}_{\mathrm{skew}}(X\times X)\underset{\mathrm{def}}{=}\{R\in\mathscr{B}(X\times X)\,|\,R(x,x')+R(x',x)=0\}$.

We emphasize the point that operators in class G are specified by a *finite* number of hyperfunctions, and that their product, as far as it is well defined, is also in class G.

Products of operators in class G are computed according to the results of § 1.4 of [1] and *V*-**3** of [2].

(*) $$\exp(\rho/2)=1+\frac{\rho}{2}+\frac{1}{2!}\left(\frac{\rho}{2}\right)^2+\cdots=1+\frac{1}{2}\iint dx_1dx_2R(x_1,x_2)\psi(x_1)\psi(x_2)$$
$$+\frac{1}{8}\iiiint dx_1dx_2dx_3dx_4R(x_1,x_2)R(x_3,x_4)\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)+\cdots.$$

Remark. The proof given in § 1. 4 of [1] is based on the finite dimensionality of the orthogonal vector space W. But an alternative proof through tedious computation of combinatorics, which is based solely on (2. 1. 13), is possible. This guarantees the applicability of the results in § 1. 4 to the infinite dimensional case.

We rewrite (1. 5. 5), (1. 5. 6) and Theorem 1. 4. 3 in the form applicable to the infinite dimensional case. We adopt the x-representation. The formulas in the u-representation are almost the same.

Let g be the operator whose norm is given by (2. 1. 14). We set $w=\int dxc(x)\psi(x)$. Then we have

$$\mathrm{Nr}(wg)=\Big(\sum_{j=1}^{k}(-)^{j-1}w_1\cdots w_{j-1}\langle ww_j\rangle w_{j+1}\cdots w_k + w^{(1)}w_1\cdots w_k\Big)\exp(\rho/2), \tag{2. 1. 15}$$

where

$$\langle ww_j\rangle_K=\iint dxdx'\,c(x)K(x,x')c_j(x'),$$

$$w^{(1)}=\int dx\{c(x)-\iint dx_1dx_2R(x,x_1)\,{}^tK(x_1,x_2)c(x_2)\}\psi(x),$$

$$\mathrm{Nr}(gw)=\Big(\sum_{j=1}^{k}(-)^{k-j}w_1\cdots w_{j-1}\langle w_jw\rangle w_{j+1}\cdots w_k + w_1\cdots w_kw^{(2)}\Big)\exp(\rho/2) \tag{2. 1. 16}$$

where

$$\langle w_jw\rangle_K=\iint dxdx'\,c_j(x)K(x,x')c(x'),$$

$$w^{(2)}=\int dx\{c(x)+\iint dx_1dx_2R(x,x_1)K(x_1,x_2)c(x_2)\}\psi(x).$$

Now let $g_\nu\,(\nu=1,\cdots,n)$ be operators in class G given by

$$\mathrm{Nr}(g_\nu)=\exp(\rho_\nu/2)$$

where $\rho_\nu=\iint dxdx'R_\nu(x,x')\psi(x)\psi(x')$, $R_\nu\in\mathscr{B}_{\mathrm{skew}}(X\times X)$. We set $R(x,x')=(R_{\mu\nu}(x,x'))_{\mu,\nu=1,\cdots,n}$, $R_{\mu\nu}(x,x')=\delta_{\mu\nu}R_\nu(x,x')$ and $A(x,x')=(A_{\mu\nu}(x,x'))_{\mu,\nu=1,\cdots,n}$,

$$A_{\mu\nu}(x, x') = \begin{cases} K(x, x') & \mu<\nu \\ 0 & \mu=\nu \\ -{}^tK(x, x') = -K(x', x) & \mu>\nu\,. \end{cases}$$

Then we have

$$(2.1.17) \quad \langle g_1\cdots g_n\rangle = \exp\Big\{-\sum_{l=2}^{\infty}\frac{1}{2l}\int\cdots\int dx_1\cdots dx_{2l}$$

$$\times\sum_{\mu_1,\cdots,\mu_l=1}^{n} A_{\mu_1\mu_2}(x_1, x_2)R_{\mu_2}(x_2, x_3)A_{\mu_2\mu_3}(x_3, x_4)\cdots R_{\mu_l\mu_1}(x_{2l}, x_1)\Big\},$$

$$(2.1.18) \quad g_1\cdots g_n = \langle g_1\cdots g_n\rangle\exp(\rho/2),$$

$$\rho = \iint dxdx'\{\sum_{l=0}^{\infty}\int\cdots\int dx_1\cdots dx_{2l}$$

$$\times\sum_{\mu_0,\mu_1,\cdots,\mu_l=1}^{n} R_{\mu_0}(x, x_1)A_{\mu_0\mu_1}(x_1, x_2)R_{\mu_1}(x_2, x_3)\cdots$$

$$\cdots A_{\mu_{l-1}\mu_l}(x_{2l-1}, x_{2l})R_{\mu_l}(x_{2l}, x')\}\psi(x)\psi(x').$$

We also remark about the basic formula (1.5.8). Let T be an orthogonal transformation. We assume that T is given by a kernel function $T(x, x')$ through

$$(2.1.19) \qquad T\psi(x') = \int dx\psi(x)T(x, x').$$

We seek for an operator φ in class G satisfying

$$(2.1.20) \qquad \varphi\cdot\psi(x') = \int dx\psi(x)\cdot\varphi T(x, x').$$

If we assume that φ is given by a kernel function $R(x, x')$ through

$$(2.1.21) \quad \mathrm{Nr}(\varphi) = \exp(\rho/2)$$

$$\rho = \iint dxdx'R(x, x')\psi(x)\psi(x'),$$

(2.1.20) is equivalent to

$$(2.1.22) \quad \int dx_1R(x, x_1)K(x_1, x') + \iint dx_1dx_2R(x, x_1)\,{}^tK(x_1, x_2)\times T(x_2, x')$$

$$= T(x, x') - \delta(x-x').$$

Hence our problem reduces to an integral equation. We remark that neither the existence nor the uniqueness of such a solution $R(x, x')$ is proved in general. Our approach is the following. Find an explicit operator solution $\varphi_\mu (\mu=1, \cdots, n)$ for elementary orthogonal transformations $T_\mu (\mu=1, \cdots, n)$. Then, if the product $\varphi_1 \cdots \varphi_n / \langle \varphi_1 \cdots \varphi_n \rangle$ is well-defined, it will serve as the operator corresponding to the product $T_1 \cdots T_n$.

The prescription (2. 1. 9) is equivalent to considering $(\psi(u))_{u \in M_+}$ and $(\psi^\dagger(u))_{u \in M_+} (\psi^\dagger(u) \underset{\text{def}}{=} \psi(-u))$ as annihilation and creation operators, respectively. (See Remark 1 of Definition 1. 5. 1 [1].) Let $A^{\text{ann}}(W)$ (resp. $A^{\text{cre}}(W)$) denote the vector subspace of $A(W)$ consisting of operators satisfying for all $m=0, 1, 2, \cdots$

$$\rho_m(u_1, \cdots, u_m) |_{M_-^m} \equiv 0 , \tag{2. 1. 24}$$

$$(\text{resp. } \rho_m(u_1, \cdots, u_m) |_{M_+^m} \equiv 0) .$$

We denote by $|\text{vac}\rangle$ (resp. $\langle\text{vac}|$) the residue class of 1 in $A(W)/A^{\text{ann}}(W)$ (resp. $A(W)/A^{\text{cre}}(W)$). We also define the following state vectors:

$$|u_1, \cdots, u_m\rangle = \psi^\dagger(u_1) \cdots \psi^\dagger(u_m) |\text{vac}\rangle , \tag{2. 1. 24}$$

$$\langle u_1, \cdots, u_m| = \langle\text{vac}| \psi(u_1) \cdots \psi(u_m) .$$

We note that if $\varphi \in A(W)$, $\psi^\dagger(v_1) \cdots \psi^\dagger(v_l) \varphi \psi(u_1) \cdots \psi(u_m)$ is always well-defined. (Notice that in our definition every operator is *a priori* normally ordered.) We set

$$\langle v_1, \cdots, v_l | \varphi | u_1, \cdots, u_m \rangle \tag{2. 1. 25}$$

$$= \langle \psi^\dagger(v_1) \cdots \psi^\dagger(v_l) \varphi \psi(u_1) \cdots \psi(u_m) \rangle ,$$

and call it a matrix element of φ. The relation between $\rho_m(u_1, \cdots, u_m)$'s and matrix elements of an operator is given by Proposition 1. 2. 11, where $r=\infty$ and the sums over the indices μ_i, ν_j are replaced by integrals over u_i, v_j. We omit the proof.

We shall give an example of operators in class G.

Let I be a union of intervals in M_+. We denote by $\theta_I(u)$ the characteristic function of I;

$$\theta_I(u) = \begin{cases} 1 & \text{if } u \in I , \\ 0 & \text{if } u \notin I . \end{cases}$$

We define N_I by

$$\mathrm{Nr}(\mathsf{N}_I) = \int_0^\infty \underline{du}\, \theta_I(u)\psi^\dagger(u)\psi(u). \tag{2.1.26}$$

Then we have

$$\langle v_1, \cdots, v_l|\mathsf{N}_I|u_1, \cdots, u_m\rangle = \sum_{j=1}^m \theta_I(u_j)\langle v_1, \cdots, v_l|u_1, \cdots, u_m\rangle \tag{2.1.27}$$

and

$$[\mathsf{N}_I, \psi(u)] = -\varepsilon(u)\theta_I(|u|)\psi(u)^{(*)}. \tag{2.1.28}$$

Now using Theorem 1.5.3 in [1] we compute the norm of an operator ϕ_I which induces the rotation given by

$$T(u, u') = a^{-\varepsilon(u)\theta_I(|u|)} 2\pi|u|\delta(u-u'), a \in \mathbf{C}. \tag{2.1.29}$$

The answer is

$$R(u, u') = (a-1)2\pi|u|\delta(u+u')(\theta_I(u') - \theta_I(u)). \tag{2.1.30}$$

In fact it is easy to check

$$T(u, u') - 2\pi|u|\delta(u-u')$$

$$= \int \underline{dv_1}\, R(u, v_1)\left\{\int \underline{dv_2}\, K(v_2, v_1)T(v_2, u') + K(v_1, u')\right\}.$$

Thus we have

$$\mathrm{Nr}(\phi_I) = e^{(a-1)\mathsf{N}_I} \tag{2.1.31}$$

We see directly from (2.1.28) that

$$\phi_I = a^{\mathsf{N}_I}, \tag{2.1.32}$$

and (2.1.31) also follows from Proposition 1.2.9 in [1]. We have

$$\langle v_1, \cdots, v_l|a^{\mathsf{N}_I}|u_1, \cdots, u_m\rangle = a^{\sum_{j=1}^m \theta_I(u_j)}\langle v_1 \cdots v_l|u_1 \cdots u_m\rangle. \tag{2.1.33}$$

Let N_u^+ (resp. N_u^-) denote $\mathsf{N}_{(0,|u|)}$ (resp. $\mathsf{N}_{(|u|,\infty)}$). We define

$$\phi_\pm(u) = :\psi(u)e^{-2\mathsf{N}_u^\pm}:. \tag{2.1.34}$$

Then we have

$$\phi_\pm(u) = \begin{cases} (-)^{\mathsf{N}_u^\pm}\psi(u) & \text{if } u \in M_+, \\ \psi(u)(-)^{\mathsf{N}_u^\pm} & \text{if } u \in M_-. \end{cases} \tag{2.1.35}$$

(*) $\varepsilon(u) = 1\ (u > 0),\ = -1\ (u < 0)$.

(2. 1. 33) implies that

$$(2.1.36) \qquad \langle v_1, \cdots, v_l | \phi_\pm(u) | u_1, \cdots, u_m \rangle$$

$$= \begin{cases} \prod_{j=1}^{l} \varepsilon(\pm(v_j - u)) \langle v_1, \cdots, v_l, u | u_1, \cdots, u_m \rangle & \text{if } u \in M_+ , \\ \prod_{j=1}^{m} \varepsilon(\pm(u_j + u)) \langle v_1, \cdots, v_l | -u, u_1, \cdots, u_m \rangle & \text{if } u \in M_- . \end{cases}$$

A little computation shows that

$$(2.1.37) \qquad [\phi_\varepsilon(u), \phi_\varepsilon(v)] = 2\pi u \delta(u+v), \quad \varepsilon = + \text{ or } - .$$

Namely, for $\varepsilon = +$ or $-$, $(\phi_\varepsilon(u))_{u \in M_-}$ and $(\phi_\varepsilon(u))_{u \in M_+}$ are creation and annihilation operators of free bosons. We shall see later that they coincide with asymptotic fields of $\varphi^F(x)$.

Remark. The relation between $\psi(u)$ and $\phi_\pm(u)$ is reciprocal. If we set $\phi_\pm^\dagger(u) = \phi_\pm(-u)$ for $u \in M_+$, we have $\psi^\dagger(u)\psi(u) = \phi_\pm^\dagger(u)\phi(u)$. Hence (2. 1. 35) is rewritten as

$$(2.1.38) \qquad \psi(u) = \begin{cases} (-)^{\mathsf{N}_u^\pm} \phi_\pm(u) & \text{if } u \in M_+ , \\ \phi_\pm(u)(-)^{\mathsf{N}_u^\pm} & \text{if } u \in M_- , \end{cases}$$

where

$$\mathsf{N}_u^+ = \int_0^{|u|} \underline{du}\, \phi^\dagger(v)\phi(v) \quad \text{and} \quad \mathsf{N}_u^- = \int_{|u|}^{\infty} \underline{dv}\, \phi^\dagger(v)\phi(v).$$

In the next paragraph we shall deal with free fermion operators $\psi(x)$ on the real projective line $\boldsymbol{P}_{\boldsymbol{R}}^1 = \boldsymbol{R} \sqcup \{\infty\}$, rather than on $\boldsymbol{R}^1$. To make manifest the covariance of the theory we recapitulate here its generalities.

Set $G = SL(2, \boldsymbol{R})$, $P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \in G | \alpha \neq 0 \right\}$. By identifying the coset $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P \in G/P$ with $x = \alpha/\gamma \in \boldsymbol{P}_{\boldsymbol{R}}^1$ we have $G/P \cong \boldsymbol{P}_{\boldsymbol{R}}^1$. In particular the left G-action on $\boldsymbol{P}_{\boldsymbol{R}}^1$ reads $g \cdot x = \dfrac{\alpha x + \beta}{\gamma x + \delta}$ for $x \in \boldsymbol{P}_{\boldsymbol{R}}^1$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$.

Let $\chi: P \to GL(1, \boldsymbol{R})$ be a character of P. For $(g, w) \in G \times \boldsymbol{R}^1$ and $p \in P$ we set $(g, w)p = (gp, \chi(p)^{-1}w)$, and denote by $E_\chi = (G \times \boldsymbol{R}^1)/P$ the associated homogeneous line bundle over $G/P \cong \boldsymbol{P}_{\boldsymbol{R}}^1$ thus obtained. We have the left G-action on E_χ given by $g_0 \cdot (g, w)P = (g_0 g, w)P$ $(g_0 \in G,$

$(g, w)P \in E_\chi)$.

Now we choose χ_0 to be the following character:

$$\chi_0\left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}\right) = \alpha . \tag{2.1.39}$$

This amounts to setting the following transformation property between different coordinate representations $w(x)$, $w'(x')$ of a cross section w of E_{χ_0}:

$$w'(x') = (\gamma x + \delta) w(x), \quad x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G . \tag{2.1.40}$$

For cross sections w_1, w_2 of $E^{\boldsymbol{C}}_{\chi_0}$ on $\boldsymbol{P}^1_{\boldsymbol{R}}$, we set

$$\langle w_1, w_2 \rangle = \int_{-\infty}^{+\infty} dx\, w_1(x) w_2(x) , \tag{2.1.41}$$

$$\langle w_1 w_2 \rangle = \iint_{-\infty}^{+\infty} dx dx'\, w_1(x) \frac{1}{2\pi} \frac{i}{x - x' + i0} w_2(x') . \tag{2.1.42}$$

It is readily verified that (2. 1. 41), (2. 1. 42) are independent of the choice of a coordinate, and are invariant under the action of G. As the orthogonal space W we take the space consisting of L^2-sections of $E^{\boldsymbol{C}}_{\chi_0}$ equipped with the inner product (2. 1. 41).

For $x_0 \in \boldsymbol{P}^1_{\boldsymbol{R}}$ we denote by $\psi(x_0)$ the hyperfunction section $\delta(x - x_0)$ of $E^{\boldsymbol{C}}_{\chi_0}$. From (2. 1. 39) it satisfies the transformation property under a change of coordinates:

$$\psi'(x_0') = (\gamma x_0 + \delta) \psi(x_0), \quad x_0' = \frac{\alpha x_0 + \beta}{\gamma x_0 + \delta}, \ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G . \tag{2.1.43}$$

Note in particular that $(x_0 - x)\psi(x_0)\psi(x)$ is independent of the choice of a coordinate (cf. (2. 2. 5) below).

Actually in the course of construction of field operators we shall fix a coordinate system, bearing in mind the transformation law (2. 1. 43).

§ 2. 2. The Riemann-Hilbert Problem in One Dimensional Space

In this section we shall construct a family of field operators $\{\varphi(a; L)\}$ in class G in one dimensional space $\boldsymbol{R}^1$, or more precisely in its compactification $\boldsymbol{P}^1_{\boldsymbol{R}}$. In the course we shall show the equivalence of the following: (i) to find a multi-valued analytic function with a pre-assigned monodromy property (the Riemann-Hilbert problem), and (ii) to construct an operator which induces a specified rotation.

Let P_C^1 denote the complex projective line $C\cup\{\infty\}$. We fix a coordinate x on P_C^1 and set $P_C^1-\{\infty\}=D_+\cup R^1\cup D_-$, $D_\pm=\{\mathrm{Im}\, x\gtrless 0\}$. Let $a_1, \cdots, a_n \in R^1$ be n points such that $a_1<\cdots<a_n$. We fix a reference point x_* in the upper half plane, and denote by $\gamma_\nu (\nu=1, \cdots, n)$ (resp. γ_∞) a closed path in $P_C^1-\{a_1, \cdots, a_n, \infty\}$ with the endpoint x_* such that it encircles a_ν (resp. ∞) in the *clockwise* direction as shown in Fig. 2. 2. 1:

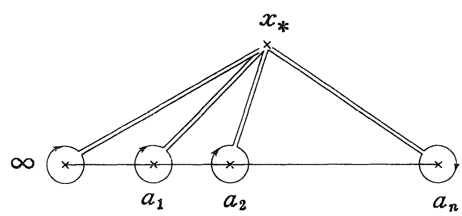

Fig. 2. 2. 1

For a multi-valued analytic function $Y(x)$ on $P_C^1-\{a_1, \cdots, a_n, \infty\}$ we denote by $\gamma Y(x)$ its analytic continuation along a closed path γ in $P_C^1-\{a_1, \cdots, a_n, \infty\}$ with the endpoint x_*. The Riemann problem on P_C^1, in the case where the branch points $a_1, \cdots, a_n, \infty$ all lie on the real line P_R^1, is then stated as follows [10]: given n matrices $M_1, \cdots, M_n \in GL(m, C)$ arbitrarily, find a matrix $Y(x)$ of multi-valued analytic functions on $P_C^1-\{a_1, \cdots, a_n, \infty\}$ such that

(2. 2. 1) (i) $Y(x)$ has at most regular singularities at

$a_1, \cdots, a_n, \infty$

(ii) $\gamma_\nu Y(x) = Y(x) M_\nu \quad (\nu=1, \cdots, n)$.

Let $Y_\pm(x)$ be a branch of $Y(x)$ on $D_\pm$, respectively, such that $Y_+(x) = Y_-(x)$ on $x>a_n$. Then (ii) is equivalent to the condition that, for $a_{\nu-1}<x<a_\nu$, $Y_-(x)=Y_+(x)M_\nu M_{\nu+1}\cdots M_n$ $(\nu=1, \cdots, n; a_0=-\infty)$. Therefore the Riemann problem is alternatively stated as: find single-valued holomorphic functions $Y_\pm(x)$ on $D_\pm$, respectively, satisfying (i) and

(2. 2. 2) (ii)' $Y_-(x-i0)=Y_+(x+i0)M(x)$, $x\in R^1-\{a_1, \cdots, a_n\}$

where we have set $M(x)=M_\nu M_{\nu+1}\cdots M_n$ for $a_{\nu-1}<x<a_\nu$ $(\nu=1, \cdots, n+1;$

$a_0=-\infty, a_{n+1}=+\infty$). In the latter formulation (with $M(x)\in GL(m, \boldsymbol{C})$ replaced by an arbitrary piecewise analytic matrix) the problem is called the Riemann-Hilbert problem [5].

First assume $M(x)=(m_{ij}(x))\in O(m, \boldsymbol{C})$. Suppose there exists a field operator φ in class G of the form

$$(2.2.3)\quad \mathrm{Nr}(\varphi)=\langle\varphi\rangle\exp(\rho/2)$$

$$\rho=\sum_{i,j=1}^{m}\iint dxdx'\psi^{(i)}(x)r_{ij}(x,x')\psi^{(j)}(x'),$$
$$r_{ij}(x,x')=-r_{ji}(x',x)$$

which satisfies the following commutation relation with ψ's:

$$(2.2.4)\qquad \varphi\psi^{(j)}(x)=\sum_{i=1}^{m}\psi^{(i)}(x)\varphi m_{ij}(x),\quad j=1,\cdots,m.$$

For $i,j=1,\cdots,m$ and $x_0>a_n$ we set

$$(2.2.5)\qquad y_{+ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{(i)}(x_0)\psi^{(j)}(x)\varphi\rangle$$
$$y_{-ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{(i)}(x_0)\varphi\psi^{(j)}(x)\rangle.$$

Proposition 2.2.1. *As a function of* x $Y_\pm(x_0;x)=(y_{\pm ij}(x_0;x))$ *is analytically prolongable to* $D_\pm$, *respectively, and their boundary values are related through* (2.2.2).

Proof. Applying the formulas (2.1.16) and (2.1.17) we have

$$(2.2.6)\quad \mathrm{Nr}(\psi^{(j)}(x)\varphi)=\sum_{i=1}^{m}\int dx_1\psi^{(i)}(x_1)\Big(\delta_{ij}\delta(x_1-x)$$
$$-\int dx_2 r_{ij}(x_1,x_2)\frac{1}{2\pi}\frac{-i}{x_2-x-i0}\Big)\cdot\mathrm{Nr}(\varphi)$$

$$\mathrm{Nr}(\varphi\psi^{(j)}(x))=\sum_{i=1}^{m}\int dx_1\psi^{(i)}(x_1)\Big(\delta_{ij}\delta(x_1-x)$$
$$+\int dx_2 r_{ij}(x_1,x_2)\frac{1}{2\pi}\frac{i}{x_2-x+i0}\Big)\cdot\mathrm{Nr}(\varphi).$$

Hence $Y_\pm(x_0;x)$ defined in (2.2.5) are expressed as

$$(2.2.7)\quad Y_\pm(x_0;x)=1\pm 2\pi i(x_0-x)\iint dx_1dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1+i0}$$
$$\times R(x_1,x_2)\frac{1}{2\pi}\frac{\mp i}{x_2-x\mp i0},$$

where $R(x,x')=(r_{ij}(x,x'))$. This implies the analytic prolongability of $Y_{\pm}(x_0;x)$. Multiplying $\psi^{(i)}(x_0)$ to both hand sides of (2. 2. 4) from the left and taking the vacuum expectation value we obtain (2. 2. 2).

Remark. From (2. 2. 7) one readily verifies the following:

(2. 2. 7)′

$$R(x,x')=\frac{1}{2\pi}\frac{i}{x-x'+i0}(Y(x+i0;x'-i0)-Y(x-i0;x'-i0))$$
$$+\frac{1}{2\pi}\frac{-i}{x-x'-i0}(Y(x+i0;x'+i0)-Y(x-i0;x'+i0)).$$

This implies that the holomorphic functions $\frac{1}{2\pi}\frac{i}{x-x'}(Y(x;x')-1)$ defined on $\{\varepsilon\,\mathrm{Im}\,x>0,\ \varepsilon'\,\mathrm{Im}\,x'>0,\ \varepsilon'\,\mathrm{Im}\,(x-x')<0\}$ $(\varepsilon,\varepsilon'=\pm)$ are defining functions of $R(x,x')$. In particular if $R(x,x')=0$ for $x>a_n$ or $x'>a_n$ (which is the case discussed below), we have $M(x)=1$ on $x>a_n$, and $Y(x_0;x)$ defined in (2. 2. 7) is continued to a single holomorphic function on $(\boldsymbol{P}_{\boldsymbol{C}}^1-[-\infty,a_n])\times(\boldsymbol{P}_{\boldsymbol{C}}^1-[-\infty,a_n])$.

Conversely we may construct an operator φ satisfying (2. 2. 4) once we know matrices $Y_{\pm}(x)=(y_{\pm ij}(x))$ of holomorphic functions on $D_{\pm}$ with the monodromic property (2. 2. 2). First note that, from (2. 2. 6), (2. 2. 4) holds if and only if $1+RK=(1-R^tK)T$, i.e. (cf. (1. 5. 8), (1. 5. 10))

(2. 2. 8) $$R(K+{}^tKT)=T-1$$

where R, K, ${}^tK$ and T denote matrices of integral operators with kernels $R(x,x')$, $\frac{1}{2\pi}\frac{i}{x-x'+i0}\cdot 1$, $\frac{1}{2\pi}\frac{-i}{x-x'-i0}\cdot 1$ and $M(x)\delta(x-x')$. The following Proposition provides us with a means to construct R from $Y_{\pm}$.

Proposition 2. 2. 2. *Let $Y_{\pm}(x)$ be matrices of holomorphic functions on $D_{\pm}$, respectively, with the properties* (2. 2. 2) *and* $\det Y_{\pm}(x)\neq 0$. *Then*

(2. 2. 9)

$$R(x,x')=(Y_+(x+i0)^{-1}-Y_-(x-i0)^{-1})\Big(\frac{1}{2\pi}\frac{i}{x-x'+i0}Y_-(x'-i0)$$

$$+\frac{1}{2\pi}\frac{-i}{x-x'-i0}Y_+(x'+i0)\Big)$$

$$=(Y_+(x+i0)^{-1}\frac{1}{2\pi}\frac{i}{x-x'+i0}+Y_-(x-i0)^{-1}\frac{1}{2\pi}\frac{-i}{x-x'-i0}$$

$$\times(Y_-(x'-i0)-Y_+(x'+i0))$$

satisfies (2. 2. 8).

Proof. Denote by $Y_\pm$ the integral operators with kernels $Y_\pm(x)\delta(x-x')$, and apply Proposition 1. 5. 4 in [1]. Since K and ${}^tK$ are projection operators onto the space of boundary values of holomorphic functions on $D_\pm$, respectively, the first two conditions of (1. 5. 11) are satisfied (J is the identity operator in the present situation). The last condition is nothing but (2. 2. 2).

Remark 1. As shown below, such $Y_\pm$ and R are not uniquely determined by the condition (2. 2. 8). Also $R(x, x')$ in (2. 2. 9) does not satisfy $R(x, x')=-{}^tR(x', x)$ in general. However we note that if $Y_\pm(x)$ satisfies (2. 2. 2), so does ${}^tY_\mp(x)^{-1}$ by virtue of the condition $M(x)\in O(m, \boldsymbol{C})$. Hence if $\det\ Y_\pm(x)\neq 0$ on $D_\pm$, $R(x, x')$ and $-{}^tR(x', x)$ simultaneously satisfy (2. 2. 8). Replacing R by $\frac{1}{2}(R-{}^tR)$, we may then assume $R=-{}^tR$.

Remark 2. For later convenience we list here some identities involving R defined in (2. 2. 9).

(2. 2. 10) $RK=(Y_+^{-1}-Y_-^{-1})\cdot J^{-1}K\cdot Y_-,$

$J^{-1}K\cdot RJ=Y_+^{-1}\cdot J^{-1}K\cdot(Y_--Y_+)$

$R^tK=(Y_+^{-1}-Y_-^{-1})\cdot J^{-1}\,{}^tK\cdot Y_+,$

$J^{-1t}K\cdot RJ=Y_-^{-1}\cdot J^{-1}\,{}^tK\cdot(Y_--Y_+)$

(2. 2. 11) $1+RK=(Y_+^{-1}\cdot J^{-1}K+Y_-^{-1}\cdot J^{-1}\,{}^tK)Y_-,$

$1+J^{-1}K\cdot RJ=Y_+^{-1}\cdot(J^{-1}K\cdot Y_-+J^{-1}\,{}^tK\cdot Y_+)$

$1-R^tK=(Y_+^{-1}\cdot J^{-1}K+Y_-^{-1}\cdot J^{-1}\,{}^tK)Y_+,$

$1-J^{-1}\,{}^tK\cdot RJ=Y_-^{-1}(J^{-1}K\cdot Y_-+J^{-1}\,{}^tK\cdot Y_+)$

So far we have assumed that $M(x)$ is an orthogonal matrix. The general case $M(x)\in GL(m, \boldsymbol{C})$ is reduced to the case of orthogonal monodromy of double size. Namely we now consider a field operator φ in class G of the form

$$(2.2.12)\qquad \mathrm{Nr}(\varphi)=\exp(\rho/2)$$

$$\rho=\sum_{i,j=1}^{m}\iint dxdx'(\psi^{(i)}(x)r_{ij}(x,x')\psi^{*(j)}(x')-\psi^{*(i)}(x)r_{ji}(x',x)\psi^{(j)}(x'))$$

$$=2\sum_{i,j=1}^{m}\iint dxdx'\psi^{(i)}(x)r_{ij}(x,x')\psi^{*(j)}(x')$$

which satisfies the commutation relation with ψ's:

$$(2.2.13)\qquad \varphi\psi^{(j)}(x)=\sum_{i=1}^{m}\psi^{(i)}(x)\varphi m_{ij}(x),\quad \varphi\psi^{*(j)}(x)=\sum_{i=1}^{m}\psi^{*(i)}(x)\varphi m^{*}_{ij}(x)$$

$$(j=1,\cdots,m;\ (m^{*}_{ij}(x))={}^{t}M(x)^{-1}).$$

Here $\psi^{(i)}(x)=\psi(x)\otimes e_i$, $\psi^{*(i)}=\psi(x)\otimes e^{*}_{i}$ with $\langle e_i,e_j\rangle=0$, $\langle e^{*}_{i},e^{*}_{j}\rangle=0$ and $\langle e_i,e^{*}_{j}\rangle=\delta_{ij}$ (see p. 11). In this case

$$(2.2.14)\qquad y_{+ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{*(i)}(x_0)\psi^{(j)}(x)\varphi\rangle$$

$$y_{-ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{*(i)}(x_0)\varphi\psi^{(j)}(x)\rangle$$

gives matrices with the monodromic property (2.2.2), while

$$(2.2.14)^{*}\qquad y^{*}_{+ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{(i)}(x_0)\psi^{*(j)}(x)\varphi\rangle$$

$$y^{*}_{-ij}(x_0;x)=-2\pi i(x_0-x)\langle\psi^{(i)}(x_0)\varphi\psi^{*(j)}(x)\rangle$$

satisfy

$$(2.2.2)^{*}\qquad Y^{*}_{-}(x-i0)=Y^{*}_{+}(x+i0)\,{}^{t}M(x)^{-1},\ x\in\boldsymbol{R}^{1}-\{a_1,\cdots,a_n\}.$$

Conversely φ satisfies (2.2.13) if $R(x,x')=(r_{ij}(x,x'))$ in (2.2.12) is given by the formula (2.2.9).

We shall now present a scheme of construction of a canonical field operator corresponding to the Riemann problem. First consider the case $n=1$. The Riemann problem then admits elementary solutions $Y_{\pm}(x)=(x-a\pm i0)^{-L}$, where L is an $m\times m$ matrix such that $e^{2\pi iL}=M$ is the given monodromy matrix. (Naturally there is an infinite number of possibilities in the choice of L). From Proposition 2.2.2 we may construct the corresponding field operator $\varphi=\varphi(a;L)$. Under the normalization

$\langle\varphi\rangle=1$, it is given by

$$\mathrm{Nr}(\varphi(a;L))=\exp(\rho(a;L)/2) \tag{2.2.15}$$

where in the case $M\in O(m, \boldsymbol{C})$

$$\rho(a;L)=\sum_{i,j=1}^{m}\iint dxdx'\psi^{(i)}(x)r_{ij}(x-a, x'-a;L)\psi^{(j)}(x') \tag{2.2.16}$$

$$\begin{aligned}R(x, x';L) &= (r_{ij}(x, x';L)) \\ &= ((x+i0)^{L}-(x-i0)^{L})\left\{\frac{1}{2\pi}\frac{i}{x-x'+i0}(x'-i0)^{-L}\right. \\ &\qquad \left.+\frac{1}{2\pi}\frac{-i}{x-x'-i0}(x'+i0)^{-L}\right\}.\end{aligned} \tag{2.2.17}$$

In the general case $M\in GL(m, \boldsymbol{C})$

$$\rho(a;L)=2\sum_{i,j=1}^{m}\iint dxdx'\psi^{(i)}(x)r_{ij}(x-a, x'-a;L)\psi^{*(j)}(x') \tag{2.2.18}$$

where $r_{ij}(x, x';L)$ is still given by (2.2.17). Making use of formula

(2.2.19)

$$\int_0^{\infty}\int_0^{\infty}dxdx'\frac{x^{L}x'^{-L}}{x-x'\pm i0}e^{-i(xu+x'u')}=\frac{-i\pi}{\sin\pi L}\frac{(u-i0)^{-L}(u'-i0)^{L}-e^{\pm\pi iL}}{u+u'-i0}$$

we obtain the u-representation of $\rho(a;L)$:

(2.2.20)

$$\rho(a;L)=\begin{cases}\displaystyle\sum_{i,j=1}^{m}\iint \underline{du}\,\underline{du'}\,\psi^{(i)}(u)r_{ij}(u, u';L)\psi^{(j)}(u')e^{ia(u+u')} \\ \qquad\qquad\qquad\text{(in the case (2.2.14))} \\ \displaystyle 2\sum_{i,j=1}^{m}\iint \underline{du}\,\underline{du'}\,\psi^{(i)}(u)r_{ij}(u, u';L)\psi^{*(j)}(u')e^{ia(u+u')} \\ \qquad\qquad\qquad\text{(in the case (2.2.16))}\end{cases}$$

where in both cases $R(u, u';L)=(r_{ij}(u, u';L))$ is given by

$$R(u, u';L)=-2\sin\pi L\cdot(u-i0)^{-L+1/2}(u'-i0)^{L+1/2}\frac{-i}{u+u'-i0}. \tag{2.2.21}$$

We remark that for an orthogonal M, $\varphi(a;L)$ given by (2.2.18) is nothing but the tensor product $\varphi(a;L)\otimes\varphi(a;L)$ of copies of $\varphi(a;L)$ given by (2.2.16). In what follows we shall mainly deal with the case $M\in GL(m, \boldsymbol{C})$ corresponding to (2.2.18).

Remark. $R(x, x';L)$ has an alternative expression

$$(2.2.17)' \quad R(x,x';L)=2i\sin\pi L\cdot x_-^L x_-'^{-L}\Big(\frac{1}{2\pi}\frac{i}{x-x'+i0}e^{\pi iL}+\frac{1}{2\pi}\frac{-i}{x-x'-i0}e^{-\pi iL}\Big)$$

$$x_-^L=\begin{cases}0 & (x>0)\\ |x|^L & (x<0),\end{cases}$$

which clearly indicates its support property. It should be noted, however, that (2.2.17)′ is not well defined at the origin $x=x'=0$ as a product of hyperfunctions. In this sence (2.2.17) is a more precise expression.

In the general case $n\geqq 1$, we choose L_ν so that $e^{2\pi iL_\nu}=M_\nu$ ($\nu=1,\cdots,n$) and set

$$(2.2.22) \quad \varphi=\varphi(a_1,\cdots,a_n;L_1,\cdots,L_n)=\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle^{-1}\varphi(a_1;L_1)\cdots\varphi(a_n;L_n).$$

Applying the product formula (1.4.11) we see that its norm takes the form

$$(2.2.23) \quad \mathrm{Nr}(\varphi(a_1,\cdots,a_n;L_1,\cdots,L_n))=\exp\Big(\frac{1}{2}\rho(a_1,\cdots,a_n;L_1,\cdots,L_n)\Big)$$

$$\frac{1}{2}\rho(a_1,\cdots,a_n;L_1,\cdots,L_n)=\sum_{\mu,\nu=1}^{n}\sum_{i,j=1}^{m}\iint dx dx'\,\psi^{(i)}(x)\hat{r}_{\mu\nu,ij}(x,x')\psi^{*(j)}(x').$$

Here $\hat{R}_{\mu\nu}(x,x')=\hat{R}_{\mu\nu}(x,x';a_1,\cdots,a_n,L_1,\cdots,L_n)=(\hat{r}_{\mu\nu,ij}(x,x'))$ denotes the (μ,ν)-th block of the $mn\times mn$ matrix

$$(2.2.24) \quad \hat{R}(x,x')=\int dx_1(1-RA)^{-1}(x,x_1)R(x_1,x')$$

where

$$(2.2.25) \quad (1-RA)^{-1}(x,x')=\delta(x-x')\cdot 1+\sum_{l=1}^{\infty}\int\cdots\int dx_1\cdots dx_{2l-1}R(x,x_1)A(x_1,x_2)\cdots\cdots R(x_{2l-2},x_{2l-1})A(x_{2l-1},x'),$$

$$(2.2.26)\qquad R(x,x')=\begin{pmatrix} R(x-a_1,x'-a_1;L_1) & & \\ & \ddots & \\ & & R(x-a_n,x'-a_n;L_n)\end{pmatrix}$$

$$A(x,x')=\begin{pmatrix} 0 & 1\cdots & 1 \\ & \ddots & \vdots \\ & \ddots & 1 \\ 0 & & 0\end{pmatrix}\frac{1}{2\pi}\frac{i}{x-x'+i0}+\begin{pmatrix} 0 & & 0 \\ -1 & \ddots & \\ \vdots & \ddots & \\ -1\cdots & -1 & 0\end{pmatrix}\frac{1}{2\pi}\frac{-i}{x-x'-i0}.$$

Accordingly $Y_{\pm}(x_0;x)$ defined by (2.2.7) is also expressed as an infinite series

(2.2.27)

$$\begin{aligned} Y_{\pm}(x_0;x) &= 1-2\pi i(x_0-x)\sum_{\mu,\nu=1}^{n}\iint dx_1dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1+i0} \\ &\qquad\times\hat{R}_{\mu\nu}(x_1,x_2)\frac{1}{2\pi}\frac{i}{x_2-x\mp i0} \\ &= 1-2\pi i(x_0-x)\sum_{\mu,\nu=1}^{n}\sum_{l=0}^{\infty}\int\cdots\int dx_1\cdots dx_{2l+2}\frac{1}{2\pi}\frac{i}{x_0-x_1+i0} \\ &\qquad\times[R(x_1,x_2)A(x_2,x_3)\cdots R(x_{2l-1},x_{2l})A(x_{2l},x_{2l+1}) \\ &\qquad\times R(x_{2l+1},x_{2l+2})]_{\mu\nu}\frac{1}{2\pi}\frac{i}{x_{2l+2}-x\mp i0}.\end{aligned}$$

The vacuum expectation value (τ-function) $\tau_n(a_1,\cdots,a_n)=\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle$ itself requires a more careful treatment. Naïve application of the product formula (2.1.18) yields an infinite series expansion of the form

$$\begin{aligned}(2.2.28)^{(*)}\quad &\tau_n(a_1,\cdots,a_n) \\ &=\exp\left\{-2\sum_{l=2}^{\infty}\frac{1}{2l}\operatorname{trace}\int\cdots\int dx_1\cdots dx_{2l}\right. \\ &\qquad\left.\times A(x_1,x_2)R(x_2,x_3)\cdots A(x_{2l-1},x_{2l})R(x_{2l},x_1)\right\}.\end{aligned}$$

Unfortunately (2.2.28) is meaningless, because $\int\cdots\int dx_2\cdots dx_{2l}A(x,x_2)R(x_2,x_3)\cdots A(x_{2l-1},x_{2l})R(x_{2l},x')$ has a singularity at $x=x'$. However we note that the series for its logarithmic derivative is termwise well-defined:

(*) Since the kernels corresponding to A and R in (2.1.18) are matrices of double size $\begin{pmatrix} & A(x,x') \\ -{}^t(A(x',x)) & \end{pmatrix}$ and $\begin{pmatrix} & R(x,x') \\ -{}^t(R(x',x)) & \end{pmatrix}$, respectively, the factor 2 in the exponential comes in.

$$(2.2.29)\quad d\log\tau_n(a_1,\cdots,a_n)=\frac{2i}{\pi}\sum_{\mu,\nu=1}^{n}\operatorname{trace}\iiint dx_1dx_2dx_3(L_\mu\sin^2 L_\mu \times(x_1-a_\mu)_-^{-L_\nu-1}(1-AR)^{-1}_{\mu\nu}(x_1,x_2)A_{\nu\mu}(x_2,x_3) \times(x_3-a_\mu)_-^{L_\mu-1}da_\mu).$$

In the next paragraph we shall show that the series (2.2.25), (2.2.27) and (2.2.29) converge, assuming each of the matrix elements of $L_\nu(\nu=1,\cdots,n)$ to be sufficiently small. These series expansions enable us to study in detail the analyticity and monodromy properties of the matrix $Y(x_0;x)$ or the kernel $\hat{R}(x,x')$, including their dependence on the parameters $a_1,\cdots,a_n$ and $L_1,\cdots,L_n$. In particular we shall verify that $Y(x_0;x)$ indeed provides a canonical solution to the Riemann problem. Here we shall be content to study its local properties in the framework of operator theory. The following arguments are rather formal but instructive, and are made precise in the next paragraph.

Consider the norm of products $\psi\varphi$ and $\psi^*\varphi$:

$$(2.2.30)\quad \operatorname{Nr}(\psi^{(j)}(x)\varphi(a;L))=\Big(\sum_{i=1}^{m}\int dx_1\psi^{(i)}(x_1)\,(1-R^tK)_{ij}(x_1,x)\Big)\times\operatorname{Nr}(\varphi(a;L))$$

$$\operatorname{Nr}(\psi^{*(j)}(x)\varphi(a;L))=\Big(\sum_{i=1}^{m}\int dx_1\psi^{*(i)}(x_1)\,(1+{}^tR^tK)_{ij}(x_1,x)\Big)\times\operatorname{Nr}(\varphi(a;L)).$$

Here $1-R^tK=((1-R^tK)_{ij})$ is given by (cf. (2.2.11))

$$(2.2.31)\quad (1-R^tK)(x_1,x)=\Big((x_1-a+i0)^L\frac{1}{2\pi}\frac{i}{x_1-x+i0} +(x_1-a-i0)^L\frac{1}{2\pi}\frac{-i}{x_1-x-i0}\Big)\cdot(x-a+i0)^{-L}.$$

Replacing L by $-{}^tL$ in (2.2.31), we obtain an expression for $1+{}^tR^tK$ $=((1+{}^tR^tK)_{ij})$. Similarly we have

$$(2.2.32)\quad \operatorname{Nr}(\varphi(a;L)\psi^{(j)}(x))=\Big(\sum_{i=1}^{m}\int dx_1\psi^{(i)}(x_1)\,(1+RK)_{ij}(x_1,x)\Big)\times\operatorname{Nr}(\varphi(a;L))$$

$$\operatorname{Nr}(\varphi(a;L)\psi^{*(j)}(x))=\Big(\sum_{i=1}^{m}\int dx_1\psi^{*(i)}(x_1)\,(1-{}^tRK)_{ij}(x_1,x)\Big)\times\operatorname{Nr}(\varphi(a;L))$$

where $(1+RK)(x_1, x)$ (resp. $(1-{}^tRK)(x_1, x)$) is given by (2. 2. 31) (resp. (2. 2. 31) with L replaced by $-{}^tL$) with the boundary value $(x-a+i0)^{-L}$ (resp. $(x-a+i0)^{{}^tL}$) replaced by $(x-a-i0)^{-L}$ (resp. $(x-a-i0)^{{}^tL}$). In a neighborhood of $x=a$, we expand (2. 2. 31) in powers of $x-a$:

$$(2.2.33)\quad (1-R{}^tK)(x_1, x)=\sum_{k=0}^{\infty}\frac{i}{2\pi}((x_1-a+i0)^{L-k-1} - (x_1-a-i0)^{L-k-1})(x-a+i0)^{-L+k}.$$

Now we introduce the following operators.

Definition 2. 2. 3.

$$(2.2.34)\quad \psi_{L'}^{(j)}(a)=\sum_{i=1}^{m}\int dx_1\psi^{(i)}(x_1)\frac{i}{2\pi}((x_1-a+i0)_{ij}^{L'-1} - (x_1-a-i0)_{ij}^{L'-1})$$

$$\psi_{L'}^{*(j)}(a)=\sum_{i=1}^{m}\int dx_1\psi^{*(i)}(x_1)\frac{i}{2\pi}((x_1-a+i0)_{ij}^{L'-1} - (x_1-a-i0)_{ij}^{L'-1}),$$

$$(2.2.35)\quad \mathrm{Nr}(\varphi_{L'}^{(j)}(a;L))=\psi_{L'}^{(j)}(a)\cdot\mathrm{Nr}(\varphi(a;L))$$

$$\mathrm{Nr}(\varphi_{L'}^{*(j)}(a;L))=\psi_{L'}^{*(j)}(a)\cdot\mathrm{Nr}(\varphi(a;L)).$$

Here we have identified $\psi_{L'}^{(j)}$ and $\psi_{L'}^{*(j)}$ with their norms, and $(x-a\pm i0)_{ij}^{L}$ denotes the (i, j)-th element of $(x-a\pm i0)^{L}$.

In terms of these operators we have the following local operator expansion formulas. (At least formally, for (2. 2. 33) is valid only for $|x-a|<|x_1-a|$.)

Proposition 2. 2. 4.

$$(2.2.36)\quad \mathrm{Nr}(\psi^{(j)}(x)\varphi(a;L))=\sum_{k=0}^{\infty}\sum_{i=1}^{m}\mathrm{Nr}(\varphi_{L-k}^{(i)}(a;L))\cdot(x-a+i0)_{ij}^{-L+k}$$

$$\mathrm{Nr}(\psi^{*(j)}(x)\psi(a;L))=\sum_{k=0}^{\infty}\sum_{i=1}^{m}\mathrm{Nr}(\varphi_{-{}^tL-k}^{*(i)}(a;L)) \times(x-a+i0)_{ij}^{{}^tL+k}.$$

Expressions for $\mathrm{Nr}(\varphi(a;L)\psi^{(j)}(x))$ (*resp.* $\mathrm{Nr}(\varphi(a;L)\psi^{*(j)}(x))$) *is obtained by replacing* $x-a+i0$ *by* $x-a-i0$ *in* (2.2.36).

$$(2.2.37)\quad \mathrm{Nr}(\psi^{*(i)}(x)\varphi_{L'}^{(j)}(a;L))=\frac{i}{2\pi}(x-a+i0)_{ij}^{L'-1}\cdot\mathrm{Nr}(\varphi(a;L))$$
$$+\sum_{k=0}^{\infty}\sum_{h=1}^{m}(x-a+i0)_{ih}^{L+k}\cdot\psi_{-tL-k}^{*(h)}(a)\psi_{L'}^{(j)}(a)\cdot\mathrm{Nr}(\varphi(a;L))$$
$$\mathrm{Nr}(\psi^{(i)}(x)\varphi_{L'}^{*(j)}(a;L))=\frac{i}{2\pi}(x-a+i0)_{ij}^{L'-1}\cdot\mathrm{Nr}(\varphi(a;L))$$
$$+\sum_{k=0}^{\infty}\sum_{h=1}^{m}(x-a+i0)_{ih}^{-tL+k}\cdot\psi_{L-k}^{(h)}(a)\psi_{L'}^{*(j)}(a)\cdot\mathrm{Nr}(\varphi(a;L)).$$

Replacing $x-a+i0$ *by* $x-a-i0$ *in* (2.2.37) *we obtain expressions for* $-\mathrm{Nr}(\varphi_{L'}^{(j)}(a;L)\psi^{*(i)}(x))$ *and* $-\mathrm{Nr}(\varphi_{L'}^{*(j)}(a;L)\psi^{(i)}(x))$, *respectively.*

Proof. Straightforward from (2.2.30) $\sim$ (2.2.34) and (2.1.16), (2.1.17).

Proposition 2.2.5. *The following commutation relations hold.*

$$(2.2.38)\quad \psi^{(j)}(x)\varphi_{L-k}^{(i)}(a;L)=\varphi_{L-k}^{(i)}(a;L)\sum_{h=1}^{m}\psi^{(h)}(x)\cdot(-m_{hj}(x))$$
$$\psi^{*(j)}(x)\varphi_{L-k}^{(i)}(a;L)=\varphi_{L-k}^{(i)}(a;L)\sum_{h=1}^{m}\psi^{*(h)}(x)\cdot(-m_{hj}^{*}(x))$$
$$(i,j=1,\cdots,m;\ k=0,1,2,\cdots).$$

Here we have set $(m_{ij}(x))=M(x)=1$ $(x>a)$, $=e^{2\pi iL}$ $(x<a)$, *and* $(m_{ij}^{*}(x))={}^{t}M(x)^{-1}$. *The same relations are valid if we replace* $\varphi_{L-k}^{(i)}(a;L)$ *by* $\varphi_{-tL-k}^{*(i)}(a;L)$ *in* (2.2.38).

Proof. For fixed x, let x' be a point sufficiently close to a. We have then

$$\psi^{(j)}(x)\psi^{(i)}(x')\varphi(a;L)=-\psi^{(i)}(x')\psi^{(j)}(x)\varphi(a;L)$$
$$=-\psi^{(i)}(x')\varphi(a;L)\sum_{h=1}^{m}\psi^{(h)}(x)m_{hj}(x).$$

Substituting this into (2.2.36) we obtain

$$(2.2.39)\quad \sum_{k=0}^{\infty}\sum_{l=1}^{m}\psi^{(j)}(x)\varphi_{L-k}^{(l)}(a;L)\cdot(x'-a+i0)_{li}^{-L+k}$$
$$=\sum_{k=0}^{\infty}\sum_{l=1}^{m}\varphi_{L-k}^{(l)}(a;L)(x'-a+i0)_{li}^{-L+k}\sum_{h=1}^{m}\psi^{(h)}(x)(-m_{hj}(x)).$$

The first relation of (2. 2. 38) is then obtained by equating the coefficients of $(x'-a+i0)_{li}^{-L+k}$ in (2. 2. 39). The rest are proved in a similar manner.

The behavior of $Y(x_0; x)$ at the branch points $x_0=a_\mu$ or $x=a_\nu$ are known from Proposition 2. 2. 4.

Proposition 2. 2. 6. *In a neighborhood of $x=a_\nu$ we have*

(2. 2. 40) $$Y(x_0; x) = \Phi_\nu(x_0; x)\cdot(x-a_\nu+i0)^{-L_\nu}$$

where $\Phi_\nu(x_0; x) = (\Phi_{\nu,ij}(x_0; x))$ is a holomorphic matrix at $x=a_\nu$ given by

(2. 2. 41) $$\Phi_{\nu,ij}(x_0; x) = -2\pi i(x_0-x)\sum_{k=0}^{\infty}(x-a_\nu)^k \times\tau_n^{-1}\langle\psi^{*(i)}(x_0)\varphi(a_1; L_1)\cdots\varphi_{L_\nu-k}^{(j)}(a_\nu; L_\nu)\cdots\varphi(a_n; L_n)\rangle$$

where $\tau_n=\langle\varphi(a_1; L_1)\cdots\varphi(a_n; L_n)\rangle$. Similarly at $x_0=a_\mu$ we have

(2. 2. 42) $$Y(x_0; x) = (x_0-a_\mu+i0)^{L_\mu}\cdot\Phi_\mu^*(x_0; x)$$

where $\Phi_\mu^*(x_0; x) = (\Phi_{\mu,ij}^*(x_0; x))$ is given by

(2. 2. 43) $$\Phi_{\mu,ij}^*(x_0; x) = 2\pi i(x_0-x)\sum_{k=0}^{\infty}(x_0-a_\mu)^k \times\tau_n^{-1}\langle\psi^{(j)}(x)\varphi(a_1; L_1)\cdots\varphi_{-{}^tL_\mu-k}^{*(i)}(a_\mu; L_\mu)\cdots\varphi(a_n; L_n)\rangle.$$

At $x_0=a_\mu$ and $x=a_\nu$,

(2. 2. 44) $$Y(x_0; x) = (x_0-a_\mu+i0)^{L_\mu}\cdot\Phi_{\mu\nu}(x_0; x)\cdot(x-a_\nu+i0)^{-L_\nu}$$

where $\Phi_{\mu\nu}(x_0; x) = (\Phi_{\mu\nu,ij}(x_0; x))$ is expressed as follows

(2. 2. 45) $$\Phi_{\mu\nu,ij}(x_0; x) = \begin{cases} -2\pi i(x_0-x)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(x_0-a_\mu)^k(x-a_\nu)^l & (\mu<\nu) \\ \quad\times\tau_n^{-1}\langle\varphi(a_1; L_1)\cdots\varphi_{-{}^tL_\mu-k}^{*(i)}(a_\mu; L_\mu)\cdots\varphi_{L_\nu-l}^{(j)}(a_\nu; L_\nu)\cdots\varphi(a_n; L_n)\rangle, & \\ \delta_{ij}-2\pi i(x_0-x)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(x_0-a_\mu)^k(x-a_\nu)^l & (\mu=\nu) \\ \quad\times\tau_n^{-1}\langle\varphi(a_1; L_1)\cdots:\psi_{-{}^tL_\nu-k}^{*(i)}(a_\nu)\psi_{L_\nu-l}^{(j)}(a_\nu)e^{(1/2)\rho(a_\nu; L_\nu)}:\cdots\varphi(a_n: L_n)\rangle, & \\ 2\pi i(x_0-x)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(x_0-a_\mu)^k(x-a_\nu)^l & (\mu>\nu) \\ \quad\times\tau_n^{-1}\langle\varphi(a_1; L_1)\cdots\varphi_{L_\nu-l}^{(j)}(a_\nu; L_\nu)\cdots\varphi_{-{}^tL_\mu-k}^{*(i)}(a_\mu; L_\mu)\cdots\varphi(a_n; L_n)\rangle. & \end{cases}$$

Proof. First consider (2. 2. 40) and (2. 2. 41). If x is sufficiently close to a_ν, $\psi(x)$ commutes with $\varphi(a_\mu; L_\mu)$ for $\mu=1, \cdots, \nu-1$, and from (2. 2. 36) we have

$$\langle\psi^{*(i)}(x_0)\psi^{(j)}(x)\varphi(a_1; L_1)\cdots\varphi(a_n; L_n)\rangle$$

$$=\langle\psi^{*(i)}(x_0)\varphi(a_1; L_1)\cdots\psi^{(j)}(x)\varphi(a_\nu; L_\nu)\cdots\varphi(a_n; L_n)\rangle$$

$$=\sum_{k=0}^{\infty}\sum_{h=1}^{m}\langle\psi^{*(i)}(x_0)\varphi(a_1; L_1)\cdots\varphi_{L_\nu-k}^{(h)}(a_\nu; L_\nu)\cdots\varphi(a_n; L_n)\rangle$$

$$\cdot(x_\nu-a_\nu+i0)_{hj}^{-L_\nu+k}.$$

This proves (2. 2. 40) - (2. 2. 41). Formulas (2. 2. 42) - (2. 2. 43) are obtained similarly by noting $\psi^{*(i)}(x_0)\psi^{(j)}(x) = -\psi^{(j)}(x)\psi^{*(i)}(x_0)$ for $x\neq x_0$. To prove (2. 2. 44) - (2. 2. 45) we start with (2. 2. 41) or (2. 2. 43). If $\mu<\nu$ similar argument leads to the expansion

$$\langle\psi^{*(i)}(x_0)\varphi(a_1; L_1)\cdots\varphi_{L_\nu-k}^{(j)}(a_\nu; L_\nu)\cdots\varphi(a_n; L_n)\rangle$$

$$=\sum_{l=0}^{\infty}\sum_{h=1}^{m}\langle\varphi(a_1; L_1)\cdots\varphi_{-{}^tL_\mu-l}^{*(h)}(a_\mu; L_\mu)\cdots\varphi_{L_\nu-k}^{(j)}(a_\nu; L_\nu)\cdots$$

$$\cdots\varphi(a_n; L_n)\rangle\cdot(x_0-a_\mu+i0)_{ik}^{L_\mu+l}.$$

Substitution into (2. 2. 41) yields (2. 2. 45) for $\mu<\nu$. The case $\mu>\nu$ is proved similarly using (2. 2. 43). In the case $\mu=\nu$ we have from (2. 2. 37)

$$\langle\psi^{*(i)}(x_0)\varphi(a_1; L_1)\cdots\varphi_{L_\nu-k}^{(j)}(a_\nu; L_\nu)\cdots\varphi(a_n; L_n)\rangle$$

$$=\frac{i}{2\pi}(x_0-a_\nu+i0)_{ij}^{L_\nu-k-1}\cdot\langle\varphi(a_1;L_1)\cdots\varphi(a_n; L_n)\rangle$$

$$+\sum_{l=0}^{\infty}\sum_{h=1}^{m}(x_0-a_\nu+i0)_{ih}^{L_\nu+l}\cdot\langle\varphi(a_1; L_1)\cdots:\psi_{-{}^tL_\nu-l}^{*(h)}(a_\nu)$$

$$\times\psi_{L_\nu-k}^{(j)}(a_\nu)e^{(1/2)\rho(a_\nu;L_\nu)}:\cdots\varphi(a_n; L_n)\rangle.$$

Noting $-2\pi i(x_0-x)\sum_{k=0}^{\infty}\frac{i}{2\pi}(x_0-a_\nu)^{-k-1}(x-a_\nu)^k=1$ we obtain (2. 2. 45) for $\mu=\nu$.

Finally we note that the norm of the derivative of the operator $\varphi(a; L)$ is expressible in terms of operators $\psi_L^{(i)}$, $\psi_L^{*(i)}$.

Proposition 2. 2. 7. *Setting* $L=(l_{ij})$ *we have*

(2. 2. 46)
$$\frac{d}{da}\psi_L^{(j)}(a)=\sum_{i=1}^{m}\psi_{L-1}^{(i)}(a)\cdot(-l_{ij}+\delta_{ij}),$$
$$\frac{d}{da}\psi_L^{*(j)}(a)=\sum_{i=1}^{m}\psi_{L-1}^{*(i)}(a)\cdot(-l_{ij}+\delta_{ij}),$$

(2. 2. 47)
$$\mathrm{Nr}\Big(\frac{d}{da}\varphi(a;L)\Big)=2\pi i\sum_{i,j=1}^{m}\psi_L^{(i)}(a)\psi_{-iL}^{*(j)}(a)l_{ij}\cdot\mathrm{Nr}(\varphi(a;L)).$$

Proof. Formula (2. 2. 46) follows immediately from the definition. To see (2. 2. 47) it suffices to note that

$$\mathrm{Nr}\Big(\frac{d}{da}\varphi(a;L)\Big)=\frac{d}{da}\Big(\frac{1}{2}\rho(a;L)\Big)\cdot e^{(1/2)\rho(a;L)}$$

and that

$$\frac{d}{da}R(x-a,x'-a,L)=\frac{1}{2\pi i}((x-a+i0)^{L-1}-(x-a-i0)^{L-1})$$
$$\times L((x'-a+i0)^{-L-1}-(x'-a-i0)^{-L-1}).$$

Corollary 2. 2. 8.

(2. 2. 48)
$$\frac{\partial}{\partial a_\nu}\log\tau_n=-\mathrm{trace}\Big(\frac{\partial\Phi_{\nu\nu}}{\partial x}(a_\nu;a_\nu)L_\nu\Big)$$

where $\Phi_{\nu\nu}(x_0;x)$ *is defined in* (2. 2. 44) - (2. 2. 45).

Proof. Straightforward from (2. 2. 45) and (2. 2. 47).

§ 2. 3. Solution to the Riemann Problem

Before proceeding to the convergence proof of (2. 2. 27) and (2. 2. 29), we must make precise their meaning, for in general an infinite series of hyperfunctions does not make sense as mentioned in § 2. 1. We shall show below that the series (2. 2. 27) is convergent in the complex domain $x_0, x\in \boldsymbol{P}_{\boldsymbol{C}}^1-[-\infty,a_n]$ and that it defines a holomorphic matrix $Y(x_0;x)$ there. It is then natural to define the series (2. 2. 27) for $Y_\pm(x_0;x)$ to be the boundary value $Y(x_0+i0;x\pm i0)$. Also the precise definition of the series (2. 2. 25) for $R(x,x')$ is given through the formula (2. 2. 7').

Apart from the field operators $\varphi(a_\nu; L_\nu)$, it is natural to extend the parameters $a_1, \cdots, a_n$ to the complex domain. Let $a_1, \cdots, a_n \in \boldsymbol{C}$ be points such that $\operatorname{Im} a_1 \geqq \cdots \geqq \operatorname{Im} a_n$, and denote by Γ_ν the half line $\{x \in \boldsymbol{C} | \operatorname{Im}(x-a_\nu)=0, \operatorname{Re}(x-a_\nu) \leqq 0\}$ (Fig. 2. 3. 1):

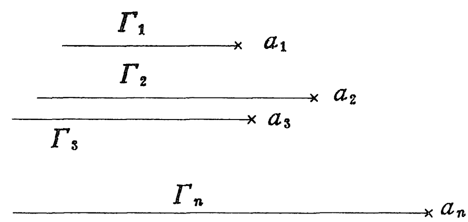

Fig. 2. 3. 1. Contiguous lines indicate those with the same imaginary part.

For $(x_0, x) \in (\boldsymbol{C}-\Gamma_\mu) \times (\boldsymbol{C}-\Gamma_\nu)$ we set

$$(2.3.1) \quad Z_{\mu\nu}(x_0; x) = \iint_{-\infty}^{0} dx_1 dx_2 \frac{1}{2\pi} \frac{i}{(x_0-a_\mu)-x_1} \times \hat{R}_{\mu\nu}(x_1+a_\mu, x_2+a_\nu) \frac{1}{2\pi} \frac{i}{x_2-(x-a_\nu)}$$

$$= \delta_{\mu\nu} \iint_{-\infty}^{0} dx_1 dx_2 \frac{1}{2\pi} \frac{i}{(x_0-a_\mu)-x_1} R(x_1, x_2; L_\nu) \frac{1}{2\pi} \frac{i}{x_2-(x-a_\nu)}$$

$$+ \sum_{l=1}^{\infty} \sum_{\nu_1, \cdots, \nu_{l-1}=1}^{n} \int_{-\infty}^{0} \cdots \int_{-\infty}^{0} dx_1 \cdots dx_{2l+2} \frac{1}{2\pi} \frac{i}{(x_0-a_\mu)-x_1} R(x_1, x_2; L_\mu)$$

$$\times A_{\mu\nu_1}(x_2, x_3) R(x_3, x_4; L_{\nu_1}) \cdots A_{\nu_{l-1}\nu}(x_{2l}, x_{2l+1}) R(x_{2l+1}, x_{2l+2}; L_\nu)$$

$$\times \frac{1}{2\pi} \frac{i}{x_{2l+2}-(x-a_\nu)},$$

where $R(x, x'; L)$ is defined by (2. 2. 17) and

$$(2.3.2) \quad A_{\mu\nu}(x, x') = \begin{cases} \dfrac{1}{2\pi} \dfrac{i}{x-x'+(a_\mu-a_\nu) \pm i0} & (\mu \lessgtr \nu) \\ 0 & (\mu=\nu). \end{cases}$$

The defining function of $Y_\pm(x_0; x)$ will then be given by

$$(2.3.3) \quad Y(x_0; x) = 1 - 2\pi i (x_0 - x) \sum_{\mu, \nu=1}^{n} Z_{\mu\nu}(x_0; x).$$

We proceed as follows. Set

$$(2.3.4)\qquad R=\begin{pmatrix} R_{L_1} & & \\ & \ddots & \\ & & R_{L_n}\end{pmatrix},\quad A=(A_{\mu\nu})=\begin{pmatrix} 0 & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & \ddots & A_{n-1n} \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix},$$

where R_{L_ν} (resp. $A_{\mu\nu}$) denotes the integral operator with the kernel $R(x, x'; L_\nu)$ (resp. $A_{\mu\nu}(x, x')$). It is shown (Proposition 2. 3. 1, 2. 3. 3) that R_{L_ν} and $A_{\mu\nu}$, regarded as linear operators on $L^2(-\infty, 0; dx)^m$, are bounded operators provided, for each $\nu=1, \cdots, n$,

$$(2.3.5)_\nu \qquad |\mathrm{Re}\,\lambda_j^{(\nu)}|<1/2 \quad (j=1, \cdots, m),$$

where $\lambda_1^{(\nu)}, \cdots, \lambda_m^{(\nu)}$ denote eigenvalues of L_ν. Convergence of (2. 3. 1) is then proved by showing that the series $\hat{R}=(1-RA)^{-1}R=\sum_{l=0}^{\infty}(RA)^l R$ converges in the operator norm for sufficiently small $|L_\nu|$ $(\nu=1, \cdots, n)$.

To begin with we note the following well known fact:

Proposition 2. 3. 1. *For* $\mathrm{Im}\,a\geqq 0$ *we set*

$$K_a^\pm: f(x)\mapsto \int \frac{dx'}{2\pi}\frac{\pm i}{x-x'\pm(a+i0)}f(x').$$

Then $K_a^\pm$ *is a bounded linear operator in* $L^2(\boldsymbol{R}^1; dx)$ *with* $\|K_a^\pm\|\leqq 1$. *It depends holomorphically on* a *for* $\mathrm{Im}\,a>0$, *and continuously for* $\mathrm{Im}\,a\geqq 0$ *in the strong topology.*

Denote by $A_{\mu\nu}$ the integral operator

$$A_{\mu\nu}: f(x)\mapsto \theta(-x)\int_{-\infty}^{0} dx' A_{\mu\nu}(x, x') f(x').$$

Proposition 2. 3. 1 implies that $A_{\mu\nu}$ is a bounded operator in $L^2(\boldsymbol{R}_-)$ $=L^2(\boldsymbol{R}_-; dx)$ $(\boldsymbol{R}_-=(-\infty, 0))$ with norm $\leqq 1$. Moreover it depends holomorphically on the parameters $(a_1, \cdots, a_n)\in V=\{(a_1, \cdots, a_n)$ $\in \boldsymbol{C}^n|\mathrm{Im}\,a_1>\cdots>\mathrm{Im}\,a_n\}$, and is continuous in the closure $\overline{V}$ in the strong topology.

Next consider the operator

$$(2.3.6)\qquad R_{L,\varepsilon}^\pm: f(x)\mapsto \int\frac{dx'}{2\pi}x_-^L\frac{\pm i}{x-x'\pm i\varepsilon}x_-'^{-L}f(x') \qquad (\varepsilon>0)$$

where $f(x)={}^t(f_1(x), \cdots, f_m(x))$ and L denotes an $m\times m$ matrix.

Making use of the Fourier transformation

$$(\mathcal{F}f)(\xi)=\int dx\, e^{-ix\xi}f(x)$$

(2. 3. 6) is alternatively written as

$$R^{\pm}_{L,\varepsilon}=x^{L}_{-}\circ\mathcal{F}^{-1}\circ\theta(\pm\xi)\,e^{-\varepsilon|\xi|}\circ\mathcal{F}\circ x^{-L}_{-} \tag{2. 3. 7}$$

where x^L_- (resp. $\theta(\pm\xi)e^{-\varepsilon|\xi|}$) denotes the multiplication operator $f(x)\mapsto x^L_-f(x)$ (resp. $g(\xi)\mapsto\theta(\pm\xi)e^{-\varepsilon|\xi|}g(\xi)$). First consider the case $m=1$ where the matrix L is a complex number $\lambda\in\boldsymbol{C}$.

Proposition 2. 3. 2. *If* $|\mathrm{Re}\,\lambda|<\frac{1}{2}$, $R^{\pm}_{\lambda,\varepsilon}$ *is a bounded linear operator in* $L^2(\boldsymbol{R}_-)$, *and* $\lim\limits_{\varepsilon\downarrow 0}R^{\pm}_{\lambda,\varepsilon}$, *which we denote by* $R^{\pm}_{\lambda}=R^{\pm}_{\lambda,0}$, *exists in the strong topology.*

The authors are grateful to Dr. K. Yajima who pointed out that the boundedness of $R^{\pm}_{\lambda}$ is implicitly proved in Lions-Magenes [18].

The following proof, divided into several steps, is essentially a modification of arguments in [18].

Let $H_\lambda(\boldsymbol{R}^1)$ denote the Hilbert space $\{g(\xi)\,|\,|x|^\lambda(\mathcal{F}^{-1}g)(x)\in L^2(\boldsymbol{R}^1)\}$ equipped with the norm $\|g\|_{H_\lambda}=\left(\int_{-\infty}^{+\infty}dx|\,|x|^\lambda(\mathcal{F}^{-1}g)(x)\,|^2\right)^{1/2}$. Note that $H_\lambda(\boldsymbol{R}^1)$ and $H_{-\lambda}(\boldsymbol{R}^1)$ are mutually dual spaces through the bilinear form

$$\begin{aligned}\langle g_1,g_2\rangle&=\int_{-\infty}^{+\infty}\frac{d\xi}{2\pi}g_1(\xi)g_2(\xi)\\&=\int_{-\infty}^{+\infty}dx|x|^\lambda(\mathcal{F}^{-1}g_1)(x)\cdot|x|^{-\lambda}(\mathcal{F}^{-1}g_2)(-x),\\&\qquad g_1\in H_\lambda(\boldsymbol{R}^1),\ g_2\in H_{-\lambda}(\boldsymbol{R}^1).\end{aligned}\tag{2. 3. 8}$$

Lemma 1. For $0<\lambda<1$ we have

$$\|g\|^2_{H_\lambda}=\frac{\sin\pi\lambda}{\pi}\Gamma(1+2\lambda)\int_0^\infty d\sigma\cdot\sigma^{-1-2\lambda}\int_{-\infty}^{+\infty}\frac{d\xi}{2\pi}|g(\xi+\sigma)-g(\xi)|^2.$$

Proof. Set $f(x)=|x|^\lambda(\mathcal{F}^{-1}g)(x)\in L^2(\boldsymbol{R}^1)$. Since $g(\xi+\sigma)-g(\xi)=\mathcal{F}((e^{-ix\sigma}-1)|x|^{-\lambda}f(x))(\xi)$, we have by the Plancherel formula

$$\int_0^\infty d\sigma\,\sigma^{-1-2\lambda}\int_{-\infty}^{+\infty}\frac{d\xi}{2\pi}|g(\xi+\sigma)-g(\xi)|^2$$

$$= \int_0^\infty d\sigma\, \sigma^{-1-2\lambda} \int_{-\infty}^{+\infty} dx |(e^{-ix\sigma}-1)|x|^{-\lambda} f(x)|^2$$

$$= \int_0^\infty \frac{d\sigma}{\sigma} \int_{-\infty}^{+\infty} dx \frac{|e^{-ix\sigma}-1|^2}{|\sigma x|^{2\lambda}} |f(x)|^2$$

$$= \int_0^\infty \frac{d\sigma}{\sigma} \frac{|e^{-i\sigma}-1|^2}{\sigma^{2\lambda}} \cdot \int_{-\infty}^{+\infty} dx |f(x)|^2 .$$

Noting the formula

$$\int_0^\infty \frac{d\sigma}{\sigma} \frac{1-\cos\sigma}{\sigma^{2\lambda}} = \frac{1}{\Gamma(1+2\lambda)} \frac{\pi}{2\sin\pi\lambda}$$

we obtain the lemma.

Lemma 2. For $g(\xi) \in H_\lambda(\boldsymbol{R}^1)$ $(0<\lambda<1/2)$, we have

$$\int_0^\infty \frac{d\xi}{2\pi} \xi^{-2\lambda} |g(\xi)|^2 \leqq 2\left(1+\frac{2}{1-2\lambda}\right) \int_0^\infty d\sigma\, \sigma^{-1-2\lambda} \int_0^\infty \frac{d\xi}{2\pi} |g(\xi+\sigma)-g(\xi)|^2 .$$

This lemma is proved in [18], pp. 58–59.

Proof of Proposition 2. 3. 2. Since multiplication by $|x|^\lambda$ is a unitary operator for $\lambda \in i\boldsymbol{R}$, we may assume that λ is real. In view of (2. 3. 7) it suffices to prove the boundedness of the map $g(\xi) \mapsto \theta(\pm\xi) e^{-\varepsilon|\xi|} g(\xi)$ in the topology of $H_\lambda(\boldsymbol{R}^1)$ for $-1/2<\lambda<1/2$. By the duality $H'_\lambda = H_{-\lambda}$ through (2. 3. 8), we see that it is sufficient to consider the case $0 \leqq \lambda < 1/2$. the case $\lambda=0$ is trivial (indeed the Proposition reduces to Proposition 2. 3. 1 in this case). Assume $0<\lambda<1/2$. From Lemma 1 we have for $\varepsilon \geqq 0$,

$$\|\theta(\pm\xi) e^{-\varepsilon|\xi|} g(\xi)\|_{H_\lambda} = c_1(\lambda) \cdot \left(\int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda} \right. \tag{2. 3. 9}$$

$$\left. \times \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} |\theta(\pm(\xi+\sigma)) e^{-\varepsilon|\xi+\sigma|} g(\xi+\sigma) - \theta(\pm\xi) e^{-\varepsilon|\xi|} g(\xi)|^2 \right)^{1/2}$$

$$\leqq c_1(\lambda) (I_1^{1/2} + I_2^{1/2} + I_3^{1/2}),$$

where $c_1(\lambda) = \left(\frac{\sin\pi\lambda}{\pi} \Gamma(1+2\lambda)\right)^{1/2}$, and

$$I_1 = \int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda} \int_0^\infty \frac{d\xi}{2\pi} e^{-2\varepsilon|\xi+\sigma|} |g(\pm(\xi+\sigma)) - g(\pm\xi)|^2$$

$$I_2 = \int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda} \int_0^\infty \frac{d\xi}{2\pi} |e^{-\varepsilon|\xi+\sigma|} - e^{-\varepsilon|\xi|}|^2 |g(\pm\xi)|^2$$

$$I_3 = \int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda} \int_0^\sigma \frac{d\xi}{2\pi} e^{-2\varepsilon|\xi|} |g(\pm\xi)|^2 .$$

Making use of the inequality

$$\varepsilon^{2\lambda} e^{-2\varepsilon\xi} \leqq \lambda^{2\lambda} e^{-2\lambda} \xi^{-2\lambda} \quad (\xi, \lambda > 0)$$

we have

$$(2.3.10) \quad I_2 = \int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda} |e^{-\varepsilon\sigma} - 1|^2 \int_0^\infty \frac{d\xi}{2\pi} e^{-2\varepsilon\xi} |g(\pm\xi)|^2$$

$$= \frac{\Gamma(1-2\lambda)}{\lambda} (1 - 2^{-1+2\lambda}) \varepsilon^{2\lambda} \int_0^\infty \frac{d\xi}{2\pi} e^{-2\varepsilon\xi} |g(\pm\xi)|^2$$

$$\leqq \Gamma(1-2\lambda) \lambda^{-1+2\lambda} e^{-2\lambda} (1 - 2^{-1+2\lambda}) \int_0^\infty \frac{d\xi}{2\pi} \xi^{-2\lambda} |g(\pm\xi)|^2 .$$

For the third term I_3 we have

$$(2.3.11) \quad I_3 \leqq \int_0^\infty \frac{d\xi}{2\pi} |g(\pm\xi)|^2 \int_\xi^\infty d\sigma \cdot \sigma^{-1-2\lambda}$$

$$= \frac{1}{2\lambda} \int_0^\infty \frac{d\xi}{2\pi} \xi^{-2\lambda} |g(\pm\xi)|^2 .$$

Combining (2. 3. 9) $\sim$ (2. 3. 11) and Lemma 2 we obtain

$$(2.3.12) \quad \|\theta(\pm\xi) e^{-\varepsilon|\xi|} g(\xi)\|_{H_\lambda} \leqq c_1(\lambda) c_2(\lambda) \Big(\int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda}$$

$$\times \int_0^\infty \frac{d\xi}{2\pi} |g(\pm(\xi+\sigma)) - g(\pm\xi)|^2 \Big)^{1/2} \leqq c_2(\lambda) \|g(\xi)\|_{H_\lambda} ,$$

$$c_2(\lambda) = 1 + \sqrt{2\Big(1 + \frac{2}{1-2\lambda}\Big)} \times \Big(\sqrt{\frac{1}{2\lambda}}$$

$$+ e^{-\lambda} \sqrt{\Gamma(1-2\lambda) \cdot \lambda^{-1+2\lambda} (1 - 2^{-1+2\lambda})} \Big).$$

This proves the boundedness of $R^{\pm}_{\lambda,\varepsilon}$. To prove the strong convergence of $R^{\pm}_{\lambda,\varepsilon}(\varepsilon \to 0)$, we note

$$(2.3.13) \quad \|\theta(\pm\xi)(e^{-\varepsilon|\xi|} - 1) g(\xi)\|_{H_\lambda} \leqq c_1(\lambda)$$

$$\times \Big\{ \Big(\int_0^\infty d\sigma \cdot \sigma^{-1-2\lambda} \int_0^\infty \frac{d\xi}{2\pi} |e^{-\varepsilon|\xi+\sigma|} - 1|^2 |g(\pm(\xi+\sigma)) - g(\pm\xi)|^2 \Big)^{1/2}$$

$$+\left(\Gamma(1-2\lambda)\cdot\lambda^{-1}(1-2^{-1+2\lambda})\int_0^\infty \frac{d\xi}{2\pi}\, e^{-2\varepsilon\xi}\varepsilon^{2\lambda}|g(\pm\xi)|^2\right)^{1/2}$$

$$+\left(\frac{1}{2\lambda}\int_0^\infty \frac{d\xi}{2\pi}\,\xi^{-2\lambda}|e^{-\varepsilon|\xi|}-1|^2|g(\pm\xi)|^2\right)^{1/2}\Bigg\}$$

which is derived by a similar argument as in (2. 3. 9). It is easy to see that each of the integrands in the right hand side of (2. 3. 13) is dominated by an integrable function independent of ε. Hence the Lebesgue's theorem is applicable and we have $\lim_{\varepsilon\downarrow 0}\|\theta(\pm\xi)\,(e^{-\varepsilon|\xi|}-1)\,g(\xi)\,\|_{H_\lambda}=0$.

Extension of Proposition 2. 3. 2 to the matrix case reads as follows:

Proposition 2. 3. 3. *Assume that the eigenvalues $\lambda_1, \cdots, \lambda_m$ of the matrix L satisfy $|\mathrm{Re}\,\lambda_j|<1/2\,(j=1, \cdots, m)$. Then $R^{\pm}_{L,\varepsilon}$ is a bounded operator in $L^2(\boldsymbol{R}_-; dx)^m$, and $\lim_{\varepsilon\downarrow 0} R^{\pm}_{L,\varepsilon}$, denoted by $R^{\pm}_L = R^{\pm}_{L,0}$, exists in the strong topology. We have, for $\varepsilon\geqq 0$,*

$$(2.\,3.\,14)\qquad R^{\pm}_{L,\varepsilon}f(x)=\frac{1}{2\pi i}\oint_C \frac{d\lambda}{\lambda-L}\,R^{\pm}_{\lambda,\varepsilon}f(x),\quad f\in L^2(\boldsymbol{R}_-; dx)^m\,.$$

Here the contour C is a simple closed curve in $|\mathrm{Re}\,\lambda|<1/2$ encircling $\lambda_1, \cdots, \lambda_m$ in the positive direction (Fig. 2. 3. 2):

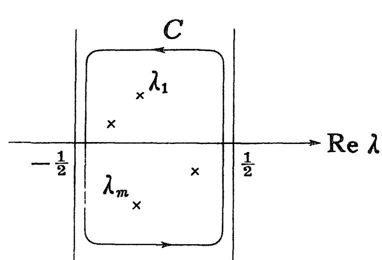

Fig. 2. 3. 2

In particular $R^{\pm}_L$ depends holomorphically on L in a neighborhood of $L=0$, and $\|R^{\pm}_L\|$ is uniformly bounded there.

Proof. From the proof of Proposition 2. 3. 2, we have an estimate

$$\|R^{\pm}_{\lambda,\varepsilon}\|=O(|\mathrm{Re}\,\lambda|^{-1/2})\qquad(|\mathrm{Re}\,\lambda|<1/2)$$

which is valid uniformly for $\varepsilon\geqq 0$. Hence, for each $f\in L^2(\boldsymbol{R}_-;dx)$, $\lambda\mapsto R^{\pm}_{\lambda,\varepsilon}f$ is an $L^2(\boldsymbol{R}_-;dx)$-valued integrable function. Making use of the formula

$$x_-^L x_-'^{-L}=\frac{1}{2\pi i}\oint_C\frac{d\lambda}{\lambda-L}x_-^\lambda x_-'^{-\lambda}$$

we have, for $\varepsilon>0$ and $f\in L^2(\boldsymbol{R}_-;dx)^m$,

$$\begin{aligned}R^{\pm}_{L,\varepsilon}f(x)&=\int\frac{dx'}{2\pi}\frac{1}{2\pi i}\oint_C\frac{d\lambda}{\lambda-L}x_-^\lambda\frac{\pm i}{x-x'\pm i\varepsilon}x_-'^{-\lambda}f(x)\\&=\frac{1}{2\pi i}\oint_C\frac{d\lambda}{\lambda-L}R^{\pm}_{\lambda,\varepsilon}f(x).\end{aligned}$$

Since $\lim_{\varepsilon\downarrow 0}R^{\pm}_{\lambda,\varepsilon}f(x)$ exists in L^2-norm, so does $\lim_{\varepsilon\downarrow 0}R^{\pm}_{L,\varepsilon}f(x)$ and (2. 3. 14) is valid also for $\varepsilon=0$. Holomorphic dependence of $R^{\pm}_{L,\varepsilon}$ on L is obvious from (2. 3. 14). This proves Proposition 2. 3. 3.

Note that from (2. 2. 17) the integral operator R_L corresponding to the kernel $R(x,x';L)$ is expressed as

(2. 3. 15) $\quad R_L=\lim_{\varepsilon\downarrow 0}R_{L,\varepsilon},\ R_{L,\varepsilon}=2i\sin\pi L\,(e^{\pi iL}R^+_{L,\varepsilon}+e^{-\pi iL}R^-_{L,\varepsilon})\,.$

By Proposition 2. 3. 1, the operator A in (2. 3. 4) is bounded in $\mathcal{H}=L^2(\boldsymbol{R}_-)^{mn}$. Moreover under the condition $(2.3.5)_\nu$ $(\nu=1,\cdots,n)$, R is also a bounded operator in $\mathcal{H}$ by Proposition 2. 3. 3. We have

(2. 3. 16) $\quad \|R\|\leqq\max_{1\leqq\nu\leqq n}2\sinh\pi|L_\nu|e^{\pi|L_\nu|}(\|R^+_{L_\nu}\|+\|R^-_{L_\nu}\|),\ \ \|A\|\leqq n-1\,.$

Note that $\|R\|$ is made as small as we please if $|L_\nu|$ $(\nu=1,\cdots,n)$ is chosen small enough. Thus we have the following.

Proposition 2. 3. 4. *In a neighborhood of* $L_\nu=0$ $(\nu=1,\cdots,n)$, $(1-RA)^{-1}$ *exists as a bounded operator in* $\mathcal{H}$, *and coincides with the Neumann series* $\sum_{l=0}^{\infty}(RA)^l$.

Proposition 2. 3. 5. *The series* (2. 3. 1) *for* $Z_{\mu\nu}(x_0;x)$ *converges absolutely and uniformly on any compact subset of* $(\boldsymbol{C}-\Gamma_\mu)\times(\boldsymbol{C}-\Gamma_\nu)$. *Moreover* $Z_{\mu\nu}(x_0;x)$ *is holomorphic with respect to* $a_1,\cdots,a_n$ *on*

$V=\{\operatorname{Im} a_1>\cdots>\operatorname{Im} a_n\}$ *and is continuous on* $\overline{V}$.

Lemma. Set $k_z(x)=\frac{1}{2\pi}\frac{i}{x-z}$, $z\in \boldsymbol{C}-(-\infty,0]$. Then $z\mapsto k_z(x)$ defines an $L^2(\boldsymbol{R}_-;dx)$-valued holomorphic function on $\boldsymbol{C}-(-\infty,0]$, and

$$\|k_z(x)\|=\frac{1}{2\pi\sqrt{r}}\sqrt{\frac{\theta}{\sin\theta}}\quad (z=re^{i\theta};\ r>0,\ |\theta|<\pi). \tag{2.3.17}$$

Proof.
$$\|k_z(x)\|^2=\int_{-\infty}^{0}\frac{1}{(2\pi)^2}\frac{dx}{|x-re^{i\theta}|^2}$$
$$=\frac{1}{(2\pi)^2}\int_{-\infty}^{0}\frac{1}{2ir\sin\theta}\left(\frac{1}{x-re^{i\theta}}-\frac{1}{x-re^{-i\theta}}\right)dx$$
$$=\frac{1}{(2\pi)^2}\frac{\theta}{r\sin\theta}.$$

(The result is valid also for $\theta=0$ by continuity.) Analyticity of $z\longmapsto k_z(x)$ is obvious.

Proof of Proposition 2.3.5. For $f,g\in L^2(\boldsymbol{R}_-;dx)^m$ we denote their inner product by $(f,g)_{L^2}=\int_{-\infty}^{0}dx\,{}^t\overline{f(x)}\,g(x)$. Then the matrix $Z_{\mu\nu}(x_0;x)$ is expressed as $\sum_{l=0}^{\infty}(k_{\bar{x}_0-\bar{a}_\mu},\ [(RA)^lR]_{\mu\nu}k_{x-a_\nu})_{L^2}$. Since $\sum_{l=0}^{\infty}(RA)^lR$ is convergent in the operator norm, we have

$$\sum_{l=0}^{\infty}|(k_{\bar{x}_0-\bar{a}_\mu},\ [(RA)^lR]_{\mu\nu}k_{x-a_\nu})_{L^2}|\leqq\sum_{l=0}^{\infty}\|[(RA)^lR]_{\mu\nu}\|\,\|k_{x_0-a_\mu}\|\,\|k_{x-a_\nu}\|<\infty .$$

This proves the first half of the Proposition. Analyticity and continuity with respect to $a_1,\cdots,a_n$ follows from that of RA in the strong topology.

Corollary. 2.3.6. *For a fixed* $x_0\in\boldsymbol{C}-\Gamma_\mu$ *we have*

$$|Z_{\mu\nu}(x_0;x)|=O\left(\frac{1}{\sqrt{|x-a_\nu|}}\right)\qquad(|x-a_\nu|\to 0) \tag{2.3.18}$$
$$=O\left(\frac{1}{\sqrt{|x|}}\right)\qquad(|x|\to\infty)$$

uniformly in any subsector $|\arg(x-a_\nu)|\leqq\pi-\varepsilon$ $(0<\varepsilon\ll 1)$ *of* $\boldsymbol{C}-\Gamma_\nu$.

Proposition 2. 3. 7. $Z_{\mu\nu}(x_0; x)$ is analytically prolongable with respect to x_0 (resp. x) across the cut Γ_μ (resp. Γ_ν).

Proof. Assume $x_0 \in \Gamma_\mu - \{a_\mu\}$. The analytic continuation of $Z_{\mu\nu}(x_0; x)$ is then obtained by deforming the overlapping paths Γ_{σ_i} (i.e. those such that $\mathrm{Im}\, a_{\sigma_i} = \mathrm{Im}\, a_\mu$) $(i = 1, \cdots, k)$ as shown in Fig. 2. 3. 3:

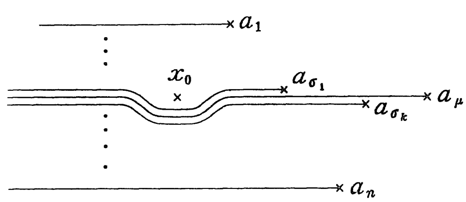

Fig. 2. 3. 3

In order to justify this procedure we must show that the convergence of $Z_{\mu\nu}$ is not affected by a slight change of the paths of integration. In other words we are to prove that R_{L_ν} and $A_{\mu\nu}$ remain bounded, and the increase in their norm is chosen as small as we please under sufficiently small modification of the path. Since the argument is essentially the same, we consider the case of R_{L_ν}.

Proposition 2. 3. 8. *Let $x(s)$ $(-\infty < s \leqq 0)$ be a C^2-curve in the x-plane satisfying*

(*i*) $x(0) = 0$

(*ii*) *there exists an $s_0 < 0$ such that $x(s) = s$ for $s \leqq s_0$*

(*iii*) *for some $c > 0$, $|x(s) - x(s')| \geqq c|s - s'|$ $(-\infty < s, s' \leqq 0)$.*

Then under the same condition on L as in Proposition 2. 3. 3,

$$\tilde{R}^{\pm}_{L,\varepsilon}: f(s) \mapsto \int_{-\infty}^{0} \dot{x}(s') \frac{ds'}{2\pi} (-x(s))^{L} \frac{\pm i}{x(s) - x(s') \pm i\varepsilon} \times (-x(s'))^{-L} f(s')$$

is a bounded operator in $L^2(\boldsymbol{R}_-; ds)^m$, and $\lim_{\varepsilon \downarrow 0} \tilde{R}^{\pm}_{L,\varepsilon}$ exists in the strong topology. Here the dot indicates differentiation with respect to s.

Proof. In view of (2. 3. 13) we may assume that L is a complex number $\lambda \in \boldsymbol{C}$. It suffices to show that the difference of two kernels

$$\chi(x, x'; \varepsilon) = (-s)^{\lambda}\frac{\pm i}{s-s'\pm i\varepsilon}(-s')^{-\lambda}$$

$$-(-x(s))^{\lambda}\frac{\pm i}{x(s)-x(s')\pm i\varepsilon}(-x(s'))^{-\lambda}\dot{x}(s')$$

belongs to $L^2(\boldsymbol{R}_-\times\boldsymbol{R}_-; dsds')$, and that $\lim_{\varepsilon\downarrow 0}\chi(s, s'; \varepsilon)$ exists in the L^2-norm.

Rewrite $\chi(s, s'; \varepsilon)$ as

$$\chi(s, s'; \varepsilon) = (\pm i)\frac{s-s'\pm i\varepsilon}{x(s)-x(s')\pm i\varepsilon}\left\{\frac{(s-s')^2}{(s-s'\pm i\varepsilon)^2}\chi_1(s, s') + \chi_2(s, s'; \varepsilon)\right.$$

$$\left. + \dot{x}(s')\frac{s-s'}{s-s'\pm i\varepsilon}\chi_3(s, s')\right\},$$

where

$$\chi_1(s, s') = \frac{(-s)^{\lambda}(-s')^{-\lambda}}{(s-s')^2}(x(s)-x(s')-(s-s')\dot{x}(s'))$$

$$\chi_2(s, s'; \varepsilon) = \frac{\pm i\varepsilon}{(s-s'\pm i\varepsilon)^2}(-s)^{\lambda}(-s')^{-\lambda}(1-\dot{x}(s'))$$

$$\chi_3(s, s') = \frac{(-s)^{\lambda}(-s')^{-\lambda}}{s-s'}\left(1-\left(\frac{-x(s)}{-s}\right)^{\lambda}\left(\frac{-x(s')}{-s'}\right)^{-\lambda}\right).$$

Since the coefficients of χ_1, χ_2 and χ_3 are bounded functions by (iii), it is sufficient to prove that $\chi_1, \chi_2, \chi_3 \in L^2(\boldsymbol{R}_-\times\boldsymbol{R}_-; dsds')$, and that $\lim_{\varepsilon\downarrow 0}\chi_2(s, s'; \varepsilon) = 0$ in the L^2-norm.

Notice that $\chi_1(s, s')$, $\chi_3(s, s') = 0$ for $s, s' < s_0$ by (ii). We note also that $\chi_i(s, s') = (-s)^{\lambda}(-s')^{-\lambda}\times$ (continuous function) for $i=1, 3$. For $i=3$ this is seen by noting $|x(s)/s| \geqq c > 0$ so that $(-x(s)/-s)^{\pm\lambda}$ is in class C^2. Hence for $i=1$ or 3

$$\iint_{-\infty}^{0} dsds'|\chi_i(s, s')|^2 = \left(\iint_{2s_0}^{0} dsds' + \int_{-\infty}^{2s_0} ds \int_{s_0}^{0} ds'\right.$$

$$\left. + \int_{s_0}^{0} ds \int_{-\infty}^{2s_0} ds'\right)|\chi_i(s, s')|^2$$

$$= J_1^{(i)} + J_2^{(i)} + J_3^{(i)}.$$

Since $\mu = |\mathrm{Re}\,\lambda| < 1/2$, $J_1^{(i)}$ is finite. As for $J_2^{(i)}$, we have

$$J_2^{(i)}=\int_{-\infty}^{2s_0}ds\int_{s_0}^{0}ds'|\chi_1(s,s')|^2$$

$$=\int_{s_0}^{0}ds'|s'|^{-2\mu}\int_{-\infty}^{2s_0}ds\frac{|s|^{2\mu}}{|s-s'|^4}|-x(s')+s'\dot{x}(s')+s(1-\dot{x}(s'))|^2$$

$$\leqq 2\left\{\int_{s_0}^{0}ds'|s'|^{-2\mu}|x(s')-s'\dot{x}(s')|^2\cdot\int_{-\infty}^{2s_0}ds\frac{|s|^{2\mu}}{|s-s_0|^4}\right.$$

$$\left.+\int_{s_0}^{0}ds'|s'|^{-2\mu}|1-\dot{x}(s')|^2\cdot\int_{-\infty}^{2s_0}ds\frac{|s|^{2(\mu+1)}}{|s-s_0|^4}\right\}$$

$$<\infty.$$

Similarly

$$J_2^{(3)}=\int_{-\infty}^{2s_0}ds\int_{s_0}^{0}ds'|\chi_3(s,s')|^2\leqq\int_{s_0}^{0}ds'|(-s')^{-\lambda}-(-x(s'))^{-\lambda}|^2$$

$$\times\int_{-\infty}^{2s_0}ds\frac{|s|^{2\mu}}{|s-s_0|^2}$$

$$<\infty.$$

The term $J_3^{(i)}$ is shown to be finite by a similar calculation. Next consider χ_2. We have

$$\iint_{-\infty}^{0}dsds'|\chi_2(s,s';\varepsilon)|^2$$

$$=\int_{s_0}^{0}ds'|s'|^{-2\mu}|1-\dot{x}(s')|^2\int_{-\infty}^{0}ds\frac{\varepsilon^2}{|s-s'\pm i\varepsilon|^2}|s|^{2\mu}$$

$$=\int_{s_0}^{0}ds'|s'|^{-2\mu}|1-\dot{x}(s')|^2\frac{\pi\varepsilon}{2i\sin 2\pi\mu}((s'+i\varepsilon)^{2\mu}-(s'-i\varepsilon)^{2\mu})$$

$$=O(\varepsilon)\qquad\qquad(\varepsilon\to 0).$$

This completes the proof of Proposition 2.3.8.

Remark 1. In the case $\mathrm{Im}\,a_1>\cdots>\mathrm{Im}\,a_n$, another way of distortion of the contour is to replace Γ'_νs simultaneously by parallel lines as shown in Fig. 2.3.4. From this we see that the estimate (2.3.18) is valid in a sector $-\pi-\varepsilon\leqq\arg(x-a_\nu)\leqq\pi+\varepsilon$ whose central angle exceeds 2π.

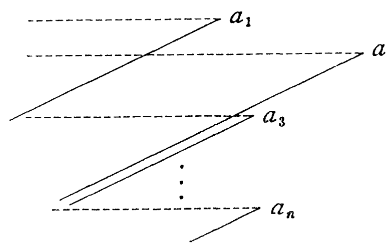

Fig. 2. 3. 4

Remark 2. By Proposition 2. 3. 8, it is possible to deform the contour as in Fig. 2. 3. 5:

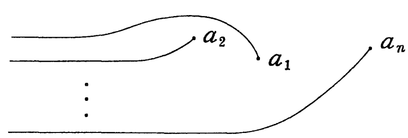

Fig. 2. 3. 5

As far as the series is convergent, this gives an analytic continuation of $Z_{\mu\nu}(x_0; x)$ with respect to $a_1, \cdots, a_n$ outside the domain $V=\{\mathrm{Im}\, a_1 > \cdots > \mathrm{Im}\, a_n\}$.

Now we shall assume $\mathrm{Im}\, a_1 > \cdots > \mathrm{Im}\, a_n$ for the moment and study the monodromic property of the matrix $Y(x_0; x)$.

Theorem 2. 3. 9. *In a neighborhood of $x=a_\nu$, we have*

(2. 3. 19) $$Y(x_0; x) = \Phi_\nu(x_0; x) \cdot (x-a_\nu)^{-L_\nu}$$

where $\Phi_\nu(x_0; x)$ is a holomorphic matrix at $x=a_\nu$ given by

(2. 3. 20) $$\Phi_\nu(x_0; x) = (x_0-a_\nu)^{L_\nu} + 2\pi i (x_0-x) \sum_{\mu=1}^{n} \sum_{\sigma(\neq\nu)} \int_{C_{\nu,x}} dx_1 Z_{\mu\sigma}(x_0; x_1) \cdot (x_1-a_\nu)^{L_\nu} \frac{1}{2\pi} \frac{i}{x_1-x}.$$

The contour $C_{\nu,x}$ is shown in Fig. 2. 3. 6.

Similarly at $x_0=a_\mu$ we have

(2. 3. 21) $$Y(x_0; x) = (x_0-a_\mu)^{L_\mu} \Phi^*_\mu(x_0; x)$$

(2. 3. 22) $$\Phi^*_\mu(x_0;x) = (x-a_\mu)^{-L_\mu} - 2\pi i(x_0-x)\sum_{\rho(\neq\mu)}\sum_{\nu=1}^{n}\int_{C_{\nu},x_0} dx_1 \frac{1}{2\pi}\frac{i}{x_0-x_1} \cdot (x_1-a_\mu)^{-L_\mu} Z_{\rho\nu}(x_1;x).$$

At $x_0 = a_\mu$ *and* $x = a_\nu$ *we have*

(2. 3. 23) $$Y(x_0;x) = (x_0-a_\mu)^{L_\mu}\cdot\Phi_{\mu\nu}(x_0;x)\cdot(x-a_\nu)^{-L_\nu}$$

(2. 3. 24)

$$\Phi_{\mu\nu}(x_0;x) = \delta_{\mu\nu}\cdot 1 + 2\pi i(x_0-x)\int_{C_{\mu},x_0} dx_1\int_{C_{\nu},x} dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1}\cdot(x_1-a_\mu)^{-L_\mu}\cdot\left(\frac{1}{2\pi}\frac{i}{x_1-x_2}(1-\delta_{\mu\nu}) + \sum_{\rho(\neq\mu)}\sum_{\sigma(\neq\nu)} Z_{\rho\sigma}(x_1;x_2)\right)\cdot (x_2-a_\nu)^{L_\nu}\frac{1}{2\pi}\frac{i}{x_2-x}.$$

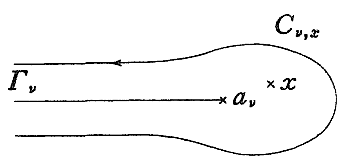

Fig. 2. 3. 6

Lemma (cf. (2. 2. 10)).

(2. 3. 25) $$\int dx_2 R(x_1,x_2;L_\nu)\frac{1}{2\pi}\frac{i}{x_2-(x-a_\nu)} = ((x_1+i0)^{L_\nu}-(x_1-i0)^{L_\nu})\frac{1}{2\pi}\frac{i}{x_1-(x-a_\nu)}(x-a_\nu)^{-L_\nu}$$

$$\int dx_1\frac{1}{2\pi}\frac{i}{(x_0-a_\mu)-x_1}R(x_1,x_2;L_\mu) = (x_0-a_\mu)^{L_\mu}\frac{1}{2\pi}\frac{i}{(x_0-a_\mu)-x_2}((x_2-i0)^{-L_\mu}-(x_2+i0)^{-L_\mu}).$$

Proof. Straightforward.

Proof of Theorem 2. 3. 9. First we note the relation $(1-RA)^{-1}R = R + (1-RA)^{-1}RAR$. This implies that $Z_{\mu\nu}(x_0;x) = (k_{\bar{x}_0-\bar{a}_\mu}, (\delta_{\mu\nu}R_{L_\nu} + \sum_{\sigma(\neq\nu)}(1-RA)^{-1}_{\mu\sigma}R_{L_\sigma}A_{\sigma\nu}R_{L_\nu})k_{x-a_\nu})$, i.e.

$$Z_{\mu\nu}(x_0; x) = \iint dx_1 dx_2 \Big\{\frac{1}{2\pi}\frac{i}{(x_0-a_\mu)-x_1}\delta_{\mu\nu}$$

$$+\sum_{\sigma(\neq\nu)} Z_{\mu\sigma}(x_0; x_1+a_\nu)\Big\} R(x_1, x_2; L_\nu)\frac{1}{2\pi}\frac{i}{x_2-(x-a_\nu)}$$

$$= \int_{\Gamma_\nu} dx_1 \Big\{\frac{1}{2\pi}\frac{i}{(x_0-a_\mu)-(x_1-a_\nu)}\delta_{\mu\nu}$$

$$+\sum_{\sigma(\neq\nu)} Z_{\mu\sigma}(x_0; x_1)\Big\} \left((x_1-a_\nu+i0)^{L_\nu}-(x_1-a_\nu-i0)^{L_\nu}\right)$$

$$\times\frac{1}{2\pi}\frac{i}{x_1-x}(x-a_\nu)^{-L_\nu}$$

where we have used (2. 3. 25). Since $Z_{\mu\sigma}(x_0; x)$ $(\sigma\neq\nu)$ is holomorphic with respect to x in a neighborhood of Γ_ν, we may deform the path of integration into a contour (Fig. 2. 3. 7) and obtain

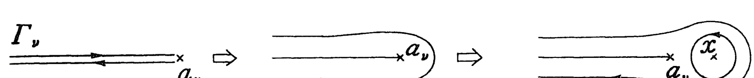

Fig. 2. 3. 7

(2. 3. 26)

$$Z_{\mu\nu}(x_0; x) = -\int_{C_{\nu,x}} dx_1 \Big(\frac{1}{2\pi}\frac{i}{x_0-x_1}\delta_{\mu\nu}+\sum_{\sigma(\neq\nu)} Z_{\mu\sigma}(x_0; x_1)\Big)\cdot(x_1-a_\nu)^{L_\nu}$$

$$\times\frac{1}{2\pi}\frac{i}{x_1-x}(x-a_\nu)^{-L_\nu}$$

$$+2\pi i\Big(\frac{1}{2\pi}\frac{i}{x_0-x}\delta_{\mu\nu}+\sum_{\sigma(\neq\nu)} Z_{\mu\sigma}(x_0; x)\Big)\cdot(x-a_\nu)^{L_\nu}\frac{i}{2\pi}(x-a_\nu)^{-L_\nu}$$

$$=\Big(\delta_{\mu\nu}\frac{1}{2\pi}\frac{i}{x_0-x}(x_0-a_\nu)^{L_\nu}-\sum_{\sigma(\neq\nu)}\int_{C_{\nu,x}} dx_1 Z_{\mu\sigma}(x_0; x_1)$$

$$\cdot(x_1-a_\nu)^{L_\nu}\frac{1}{2\pi}\frac{i}{x_1-x}\Big)(x-a_\nu)^{-L_\nu}$$

$$-\delta_{\mu\nu}\frac{1}{2\pi}\frac{i}{x_0-x}-\sum_{\sigma(\neq\nu)} Z_{\mu\sigma}(x_0; x).$$

Hence by (2. 3. 3) $Y(x_0; x)$ is expressed as

$$Y(x_0; x) = 1-2\pi i(x_0-x)\sum_{\mu=1}^{n}(Z_{\mu\nu}(x_0; x)+\sum_{\sigma(\neq\nu)} Z_{\mu\sigma}(x_0; x))$$

$$=\Phi_\nu(x_0; x)\cdot(x-a_\nu)^{-L_\nu}$$

where $\Phi_\nu(x_0; x)$ is given by (2. 3. 20). Formulas (2. 3. 21) - (2. 3. 22) are proved similarly. To prove (2. 3. 23) - (2. 3. 24) we start with (2. 3. 22). From (2. 3. 22) we have

$$\Phi_\mu^*(x_0; x) = (x-a_\mu)^{-L_\nu} - 2\pi i(x_0 - x)\sum_{\rho(\neq\mu)}\int_{C_{\mu,x_0}} dx_1 \frac{1}{2\pi}\frac{i}{x_0-x_1}(x_1-a_\mu)^{-L_\mu}$$
$$\times\sum_{\sigma=1}^{n} Z_{\rho\sigma}(x_1; x)$$
$$= (x-a_\mu)^{-L_\mu} + 2\pi i(x_0-x)\sum_{\rho(\neq\mu)}\delta_{\rho\nu}\int_{C_{\mu,x_0}} dx_1\frac{1}{2\pi}\frac{i}{x_0-x_1}(x_1-a_\mu)^{-L_\mu}$$
$$\times\frac{1}{2\pi}\frac{i}{x_1-x} + 2\pi i(x_0-x)\int_{C_{\mu,x_0}} dx_1\int_{C_{\nu,x}} dx_2\frac{1}{2\pi}\frac{i}{x_0-x_1}(x_1-a_\mu)^{-L_\mu}$$
$$\times\left(\frac{1}{2\pi}\frac{i}{x_1-x_2}(1-\delta_{\mu\nu}) + \sum_{\rho(\neq\mu)}\sum_{\sigma(\neq\nu)} Z_{\rho\sigma}(x_1; x_2)\right)\cdot(x_2-a_\nu)^{L_\nu}$$
$$\times\frac{1}{2\pi}\frac{i}{x_2-x}(x-a_\nu)^{-L_\nu}$$
$$= (x-a_\mu)^{-L_\mu} - (1-\delta_{\mu\nu})(x-a_\mu)^{-L_\mu} + \Phi_{\mu\nu}(x_0; x)\cdot(x-a_\nu)^{-L_\nu},$$

where $\Phi_{\mu\nu}(x_0; x)$ is given by (2. 3. 24). This completes the proof of Theorem 2. 3. 9.

Theorem 2. 3. 9 shows that the branch points are regular singularities for $Y(x_0; x)$, and that the latter has the monodromic property (2. 2. 1)-(ii) there. When prolonged around $x=\infty$ it satisfies

$$\gamma_\infty Y(x_0; x) = Y(x_0; x) M_\infty$$

where $M_\infty = (M_1 \cdots M_n)^{-1}$ and γ_∞ denotes a closed path encircling ∞ clockwise (see p. 17). If each $|L_\nu|$ ($\nu = 1, \cdots, n$) is sufficiently small, M_∞ is arbitrarily close to the unit matrix. We set

$$L_\infty = \frac{1}{2\pi i}\log M_\infty = \frac{1}{2\pi i}\sum_{l=1}^{\infty}\frac{(-)^{l-1}}{l}(M_\infty - 1)^l. \tag{2. 3. 27}$$

From (2. 3. 18) we have an estimate

$$|Y(x_0; x)| = 0(\sqrt{|x|}) \quad (|x|\to\infty) \tag{2. 3. 28}$$

which is valid in any finite sector $\theta_0 \leqq \arg x \leqq \theta_1$ thanks to the monodromy property (see Remark 1 below Proposition 2. 3. 8). Thus $x=\infty$ also is a regular singularity of $Y(x_0; x)$.

Theorem 2. 3. 10. *For sufficiently small* $|L_\nu|$ $(\nu=1, \cdots, n)$, *we have*

(2. 3. 29) $$Y(x_0; x) = \Phi_\infty(x_0; x) \cdot x^{L_\infty}$$

where $\Phi_\infty(x_0; x)$ *denotes an invertible holomorphic matrix at* $x=\infty$, *and* L_∞ *is given by* (2. 3. 27). *Moreover for* $\nu=1, \cdots, n$ $\Phi_\nu(x_0; x)$ *defined in* (2. 3. 20) *is invertible at* $x=a_\nu$.

Lemma. Let $M \in GL(m, \boldsymbol{C})$ satisfy $|M-1|<1$, and set $L=\frac{1}{2\pi i}\cdot \log M=\sum_{l=1}^{\infty}\frac{(-)^{l-1}}{l}(M-1)^l$. If $L' \neq L$ satisfies $e^{2\pi i L'}=M$, there exists an eigenvalue λ of L' such that $|\mathrm{Re}\,\lambda| \geqq \frac{3}{4}$.

Proof. First assume that the eigenvalues $\mu_1, \cdots, \mu_m$ of M are mutually distinct. Take a $P \in GL(m, \boldsymbol{C})$ so that $P^{-1}MP$ is diagonal. Since $P^{-1}LP$ and $P^{-1}L'P$ both commute with $P^{-1}MP$, they also must be diagonal. On the other hand the eigenvalues of L' have the form $\lambda_j=n_j+\frac{1}{2\pi i}\log \mu_j$, $n_j \in \boldsymbol{Z}$. Now if $L \neq L'$ we have $n_j \neq 0$ for some j, and $|\mathrm{Re}\,\lambda_j - n_j| = |\frac{1}{2\pi}\mathrm{Arg}\,\mu_j| < \frac{1}{4}$ since $|\mu_j-1|<1$. This implies that $|\mathrm{Re}\,\lambda_j| \geqq \frac{3}{4}$.

In the general case, there exists a sequence $L^{(k)} \to L'$ $(k\to\infty)$ such that $M^{(k)}=e^{2\pi i L^{(k)}}$ has distinct eigenvalues, $|M^{(k)}-1|<1$ and $L^{(k)} \neq \frac{1}{2\pi i}\cdot \log M^{(k)}$. Let $\lambda^{(k)}$ be an eigenvalue of $L^{(k)}$ satisfying $|\mathrm{Re}\,\lambda^{(k)}| \geqq \frac{3}{4}$. Then an accumulation point λ of $\{\lambda^{(k)}\}$ is the desired eigenvalue of L'.

Proof of Theorem 2. 3. 10. Set $y(x)=\det Y(x_0; x)\cdot \prod_{\nu=1}^{n}\left(\frac{x-a_\nu}{x_0-a_\nu}\right)^{\mathrm{trace}\,L_\nu}$, where the branch is so chosen that $y(x_0)=1$. Theorem 2. 3. 9 implies that $y(x)$ is single-valued and holomorphic everywhere in the finite x-plane. In view of (2. 3. 28), $y(x)$ must be a polynomial. Since $x=\infty$ is a regular singularity, $Y(x_0; x)$ is written in the form

(2. 3. 30) $$Y(x_0; x) = \Phi''_\infty(x_0; x) \cdot x^{L''_\infty},$$

where $\Phi''_\infty(x_0;x)$ is a holomorphic matrix at $x=\infty$ and $e^{2\pi i L''_\infty}=M_\infty$. We have then $y(x)=x^{\mathrm{trace}(L''_\infty+\sum_{\nu=1}^{n}L_\nu)}\times$ (holomorphic function at $x=\infty$), so that $\mathrm{trace}(L''_\infty+\sum_{\nu=1}^{n}L_\nu)$ is a non-negative integer. Among the possible choice of Φ''_∞ and L''_∞ satisfying (2.3.30), let Φ'_∞, L'_∞ be such that $\mathrm{trace}(L''_\infty+\sum_{\nu=1}^{n}L_\nu)$ attains its minimum. We insist that $L'_\infty=L_\infty$.

Choose $P\in GL(m,\boldsymbol{C})$ so that $P^{-1}L'_\infty P=J=J_1\oplus\cdots\oplus J_s$ is the Jordan's canonical form, where J_r is the $m_r\times m_r$ matrix $\begin{pmatrix}\lambda_r^* & \ddots & \\ & \ddots & \lambda_r^*\end{pmatrix}$ $(*=0$ or $1;\ m_1+\cdots+m_s=m)$ and $\lambda_r\ (r=1,\cdots,s)$ denote the distinct eigenvalues of L'_∞ satisfying $\mathrm{Re}\,\lambda_1\geqq\cdots\geqq\mathrm{Re}\,\lambda_s$. Let $\phi_j(x)$ denote the j-th column vector of $\Phi'_\infty P$. From the estimate $|\Phi'_\infty(x_0;x)Px^J|=|Y(x_0;x)P|=0(\sqrt{|x|})$, we have

$$|(\phi_1(x),\cdots,\phi_m(x))x^J|=0(\sqrt{|x|}). \tag{2.3.31}$$

Assume $L'_\infty\neq L_\infty$. From the lemma there exists an eigenvalue λ_r of L'_∞ such that $|\mathrm{Re}\,\lambda_r|\geqq\frac{3}{4}$. Since $\sum_{r=1}^{s}m_r\lambda_r+\sum_{\nu=1}^{n}\mathrm{trace}L_\nu\geqq 0$ and $|L_\nu|$ is sufficiently small $(\nu=1,\cdots,n)$, we have $\mathrm{Re}\,\lambda_1\geqq\frac{3}{4}$.

On the other hand we have from (2.3.31) $|\phi_1(x)x^{\lambda_1}|=0(\sqrt{|x|})$, which implies $\phi_1(\infty)=0$. From the estimate for the second column $|\phi_1(x)\cdot x^{\lambda_1}\log x+\phi_2(x)\cdot x^{\lambda_1}|=0(\sqrt{|x|})$ we then conclude $\phi_2(\infty)=0$. Continuing this process we find that the first m_1-column of $\Phi'_\infty(x_0;x)P$ is divisible by x^{-1}. Therefore

$$\begin{aligned}Y(x_0;x)&=\Phi''_\infty(x_0;x)P\cdot x^{(-1_{m_1})\oplus 0}x^J P^{-1}\\ &=\Phi''_\infty(x_0;x)\cdot x^{L''_\infty}\end{aligned}$$

where Φ''_∞ is holomorphic at $x=\infty$ and $L''_\infty=L'_\infty+P\begin{pmatrix}-1_{m_1} & \\ & 0\end{pmatrix}P^{-1}$. This contradicts to the choice of L'_∞.

Since $|L_\infty|$ and $|L_\nu|$'s are sufficiently small, it follows from the relation $e^{2\pi iL_\infty}e^{2\pi iL_1}\cdots e^{2\pi iL_n}=1$ that trace $(L_\infty+\sum_{\nu=1}^{n}L_\nu)=0$. Hence $y(x)$ reduces to a constant $y(x_0)=1$, and in particular $\det\Phi_\infty(x_0;\infty)\neq 0$. From (2.3.19) we have $1=y(x)=\det\Phi_\nu(x_0;x)\cdot\prod_{\mu(\neq\nu)}(x-a_\mu)^{\mathrm{trace}\,L_\mu}\prod_{\mu=1}^{n}(x_0-a_\mu)^{-\mathrm{trace}\,L_\mu}$. This implies $\det\Phi_\nu(x_0;a_\nu)\neq 0$.

Corollary 2.3.11. $\det Y(x_0;x)=\prod_{\nu=1}^{n}\left(\frac{x-a_\nu}{x_0-a_\nu}\right)^{-\mathrm{trace}\,L_\nu}$.

In general, let $a_1, \cdots, a_n$ and x_0 be distinct points of $\boldsymbol{P}_{\boldsymbol{C}}^1$, and let $L_1, \cdots, L_n$ be $m \times m$ matrices subject to the condition

$$e^{2\pi i L_1} \cdots e^{2\pi i L_n} = 1 . \tag{2.3.32}$$

Consider the following precise version of the Riemann problem: find a matrix $Y(x)$ with the properties

(2. 3. 33)

$$\begin{cases} (0) \quad Y(x) \text{ is a multi-valued analytic matrix on } P_C^1 - \{a_1, \cdots, a_n\}, \\ (1)^{(*)} \quad Y(x) = \Phi_\nu(x) \cdot (x - a_\nu)^{-L_\nu} \text{ at } x = a_\nu \ (\nu = 1, \cdots, n), \text{ where} \\ \quad \Phi_\nu(x) \text{ denotes an invertible holomorphic matrix at } x = a_\nu, \\ (2) \quad \det Y(x) \neq 0 \text{ for } x \neq a_1, \cdots, a_n, \\ (3) \quad Y(x_0) = 1. \end{cases}$$

Here if $a_\nu = \infty$ for some ν, $x - a_\nu$ is to be replaced by $1/x$ in (1). Such a matrix does not exist in general, but if it does it is uniquely determined. For if $Y_1(x)$, $Y_2(x)$ both satisfy the properties (0)～(3), one verifies easily that their ratio $C = Y_1(x) Y_2(x)^{-1}$ is single valued and holomorphic on $\boldsymbol{P}_{\boldsymbol{C}}^1$. Hence it reduces to a constant $C = Y_1(x_0) Y_2(x_0)^{-1} = 1$ by (3). To make explicit the dependence on parameters we denote this matrix by $Y\left(x_0; x; \begin{matrix} a_1 \cdots a_n \\ L_1 \cdots L_n \end{matrix}\right)$. Theorems 2. 3. 9 and 2. 3. 10 show the existence of $Y\left(x_0; x; \begin{matrix} a_1 \cdots a_n & \infty \\ L_1 \cdots L_n & L_\infty \end{matrix}\right)$ for sufficiently small $|L_\nu|$ $(\nu = 1, \cdots, n)$. This result is also proved by Lappo-Danilevski [12] by a quite different method. We emphasize the point that in our solution (2. 2. 27), (2. 3. 3) the dependence on the exponent matrices $L_1, \cdots, L_n$ is more explicit and manageble than in the expression given by Lappo-Danilevski.

We now consider some elementary properties of the matrix $Y(y; x) = Y\left(y; x; \begin{matrix} a_1 \cdots a_n \\ L_1 \cdots L_n \end{matrix}\right)^{(**)}$. In what follows we choose a projective coordinate so that

$$a_\nu \neq \infty \ (\nu = 1, \cdots, n). \tag{2.3.34}$$

Observe first that it is invariant under projective transformations in the following sense:

(*) In order to specify the branch we choose as branch cuts mutually non-intersecting smooth curves joining a_ν and, say, a_n.

(**) Hereafter we denote x_0 by y so as to regard it as a variable.

(2. 3. 35) $$Y\left(h(y);h(x);\begin{matrix}h(a_1)\\L_1\end{matrix}\cdots\begin{matrix}h(a_n)\\L_n\end{matrix}\right)=Y\left(y;x;\begin{matrix}a_1\\L_1\end{matrix}\cdots\begin{matrix}a_n\\L_n\end{matrix}\right)$$

where

$$h(x)=\frac{\alpha x+\beta}{\gamma x+\delta},\quad h=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in SL(2,\mathbf{C}).$$

Formula (2. 3. 35) is seen by noting that the left hand side has the same characteristic properties (0) ~ (3) of (2. 3. 33). Also we note that for any $Y_1(x)$ satisfying (0) ~ (2) of (2. 3. 33) we have

(2. 3. 36) $$Y(y;x)=Y_1(y)^{-1}Y_1(x).$$

In particular the y-dependence of $Y(y;x)$ is known from (2. 3. 36); namely at $y=a_\mu$ it behaves like

(2. 3. 37) $$Y(y;x)=(y-a_\mu)^{L_\mu}\Phi_\mu^*(y;x)$$

where $\Phi_\mu^*(y;x)$ is a holomorphic invertible matrix at $y=a_\mu$.

Proposition 2. 3. 12. *Under the condition* (2. 3. 34)

$$Y=Y\left(y;x;\begin{matrix}a_1\\L\end{matrix}\cdots\begin{matrix}a_n\\L_n\end{matrix}\right)$$

satisfies the following linear total differential equations

(2. 3. 38) $$dY=\Omega Y,$$

(2. 3. 39) $$\Omega=\sum_{\nu=1}^{n}A_\nu d\log\frac{x-a_\nu}{y-a_\nu}$$
$$=\sum_{\nu=1}^{n}A_\nu\left(\frac{d(x-a_\nu)}{x-a_\nu}-\frac{d(y-a_\nu)}{y-a_\nu}\right).$$

Here

(2. 3. 40) $$A_\nu=A_\nu\left(y;\begin{matrix}a_1\\L_1\end{matrix}\cdots\begin{matrix}a_n\\L_n\end{matrix}\right)=-\Phi_\nu(a_\nu)L_\nu\Phi_\nu(a_\nu)^{-1}\quad(\nu=1,\cdots,n)$$

denote matrices independent of x satisfying

(2. 3. 41) $$\sum_{\nu=1}^{n}A_\nu=0.$$

In particular, as a function of x, Y satisfies the Fuchsian system of ordinary differential equations

(2. 3. 42) $$\frac{dY}{dx}=\sum_{\nu=1}^{n}\frac{A_\nu}{x-a_\nu}\cdot Y .$$

The coefficients A_ν, *regarded as functions of* y *and* $a=(a_1, \cdots, a_n)$, *satisfy the Schlesinger's equations*

(2. 3. 43) $$dA_\mu=-\sum_{\nu(\neq\mu)}[A_\mu, A_\nu]d\log\frac{a_\mu-a_\nu}{y-a_\nu} \quad (\mu=1, \cdots, n).$$

Proof. Denote the 1-form $dY\cdot Y^{-1}$ by Ω. Clearly Ω is homorphic in x outside $x=a_1, \cdots, a_n$, and from (2. 3. 33) -(1) it is written there as

$$\Omega=dY\cdot Y^{-1}=d\Phi_\nu\cdot\Phi_\nu^{-1}-\Phi_\nu L_\nu\frac{d(x-a_\nu)}{x-a_\nu}\Phi_\nu^{-1} \quad (\nu=1, \cdots, n).$$

This implies that Ω is of the form

$$\Omega=\sum_{\nu=1}^{n}A_\nu\frac{d(x-a_\nu)}{x-a_\nu}+\Omega'$$

$$A_\nu=\operatorname*{Res}_{x=a_\nu}(dY\cdot Y^{-1})=-\Phi_\nu(a_\nu)L_\nu\Phi_\nu(a_\nu)^{-1}.$$

Here Ω' is a matrix of 1-forms in y and a. The relation (2. 3. 41) follows from the residue theorem. Since $Y|_{x=y}=1$, the pullback of Ω to the submanifold $x=y$ vanishes identically, and we have

$$0=\sum_{\nu=1}^{n}A_\nu\frac{d(y-a_\nu)}{y-a_\nu}+\Omega' .$$

This proves (2. 3. 39). Differentiation of (2. 3. 38) yields

(2. 3. 44) $$0=d(dY)=d\Omega\cdot Y-\Omega\wedge dY=(d\Omega-\Omega\wedge\Omega)Y.$$

On the other hand, we have

$$d\Omega=\sum_{\nu=1}^{n}dA_\nu\wedge d\log\frac{x-a_\nu}{y-a_\nu}$$

$$\Omega\wedge\Omega=\sum_{\mu,\nu=1}^{n}A_\mu A_\nu d\log\frac{x-a_\mu}{y-a_\mu}\wedge d\log\frac{x-a_\nu}{y-a_\nu}.$$

Noting

$$d\log(x-a_\mu)\wedge d\log(x-a_\nu)$$

$$=\left(\frac{1}{x-a_\mu}-\frac{1}{x-a_\nu}\right)\frac{1}{a_\mu-a_\nu}d(x-a_\mu)\wedge d(x-a_\nu)$$

$$=\frac{d(x-a_\mu)}{x-a_\mu}\wedge\frac{d((x-a_\nu)-(x-a_\mu))}{a_\mu-a_\nu}$$

$$-\frac{d((x-a_\mu)-(x-a_\nu))}{a_\mu-a_\nu}\wedge\frac{d(x-a_\nu)}{x-a_\nu}$$

$$=d\log(a_\nu-a_\mu)\wedge d\log(x-a_\nu)-d\log(a_\mu-a_\nu)\wedge d\log(x-a_\mu)$$

$$(\mu\neq\nu),$$

we have

$$\Omega\wedge\Omega=\sum_{\mu\neq\nu}[A_\mu, A_\nu]d\log(a_\nu-a_\mu)\wedge d\log(x-a_\nu)$$

$$-\sum_{\mu\neq\nu}A_\mu\cdot A_\nu(d\log(y-a_\mu)\wedge d\log(x-a_\nu)$$

$$-d\log(y-a_\nu)\wedge d\log(x-a_\mu))$$

$$+2\sum_{\mu\neq\nu}A_\mu\cdot A_\nu d\log(y-a_\mu)\wedge d\log(y-a_\nu)$$

$$-\sum_{\mu\neq\nu}[A_\mu, A_\nu]d\log(a_\nu-a_\mu)\wedge d\log(y-a_\nu)$$

$$=\sum_{\mu\neq\nu}[A_\mu, A_\nu](d\log(a_\nu-a_\mu)-d\log(y-a_\mu))$$

$$\wedge(d\log(x-a_\nu)-d\log(y-a_\nu)),$$

and hence

(2.3.45)

$$d\Omega-\Omega\wedge\Omega=\sum_{\nu=1}^{n}\Big(dA_\nu+\sum_{\mu(\neq\nu)}[A_\nu, A_\mu]d\log\frac{a_\nu-a_\mu}{y-a_\mu}\Big)\wedge d\log\frac{x-a_\nu}{y-a_\nu}.$$

Combining (2.3.44) and (2.3.45) we obtain (2.3.43).

Remark. Similar argument shows that $Y\begin{pmatrix}y; x; \begin{matrix}a_1 & \cdots & a_n & \infty\\ L_1 & \cdots & L_n & L_\infty\end{matrix}\end{pmatrix}$ satisfies equation of the form (2.3.38), where the coefficients A_ν need not satisfy the condition (2.3.41).

So far in studying the properties of the matrix $Y(x_0; x)$ defined by (2.3.3) we have assumed $\mathrm{Im}\, a_1>\cdots>\mathrm{Im}\, a_n$. We now return to the original situation where $a_1, \cdots, a_n\in \boldsymbol{R}^1$; namely we consider

$$(2.3.46)\quad Y_+(x_0;x) = -2\pi i(x_0-x) \times \left(\left\langle \psi^{*(i)}(x_0)\psi^{(j)}(x)\frac{\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)}{\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle}\right\rangle\right).$$

Proposition 2.3.13. *Assume that $a_1, \cdots, a_n$ are distinct points of $\boldsymbol{R}^1$. For fixed $\nu\,(1\leqq\nu\leqq n)$, let $\{\mu_1, \cdots, \mu_k\}$ $(\mu_1<\cdots<\mu_k)$ be the set of indices satisfying $\mu_j<\nu$, $a_\nu<a_{\mu_j}$. Then at $x=a_\nu$ $Y_+(x_0;x)$ defined in* (2.3.46) *has the behavior*

$$(2.3.47)\quad Y_+(x_0;x) = (\text{invertible holomorphic matrix}) \times (x-a_\nu+i0)^{-L'_\nu},$$

where

$$(2.3.48)\quad L'_\nu = (M_{\mu_1}\cdots M_{\mu_k})L_\nu(M_{\mu_1}\cdots M_{\mu_k})^{-1}.$$

In particular if $a_1<\cdots<a_n$, we have $L'_\nu=L_\nu$ $(\nu=1, \cdots, n)$. At $x=\infty$ $Y_\pm$ has the behavior (2.3.29).

Proof. Suppose x belongs to a neighborhood U of $a_\nu\in\boldsymbol{R}$ in the upper half plane. Choose $a'_\mu\in\boldsymbol{C}$ sufficiently close to a_μ $(\mu=1, \cdots, n)$ so that $\operatorname{Im} a'_1>\cdots>\operatorname{Im} a'_n$. Let $Y'(x)=Y\begin{pmatrix}x_0;x; \begin{matrix}a'_1 & \cdots & a'_n & \infty\\ L_1 & \cdots & L_n & L_\infty\end{matrix}\end{pmatrix}$ be given by (2.3.1)-(2.3.3), where the integration paths Γ'_μ involved are as shown in Fig. 2.3.8 (a). By the continuity with respect to a'_μ we have

$$(2.3.49)\quad Y_+(x_0;x) = \lim_{a'_1\to a_1\cdots a'_n\to a_n} Y'(x).$$

Next we deform the paths $\Gamma'_{\mu_1}, \cdots, \Gamma'_{\mu_k}$ into $\Gamma''_{\mu_1}, \cdots, \Gamma''_{\mu_k}$ so that x is contained inside the region bounded by Γ''_{μ_k} and Γ'_ν (Fig. 2.3.8(b)).

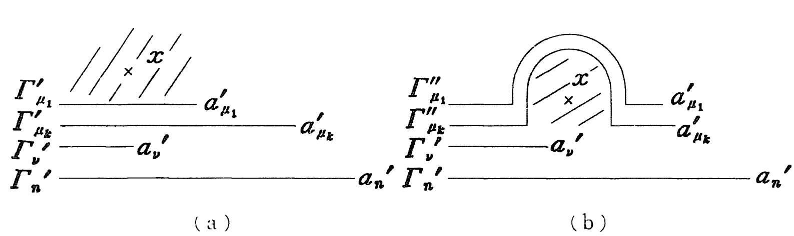

Fig. 2.3.8

Clearly the corresponding matrix $Y''(x)$ is obtained by analytic continuation of $Y'(x)$. Indeed, the monodromic property of the latter implies

(2. 3. 50) $$Y''(x)=Y'(x)M_{\mu_1}\cdots M_{\mu_k}.$$

On the other hand, from Theorem 2. 3. 9 we have

(2. 3. 51) $$Y''(x)=\Phi''_{\nu}(x_0;x)\cdot(x-a)^{-L_\nu}\ \text{at}\ x=a_\nu$$

where $\Phi''_{\nu}(x_0;x)$ is given by (2. 3. 20) with Γ'''_{μ_j} in place of Γ'_{μ_j} ($j=1,\cdots,k$). Making use of the integral representation (2. 3. 20) we can prove

Lemma. $\lim\limits_{a'_1\to a_1\cdots a'_n\to a_n}\Phi''_{\nu}(x_0;x)$ is holomorphic at $x=a_\nu$.

Combining (2. 3. 49) - (2. 3. 51) and the above lemma we conclude that

(2. 3. 52) $$\begin{aligned}Y_+(x_0;x)&=\lim_{a'_1\to a_1\cdots a'\to a_n}\Phi''_{\nu}(x_0;x)\cdot(M_{\mu_1}\cdots M_{\mu_k})^{-1}\cdot(x-a_\nu)^{-L'_\nu}\\&=(\text{holomorphic matrix})\times(x-a_\nu)^{-L'_\nu}\end{aligned}$$

at $x=a_\nu(\nu=1,\cdots,n)$. Likewise (2. 3. 52) holds also at $x=\infty$, with $L'_\infty=L_\infty$. By virtue of the relation $\operatorname{trace}(L'_\infty+\sum_{\nu=1}^{n}L'_\nu)=0$, it then follows that the holomorphic matrix in (2. 3. 5) is in fact invertible at $x=a_\nu$ (see the proof of Theorem 2. 3. 10). This completes the proof of Proposition 2. 3. 13.

As a consequence of Proposition 2. 3. 13, we have the following commutation relation for $\varphi(a;L)$'s:

Proposition 2. 3. 14.

(2. 3. 53)
$$\frac{\varphi(a_1;L_1)\varphi(a_2;L_2)}{\langle\varphi(a_1;L_1)\varphi(a_2;L_2)\rangle}=\begin{cases}\dfrac{\varphi(a_2;L_2)\varphi(a_1;M_2^{-1}L_1M_2)}{\langle\varphi(a_2;L_2)\varphi(a_1;M_2^{-1}L_1M_2)\rangle} & (a_1<a_2)\\[2ex]\dfrac{\varphi(a_2;M_1L_2M_1^{-1})\varphi(a_1;L_1)}{\langle\varphi(a_2;M_1L_2M_1^{-1})\varphi(a_1;L_1)\rangle} & (a_1>a_2)\end{cases}$$

Proof. From the remark below Proposition 2. 2. 1, it suffices to prove that the corresponding matrix $Y_+(x_0;x)$ for both hand sides coincide. We are only to show that they share the common characteristic properties, namely that they have the same exponent matrices at $x=a_1, a_2$. But this is a direct corollary of Proposition 2. 3. 13.

Finally we mention about the coalescence of branch points. For simplicity assume $\mathrm{Im}\, a_1 > \cdots > \mathrm{Im}\, a_n$ and consider the limit $a_{\nu_0+1}, \cdots, a_{\nu_0+k} \to a_{\nu_0}$.

Proposition 2. 3. 15. *For sufficiently small* $|L_\nu|$ $(\nu=1, \cdots, n)$, *we have*

$$(2.3.54)\qquad \lim_{a_{\nu_0+1}, \cdots, a_{\nu_0+k} \to a_{\nu_0}} Y\left(x_0; x; \begin{matrix} a_1 & \cdots & a_n \\ L_1 & & L_n \end{matrix}\right)$$

$$= Y\left(x_0; x; \begin{matrix} a_1 & \cdots & a_{\nu_0-1} & a_{\nu_0} & a_{\nu_0+k+1} & \cdots & a_n \\ L_1 & & L_{\nu_0-1} & \tilde{L}_{\nu_0} & L_{\nu_0+k+1} & & L_n \end{matrix}\right)$$

where $\tilde{L}_{\nu_0}$ *is given by*

$$(2.3.55)\qquad \tilde{L}_{\nu_0} = \frac{1}{2\pi i} \log \tilde{M}_{\nu_0} = \frac{1}{2\pi i} \sum_{l=1}^{\infty} \frac{(-)^{l-1}}{l} (\tilde{M}_{\nu_0} - 1)^l,$$

$$\tilde{M}_{\nu_0} = M_{\nu_0} M_{\nu_0+1} \cdots M_{\nu_0+k}.$$

Proof. We already know that the limit $Y(x_0; x) = \lim_{a_{\nu_0+1}, \cdots, a_{\nu_0+k} \to a_{\nu_0}} Y\left(x_0; x; \begin{matrix} a_1 & \cdots & a_n \\ L_1 & & L_n \end{matrix}\right)$ exists and is given by the series (2. 3. 1), (2. 3. 3). By the same argument as in Theorem 2. 3. 9 we see that it behaves like (2. 3. 19) at $x = a_\nu \neq a_{\nu_0}$ $(= a_{\nu_0+1} = \cdots = a_{\nu_0+k})$ with a holomorphic matrix (2. 3. 20). At $x = \infty$ $Y(x_0; x) x^{-L_\infty}$ is clearly single-valued. Hence the estimate (2. 3. 28) guarantees the behavior (2. 3. 29) at $x = \infty$ with some holomorphic matrix Φ_∞. We see also that it has the following monodromic property around the point $a_{\nu_0} = a_{\nu_0+1} = \cdots = a_{\nu_0+k}$:

$$(2.3.56)\qquad \gamma_{\nu_0} Y = Y \tilde{M}_{\nu_0}$$

where $\tilde{M}_{\nu_0}$, given by (2. 3. 55), is sufficiently close to the unit matrix by assumption. Note that the growth order of $Y(x_0; x)$ at $x = a_{\nu_0}$ is estimated as

$$(2.3.57)\qquad |Y(x_0; x)| = 0\left(\frac{1}{\sqrt{|x - a_{\nu_0}|}}\right) \quad (|x - a_{\nu_0}| \to 0)$$

uniformly in any finite sector $\theta_0 \leqq \arg(x - a_{\nu_0}) \leqq \theta_1$. The rest of the arguments is the same in the proof of Theorem 2. 3. 10. In particular we conclude that Φ_ν (resp. Φ_∞) appearing in (2. 3. 19) (resp. (2. 3. 20)) is necessarily invertible. This completes the proof.

Corollary 2.3.16. *Under the same condition as above, we have*

$$(2.3.58)\qquad \lim_{a_{\nu_0+1},\cdots,a_{\nu_0+k}\to a_{\nu_0}} \frac{\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)}{\langle\varphi(a_1;L_1)\cdots\varphi(a_n;L_n)\rangle}$$

$$=\frac{\varphi(a_1;L_1)\cdots\varphi(a_{\nu_0-1};L_{\nu_0-1})\varphi(a_{\nu_0};L_{\nu_0})\varphi(a_{\nu_0+k+1};L_{\nu_0+k+1})\cdots\varphi(a_n;L_n)}{\langle\varphi(a_1;L_1)\cdots\varphi(a_{\nu_0-1};L_{\nu_0-1})\varphi(a_{\nu_0};L_{\nu_0})\varphi(a_{\nu_0+k+1};L_{\nu_0+k+1})\cdots\varphi(a_n;L_n)\rangle}.$$

§ 2.4. τ Functions

Theorem 2.4.1. *Let $A_\mu(y;a)$ $(\mu=1,\cdots,n)$ be solutions of the Schlesinger's equations* (2.3.43). *We denote by ω the following* 1-*form:*

$$(2.4.1)\qquad \omega=\frac{1}{2}\sum_{\mu\neq\nu}\text{trace } A_\mu(y;a)A_\nu(y;a)\,d\log(a_\mu-a_\nu).$$

Then ω is independent of y, and it is closed.

$$d\omega=0.$$

Proof. From (2.3.38) and (2.3.43) we have

$$\frac{\partial}{\partial y}(Y^{-1}A_\mu Y)=Y^{-1}\left(\left[A_\mu,\frac{\partial}{\partial y}Y\cdot Y^{-1}\right]+\frac{\partial}{\partial y}A_\mu\right)Y=0.$$

Hence we have $Y(y;x;a)^{-1}A_\mu(y;a)\,Y(y;x;a)=Y(y';x;a)^{-1}A_\mu(y';a)$ $Y(y';x;a)$, and ω is independent of y.

We have

$$d\omega=\frac{1}{2}\sum_{\mu\neq\nu}\text{trace}\left(-\sum_{\lambda(\neq\mu)}[A_\nu,A_\lambda]\,d\log\frac{a_\mu-a_\lambda}{y-a_\lambda}\right)A_\nu d\log(a_\mu-a_\nu)$$

$$+\frac{1}{2}\sum_{\mu\neq\nu}\text{trace } A_\mu\left(-\sum_{\lambda(\neq\nu)}[A_\nu,A_\lambda]\,d\log\frac{a_\nu-a_\lambda}{y-a_\lambda}\right)d\log(a_\mu-a_\nu)$$

$$=\sum_{\mu,\nu,\lambda:\text{distinct}}(-\text{trace}[A_\mu,A_\lambda]A_\nu)\,d\log(a_\mu-a_\lambda)\,d\log(a_\mu-a_\nu)$$

$$+\frac{1}{2}\sum_{\mu,\nu,\lambda:\text{distinct}}\text{trace}([A_\mu,A_\lambda]A_\nu+A_\mu[A_\nu,A_\lambda])\,d\log(y-a_\lambda)$$

$$\times d\log(a_\mu-a_\nu).$$

Since we have $\text{trace}[A_\mu,A_\lambda]A_\nu=\text{trace}[A_\nu,A_\mu]A_\lambda=\text{trace}[A_\lambda,A_\nu]A_\mu$, $\text{trace}[A_\mu,A_\lambda]A_\nu+\text{trace } A_\mu[A_\nu,A_\lambda]=0$ and $d\log(a_\lambda-a_\mu)\,d\log(a_\mu-a_\nu)+$

$d\log(a_\mu-a_\nu)\,d\log(a_\nu-a_\lambda)+d\log(a_\nu-a_\lambda)\,d\log(a_\lambda-a_\mu)=0$, we see easily that $d\omega=0$.

Let us consider the transformation property of ω under a projective transformation $h=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2, \boldsymbol{C})$. We assume one of the following.

$$\sum_\mu A_\mu(y;a)=0. \tag{2.4.2}$$

$$\gamma=0. \tag{2.4.3}$$

Remark. (2.3.43) implies that

$$d(\sum_\mu A_\mu)=[\sum_\mu A_\mu, \sum_\nu A_\nu d\log(y-a_\nu)].$$

Hence the algebraic equation (2.4.2) is compatible with (2.3.43).

Proposition 2.4.2. *We assume* (2.4.2) *or* (2.4.3). *Then we have*

$$A_\mu(h(y);h(a))=A_\mu(y;a)\qquad(\mu=1,\cdots,n). \tag{2.4.4}$$

Proof. We fix y and a and consider h as variable. We have

$$d\log\frac{h(a_\mu)-h(a_\nu)}{h(y)-h(a_\nu)}=d\log\frac{a_\mu-a_\nu}{(\gamma a_\mu+\delta)(\gamma a_\nu+\delta)}\,\frac{(\gamma y+\delta)(\gamma a_\nu+\delta)}{y-a_\nu}$$
$$=d\log\frac{\gamma y+\delta}{\gamma a_\mu+\delta}.$$

Hence from (2.3.43) it follows that

$$\begin{aligned}
&dA_\mu(h(y);h(a))\\
&=-\sum_{\nu(\neq\mu)}[A_\mu(h(y);h(a)), A_\nu(h(y);h(a))]\,d\log\frac{h(a_\mu)-h(a_\nu)}{h(y)-h(a_\nu)},\\
&=-[A_\mu(h(y);h(a)), \sum_\nu A_\nu(h(y);h(a))]\,d\log\frac{\gamma y+\delta}{\gamma a_\mu+\delta},\\
&=0.
\end{aligned}$$

Remark. As a corollary of Proposition 2.4.2 we see easily that

$$Y(h(y);h(x);h(a))=Y(y;x;a) \tag{2.4.5}$$

under the assumption (2. 4. 2) or (2. 4. 3). This is the general form of (2. 3. 35).

Proposition 2. 4. 3. *Assuming* (2. 4. 2), *we have*

(2. 4. 6) $$h^*\omega-\omega=\sum_{\mu}\text{trace }A_\mu^2\, d\log(\gamma a_\nu+\delta),$$

where $h^*\omega=\frac{1}{2}\sum_{\mu\neq\nu}\text{trace }A_\mu(h(y);h(a))A_\nu(h(y);h(a))d\log(h(a_\mu)-h(a_\nu))$.
Assuming (2. 4. 3), *we have also*

(2. 4. 7) $$h^*\omega-\omega=(\sum_{\mu}\text{trace }A_\mu^2-\text{trace }A_\infty^2)\, d\log\delta,$$

where $A_\infty=-\sum_{\mu}A_\mu$.

Remark. From (2. 3. 43) we see easily that $d(\text{trace }A_\mu^k)=0$ and $d(\text{trace }A_\infty^k)=0$. In particular, for A_μ given by (2. 3. 40) we have trace $A_\mu^k=$ trace L_μ^k and trace $A_\infty^k=$ trace L_∞^k.

Proof of Proposition 2. 4. 3. From Proposition 2. 4. 3 we have

$$h^*\omega=\frac{1}{2}\sum_{\mu\neq\nu}\text{trace }A_\mu(y;a)A_\nu(y;a)d\log\frac{a_\mu-a_\nu}{(\gamma a_\mu+\delta)(\gamma a_\nu+\delta)}$$

$$=\omega-\sum_{\mu\neq\nu}\text{trace }A_\mu(y;a)A_\nu(y;a)d\log(\gamma a_\mu+\delta)$$

$$=\omega+\sum_{\mu}\text{trace }A_\mu^2 d\log(\gamma a_\mu+\delta)$$

$$+\text{trace}\{A_\infty(y;a)\sum_{\mu}A_\mu(y;a)d\log(\gamma a_\mu+\delta)\}.$$

Under the assumption (2. 4. 2) or (2. 4. 3) we have (2. 4. 6) or (2. 4. 7), respectively.

Let us introduce an equivalence relation among all $A_\mu(a)$ ($\mu=1,\cdots,n$) which satisfy the Schlesinger's equation

$$d_aA_\mu=-\sum_{\nu(\neq\mu)}[A_\mu,A_\nu]d\log\frac{a_\mu-a_\nu}{y-a_\nu},$$

where d_a denotes the exterior differentiation with respect to $a_1,\cdots,a_n$ for some fixed y.

We say $A_\mu(a)$ $(\mu=1, \cdots, n)$ and $\tilde{A}_\mu(a)$ $(\mu=1, \cdots, n)$ are equivalent if and only if there exist an invertible holomorphic matrix $P(a)$ satisfying $\tilde{A}_\mu(a)=P(a)^{-1}A_\mu(a)P(a)$. We call an equivalence class $\mathscr{S}$ an inner automorphism class. The 1-form ω is determined by inner automorphism classes. We denote by $\omega_{\mathscr{S}}$ the 1-form (2. 4. 1) determined by $\mathscr{S}$.

Definition 2. 4. 4. We denote by $\tau_{\mathscr{S}}(a_1, \cdots, a_n)$ the (multi-valued) analytic function defined by

$$d\log\tau_{\mathscr{S}}=\omega_{\mathscr{S}}.$$

We leave a constant multiple undetermined in this definition of $\tau_{\mathscr{S}}(a_1, \cdots, a_n)$.

The following proposition follows directly from Proposition 2. 4. 3.

Proposition 2. 4. 5. *We denote by* $\mathscr{S}$ *the inner automorphism class containing* $A_\mu(y, a)$ $(\mu=1, \cdots, n)$. *Assuming* (2. 4. 2), *we have*

$$\frac{\tau_{\mathscr{S}}(h(a_1), \cdots, h(a_n))}{\tau_{\mathscr{S}}(a_1, \cdots, a_n)}=\prod_\mu(\gamma a_\mu+\delta)^{\mathrm{trace}\,A_\mu^2}. \tag{2. 4. 8}$$

We have also

$$\frac{\tau_{\mathscr{S}}(ta_1+a, \cdots, ta_n+a)}{\tau_{\mathscr{S}}(a_1, \cdots, a_n)}=t^{-1/2(\sum_\mu \mathrm{trace}\,A_\mu^2-\mathrm{trace}\,A_\infty^2)}. \tag{2. 4. 9}$$

If we assume (2. 4. 2), $a_1=\infty$ $(a_{\mu'}\neq\infty, \mu'=2, \cdots, n)$ is not a singular point of the Schlesinger's equations. Let $A_\mu(y; a_1, \cdots, a_n)$ $(\mu=1, \cdots, n)$ be solutions with initial values $A_{\mu'}(y; \infty, a_2, \cdots, a_n)=A'_{\mu'}(y; a_2, \cdots, a_n)$ $(\mu'=2, \cdots, n)$. $A'_{\mu'}(y; a_2, \cdots, a_n)$ $(\mu'=2, \cdots, n)$ themselves satisfy the Schlesinger's equations with $n-1$ branch points. Conversely, if $A'_{\mu'}(y; a_2, \cdots, a_n)$ $(\mu'=2, \cdots, n)$ satisfy the Schlesinger's equations, there exist unique solution matrices $A_\mu(y; a_1, \cdots, a_n)$ $(\mu=1, \cdots, n)$ such that $\sum_\mu A_\mu=0$ and $A_\mu(y; \infty, a_2, \cdots, a_n)=A'_{\mu'}(y; a_2, \cdots, a_n)$. We denote by $\mathscr{S}$ or $\mathscr{S}'$ the inner automorphism class determined by A_μ's or $A'_{\mu'}$'s, respectively.

Proposition 2. 4. 6. *We have*

$$\lim_{a_1\to\infty}\tau_{\mathscr{S}}(a_1, \cdots, a_n)\,a_1^{\mathrm{trace}\,A_1^2}=\mathrm{const.}\ \tau_{\mathscr{S}'}(a_2, \cdots, a_n). \tag{2. 4. 10}$$

Proof. Choosing $h=\begin{pmatrix} 0 & 1 \\ -1 & a_1^{-1} \end{pmatrix}$ in (2.4.8),

we have

$$\frac{\tau_{\mathscr{S}}(a_1, \cdots, a_n)}{\tau_{\mathscr{S}}(0, a_1^{-1}-a_2^{-1}, \cdots, a_1^{-1}-a_n^{-1})} = \prod_{1\leqq\mu\leqq n} a_\mu^{-\mathrm{trace}\, A_\mu^2}.$$

Hence we have the following finite limit.

$$\lim_{a_1\to\infty} \tau_{\mathscr{S}}(a_1, \cdots, a_n)\, a_1^{\mathrm{trace}\, A_1^2} = \tau_{\mathscr{S}}(0, -a_2^{-1}, \cdots, -a_n^{-1}) \prod_{2\leqq\mu\leqq n} a_\mu^{-\mathrm{trace}\, A_\mu^2}$$

If we denote by d' the exterior differentiation with respect to $a_2, \cdots, a_n$, we have

$$\lim_{a_1\to\infty} d' \log \tau_{\mathscr{S}}(a_1, \cdots, a_n)\, a_1^{\mathrm{trace}\, A_1^2}$$

$$= \lim_{a_1\to\infty} \frac{1}{2} \sum_{1\leqq\mu\neq\nu\leqq n} \mathrm{trace}\, A_\mu(a_1, \cdots, a_n) A_\nu(a_1, \cdots, a_n)\, d' \log(a_\mu - a_\nu)$$

$$= \frac{1}{2} \sum_{2\leqq\mu\neq\nu\leqq n} \mathrm{trace}\, A_\mu(\infty, a_2, \cdots, a_n) A_\nu(\infty, a_2, \cdots, a_n)\, d'\log(a_\mu - a_\nu)$$

$$= d' \log \tau_{\mathscr{S}'}(a_2, \cdots, a_n).$$

The main result in this section is the following.

Theorem 2.4.7. *If* $A_\mu(y, a)$ $(\mu=1, \cdots, n)$ *are given by* (2.3.40), *we have*

(2.4.11) $$d \log \langle \varphi(a_1; L_1) \cdots \varphi(a_n; L_n) \rangle = \omega_{\mathscr{S}}$$

where the left hand side is defined by (2.2.29). *We define* $\langle \varphi(a_1; L_1) \cdots \varphi(a_n; L_n) \rangle$ *up to a constant multiple by integrating* (2.4.11), *namely we set*

$$\langle \varphi(a_1; L_1) \cdots \varphi(a_n; L_n) \rangle = \mathrm{const.}\ \tau_{\mathscr{S}}(a_1, \cdots, a_n).$$

Proof. From (2.2.29) and (2.3.24) we have (cf. (2.2.48))

(2.4.12) $$\frac{\partial}{\partial a_\mu} \log \langle \varphi(a_1; L_1) \cdots \varphi(a_n; L_n) \rangle = -\mathrm{trace}\left(\frac{\partial \Phi_{\mu\mu}}{\partial x}(a_\mu; a_\mu) L_\mu\right).$$

On the other hand $\Phi_\mu^*(a_\mu; x) = \Phi_\mu^*(a_\mu; y) Y(y; x; a)$ satisfies the following Fuchsion system with respect to x

$$(2.4.13) \qquad \frac{d\Phi_\mu^*(a_\mu; x)}{dx} = \sum_\nu \frac{A_\nu^{(\mu)}(a)}{x-a_\nu} \Phi_\mu^*(a_\mu; x),$$

$$(2.4.14) \qquad A_\nu^{(\mu)}(a) = \Phi_\mu^*(a_\mu; y) A_\mu(y; a) \Phi_\mu^*(a_\mu; y)^{-1}.$$

The local expansion of $\Phi_\mu^*(a_\mu; x)$ at $x = a_\mu$ is as follows:

$$(2.4.15) \qquad \Phi_\mu^*(a_\mu; x) = \left(1 + \frac{\partial \Phi_{\mu\mu}}{\partial x}(a_\mu; a_\mu)(x-a_\mu) + \cdots\right)(x-a_\mu)^{-L_\mu}.$$

From (2.4.13) and (2.4.15) we have

$$(2.4.16) \qquad A_\mu^{(\mu)} = -L_\mu,$$

$$(2.4.17) \qquad \frac{\partial \Phi_{\mu\mu}}{\partial x}(a_\mu; a_\mu) - \left[\frac{\partial \Phi_{\mu\mu}}{\partial x}(a_\mu; a_\mu), L_\mu\right] = \sum_{\nu(\neq\mu)} \frac{A_\nu^{(\mu)}}{a_\mu - a_\nu}.$$

(2.4.11) follows from (2.4.12), (2.4.14), (2.4.16) and (2.4.17).

Example 1. Let L_μ $(\mu = 1, \cdots, n)$ be commutative matrices. Then we have

$$\frac{\langle\varphi(a_1, L_1)\cdots\varphi(a_n, L_n)\rangle}{\langle\varphi(a_1^0, L_1)\cdots\varphi(a_n^0, L_n)\rangle} = \prod_{\mu<\nu}\left(\frac{a_\mu - a_\nu}{a_\mu^0 - a_\nu^0}\right)^{\mathrm{trace}\, L_\mu L_\nu}.$$

Example 2. In the case $n=2$, we have

$$\frac{\langle\varphi(a_1, L_1)\varphi(a_2, L_2)\rangle}{\langle\varphi(a_1^0, L_1)\varphi(a_2^0, L_2)\rangle} = \left(\frac{a_1 - a_2}{a_1^0 - a_2^0}\right)^{-1/2\, \mathrm{trace}(L_1^2 + L_2^2 + L_\infty^2)}$$

Example 3. We assume that $n=3$ and $L_\infty = 0$. Then we have

$$\frac{\langle\varphi(a_1, L_1)\varphi(a_2, L_2)\varphi(a_3, L_3)\rangle}{\langle\varphi(a_1^0, L_1)\varphi(a_2^0, L_2)\varphi(a_3^0, L_3)\rangle} = \left(\frac{a_1 - a_2}{a_1^0 - a_2^0}\right)^{1/2\, \mathrm{trace}(-L_1^2 - L_2^2 + L_3^2)}.$$

$$\times \left(\frac{a_2 - a_3}{a_2^0 - a_3^0}\right)^{1/2\, \mathrm{trace}(L_1^2 - L_2^2 - L_3^2)} \left(\frac{a_3 - a_1}{a_3^0 - a_1^0}\right)^{1/2\, \mathrm{trace}(-L_1^2 + L_2^2 - L_3^2)}.$$

Now we shall study the behavior of $\tau_{\mathscr{S}}(a_1, \cdots, a_n)$ when some of the branch points meet at one point. The bahavior of $\tau_{\mathscr{S}}(ta_1 + a_0, \cdots, ta_n + a_0)$ in the limit $t \to 0$ is known by (2.4.9). We shall show below

$$\lim_{t\to 0}\frac{\tau_{\mathscr{S}}(a_1,\cdots,a_{n_1},tb_1+a_0,\cdots,tb_{n_2}+a_0)}{\tau_{\mathscr{S}_2}(tb_1+a_0,\cdots,tb_{n_2}+a_0)}=\text{const.}\ \tau_{\mathscr{S}_1}(a_1,\cdots,a_{n_1},a_0)$$

for appropriate choice of $\mathscr{S}$, $\mathscr{S}_1$ and $\mathscr{S}_2$. For this purpose we study the Schlesinger's equations at "fixed singularities" ([8], [15]).

Let A_μ ($\mu=1,\cdots,n$) satisfy (2.3.43). We set $A_\mu(t)=A_\mu(y;a_1,\cdots,a_{n_1},tb_1,\cdots,tb_{n_2})$ ($\mu=1,\cdots,n_1$, $n_2=n-n_1$) and $B_\nu(t)=A_{\nu-n_1}(y;a_1,\cdots,a_{n_1},tb_1,\cdots,tb_{n_2})$ ($\nu=1,\cdots,n_2$). Then we have the following Briot-Bouquet type ordinary differential equations for $A_\mu(t)$ ($\mu=1,\cdots,n_1$) and $B_\nu(t)$ ($\nu=1,\cdots,n_2$).

$$(2.4.18)\qquad \frac{dA_\mu(t)}{dt}=-\sum_\nu[A_\mu(t),B_\nu(t)]\frac{b_\nu(a_\mu-y)}{(a_\mu-tb_\nu)(y-tb_\nu)}$$

$$\frac{dA_\infty(t)}{dt}=-\sum_\nu[A_\infty(t),B_\nu(t)]\frac{b_\nu}{y-tb_\nu}$$

$$\frac{dB_\nu(t)}{dt}=\frac{1}{t}[B_\nu(t),A_\infty(t)+\sum_\mu A_\mu(t)]$$

$$+\sum_\mu[B_\nu(t),A_\mu(t)]\frac{b_\nu}{a_\mu-tb_\nu}-\sum_{\nu'(\neq\nu)}[B_\nu(t),B_{\nu'}(t)]\frac{b_{\nu'}}{y-tb_{\nu'}}.$$

Likewise, if we set $A_\mu(t)=A_\mu(y;a_1,\cdots,a_{n_1},\frac{b_1}{t},\cdots,\frac{b_{n_2}}{t})$ ($\mu=1,\cdots,n_1$) and $B_\nu(t)=A_{\nu+n_1}(y;a_1,\cdots,a_{n_1},\frac{b_1}{t},\cdots,\frac{b_{n_2}}{t})$ ($\nu=1,\cdots,n_2$), we have

$$(2.4.19)\qquad \frac{dA_\mu(t)}{dt}=\sum_\nu[A_\mu(t),B_\nu(t)]\frac{b_\nu(a_\mu-y)}{(ta_\mu-b_\nu)(ty-b_\nu)}$$

$$\frac{dB_\nu(t)}{dt}=\frac{1}{t}[B_\nu(t),\sum_\mu A_\mu(t)]$$

$$-\sum_\mu[B_\nu(t),A_\mu(t)]\frac{a_\mu}{ta_\mu-b_\nu}+\sum_{\nu'(\neq\nu)}[B_\nu(t),B_{\nu'}(t)]\frac{y}{ty-b_{\nu'}}.$$

In general, let $f_{\mu\nu}(t)$, $g_{\mu\nu}(t)$ and $h_{\nu\nu'}(t)$ ($1\leqq\mu\leqq n_1$, $1\leqq\nu\neq\nu'\leqq n_2$) be holomorphic functions defined in $\{t\,|\,|t|<\varepsilon_0\}$, and consider the following system of ordinary differential equations for $m\times m$ matrices $A_\mu(t)$ ($\mu=1,\cdots,n_1$) and $B_\nu(t)$ ($\nu=1,\cdots,n_2$)

$$(2.4.20)\qquad \frac{dA_\mu(t)}{dt}=\sum_\nu[A_\mu(t),B_\nu(t)]f_{\mu\nu}(t)$$

$$\frac{dB_\nu(t)}{dt}=\frac{1}{t}[B_\nu(t), \sum_\mu A_\mu(t)]$$
$$+\sum_\mu [B_\nu(t), A_\mu(t)]g_{\mu\nu}(t)+\sum_{\nu'(\neq\nu)}[B_\nu(t), B_{\nu'}(t)]h_{\nu\nu'}(t).$$

Let A_μ^0 $(\mu=1, \cdots, n_1)$ be $m\times m$ constant matrices and let $\mu_1, \cdots, \mu_m$ denote the eigenvalues of $\Lambda^0 \underset{\text{def}}{=} \sum_\mu A_\mu^0$. We shall study (2. 4. 20) in the neighborhood of $t=0$ and $A_\mu=A_\mu^0$ $(\mu=1, \cdots, n_1)$, assuming

(2. 4. 21) $$\operatorname{Re}(\mu_j-\mu_k)<1 \quad (j, k=1, \cdots, m).$$

Theorem 2. 4. 8. *Let U be a relatively compact subset of $\{\Lambda^0 \in M(m, \boldsymbol{C}) \mid \max_{j,k=1,\cdots,m} \operatorname{Re}(\mu_j(\Lambda^0)-\mu_k(\Lambda^0))<1\}$ where $\mu_j(\Lambda^0)$ $(j=1, \cdots, m)$ denote the eigenvalues of an $m\times m$ matrix Λ^0. We set $\sigma=\max_{\substack{j,k=1,\cdots,m\\ \Lambda^0\in U}} \operatorname{Re}(\mu_j(\Lambda^0)-\mu_k(\Lambda^0))$ and choose σ_1, σ_2 so that $\sigma<\sigma_2<\sigma_1<1$. Let $C>1$ and θ be positive constants and let A_μ^0 $(\mu=1, \cdots, n_1)$ and B_ν^0 $(\nu=1, \cdots, n_2)$ be $m\times m$ matrices satisfying $\Lambda^0=\sum_\mu A_\mu^0 \in U$ and*

(2. 4. 22) $$|A_\mu^0|, \quad |B_\nu^0|<C.$$

There exist constants K and K' independent of A_μ^0, B_0^ν and C such that for any ε satisfying

(2. 4. 23) $$C^2\varepsilon^{\sigma_1-\sigma_2}<K, \quad 0<\varepsilon<K'$$

there exist unique solutions $A_\mu(t)$ $(\mu=1, \cdots, n_1)$ and $B_\nu(t)$ $(\nu=1, \cdots, n_2)$ of (2. 4. 20) which are holomorphic in the sector $S_{\varepsilon,\theta}=\{t \mid 0<|t|<\varepsilon, |\arg t|<\theta\}$ and satisfy the asymptotic conditions

(2. 4. 24) $$|A_\mu(t)-A_\mu^0|<C|t|^{1-\sigma_1},$$
$$|t^{\Lambda^0}B_\nu(t)t^{-\Lambda^0}-B_\nu^0|<C|t|^{1-\sigma_1}.$$

Conversely, for any $\varepsilon_1>0$ there exists $\varepsilon_2>0$ such that if $A_\mu(t)$ $(\mu=1, \cdots, n_1)$ and $B_\nu(t)$ $(\nu=1, \cdots, n_2)$ satisfy (2. 4. 20) and if $|A_\mu(1)|<\varepsilon_2$ $(\mu=1, \cdots, n_1)$ and $|B_\nu(1)|<\varepsilon_2$ $(\nu=1, \cdots, n_2)$ then the limits as $t\to 0$ exist in the sense of (2. 4. 24) with $|A_\mu^0|<\varepsilon_1$ $(\mu=1, \cdots, n_1)$ and $|B_\nu^0|<\varepsilon_1$ $(\nu=1, \cdots, n_2)$.

Proof. We set $\widetilde{B}_\nu(t)=t^{\Lambda^0}B_\nu(t)t^{-\Lambda^0}$ and rewrite (2. 4. 20) with $A_\mu(t)$ and $\widetilde{B}_\nu(t)$ as unknown matrices.

$$\frac{dA_\mu(t)}{dt}=\sum_\nu [A_\mu(t), t^{-\Lambda^0}\tilde{B}_\nu(t)t^{\Lambda^0}]f_{\mu\nu}(t)$$

$$\frac{d\tilde{B}_\nu(t)}{dt}=\frac{1}{t}[\tilde{B}_\nu(t), \sum_\mu t^{\Lambda^0}(A_\mu(t)-A^0_\mu)t^{-\Lambda^0}]$$

$$+\sum_\mu [\tilde{B}_\nu(t), t^{\Lambda^0}A_\mu(t)t^{-\Lambda^0}]g_{\mu\nu}(t)+\sum_{\nu'(\neq\nu)}[\tilde{B}_\nu(t), \tilde{B}_{\nu'}(t)]h_{\nu\nu'}(t).$$

set $A^{(0)}_\mu(t)=A^0_\mu$, $\tilde{B}^{(0)}_\nu(t)=B^0_\nu$ and define $A^{(k)}_\mu(t)$, $\tilde{B}^{(k)}_\nu(t)$ $(k=0, 1, 2, \cdots)$ recursively by

$$(2.4.25)_k \quad A^{(k)}_\mu(t)=A^0_\mu+\int_0^t \sum_\nu [A^{(k-1)}_\mu(s), s^{-\Lambda^0}\tilde{B}^{(k-1)}_\nu(s)s^{\Lambda^0}]f_{\mu\nu}(s)\,ds$$

$$\tilde{B}^{(k)}_\nu(t)=B^0_\nu+\int_0^t \Big\{\frac{1}{s}[\tilde{B}^{(k-1)}_\nu(s), \sum_\mu s^{\Lambda^0}(A^{(k-1)}_\mu(s)-A^0_\mu)s^{-\Lambda^0}]$$

$$+\sum_\mu [\tilde{B}^{(k-1)}_\nu(s), s^{\Lambda^0}A^{(k-1)}_\mu(s)s^{-\Lambda^0}]g_{\mu\nu}(s)$$

$$+\sum_{\nu'(\neq\nu)}[\tilde{B}^{(k-1)}_\nu(s), \tilde{B}^{(k-1)}_{\nu'}(s)]h_{\nu\nu'}(s)\Big\}\,ds.$$

Here the path of integration is $\{s=re^{i\theta}|0<r<|t|, \theta=\arg t\}$.

Let δ be a constant such that $0<\delta<1$. For an appropriate choice of K, K' in (2.4.23) we claim the following:

$$(2.4.26)_k \qquad |A^{(k)}_\mu(t)-A^0_\mu|\leqq C|t|^{1-\sigma_1}$$

$$(2.4.27)_k \qquad |t^{\Lambda^0}(A^{(k)}_\mu(t)-A^0_\mu)t^{-\Lambda^0}|\leqq C^2|t|^{1-\sigma_2}$$

$$(2.4.28)_k \qquad |\tilde{B}^{(k)}_\nu(t)-B^0_\nu|\leqq C|t|^{1-\sigma_1}$$

$$(2.4.29)_k \qquad |A^{(k)}_\mu(t)-A^{(k-1)}_\mu(t)|\leqq C\delta^{k-1}|t|^{1-\sigma_1}$$

$$(2.4.30)_k \qquad |t^{\Lambda^0}(A^{(k)}_\mu(t)-A^{(k-1)}_\mu(t))t^{-\Lambda^0}|\leqq C^2\delta^{k-1}|t|^{1-\sigma_2}$$

$$(2.4.31)_k \qquad |\tilde{B}^{(k)}_\nu(t)-\tilde{B}^{(k-1)}_\nu(t)|\leqq C\delta^{k-1}|t|^{1-\sigma_1},$$

for $t\in S_{\varepsilon,\theta}$.

We choose K' so that $0<K'<1$. Then we have from $(2.4.26)_k$ and $(2.4.28)_k$

$$(2.4.32)_k \qquad |A^{(k)}_\mu(t)|\leqq 2C, \ |\tilde{B}^{(k)}_\nu(t)|\leqq 2C.$$

Making use of the formula

$$f(\Lambda^0)=\frac{1}{2\pi i}\oint \frac{f(\lambda)}{\lambda-\Lambda^0}d\lambda$$

we have the following lemma.

Lemma. Let $A(t)$ and $B(t)$ be $m\times m$ matrices which satisfy $|A(t)|\leqq 2C_1$, $|B(t)|\leqq 2C_2$ for $t\in S_{\varepsilon,\theta}$ and let $f(t)$ be a holomorphic function in $\{t||t|<1\}$. There exists a constant K' independent of C_1, C_2 nor $\Lambda^0\in U$ such that for any ε satisfying $0<\varepsilon<K'$, the following are valid.

$$|t^{\Lambda^0}A(t)t^{-\Lambda^0}|\leqq C_1|t|^{-\sigma_2},\quad |t^{-\Lambda^0}B(t)t^{\Lambda^0}|\leqq C_2|t|^{-\sigma_2}$$

$$|t^{\Lambda^0}\int_0^t A(s)s^{-\Lambda^0}B(s)s^{\Lambda^0}f(s)\,ds\,t^{-\Lambda^0}|\leqq\frac{C_1C_2}{2n_2}|t|^{1-\sigma_2}$$

$$|t^{\Lambda^0}\int_0^t s^{-\Lambda^0}B(s)s^{\Lambda^0}A(s)f(s)\,ds\,t^{-\Lambda^0}|\leqq\frac{C_1C_2}{2n_2}|t|^{1-\sigma_2}$$

for $t\in S_{\varepsilon,\theta}$ and $\Lambda^0\in U$.

By virtue of this lemma $(2.4.32)_k$ now implies

$$|t^{\Lambda^0}A_\mu^{(k)}(t)t^{-\Lambda^0}|\leqq C|t|^{-\sigma_2},\quad |t^{-\Lambda^0}\widetilde{B}_\nu^{(k)}(t)t^{\Lambda^0}|\leqq C|t|^{-\sigma_2}.$$

Assuming that $(2.4.26)_k$, $(2.4.27)_k$ and $(2.4.28)_k$ are valid, we see that $(2.4.25)_{k+1}$ is well-defined. Moreover we have

$$|A_\mu^{(k+1)}(t)-A_\mu^0|\leqq\int_0^{|t|}\sum_\nu 2|A_\mu^{(k)}(s)||s^{-\Lambda^0}\widetilde{B}_\nu^{(k)}(s)s^{\Lambda^0}||f_{\mu\nu}(s)|d|s|$$

$$\leqq C|t|^{1-\sigma_1}\cdot\frac{4n_2C\max|f_{\mu\nu}(s)|}{1-\sigma_2}|t|^{\sigma_1-\sigma_2},$$

$$|t^{\Lambda^0}(A_\mu^{(k+1)}(t)-A_\mu^0)t^{-\Lambda^0}|$$

$$\leqq\sum_\nu|t^{\Lambda^0}\int_0^t A_\mu^{(k)}(s)s^{-\Lambda^0}\widetilde{B}_\nu^{(k)}(s)s^{\Lambda^0}f_{\mu\nu}(s)\,ds\,t^{-\Lambda^0}|$$

$$+\sum_\nu|t^{\Lambda^0}\int_0^t s^{-\Lambda^0}\widetilde{B}_\nu^{(k)}(s)s^{\Lambda^0}A_\mu^{(k)}(s)f_{\mu\nu}(s)\,ds\,t^{-\Lambda^0}|$$

$$\leqq C^2|t|^{1-\sigma_2},$$

$$|\widetilde{B}_\nu^{(k)}(t)-B_\nu^0|\leqq\int_0^{|t|}\sum_\mu 2\frac{1}{|s|}\cdot|\widetilde{B}_\nu^{(k)}(s)||s^{\Lambda^0}(A_\mu^{(k)}(s)-A_\mu^0)s^{-\Lambda^0}|d|s|$$

$$+\int_0^{|t|}\sum_\mu 2|\widetilde{B}_\nu^{(k)}(s)||s^{\Lambda^0}A_\mu^{(k)}(s)s^{-\Lambda^0}||g_{\mu\nu}(s)|d|s|$$

$$+\int_0^{|t|}\sum_{\nu'(\neq\nu)}2|\widetilde{B}_\nu^{(k)}(s)||\widetilde{B}_{\nu'}^{(k)}(s)||h_{\nu\nu'}(s)|d|s|$$

$$\leqq C|t|^{1-\sigma_1}\Big(\frac{4n_1C^2}{1-\sigma_2}|t|^{\sigma_1-\sigma_2}+\frac{4n_1C\max|g_{\mu\nu}(s)|}{1-\sigma_2}|t|^{\sigma_1-\sigma_2}$$

$$+8n_2C\max|h_{\nu\nu'}(s)||t|^{\sigma_1}\Big).$$

These estimates prove our claim for $(2.4.26)_{k-1}\sim(2.4.28)_{k-1}$. A similar calculation shows that our claim is valid for $(2.4.29)_k\sim(2.4.31)_k$.

Thus we have proved the existence of solutions

$$A_\mu(t)=\lim_{k\to\infty}A_\mu^{(k)}(t),$$

$$B_\nu(t)=\lim_{k\to\infty}B_\nu^{(k)}(t),$$

which satisfies the asymptotic condition (2. 4. 24) and

(2. 4. 33) $$|t^{A^0}(A_\mu(t)-A_\mu^0)t^{-A^0}|\leqq C|t|^{1-\sigma_2}.$$

The uniqueness of the solution $A_\mu(t)$, $B_\nu(t)$ satisfying (2. 4. 24) and (2. 4. 33) is also proved by iteration. Since (2. 4. 33) follows from (2. 4. 24), we can omit it in the statement of Theorem 2. 4. 8.

To prove the last statement of Theorem 2. 4. 8, we start with the following iteration.

$$A_\mu^{(k)}(t)=A_\mu(1)-\int_t^1\sum_\nu[A_\mu^{(k-1)}(s),\ B_\nu^{(k-1)}(s)]f_{\mu\nu}(s)ds,$$

$$B_\nu^{(k)}(t)=B_\nu(1)-\int_t^1\Big\{\frac{1}{s}[B_\nu^{(k-1)}(s),\ \sum_\mu A_\mu^{(k-1)}(s)]$$

$$+\sum_\mu[B_\nu^{(k-1)}(s),A_\mu^{(k-1)}(s)]g_{\mu\nu}(s)$$

$$+\sum_{\nu'(\neq\nu)}[B_\nu^{(k-1)}(s),B_{\nu'}^{(k-1)}(s)]h_{\nu\nu'}(s)\Big\}ds,$$

and $A_\mu^{(0)}(t)=A_\mu(1)$, $B_\nu^{(0)}(t)=B_\nu(1)$. By a similar estimation as above we see that $A_\mu(t)=\lim_{k\to\infty}A_\mu^{(k)}(t)$ has a continuous limit $A_\mu(0)=A_\mu^0$ at $t=0$ such that $|A_\mu^0|\leqq\varepsilon_1$. Also we have the estimate $|B_\nu(t)|\leqq\varepsilon_1|t|^{-\sigma}$ with $0<\sigma\ll1$. Since $A_\mu(t)$ satisfy the integral equation

$$A_\mu(t)=A_\mu^0+\int_0^t\sum_\nu[A_\mu(s),B_\nu(s)]f_{\mu\nu}(s)ds,$$

$A_\mu(t)$ satisfy the asymptotic condition

$$|A_\mu(t)-A_\mu^0|\leqq\varepsilon_1|t|^{1-\sigma}.$$

Now we define another iterative approximation.

$$A_{\mu}^{[k]}(t)=A_{\mu}^{1}-\int_{t}^{1}\sum_{\nu}[A_{\mu}^{[k-1]}(s),s^{-A^{0}}\widetilde{B}_{\nu}^{[k-1]}(s)s^{A^{0}}]f_{\mu\nu}(s)\,ds$$

$$\widetilde{B}_{\nu}^{[k]}(t)=B_{\nu}^{1}-\int_{t}^{1}\left\{\frac{1}{s}[\widetilde{B}_{\nu}^{[k-1]}(s),\sum_{\mu}s^{A^{0}}(A_{\mu}^{[k-1]}(s)-A_{\mu}^{0})s^{-A^{0}}]\right.$$

$$+\sum_{\mu}[\widetilde{B}_{\nu}^{[k-1]}(s),s^{A^{0}}A_{\mu}^{[k-1]}(s)s^{-A^{0}}]g_{\mu\nu}(s)$$

$$\left.+\sum_{\nu'(\neq\nu)}[\widetilde{B}_{\nu}^{[k-1]}(s),\widetilde{B}_{\nu'}^{[k-1]}(s)]h_{\nu\nu'}(s)\right\}ds\,.$$

By a similar estimation we see that $\widetilde{B}_{\nu}(t)=\lim_{k\to\infty}\widetilde{B}_{\nu}^{[k]}(t)$ has a continuous limit $\widetilde{B}_{\nu}(0)=B_{\nu}^{0}$ at $t=0$ such that $|B_{\nu}^{0}|\leqq\varepsilon_{1}$ and $|\widetilde{B}_{\nu}(t)-B_{\nu}^{0}|\leqq\varepsilon_{1}|t|^{1-\sigma}$.

The asymptotic expansions of $A_{\mu}(t)$ and $B_{\nu}(t)$ are obtained as follows. $X=(X_{1},\cdots,X_{n_{1}})$ and $Y=(Y_{1},\cdots,Y_{n_{2}})$ will denote n_{1}- and n_{2}-tuples of $m\times m$ matrix variables, respectively.

Proposition 2.4.9. *There exist holomorphic $m\times m$ matrices $F_{\mu}^{k}(X,Y,t)$ $(\mu=1,\cdots,n_{1};k=0,1,\cdots)$ and $G_{\nu}^{k}(X,Y,t)$ $(\nu=1,\cdots,n_{2};k=0,1,\cdots)$ which satisfy the following.*

(i) *They are polynomials in $(X_{\mu})_{jl}$ and $(Y_{\nu})_{jl}$ $(\mu=1,\cdots,n_{1};\nu=1,\cdots,n_{2};j,l=1,\cdots,m)$.*

(ii) *Each monomial in $F_{\mu}^{k}(X,Y,t)$ $(k\geqq1)$ has a degree at most $2k$ in $\{(X_{\mu})_{jl},(Y_{\nu})_{jl}\}_{\substack{\mu=1,\cdots,n_{1}\\ \nu=1,\cdots,n_{2}\\ j,l=1,\cdots,m}}$, at most k in $\{(X_{\mu})_{jl}\}_{\substack{\mu=1,\cdots,n_{1}\\ j,l=1,\cdots,m}}$ and at most k in $\{(Y_{\nu})_{jl}\}_{\substack{\nu=1,\cdots,n_{2}\\ j,l=1,\cdots,m}}$.*

(iii) *Each monomial in $G_{\nu}^{k}(X,Y,t)$ $(k\geqq1)$ has a degree at most $2k+1$ in $\{(X_{\mu})_{jl},(Y_{\nu})_{jl}\}_{\substack{\mu=1,\cdots,n_{1}\\ \nu=1,\cdots,n_{2}\\ j,l=1,\cdots,m}}$, at most k in $\{(X_{\mu})_{jl}\}_{\substack{\mu=1,\cdots,n_{1}\\ j,l=1,\cdots,m}}$ and at most $k+1$ in $\{(Y_{\nu})_{jl}\}_{\substack{\nu=1,\cdots,n_{2}\\ j,l=1,\cdots,m}}$.*

(iv) *The coefficients of $F_{\mu}^{k}(X,Y,t)$ and $G_{\nu}^{k}(X,Y,t)$ are t^{k} times holomorphic functions of t defined in $\{t\,|\,|t|<\varepsilon_{0}\}$.*

(v) *$A_{\mu}(t)$ and $B_{\nu}(t)$ of Theorem 2.4.8 have the following asymptotic expansions.*

$$A_{\mu}(t)=\sum_{k=0}^{\infty}F_{\mu}^{k}(A_{1}^{0},\cdots,A_{n_{1}}^{0},t^{-A^{0}}B_{1}^{0}t^{A^{0}},\cdots,t^{-A^{0}}B_{n_{2}}^{0}t^{A^{0}},t)$$

$$B_\nu(t)=\sum_{k=0}^{\infty} G_\nu^k(A_1^0,\cdots,A_{n_1}^0,t^{-\Lambda^0}B_1^0t^{\Lambda^0},\cdots,t^{-\Lambda^0}B_{n_2}^0t^{\Lambda^0},t).$$

Proof. We set $F_\mu^0(X,Y,t)=X_\mu$ and $G_\nu^0(X,Y,t)=Y_\nu$ and define $F_\mu^k(X,Y,t)$ and $G_\nu^k(X,Y,t)$ recursively as follows.

$$F_\mu^k(X,Y,t)=\int_0^t\sum_\nu\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}\Big[F_\mu^j\Big(X,\Big(\frac{s}{t}\Big)^{-\Lambda^0}Y\Big(\frac{s}{t}\Big)^{\Lambda^0},s\Big),$$

$$G_\nu^l\Big(X,\Big(\frac{s}{t}\Big)^{-\Lambda^0}Y\Big(\frac{s}{t}\Big)^{\Lambda^0},s\Big)\Big]f_{\mu\nu}(s)\,ds$$

$$G_\nu^k(X,Y,t)=\int_0^t\Big\{\frac{1}{s}\sum_{\substack{j+l=k\\0\leq j\leq k-1\\1\leq l\leq k}}\Big[G_\nu^j\Big(\Big(\frac{s}{t}\Big)^{\Lambda^0}X\Big(\frac{s}{t}\Big)^{-\Lambda^0},Y,s\Big),$$

$$\sum_\mu F_\mu^l\Big(\Big(\frac{s}{t}\Big)^{\Lambda^0}X\Big(\frac{s}{t}\Big)^{-\Lambda^0},Y,s\Big)\Big]$$

$$+\sum_\mu\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}\Big[G_\nu^j\Big(\Big(\frac{s}{t}\Big)^{\Lambda^0}X\Big(\frac{s}{t}\Big)^{-\Lambda^0},Y,s\Big),$$

$$F_\mu^l\Big(\Big(\frac{s}{t}\Big)^{\Lambda^0}X\Big(\frac{s}{t}\Big)^{-\Lambda^0},Y,s\Big)\Big]g_{\mu\nu}(s)$$

$$+\sum_{\nu'(\neq\nu)}\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}\Big[G_\nu^j\Big(\Big(\frac{s}{t}\Big)^{\Lambda^0}X\Big(\frac{s}{t}\Big)^{-\Lambda^0},Y,s\Big),$$

$$G_{\nu'}^l\Big(\Big(\frac{s}{t}\Big)^{\Lambda^0}X\Big(\frac{s}{t}\Big)^{-\Lambda^0},Y,s\Big)\Big]h_{\nu\nu'}(s)\Big\}\,ds\,.$$

The assertions (i) $\sim$ (iii) are obvious from the definition. We shall prove (iv) by induction on k. The case $k=0$ is trivial. We assume that the assertion is valid up to $k-1$. Then there exist holomorphic matrices $\widetilde{F}_\mu^j(X,Y,t)$ $(0\leqq j\leqq k-1)$ and $\widetilde{G}_\nu^l(X,Y,t)$ $(0\leqq l\leqq k-1)$ such that $F_\mu^j(X,Y,t)=t^j\widetilde{F}_\mu^j(X,Y,t)$ and $G_\nu^l(X,Y,t)=t^l\widetilde{G}_\nu^l(X,Y,t)$. Then we have

$$F_\mu^k(X,Y,t)=\int_0^t\sum_\nu\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}\Big[s^j\widetilde{F}_\mu^j\Big(X,\Big(\frac{s}{t}\Big)^{-\Lambda^0}Y\Big(\frac{s}{t}\Big)^{\Lambda^0},s\Big),$$

$$s^l\widetilde{G}_\nu^l\Big(X,\Big(\frac{s}{t}\Big)^{-\Lambda^0}Y\Big(\frac{s}{t}\Big)^{\Lambda^0},s\Big)\Big]f_{\mu\nu}(s)\,ds$$

$$=t^k\int_0^1\sum_\nu\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}[s^j\widetilde{F}^j_\mu(X,s^{-A^0}Ys^{A^0},ts),\ s^l\widetilde{G}^l_\nu(X,s^{-A^0}Ys^{A^0},ts)]f_{\mu\nu}(ts)\,ds\,.$$

Hence (iv) is valid for $F^k_\mu(X, Y, t)$. Likewise it is valid for $G^k_\nu(X, Y, t)$.

By induction we see also that for any matrix M which commutes with A^0, we have

$$M^{-1}F^k_\mu(X,Y,t)M=F^k_\mu(M^{-1}XM,M^{-1}YM,t),$$

$$M^{-1}G^l_\nu(X,Y,t)M=G^l_\nu(M^{-1}XM,M^{-1}YM,t).$$

Now we define $A^k_\mu(t)$ $(\mu=1,\cdots,n_1;k=0,1,\cdots)$ and $B^k_\nu(t)$ $(\nu=1,\cdots,n_2;k=0,1,\cdots)$ by

$$A^k_\mu(t)=F^k_\mu(A^0_1,\cdots,A^0_{n_1},t^{-A^0}B^0_1t^{A^0},\cdots,t^{-A^0}B^0_{n_2}t^{A^0},t)$$

$$B^k_\nu(t)=G^k_\nu(A^0_1,\cdots,A^0_{n_1},t^{-A^0}B^0_1t^{A^0},\cdots,t^{-A^0}B^0_{n_2}t^{A^0},t).$$

By the above remark we have

$$t^{A^0}A^k_\mu(t)t^{-A^0}=F^k_\mu(t^{A^0}A^0_1t^{-A^0},\cdots,t^{A^0}A^0_{n_1}t^{-A^0},B^0_1,\cdots,B^0_{n_2},t)$$

$$\widetilde{B}^k_\nu(t)=t^{A^0}B^k_\nu(t)t^{-A^0}=G^k_\nu(t^{A^0}A^0_1t^{-A^0},\cdots,t^{A^0}A^0_{n_1}t^{-A^0},B^0_1,\cdots,B^0_{n_2},t)$$

These matrices satisfy the following recursive equations.

$$A^k_\mu(t)=\int_0^t\sum_\nu\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}[A^j_\mu(s),B^l_\nu(s)]f_{\mu\nu}(s)\,ds\,,$$

$$\begin{aligned}\widetilde{B}^k_\nu(t)=\int_0^t\Big\{&\frac{1}{s}\sum_{\substack{j+l=k\\0\leq j\leq k-1\\1\leq l\leq k}}[\widetilde{B}^j_\nu(s),\sum_\mu s^{A^0}A^l_\mu(s)s^{-A^0}]\\&+\sum_\mu\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}[\widetilde{B}^j_\nu(s),s^{A^0}A^l_\mu(s)s^{-A^0}]g_{\mu\nu}(s)\\&+\sum_{\nu'(\neq\nu)}\sum_{\substack{j+l=k-1\\0\leq j\leq k-1\\0\leq l\leq k-1}}[\widetilde{B}^j_\nu(s),\widetilde{B}^l_{\nu'}(s)]h_{\nu\nu'}(s)\Big\}\,ds\,.\end{aligned}$$

Note that (ii) and (iii) implies that

$$A^k_\mu(t)=0(|t|^{k(1-\sigma_1)})\qquad(k\geqq 0)$$

$$t^{A_0}A^k_\mu(t)t^{-A_0}=\begin{cases}0(|t|^{-\sigma_1})&(k=0)\\0(|t|^{k(1-\sigma_1)})&(k\geqq 1)\end{cases}$$

$$\tilde{B}^k_\nu(t) = 0(|t|^{k(1-\sigma_1)}) \qquad (k \geqq 0)$$

$$B^k_\nu(t) = 0(|t|^{k-(k+1)\sigma_1}) \qquad (k \geqq 0).$$

Let $A_\mu(t)$, $B_\nu(t)$ be solutions of (2. 4. 20) which satisfy (2. 4. 24) and (2. 4. 33). We claim that

$$A_\mu(t) - \sum_{0 \leqq j \leqq k} A^j_\mu(t) = 0(|t|^{k(1-\sigma_1)}),$$

$$t^{A^0}(A_\mu(t) - \sum_{0 \leqq j \leqq k} A^j_\mu(t))\, t^{-A^0} = 0(|t|^{k(1-\sigma_1)}),$$

$$\tilde{B}_\nu(t) - \sum_{0 \leqq j \leqq k} \tilde{B}^j_\nu(t) = 0(|t|^{k(1-\sigma_1)}),$$

$$B_\nu(t) - \sum_{0 \leqq j \leqq k} B^j_\nu(t) = 0(|t|^{k-(k+1)\sigma_1}).$$

We assume that our claim is valid up to $k-1$. Then we have

$$A_\mu(t) - A^0_\mu - \cdots - A^k_\mu(t)$$

$$= \int_0^t \sum_\nu [(A_\mu(s) - \sum_{0 \leqq j \leqq k-1} A^j_\mu(s)) + \sum_{0 \leqq j \leqq k-1} A^j_\mu(s),$$

$$(B_\nu(s) - \sum_{0 \leqq j \leqq k-1} B^j_\nu(s)) + \sum_{0 \leqq j \leqq k-1} B^j_\nu(s)] f_{\mu\nu}(s)\, ds$$

$$- \int_0^t \sum_\nu \sum_{\substack{j+l \leqq k-1 \\ 0 \leqq j \leqq k-1 \\ 0 \leqq l \leqq k-1}} [A^j_\mu(s), B^l_\nu(s)] f_{\mu\nu}(s)\, ds$$

$$= 0(|t|^{k(1-\sigma_1)}).$$

Likewise we can prove other claims. Hence we have proved Proposition 2. 4. 9.

By the formal transformation

$$(2.4.34) \qquad A_\mu = \sum_{k=0}^{\infty} F^k_\mu(X, Y, t),$$

$$B_\nu = \sum_{k=0}^{\infty} G^k_\nu(X, Y, t),$$

the system (2. 4. 20) is transformed into the following linear system.

$$(2.4.35) \qquad \frac{dX_\mu(t)}{dt} = 0, \quad \frac{dY_\nu(t)}{dt} = \frac{1}{t}[Y_\nu(t), \sum_\mu X_\mu(t)].$$

In the following we discuss the convergence of the series (2. 4. 34). We denote by Δ_σ the following subset of $(X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}, t)$ space.

$$\Delta_\sigma = \{(X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}, t) \mid t=0,$$
$$\mathrm{Re}(\mu_j - \mu_k) < \sigma\ (j, k=1, \cdots, m)\},$$

where μ_j $(j=1, \cdots, m)$ denote the eigenvalues of $\sum_\mu X_\mu$.

Theorem 2.4.10. *There exist holomorphic functions* $F_\mu(X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}, t)$ $(\mu=1, \cdots, n_1)$, $G_\nu(X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}, t)$ $(\nu=1, \cdots, n_2)$ *defined in a neighborhood of* $\Delta_{1/3}$ *such that the system* (2.4.20) *is transformed into* (2.4.35) *by the non-linear transformation*

$$A_\mu = F_\mu(X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}, t) \tag{2.4.36}$$
$$B_\nu = G_\nu(X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}, t).$$

Proof. We denote by $A_\mu(t; X, Y, s)$, $B_\nu(t; X, Y, s)$ the unique solution of (2.4.20) satisfying the asymptotic conditions

$$|A_\mu(t; X, Y, s) - X_\mu| < C|t|^{1-\sigma_1}$$
$$|t^{\Lambda^0} B_\nu(t; X, Y, s) t^{-\Lambda^0} - s^{\Lambda^0} Y_\nu s^{-\Lambda^0}| < C|t|^{1-\sigma_1}.$$

The condition (2.4.23) for ε implies that there exists a neighborhood $D_{1/3}$ of $\Delta_{1/3}$ such that, for any $(X, Y, s) \in D_{1/3}$, $A_\mu(t; X, Y, s)$ and $B_\nu(t; X, Y, s)$ are holomorphic in a sector containing s. (Choose σ_1 and σ_2 so that $\sigma_1 - \sigma_2 > \frac{2}{3}$.) We set

$$F_\mu(X, Y, s) = A_\mu(s; X, Y, s)$$
$$G_\nu(X, Y, s) = B_\nu(s; X, Y, s)$$

for $(X, Y, s) \in D_{1/3}$ and claim that F_μ and G_ν are single-valued. We have

$$|A_\mu(t; X, Y, e^{2\pi i}s) - X_\mu| < C|t|^{1-\sigma_1}$$
$$|t^{\Lambda^0} B_\nu(t; X, Y, e^{2\pi i}s) t^{-\Lambda^0} - e^{2\pi i\Lambda^0} s^{\Lambda^0} Y_\nu s^{-\Lambda^0} e^{-2\pi i\Lambda^0}| < C|t|^{1-\sigma_1}.$$

If we set $t = e^{2\pi i}t'$, $A_\mu(e^{2\pi i}t'; X, Y, e^{2\pi i}s)$ and $B_\nu(e^{2\pi i}t'; X, Y, e^{2\pi i}s)$ satisfy (2.4.25) and the following.

$$|A_\mu(e^{2\pi i}t'; X, Y, e^{2\pi i}s) - X_\mu| < C|t|^{1-\sigma_1}$$
$$|t'^{\Lambda^0} B_\nu(e^{2\pi i}t'; X, Y, e^{2\pi i}s) t'^{-\Lambda^0} - s^{\Lambda^0} Y_\nu s^{-\Lambda^0}| < C'|t|^{1-\sigma_1},$$

for some constant C'. Hence the uniqueness of solution implies

$$A_{\mu}(e^{2\pi i}t';X,Y,e^{2\pi i}s)=A_{\mu}(t';X,Y,s)$$

$$B_{\nu}(e^{2\pi i}t';X,Y,e^{2\pi i}s)=B_{\nu}(t';X,Y,s).$$

This proves our claim. Moreover (2. 4. 24) implies that

$$\lim_{s\to 0}F_{\mu}(X,Y,s)=X_{\mu},$$

$$\lim_{s\to 0}G_{\nu}(X,Y,s)=Y_{\nu}.$$

Hence F_{μ}, G_{ν} are holomorphic at $s=0$ and the transformation (2. 4. 36) is invertible. Since $A_{\mu}(t;X,s^{-A^0}Ys^{A^0},s)$ and $B_{\nu}(t;X,s^{-A^0}Ys^{A^0},s)$ are independent of s, the substitution $X_{\mu}=A_{\mu}^0$, $Y_{\nu}=t^{-A^0}B_{\nu}^0t^{A^0}$ into (2. 4. 36) gives a solution of (2. 4. 20).

So far, we have fixed $a_1,\cdots,a_{n_1},b_1,\cdots,b_{n_2}$ and y in (2. 4. 18) and (2. 4. 19). Now from (2. 3. 43) we derive the system of total differential equation satisfied by $A_1^0,\cdots,A_{n_1}^0$, $B_1^0,\cdots,B_{n_2}^0$ as functions of $a_1,\cdots,a_{n_1}$, $b_1,\cdots,b_{n_2}$ and y.

Proposition 2. 4. 11. *In the case of* (2. 4. 18) *we have*

$$(2.4.37)\quad dA_{\mu}^0=-\sum_{\mu'(\neq\mu)}[A_{\mu}^0,A_{\mu'}^0]d\log\frac{a_{\mu}-a_{\mu'}}{y-a_{\mu'}}+[A_{\mu}^0,A^0]d\log\frac{a_{\mu}}{y},$$

$$dA^0=-\sum_{\mu}[A^0,A_{\mu}^0]d\log\frac{a_{\mu}}{y-a_{\mu}},$$

$$(2.4.38)\quad dB_{\nu}^0=-\sum_{\nu'(\neq\nu)}[B_{\nu}^0,B_{\nu'}^0]d\log\frac{b_{\nu}-b_{\nu'}}{y}-\sum_{\mu}[B_{\nu}^0,A_{\mu}^0]d\log\frac{a_{\mu}}{y-a_{\mu}}$$

where $A^0=\sum_{\mu}A_{\mu}^0+A_{\infty}^0$. *In the case of* (2. 4. 19), *we have*

$$(2.4.39)\quad dA_{\mu}^0=-\sum_{\mu'(\neq\mu)}[A_{\mu}^0,A_{\mu'}^0]d\log\frac{a_{\mu}-a_{\mu'}}{y-a_{\mu'}},$$

$$(2.4.40)\quad dB_{\nu}^0=-\sum_{\mu}[B_{\nu}^0,A_{\mu}^0]d\log\frac{b_{\nu}}{y-a_{\mu}}-\sum_{\nu'(\neq\nu)}[B_{\nu}^0,B_{\nu'}^0]d\log\frac{b_{\nu}-b_{\nu'}}{b_{\nu'}}.$$

Proof. We shall prove (2. 4. 39) and (2. 4. 40). (2. 4. 37) and (2. 4. 38) are proved similarly. From (2. 3. 43) we have

$$(2.4.41)\quad dA_\mu = -\sum_{\mu'(\neq\mu)}[A_\mu, A_{\mu'}]d\log\frac{a_\mu - a_{\mu'}}{y-a_{\mu'}} - \sum_\nu [A_\mu, B_\nu]d\log\frac{ta_\mu - b_\nu}{ty-b_\nu},$$

$$(2.4.42)\quad dB_\nu = -\sum_\mu [B_\nu, A_\mu]d\log\frac{b_\nu - ta_\mu}{y-a_\mu} - \sum_{\nu'(\neq\nu)}[B_\nu, B_{\nu'}]d\log\frac{b_\nu - b_{\nu'}}{ty-b_{\nu'}}.$$

(2.4.39) follows directly from (2.4.41) in the limit $t\to 0$. Especially for $\Lambda^0 = \sum_\mu A^0_\mu$ we have

$$d\Lambda^0 = \sum_\mu [\Lambda^0, A^0_\mu]d\log(y-a_\mu).$$

Hence we have also

$$(2.4.43)\qquad d't^{\Lambda^0} = \sum_\mu [t^{\Lambda^0}, A^0_\mu]d\log(y-a_\mu),$$

where d' denotes the exterior differentiations with respect to $a_1, \cdots, a_{n_1}$, $b_1, \cdots, b_{n_2}$ and y. Using (2.4.43) we can rewrite (2.4.42) as follows.

$$d'\tilde{B}_\nu = -\sum_\mu [\tilde{B}_\nu, t^{\Lambda^0}A_\mu t^{-\Lambda^0}]d'\log(b_\nu - ta_\mu) - \sum_\mu [A^0_\mu, \tilde{B}_\nu]d'\log(y-a_\mu) - \sum_{\nu'(\neq\nu)}[\tilde{B}_\nu, \tilde{B}_{\nu'}]d'\log\frac{b_\nu - b_{\nu'}}{ty-b_{\nu'}}.$$

Taking the limit $t\to 0$, we have (2.4.40). Here we also use (2.4.33).

So far, we have considered the behaviour of A_μ and B_ν in the limit $t\to 0$. Let us now consider that of Y.

Proposition 2.4.12. *Let* $A_\mu(t)$ $(\mu=1, \cdots, n_1)$, $B_\nu(t)$ $(\nu=1, \cdots, n_2)$ *be a solution of* (2.4.19) *satisfying* (2.4.24). *The following limits exist and satisfy the linear total differential equations below.*

$$(2.4.44)\quad Y_1(y, x; a_1, \cdots, a_{n_1}) = \lim_{t\to 0} Y\left(y, x; a_1, \cdots, a_{n_1}, \frac{b_1}{t}, \cdots, \frac{b_{n_2}}{t}\right),$$

$$(2.4.45)\quad dY_1 = \left(\sum_\mu A^0_\mu d\log\frac{x-a_\mu}{y-a_\mu}\right)Y_1,$$

(2. 4. 46) $$Y_2(y, x, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2}) = \lim_{t\to 0} t^{\Lambda^0} Y\left(y, \frac{x}{t}, a_1, \cdots, a_{n_1}, \frac{b_1}{t}, \cdots, \frac{b_{n_2}}{t}\right)$$

(2. 4. 47) $$dY_2 = \left(\sum_\mu A_\mu^0 d \log \frac{x}{y-a_\mu} + \sum_\nu B_\nu^0 d \log \frac{x-b_\nu}{b_\nu}\right) Y_2 .$$

Similarly, for the system (2. 4. 18) *we have the following limit.*

(2. 4. 48) $$Y_1(y, x; a_1, \cdots, a_{n_1}) = \lim_{t\to 0} Y(y, x; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})$$

(2. 4. 49) $$dY_1 = \left(\sum_\mu A_\mu^0 d \log \frac{x-a_\mu}{y-a_\mu} - \Lambda^0 d \log \frac{x}{y}\right) Y_1$$

where $\Lambda^0 = \sum_\mu A_\mu^0 + A_\infty^0$.

(2. 4. 50) $$Y_2(y, x; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2}) = \lim_{t\to 0} t^{\Lambda^0} Y(y, tx; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})$$

(2. 4. 51) $$dY_2 = \left(\sum_\mu A_\mu^0 d \log \frac{a_\mu}{y-a_\mu} - \sum_\nu B_\nu^0 d \log \frac{x-b_\nu}{y}\right) Y_2 .$$

Proof. We shall prove (2. 4. 46) and (2. 4. 47). Other cases are proved similarly. From (2. 3. 43) we have

(2. 4. 52) $$dY\left(y, \frac{x}{t}, a_1, \cdots, a_{n_1}, \frac{b_1}{t}, \cdots, \frac{b_{n_2}}{t}\right) = \left(\sum_\mu A_\mu d \log \frac{x-ta_\mu}{t(y-a_\mu)} + \sum_\nu B_\nu d \log \frac{x-b_\nu}{ty-b_\nu}\right) \times Y\left(y, \frac{x}{t}, a_1, \cdots, a_{n_1}, \frac{b_1}{t}, \cdots, \frac{b_{n_2}}{t}\right)$$

If we abbreviate $Y\left(y, \frac{x}{t}, a_1, \cdots, a_{n_1}, \frac{b_1}{t}, \cdots, \frac{b_{n_2}}{t}\right)$ to $Y(t)$, we have

(2. 4. 53) $$t\frac{dY(t)}{dt} = A(t) Y(t),$$

where

$$A(t) = -\Lambda^0 + B(t) + tC(t)$$

$$B(t) = -\sum_\mu (A_\mu(t) - A_\mu^0)$$

$$C(t)=\sum_{\mu} A_{\mu}(t)\frac{a_{\mu}}{ta_{\mu}-x}+\sum_{\nu} B_{\nu}(t)\frac{y}{b_{\nu}-ty}.$$

We claim that there exists a holomorphic matrix $Q(t)$ in a sector such that

$$Y(t)=t^{-A^0}Q(t)Y_0, \tag{2.4.54}$$

where Y_0 is a constant matrix and

$$|Q(t)-1|=0(|t|^{1-\sigma_1}).$$

Substituting (2.4.54) into (2.4.53) we have

$$t\frac{dQ}{dt}=t^{A^0}(B(t)+tC(t))t^{-A^0}Q(t).$$

Again (2.4.33) assures the existence of such $Q(t)$. Thus we have proved the existence of the limit (2.4.46). The proof of (2.4.47) is similar to that of (2.4.42), so we omit it.

Let us assume that $A^0_{\mu}(y; a_1, \cdots, a_{n_1})$ $(\mu=1, \cdots, n_1)$, $A^0(y; a_1, \cdots, a_{n_1})$ and $B^0_{\nu}(y; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})$ $(\nu=1, \cdots, n_2)$ satisfy (2.4.37) and (2.4.38). We also assume (2.4.21). For fixed $a_1, \cdots, a_{n_1}$ and y, B^0_{ν} $(\nu=1, \cdots, n_2)$ satisfy the Schlesinger's equation. If we denote by d' the exterior differentiation with respect to $a_1, \cdots, a_{n_1}$ and y, from (2.4.38) and (2.4.51) we see easily that $d'(Y_2^{-1}B^0_{\nu}Y_2)=0$. This implies that trace $B^0_{\mu}B^0_{\nu}$ is independent of $a_1, \cdots, a_{n_1}$ and y. Hence $B^0_{\nu}(\nu=1, \cdots, n_2)$ determine a unique inner automorphism class, which we denote by $\mathscr{S}_1$. We also denote by $\mathscr{S}_2$ the inner automorphism class determined by A^0_{μ} $(\mu=1, \cdots, n_1)$ and $-A^0$.

Let $A_{\mu}(t; y, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})$ $(\mu=1, \cdots, n_1, \infty)$, $B_{\nu}(t; y, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})$ $(\nu=1, \cdots, n_2)$ denote the unique solution of (2.4.18) satisfying the asymptotic conditions

$$|A_{\mu}(t; y, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})-A^0_{\mu}(y; a_1, \cdots, a_{n_1})|<C|t|^{1-\sigma_1} \qquad (\mu=1, \cdots, n_1, \infty)$$

$$|\widetilde{B}_{\nu}(t; y, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})-B^0_{\nu}(y; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})| <C|t|^{1-\sigma_1} \qquad (\nu=1, \cdots, n_2)$$

where $A_\infty^0 = A^0 - \sum_{\mu=1}^{n_1} A_\mu^0$. (See Theorem 2.4.8.) There exists a unique solution $A_\mu(y; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})$ $(\mu=1, \cdots, n_1)$, $B_\nu(y; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})$ $(\nu=1, \cdots, n_2)$ of (2.3.43) satisfying

$$A_\mu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2}) = A_\mu(t; y, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})$$

$$B_\nu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2}) = B_\nu(t; y, a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2}).$$

We denote by $\mathscr{S}_0$ the inner automorphism class of this solution.

Theorem 2.4.13. *Under the above assumptions, we have*

$$\lim_{t\to 0} \frac{\tau_{\mathscr{S}_0}(a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})}{\tau_{\mathscr{S}_1}(tb_1, \cdots, tb_{n_2})} = \text{const. } \tau_{\mathscr{S}_2}(a_1, \cdots, a_{n_1}, 0) \tag{2.4.55}$$

Proof. We have

$$\begin{aligned}
&d\log\{\tau_{\mathscr{S}_0}(a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})/\tau_{\mathscr{S}_1}(tb_1, \cdots, tb_{n_2})\}\\
&\quad = \sum_{\mu<\mu'} \text{trace } A_\mu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\\
&\qquad \times A_{\mu'}(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\, d\log(a_\mu - a_{\mu'})\\
&\qquad + \sum_{\mu,\nu} \text{trace } A_\mu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\\
&\qquad \times B_\nu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\, d\log(a_\mu - tb_\nu)\\
&\qquad + \sum_{\nu<\nu'} (\text{trace } B_\nu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\\
&\qquad \times B_{\nu'}(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\\
&\qquad - \text{trace } B_\nu^0(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\\
&\qquad \times B_{\nu'}^0(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2}))\, d\log t(b_\nu - b_{\nu'})
\end{aligned}$$

Noting that

$$\begin{aligned}
&\text{trace } B_\nu^0(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\, B_\nu^0(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\\
&\quad = \text{trace } B_\nu^0\left(\frac{y}{t}; \frac{a_1}{t}, \cdots, \frac{a_{n_1}}{t}, b_1, \cdots, b_{n_2}\right) B_{\nu'}^0\left(\frac{y}{t}; \frac{a_1}{t}, \cdots, \frac{a_{n_1}}{t}, b_1, \cdots, b_{n_2}\right)\\
&\quad = \text{trace } B_\nu^0(y; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2})\, B_{\nu'}^0(y; a_1, \cdots, a_{n_1}, b_1, \cdots, b_{n_2}),
\end{aligned}$$

and that

$$\text{trace } B_\nu(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})\, B_{\nu'}(y; a_1, \cdots, a_{n_1}, tb_1, \cdots, tb_{n_2})$$

$$=\operatorname{trace}\widetilde{B}(t;y,a_1,\cdots,a_{n_1},b_1,\cdots,b_{n_2})$$
$$\times\widetilde{B}_{\nu'}(t;y,a_1,\cdots,a_{n_1},b_1,\cdots,b_{n_2}),$$

we have

$$\frac{d}{dt}\log\frac{\tau_{\mathscr{S}0}(a_1,\cdots,a_{n_1},tb_1,\cdots,tb_{n_2})}{\tau_{\mathscr{S}_1}(tb_1,\cdots,tb_{n_2})}=0(|t|^{-\sigma_1}).$$

This implies that $\log\{\tau_{\mathscr{S}_0}(a_1,\cdots,a_{n_1},tb_1,\cdots,tb_{n_2})/\tau_{\mathscr{S}_1}(tb_1,\cdots,tb_{n_2})\}$ has a finite limit when $t\to 0$. If we denote by d' the exterior differentiation with respect to $a_1,\cdots,a_{n_1},b_1,\cdots,b_{n_2}$, we have in this limit

$$d'(\lim_{t\to 0}\log\{\tau_{\mathscr{S}_0}(a_1,\cdots,a_{n_1},tb_1,\cdots,tb_{n_2})/\tau_{\mathscr{S}_1}(tb_1,\cdots,tb_{n_2})\})$$
$$=\sum_{\mu<\mu'}\operatorname{trace}A_\mu^0(y;a_1,\cdots,a_{n_1})A_{\mu'}^0(y;a_1,\cdots,a_{n_1})\log(a_\mu-a_{\mu'})$$
$$-\sum_{\mu}\operatorname{trace}A_\mu^0(y;a_1,\cdots,a_{n_1})\Lambda^0(y;a_1,\cdots,a_{n_1})\log a_\mu$$
$$=d\log\tau_{\mathscr{S}_2}(a_1,\cdots,a_{n_1},0).$$

The last statement of Theorem 2. 4. 8 implies the following corollary to Theorem 2. 4. 13.

Corollary 2. 4. 14. *For sufficiently small* $|L_\mu|$ $(\mu=1,\cdots,n_1+n_2)$, *we have*

$$\lim_{t\to 0}\frac{\langle\varphi(a_1;L_1)\cdots\varphi(a_{n_1};L_{n_1})\varphi(tb_1+a_0;L_{n_1+1})\cdots\varphi(tb_{n_2}+a_0;L_{n_1+n_2})\rangle}{\langle\varphi(tb_1+a_0;L_{n_1+1})\cdots\varphi(tb_{n_2}+a_0;L_{n_1+n_2})\rangle}$$
$$=\text{const.}\ \langle\varphi(a_1;L_1)\cdots\varphi(a_{n_1};L_{n_1})\varphi(a_0;L_0)\rangle$$

where L_0 *is uniquely determined by* $e^{2\pi iL_0}=e^{2\pi iL_{n_1+1}}\cdots e^{2\pi iL_{n_1+n_2}}$ *and* $|L_0|\ll 1$.

Errata in [1].

Page 231, line -15, $\Lambda(V)|\text{vac}\rangle=0$ $\to V|\text{vac}\rangle=0$

line -12, $\langle\text{vac}|\Lambda(V^\dagger)=0$ $\to\langle\text{vac}|V^\dagger=0$

Page 250, line -11, $\rho=\frac{1}{2}(v_1,\cdots$ $\to\rho=(v_1,\cdots$

Page 255, line 10, $\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle\left(\begin{array}{c|ccc} & {}^t\boldsymbol{e} & & \\ & & & {}^t\boldsymbol{r} \\ \hline -\boldsymbol{e} & & -A(\Lambda) & 1 \\ & -\boldsymbol{r} & -1 & -R \end{array}\right)$

$$\rightarrow\langle g^{(1)}\rangle\cdots\langle g^{(n)}\rangle\ \text{Pfaffian}\left(\begin{array}{c|ccc} & {}^t\boldsymbol{e} & & \\ & & & {}^t\boldsymbol{r} \\ \hline -\boldsymbol{e} & & -A(\Lambda) & 1 \\ & -\boldsymbol{r} & -1 & -R \end{array}\right)$$

line 11, $\boldsymbol{e}=(e_{\mu_1},\cdots,e_{\mu_m},e^{\dagger}_{\nu_1},\cdots,e^{\dagger}_{\nu_m})$

$$\rightarrow\boldsymbol{e}=(\hat{e}_{\mu_1},\cdots,\hat{e}_{\mu_m},\hat{e}^{\dagger}_{\nu_1},\cdots,\hat{e}^{\dagger}_{\nu_m})$$

Page 260, line 11, The sign "=" should be inserted at the top of the line.

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List of symbols

$\mathscr{B}(M)$: the space of hyperfunctions on a real analytic manifold M ([16], [17]).

$$\underline{du} = \frac{du}{2\pi |u|} .$$

$$\sqrt{0 \pm iu} = \begin{cases} e^{\pm \pi i/4} |u|^{1/2} & u > 0 , \\ e^{\mp \pi i/4} |u|^{1/2} & u < 0 . \end{cases}$$

$$u_{\pm} = \begin{cases} |u| & u \gtrless 0 , \\ 0 & u \lessgtr 0 . \end{cases}$$

$$\theta(u) = \begin{cases} 1 & u > 0 , \\ 0 & u < 0 . \end{cases}$$

$$\varepsilon(u) = \begin{cases} 1 & u > 0 , \\ -1 & u < 0 . \end{cases}$$