

A Characterization of the Hyperbolic Mixed Problems in a Quarter Space for Differential Operators with Constant Coefficients

By

Yoshihiro SHIBATA*

§ 1. Introduction

L. Gårding showed in his papers [2] and [3] that the following three statements on a differential operator $P(D)$ with constant coefficients are equivalent (see also Atiyah, Bott and Gårding [1]).

(I) The Cauchy problem for $P(D)$ in the half space $\{x \in \mathbf{R}^n; x \cdot \vartheta > 0\}$ with data on the plane $x \cdot \vartheta = 0$ is C^∞ (or \mathcal{E}) well posed, where ϑ is a non-zero vector in the real dual \mathbf{E}^n of the real n -dimensional Euclidean space \mathbf{R}^n by the inner product $x \cdot \xi = x_1 \cdot \xi_1 + \cdots + x_n \cdot \xi_n$.

(II) $P(D)$ has a fundamental solution E satisfying $P(D)E = \delta$ and having support in some closed proper¹⁾ cone K with its vertex at the origin such that $x \cdot \vartheta > 0$ on $K - \{0\}$.

(III) The characteristic polynomial $P(\xi)$ of the operator $P(D)$ has the properties:

$$(1.1) \quad P^0(\vartheta) \neq 0 \text{ where } P^0(\xi) \text{ is the principal part of } P(\xi),$$

and

$$(1.2) \quad P(\xi + s\vartheta) \neq 0 \text{ when } \xi \in \mathbf{E}^n \text{ and } \text{Im } s \text{ is less than some fixed number } \gamma_0.$$

Such a differential operator $P(D)$ (resp. a polynomial $P(\xi)$) is said to be *hyperbolic* with respect to ϑ .

Now consider the mixed initial-boundary value problem in the quarter space $\{x = (x_1, \dots, x_n) \in \mathbf{R}^n; x_1 > 0, x_n > 0\}$ for a system $\{P(D); B_j(D)\}$,

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* Institute of Mathematics, The University of Tsukuba.

1) A cone which does not contain any straight lines is said to be *proper*.

$j=1, \dots, q\}$ of differential operators with constant coefficients:

$$(1.3) \quad P(D)u(x) = f(x), \quad x_1 > 0, x_n > 0,$$

$$(1.4) \quad D_1^j u(x)|_{x_1=0} = g_j(x''), \quad j=0, 1, \dots, m-1, x_n > 0,$$

$$(1.5) \quad B_j(D)u(x)|_{x_n=0} = h_j(x'), \quad j=1, \dots, q, x_1 > 0,$$

where m is the order of $P(D)$. Then the corresponding question for the system $\{P(D); B_j(D), j=1, \dots, q\}$ can be proposed. After the works [5], [6] by R. Hersh and [11] by T. Shirota, R. Sakamoto [9] proved the following.

Assume that $P(\xi)$ is a hyperbolic polynomial with respect to $\vartheta = (1, 0, \dots, 0)$ and the boundary $\partial R_+^n = \{x \in R_+^n; x_n = 0\}$ ($R_+^n = \{x = (x_1, \dots, x_n) \in R^n; x_n > 0\}$) is non-characteristic for $P(D)$ and $B_j(D)$ ($j=1, \dots, q$) and that q is equal to the constant number p of the roots with positive imaginary part of the equation $P(\xi' - is\vartheta', \lambda) = 0$ in λ with s large enough where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Then the mixed problem (1.3)-(1.5) is C^∞ (or \mathcal{E}) well posed if and only if the Lopatinski determinant $R(\zeta')$ ²⁾ of $\{P(D); B_j(D), j=1, \dots, p\}$ has the properties corresponding to (1.1) and (1.2):

$$(1.6) \quad R^0(\vartheta') \neq 0, \text{ where } R^0(\zeta') \text{ is the principal part of } R(\zeta').$$

$$(1.7) \quad R(\xi' + s\vartheta') \neq 0 \text{ when } \xi' \in \mathbf{E}^{n-1} \text{ and } \text{Im } s \text{ is less than some fixed number.}$$

Such a Lopatinski determinant $R(\zeta')$ is said to be *hyperbolic* with respect to ϑ' . Moreover she showed the existence of a unique system $\{E(x, y); F_j(x), j=1, \dots, p\}$ of fundamental solutions having support in some closed proper cones.

A system $\{E(x, y); F_j(x), j=1, \dots, q\}$ of fundamental solutions of the mixed problems (1.3)-(1.5) for $\{P(D); B_j(D), j=1, \dots, q\}$ is defined as a system of solutions of the following equations such that $E(x, y) \in \mathcal{D}'(R_+^n) \cap C^\infty([0, y_n]; \mathcal{D}'(R^{n-1}))$ ³⁾ $\cap C^\infty((y_n, \infty); \mathcal{D}'(R^{n-1}))$ and $F_j(x), j=1, \dots, q$, belong to $C^\infty([0, \infty); \mathcal{D}'(R^{n-1})) \subset \mathcal{D}'(\bar{R}_+^n)$.

$$(1.8) \quad P(D_x)E(x, y) = \delta(x - y), \quad x_n > 0, y = (0, y'') \in R_+^n,$$

2) This will be defined in Section 4.
 3) This space will be defined in Section 3.

$$(1.9) \quad B_j(D_x) E(x, y)|_{x_n=0} = 0, \quad j=1, \dots, q,$$

$$(1.10) \quad P(D) F_k(x) = 0, \quad x_n > 0, \quad k=1, \dots, q,$$

$$(1.11) \quad B_j(D) F_k(x)|_{x_n=0} = \delta_{jk}(x'), \quad j, k, = 1, \dots, q,$$

where δ is the Dirac delta function, $y'' = (y_2, \dots, y_n)$, $x' = (x_1, \dots, x_{n-1})$ and δ_{jk} is the Kronecker delta symbol (i.e., $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$). D_j denotes $i^{-1}\partial/\partial x_j$ and D denotes (D_1, \dots, D_n) .

We say that a system $\{P(D); B_j(D), j=1, \dots, q\}$ of differential operators is *hyperbolic* in \mathbf{R}_+^n with respect to $\vartheta = (1, 0, \dots, 0)$ if it has a unique system $\{E(x, y); F_j(x), j=1, \dots, q\}$ of fundamental solutions of the mixed problem (1.3)-(1.5) such that the support of $E(x, y)$ with respect to x is contained in $(K + y) \cap \overline{\mathbf{R}_+^n}$ and the supports of the $F_k(x)$ are contained in $K \cap \overline{\mathbf{R}_+^n}$. Here K is a closed proper cone with its vertex at the origin such that $x \cdot \vartheta > 0$ on $K - \{0\}$.

The purpose of this paper is to characterize the hyperbolicity for the mixed problem (1.3)-(1.5) by showing that $\{P(D); B_j(D), j=1, \dots, q\}$ is hyperbolic in \mathbf{R}_+^n with respect to $\vartheta = (1, 0, \dots, 0)$ if and only if $P(\xi)$ and $R(\zeta')$ are both hyperbolic with respect to ϑ and ϑ' , respectively. Note that we do not assume that the hyperplane $x_n = 0$ is non-characteristic for $P(D)$ and $B_j(D)$. We shall show

Main Theorem. *If a system $\{P(D); B_j(D), j=1, \dots, q\}$ of differential operators is hyperbolic in \mathbf{R}_+^n with respect to $\vartheta = (1, 0, \dots, 0)$, then the characteristic polynomial $P(\xi)$ has the properties (1.1) and (1.2) with respect to ϑ and the number q of the boundary operators $B_j(D)$ must be equal to the constant number p of the roots with positive imaginary part of the equation $P(\xi' - is\vartheta', \lambda) = 0$ in λ with s large enough and the Lopatinski determinant $R(\zeta')$ has the properties (1.6) and (1.7).*

The plan of the remainder is as follows. In Section 2 we prove the hyperbolicity of the characteristic polynomial $P(\xi)$ with respect to ϑ . We give some preliminaries for the latter sections in Section 3. In Section 4 we prove the hyperbolicity of the Lopatinski determinant with respect to ϑ' under the assumption $p = q \geq 1$. In Section 5 we complete our proof

of the main theorem. In Section 6 the system of fundamental solutions $\{E(x, y); F_j(x), j=1, \dots, q\}$ are constructed under the assumptions:

- (I) $P(\xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \dots, 0)$,
- (II) $p = q$,
- (III) The Lopatinski determinant of the system $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' .

Moreover we shall prove that $E(x, y)$ belongs to $C^\infty([0, y_n]; \mathcal{D}'(\mathbf{R}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ and $F_j(x)$ belongs to $C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$, $j=1, \dots, q$ and that such a system is unique. When the boundary $x_n=0$ is non-characteristic for $P(D)$, we give a proof of the uniqueness of this system. The proof is done by using the Paley-Wiener-Schwartz theorem. When the condition is not assumed, we prove it by a standard method employing the adjoint system of $\{P(D); B_j(D), j=1, \dots, p\}$ and by modifying the proof given in [9].

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§ 2. Hyperbolicity of the Characteristic Polynomial

In this section we shall prove that the existence of certain fundamental solution $E(x, y)$ implies hyperbolicity of the characteristic polynomial $P(\xi)$. This can be done by a slight modification of the proof given in Atiyah, Bott and Gårding [1].

Proposition 2.1. *Assume that for some $y = (0, y'') \in \mathbf{R}_+^n$ there exists a fundamental solution $E(x, y)$ of the mixed problem (1.3)–(1.5) (i.e., solution of the equations (1.8) and (1.9)) such that the support of $E(x, y)$ with respect to x is contained in $(K + y) \cap \overline{\mathbf{R}_+^n}$. Here K is a closed proper cone with its vertex at the origin such that $x \cdot \vartheta > 0$ on $K - \{0\}$. Then the characteristic polynomial $P(\xi)$ is hyperbolic with respect to $\vartheta = (1, 0, \dots, 0)$.*

Proof. Let U be a neighborhood of the point y such that $\bar{U} \subset \mathbf{R}_+^n$

and $\phi \in C_0^\infty(\mathbf{R}_+^n)$ be equal to 1 in U . Then we have

$$\phi(x)P(D_x)E(x, y) = \delta(x - y) \quad \text{in } \mathbf{R}^n,$$

so it follows from Leibniz's formula that

$$(2.1) \quad P(D)f(x) = \delta(x - y) + g(x) \quad \text{in } \mathbf{R}^n,$$

where $f(x) = \phi(x)E(x, y)$ and $g(x) = \sum_{|\alpha| \geq 1} (D^\alpha \phi(x)) (P^{(\alpha)}(D_x)E(x, y)) / \alpha!$ are distributions with compact support. Taking Fourier-Laplace transforms we have

$$(2.2) \quad P(\zeta)\hat{f}(\zeta) = \exp\{-iy \cdot \zeta\} + \hat{g}(\zeta), \quad \zeta \in \mathbf{C}^n = \mathfrak{E}^n + i\mathfrak{E}^n,$$

where $\hat{f}(\zeta) = \mathcal{F}f$, $\hat{g}(\zeta) = \mathcal{F}g$ are entire functions. If $P(\zeta)$ is a non-vanishing constant, the conditions (1.1) and (1.2) are satisfied, so we assume that $P(\zeta)$ is not a constant. Then g cannot identically vanish because $P(\zeta)$ has zeros. Since $g \in \mathcal{E}'(\mathbf{R}^n)$ we have for some constants C and N

$$(2.3) \quad |\hat{g}(\zeta)| \leq C(1 + |\zeta|)^N \exp h(\text{Im } \zeta),$$

where

$$(2.4) \quad h(\eta) = \text{Max } x \cdot \eta, \quad x \in \text{supp } g, \quad \eta \in \mathfrak{E}^n.$$

We note that

$$(2.5) \quad h(-\vartheta) > 0.$$

In fact, this follows from the fact that the support of g is contained in $(\text{supp } \phi - U) \cap (K + y)$. From (2.2) and (2.3) we find for any $\zeta \in \mathbf{C}^n$ with $P(\zeta) = 0$

$$|\exp\{-iy \cdot \zeta\}| \leq C(1 + |\zeta|)^N \exp h(\text{Im } \zeta).$$

After taking logarithms of both sides, we get for any $\zeta \in \mathbf{C}^n$ with $P(\zeta) = 0$

$$(2.6) \quad -h(\text{Im } \zeta) + y \cdot \text{Im } \zeta \leq C_1 + C_2 \log(1 + |\zeta|),$$

where C_1 and C_2 are some positive constants independent of ζ .

We shall now prove that the plane $x \cdot \vartheta = 0$ is non-characteristic with respect to $P(D)$, i.e., $P^0(\vartheta) \neq 0$. We will argue by contradiction. Suppose that $P^0(\vartheta) = 0$. Since $P^0(\xi) \neq 0$, we take a $\xi^0 \in \mathfrak{E}^n$ such that

$P^0(\xi^0) \neq 0$, and consider the polynomial in two variable t and s :

$$P(st(\xi^0 - \vartheta) + t\vartheta) = P^0(s(\xi^0 - \vartheta) + \vartheta)t^m + \cdots = c_0(s)t^m + \cdots.$$

Since $c_0(0) = P^0(\vartheta) = 0$ and $c_0(1) = P^0(\xi^0) \neq 0$, the algebraic equation in t : $P(st(\xi^0 - \vartheta) + t\vartheta) = 0$ has a solution $t(s)$ in a neighborhood of $s=0$ such that

$$(2.7) \quad t(s) = as^p(1 + o(1)), \quad s \rightarrow 0$$

where a is a non-zero complex number and p is a negative rational number. In fact, the assumption $P^0(\vartheta) = 0$ implies that p is negative. If we choose the path $\gamma: |s| \leq \delta$ and $\arg s = (-\pi/2 - \arg a)/p$, we have

$$\operatorname{Im} t(s) = -|a| \cdot |s|^p(1 + o(1)) \quad \text{as } s \rightarrow 0 \text{ along the path } \gamma$$

and therefore

$$(2.8) \quad \operatorname{Im} \zeta(s) = \operatorname{Im} (st(s)(\xi^0 - \vartheta) + t(s)\vartheta) = -|a||s|^p\vartheta(1 + o(1))$$

as $s \rightarrow 0$ along the path γ , because $st(s) = as^{p+1}(1 + o(1))$, $s \rightarrow 0$. Since $y \cdot \operatorname{Im} \zeta(s) = y'' \cdot \operatorname{Im} \zeta''(s) = \operatorname{Im} st(s)(\xi^0)'' \cdot y''$, it follows from (2.5), (2.6) and (2.8) that

$$|s|^p \leq C_3 + C_4 \log(1 + |s|^p) \quad \text{as } s \rightarrow 0 \text{ along the path } \gamma,$$

where C_3 and C_4 are some positive constants. Then we have a contradiction. This proves that $P^0(\vartheta) \neq 0$.

Next we proceed to show (1.2) with $\vartheta = (1, 0, \dots, 0)$. Since $P^0(\vartheta) \neq 0$, the algebraic equation $P(\xi + s\vartheta) = 0$ in s has m roots $s_j(\xi)$, $j=1, \dots, m$, (counted according to multiplicity). Set

$$\mu(r) = \sup_{\substack{|\xi|=r \\ j=1, \dots, m}} -\operatorname{Im} s_j(\xi).$$

For every $s_j(\xi)$ we have from (2.6)

$$(2.9) \quad h(-\vartheta) \operatorname{Im} s_j(\xi) \leq C_1 + C_2 \log(1 + |\xi| + s_j(\xi))$$

because $y \cdot \vartheta = 0$. Then we have

$$(2.10) \quad \mu(r) = ar^p(1 + o(1)), \quad r \rightarrow +\infty,$$

where p is rational and a is real. For we write

$$P(\xi + s\vartheta) = R_1(\xi, \operatorname{Re} s, \operatorname{Im} s) + iR_2(\xi, \operatorname{Re} s, \operatorname{Im} s).$$

Here R_1 and R_2 are polynomials with constant coefficients in the $n + 2$ real variables $\xi = (\xi_1, \dots, \xi_n)$, $\text{Re } s$ and $\text{Im } s$. If we set

$$M_r = \{(\xi, \text{Re } s, \text{Im } s); |\xi|^2 = r^2, R_k(\xi, \text{Re } s, \text{Im } s) = 0, k = 1, 2\},$$

then $\mu(r) = \sup_{(\xi, \text{Re } s, \text{Im } s) \in M_r} -\text{Im } s$. By the Tarski-Seidenberg theorem we have either (2.10) or $\mu(r) \equiv +\infty$ for all large r . Since $P^0(\vartheta) \neq 0$ as proved above, the roots $s_j(\xi)$ are multivalued continuous functions. Hence the function $\sup_{j=1, \dots, m} |s_j(\xi)|$ is bounded when ξ remains bounded. Thus $\mu(r)$ is finite for any finite $r \rightarrow 0$, so we have (2.10). From (2.9) and (2.10), it follows that

$$-h(-\vartheta)\mu(r) \leq C_1 + C_2 \log(1 + C_3 r),$$

where C_3 is a new positive constant. In view of (2.5) and (2.10) this inequality means that $\mu(r)$ remains bounded when $r \rightarrow +\infty$, namely there exists a real γ_0 such that

$$-\text{Im } s_j(\xi) \leq \gamma_0 \text{ for any } j \text{ (} 1 \leq j \leq m \text{) and for any } \xi \in \mathfrak{E}^n.$$

Therefore, $P(\xi + s\vartheta) \neq 0$ when $\text{Im } s < -\gamma_0$ and $\xi \in \mathfrak{E}^n$. So the theorem is proved.

§ 3. Preliminaries for Latter Sections

We denote by $\overline{\mathbf{R}}_+^n$ the space $\{x = (x', x_n) \in \mathbf{R}^n; x_n \geq 0\}$, i.e., the closure of \mathbf{R}_+^n and by $\partial\mathbf{R}_+^n$ the boundary plane $x_n = 0$ of \mathbf{R}_+^n . Let \mathcal{Q} be a non empty open set in \mathbf{R}^n and let $\mathcal{D}(\mathcal{Q})$ denote the space of all complex C^∞ functions with compact support in \mathcal{Q} , which is equipped with the Schwartz topology. We set

$$(3.1) \quad \mathring{\mathcal{D}}(\overline{\mathbf{R}}_+^n) = \{\phi \in \mathcal{D}(\mathbf{R}^n); \text{supp } \phi \subset \overline{\mathbf{R}}_+^n\}.$$

This is a closed subspace of $\mathcal{D}(\mathbf{R}^n)$ and we give it the induced topology. We denote by $\mathcal{D}'(\overline{\mathbf{R}}_+^n)$ the dual space of $\mathring{\mathcal{D}}(\overline{\mathbf{R}}_+^n)$. By the Hahn-Banach theorem, any element of $\mathcal{D}'(\overline{\mathbf{R}}_+^n)$ can be extended to a continuous linear form of $\mathcal{D}(\mathbf{R}^n)$, that is, a distribution on \mathbf{R}^n . On the other hand $\mathcal{D}(\mathbf{R}_+^n)$ is a dense subset of $\mathring{\mathcal{D}}(\overline{\mathbf{R}}_+^n)$ and therefore a continuous linear form on $\mathring{\mathcal{D}}(\overline{\mathbf{R}}_+^n)$ is uniquely determined by its restriction to $\mathcal{D}(\mathbf{R}_+^n)$ and can be identified with an element of $\mathcal{D}'(\mathbf{R}_+^n)$ consists of all distributions in \mathbf{R}_+^n .

Thus the space $\mathcal{D}'(\overline{\mathbf{R}}_+^n)$ consists of all distributions in \mathbf{R}_+^n which are extendable to distributions on \mathbf{R}^n . Namely

$$(3.2) \quad \mathcal{D}'(\overline{\mathbf{R}}_+^n) = \mathcal{D}'(\mathbf{R}^n) / \mathcal{D}'(\overline{\mathbf{R}}_-^n),$$

where $\mathcal{D}'(\overline{\mathbf{R}}_-^n) = \{u \in \mathcal{D}'(\mathbf{R}^n); \text{supp } u \subset \overline{\mathbf{R}}_-^n = \{x = (x', x_n); x_n \leq 0\}\}$. We denote by $C^\infty(I; \mathcal{D}'(\mathbf{R}^{n-1}))$ the space of infinitely differentiable functions which are defined on interval I in \mathbf{R}^1 with values in $\mathcal{D}'(\mathbf{R}^{n-1})$ equipped with the weak topology. When $I = [a, b]$, $u \in C^\infty(I; \mathcal{D}'(\mathbf{R}^{n-1}))$ means by definition that for any $\phi \in \mathcal{D}(\mathbf{R}^{n-1})$, $\langle u(t), \phi \rangle$ is a C^∞ function on $[a, b]$, that is, there exists a C^∞ function $f_\phi(t)$ defined in $(a - \delta, b)$ for some $\delta > 0$ such that $\langle u(t), \phi \rangle = f_\phi(t)$ on $[a, b]$. Then, for any $u \in C^\infty([a, b]; \mathcal{D}'(\mathbf{R}^{n-1}))$ there exists a $v \in C^\infty((a - \delta', b); \mathcal{D}'(\mathbf{R}^{n-1}))$ such that $u(t) = v(t)$ on $[a, b]$. This follows from a theorem due to Seeley [10] which guarantees the existence of a linear operator T from $C^\infty([0, A])$ to $C^\infty((-A, A))$ ($A > 0$) such that for any $f \in C^\infty([0, A])$, $(Tf)(t) = f(t)$ on $[0, A]$ and

$$\text{Max}_{t \in [-A, A]} |D_t^k (Tf)(t)| \leq C_{k,A} \text{Max}_{\substack{t \in [0, A] \\ 0 \leq j \leq k}} |D_t^j f(t)|,$$

where $C_{k,A}$ is a positive constant depending on k and A but not on f . In fact, define a linear form $\langle v(t), \phi \rangle$ on $\mathcal{D}(\mathbf{R}^{n-1})$ by $T\langle u(t), \phi \rangle$. Then it is easy to see that $v(t)$ is a C^∞ function on $(-A, A)$ with values in $\mathcal{D}'(\mathbf{R}^{n-1})$. Clearly we have

$$(3.3) \quad C^\infty([0, \infty), \mathcal{D}'(\mathbf{R}^{n-1})) \subset \mathcal{D}'(\overline{\mathbf{R}}_+^n) \subset \mathcal{D}'(\mathbf{R}_+^n).$$

In our definition of a system $\{E(x, y); F_j(x), j=1, \dots, q\}$ of fundamental solutions of the mixed problem (1.3)-(1.5), we assumed that $E(x, y) \in C^\infty([0, y_n]; \mathcal{D}'(\mathbf{R}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ and $F_j(x) \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$.⁴⁾ These assumptions are reasonable because the fundamental solutions which we shall construct belong to such a space (see section 6). As already remarked, we do not assume that $\partial\mathbf{R}_+^n$ is non-characteristic for $P(D)$. However if $\partial\mathbf{R}_+^n$ is non-characteristic for $P(D)$, we need not assume that $E(x, y) \in C^\infty([0, y_n]; \mathcal{D}'(\mathbf{R}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ and $F_j(x) \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$. It is sufficient

4) See introduction.

to assume that $E(x, y)$ and $F_j(x)$ belong to $\mathcal{D}'(\overline{\mathbf{R}}_+^n)$, in view of the following.

Proposition 3.1. *Let $P(x, D)$ be a differential operator of order m with coefficients in $C^\infty(\mathbf{R}^n)$ and assume that $\partial\mathbf{R}_+^n$ is non-characteristic with respect to the differential operator. If $u \in \mathcal{D}'(\overline{\mathbf{R}}_+^n)$ and $P(x, D)u \in C^\infty([0, \delta]; \mathcal{D}'(\mathbf{R}^{n-1}))$, then we have that u belongs to $C^\infty([0, \delta]; \mathcal{D}'(\mathbf{R}^{n-1}))$.*

Proof. Since $u \in \mathcal{D}'(\overline{\mathbf{R}}_+^n)$, there exists a distribution $\tilde{u} \in \mathcal{D}'(\mathbf{R}^n)$ with $\tilde{u} = u$ in \mathbf{R}_+^n . Let \mathcal{Q} be an open set of $\{x \in \mathbf{R}^n; x_n < \delta\}$ such that the closure $\overline{\mathcal{Q}}$ is compact. Then there exists a real s such that $\phi \cdot \tilde{u} \in \mathcal{H}_{(s,0)}(\mathbf{R}^n)$ for any $\phi \in C_0^\infty(\mathcal{Q})$, where s is independent of ϕ and $\mathcal{H}_{(s,\sigma)} = \{f \in \mathcal{S}'(\mathbf{R}^n); (1 + |\xi|^2)^{s/2} (1 + |\xi'|^2)^{\sigma/2} \hat{f}(\xi) \in L^2(\mathbf{E}^n)\}$. If we set $v = \phi \tilde{u}$, $g = \phi P(x, D)\tilde{u}$, Leibniz's formula gives

$$(3.4) \quad P(x, D)v = g + Q(x, D)\tilde{u} \quad \text{in } \mathbf{R}^n$$

where $Q(x, D)$ is a differential operator of order $m - 1$ with coefficients in $C_0^\infty(\mathcal{Q})$:

$$Q(x, D)\tilde{u} = \sum_{\substack{|\alpha| + |\beta| \leq m \\ |\alpha| > 0}} b_{\alpha\beta}(x) (D^\alpha \phi) (D^\beta \tilde{u}).$$

Let $\psi(x') \in C_0^\infty(\mathbf{R}^{n-1})$. Then we get from (3.4)

$$(3.5) \quad P(x, D)(\psi_x^* v) = \psi_x^* g + \psi_x^* (Q(x, D)\tilde{u}) + [P(x, D), \psi_x^*]v,$$

where $[P(x, D), \psi_x^*]v = P(x, D)(\psi_x^* v) - \psi_x^* (P(x, D)v)$. Since $\hat{\psi}(\xi')$ is rapidly decreasing C^∞ function of ξ' , we have $\psi_x^* (\mathcal{H}_{(s,\sigma)}) \subset \mathcal{H}_{(s,\infty)}$ for any reals s and σ . Thus $\psi_x^* v \in \mathcal{H}_{(s,\infty)}$ and so $\psi_x^* v|_{\mathbf{R}_+^n} = \psi_x^* (\phi u) \in \mathcal{H}_{(s,\infty)}(\overline{\mathbf{R}}_+^n) = \{w \in \mathcal{D}'(\mathbf{R}_+^n); \text{There exists } \tilde{w} \in \mathcal{H}_{(s,\infty)} \text{ such that } \tilde{w} = w \text{ in } \mathbf{R}_+^n\}$. Here we denoted $\phi|_{\mathbf{R}_+^n} u$ by ϕu for simplicity. From the assumption on $P(x, D)u$ and the relation

$$D^\alpha (g_x^* \psi)(x) = D_n^{\alpha_n} \langle g(y', x_n), D_x^{\alpha'} \psi(x' - y') \rangle,$$

it follows that $\psi_x^* g \in C_{(0)}^\infty(\mathbf{R}_+^n) = \{\phi \in C^\infty(\mathbf{R}_+^n); \text{There exists } \tilde{\phi} \in C_0^\infty(\mathbf{R}^n) \text{ such that } \tilde{\phi} = \phi \text{ in } \mathbf{R}_+^n\}$. Consider the term $\psi_x^* (Q(x, D)u)$ in (3.5). Take $\chi \in C_0^\infty(\mathcal{Q})$ such that $\chi(x) = 1$ on $\text{supp } \phi$. Since $\chi \tilde{u} \in \mathcal{H}_{(s,0)}$ and

$Q(x, D)$ is of order $m - 1$, we have $Q(x, D) \tilde{u} = Q(x, D) (\chi \tilde{u}) \in \mathcal{H}_{(s-m+1, 0)}$. Hence $\psi_{x'}^* (Q(x, D) u) \in \mathcal{H}_{(s-m+1, \infty)}(\overline{\mathbf{R}}_+^n)$. We have also $[P(x, D), \psi_{x'}^*] \phi u \in \mathcal{H}_{(s-m+1, \infty)}(\overline{\mathbf{R}}_+^n)$. In fact, we may assume from the assumption on $P(x, D)$ that $P(x, D)$ has the form

$$P(x, D) = D_n^m + \sum_{\substack{|\alpha| \leq m \\ \alpha_n < m}} a_\alpha(x) D^\alpha = D_n^m + R(x, D).$$

Then $[P(x, D), \psi_{x'}^*] v = [R(x, D), \psi_{x'}^*] v$. Since the order of $R(x, D)$ with respect to D_n is at most $m - 1$, we have $R(x, D) (\psi_{x'}^* v), \psi_{x'}^* (R(x, D) v) \in \mathcal{H}_{(s-m+1, \infty)}$ by using the fact the weight function $(1 + |\xi|^2)^{\sigma/2}$ is equivalent to $(1 + |\xi'|^2)^{\sigma/2} (1 + |\xi_n|^2)^{\sigma/2}$ and so $[P(x, D), \psi_{x'}^*] v \in \mathcal{H}_{(s-m+1, \infty)}$. This also follows from the fact that $[R(x, D), \psi_{x'}^*]$ is a pseudo-differential operator whose symbol is

$$-\sum_{\alpha} D_x^\alpha R(x, \xi) (\partial/\partial \xi)^\alpha \psi(\xi') / \alpha!.$$

Summing up, we have proved

$$P(x, D) (\psi_{x'}^* (\phi u)) \in \mathcal{H}_{(s-m+1, \infty)}(\overline{\mathbf{R}}_+^n).$$

By Theorem 4.3.1 in Hörmander [7], we get

$$\psi_{x'}^* (\phi u) \in \mathcal{H}_{(s+1, \infty)}(\overline{\mathbf{R}}_+^n).$$

Repeated application of the same argument gives that

$$\psi_{x'}^* (\phi u) \in \mathcal{H}_{(\infty, \infty)}(\overline{\mathbf{R}}_+^n).$$

Then

$$(\psi_{x'}^* (\phi u)) (0, x') = \langle (\phi u) (y', x_n), \check{\psi}(y') \rangle_{y'} \in C^\infty([0, \infty)).$$

Since $\phi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\mathbf{R}^{n-1})$ are arbitrary, we conclude that $u \in C^\infty([0, \delta); \mathcal{D}'(\mathbf{R}^{n-1}))$. This completes the proof.

Lemma 3.2. *Let $u(x_n)$ be a continuous function defined on the interval $[0, b]$ with values in $\mathcal{D}'(\mathbf{R}_x^{n-1})$. Then there exist constants C, r and a compact set $K \subset \mathbf{R}^{n-1}$ such that*

$$(3.6) \quad |\mathcal{F}_{x'} u(x_n) (\zeta')| \leq C (1 + |\zeta'|)^r \exp h(\operatorname{Im} \zeta'),$$

$$\zeta' = (\zeta_1, \dots, \zeta_{n-1}) \in \mathbf{C}^{n-1}$$

where $h(\text{Im } \zeta') = \text{Max}_{x' \in K} x' \cdot \text{Im } \zeta'$ and C, r are independent of $x_n \in [0, b]$.

Proof. For any $\phi \in \mathcal{E}(\mathbf{R}_{x'}^{n-1})$ we have

$$\sup_{x_n \in [0, b]} |\langle u(x_n), \phi \rangle| < \infty,$$

that is, $\{u(x_n); x_n \in [0, b]\}$ is a weakly bounded set in $\mathcal{E}'(\mathbf{R}_{x'}^{n-1})$. Hence $\{u(x_n); x_n \in [0, b]\}$ forms a strongly bounded set in $\mathcal{E}'(\mathbf{R}_{x'}^{n-1})$ and so there exists a compact subset K of $\mathbf{R}_{x'}^{n-1}$, a positive constant C and an integer $r \geq 0$ such that $\bigcup_{x_n \in [0, b]} \text{supp } u(x_n) \subset K$ and that

$$(3.7) \quad \sup_{x_n \in [0, b]} |\langle u(x_n), \phi \rangle| \leq C \cdot \sup_{\substack{x' \in K \\ |\alpha| \leq r}} |D^\alpha \phi(x')|, \quad \phi \in \mathcal{E}(\mathbf{R}_{x'}^{n-1}),$$

where C is independent of ϕ . On the other hand, we have

$$(\mathcal{F}_{x'} u(x_n))(\zeta') = \langle u(x_n), \exp ix' \cdot \zeta' \rangle.$$

Applying (3.7) to this relation, we get (3.6).

Q.E.D.

Now we present a basic lemma concerning a representation of solutions of equations which will be used in Sections 4 and 5 (see Hörmander [7]).

Let $P(D_t) = D_t^m + a_{m-1}D_t^{m-1} + \dots + a_0$, where $D_t = -id/dt$, be an ordinary differential operator with constant coefficients of order m . Assume that the zeros of the equation $P(\tau) = 0$ are all non-real. Then let us denote them by $\tau_1^+, \dots, \tau_p^+, \tau_1^-, \dots, \tau_{m-p}^-$, where $\text{Im } \tau_j^+ > 0, j = 1, \dots, p$ and $\text{Im } \tau_j^- < 0, j = 1, \dots, m-p$. Assume that $p \geq 1$. Let $B_j(D_t), j = 1, \dots, p$, be some other ordinary differential operators. Put

$$P_+(\tau) = \prod_{j=1}^p (\tau - \tau_j^+).$$

We define the Lopatinski determinant of the system $\{P(D_t), B_j(D_t), j = 1, \dots, p\}$ by

$$R = \det \left((2\pi i)^{-1} \int_\gamma B_j(\tau) \tau^{k-1} P_+(\tau)^{-1} d\tau \right)_{j, k=1, \dots, p},$$

where γ is a positively oriented contour in the complex τ -plane enclosing the zeroes $\tau_1^+, \dots, \tau_p^+$. We have

Lemma 3.3. Assume that $R \neq 0$. If $f \in C_\emptyset^\infty(\overline{\mathbf{R}}_+^1), g_j, j = 1, \dots, p$,

are complex numbers and u is a solution of the inhomogeneous boundary problem:

$$P(D_t)u = f, \quad t > 0, \quad B_j(D_t)u|_{t=0} = g_j, \quad j = 1, \dots, p,$$

which belongs to $C^\infty_{(0)}(\overline{\mathbf{R}^1_+})$, then it follows $u = v_1 + v_2$, where

$$v_1 = (2\pi)^{-1} \int \exp\{it\tau\} \mathcal{F}[f_0](\tau) P(\tau)^{-1} d\tau,$$

$$v_2 = (2\pi i)^{-1} \int \sum_{j,k=1}^p \exp\{it\tau\} \tau^{k-1} R_{jk}(g_j - B_j(D_t)v_1|_{t=0})(P_+(\tau)R)^{-1} d\tau.$$

Here f_0 denotes a compactly supported C^∞ extension of f to $t < 0$,⁵⁾ $\mathcal{F}[f_0]$ denotes the Fourier transform of f_0 , and R_{jk} denote the (j, k) cofactor of the matrix $((2\pi i)^{-1} \int B_j(\tau) \tau^{k-1} (P_+(\tau))^{-1} d\tau)_{j,k=1,\dots,p}$.

§ 4. Hyperbolicity of the Lopatinski Determinant

Let $P(\xi)$ be a hyperbolic polynomial of order m with respect to ϑ . Then it follows that the principal part $P^0(\xi)$ is hyperbolic with respect to ϑ and that a homogeneous hyperbolic polynomial has only real characteristics. The component $\Gamma = \Gamma(P) = \Gamma(P, \vartheta)$ of the set $\mathbf{E}^n - \{\xi \in \mathbf{E}^n; P^0(\xi) = 0\}$ which contains ϑ is an open convex cone with its vertex at the origin and we have

$$(4.1) \quad P(\xi + i\eta) \neq 0 \text{ when } \xi \in \mathbf{E}^n \text{ and } \eta \in -\Gamma - s \text{ with } s \text{ large enough.}$$

The localization $P_{\xi^0}(\xi)$ of $P(\xi)$ at ξ^0 is defined by

$$(4.2) \quad \tau^m P(\tau^{-1}\xi^0 + \xi) = \tau^{m_{\xi^0}} P_{\xi^0}(\xi) + O(\tau^{m_{\xi^0}+1}) \text{ as } \tau \rightarrow 0,$$

where $P_{\xi^0}(\xi)$ is the first coefficients that does not vanish identically in ξ . Then it follows that the localization $P_{\xi^0}(\xi)$ of $P(\xi)$ at any real point ξ^0 is hyperbolic with respect to ϑ and that

$$(4.3) \quad \Gamma = \Gamma(P, \vartheta) \subset \Gamma_{\xi^0} = \Gamma(P_{\xi^0}, \vartheta),$$

For further details we refer to Atiyah, Bott and Gårding [1].

Now we write the polynomial $P(\xi)$ in terms of power of ξ_n .

$$P(\xi) = P(\xi', \xi_n) = \sum_{j=1}^{m'} q_j(\xi') \xi_n^{m'-j},$$

5) See Seeley [10].

where $q_0(\xi')$ is not identically 0. So $m' \leq m$, in particular $m = m'$ if the hyperplane $x_n = 0$ ($:\partial \mathbf{R}_+^n$) is non-characteristic for $P(\xi)$. Then we see that $q_0(\xi')$ is the localization of $P(\xi)$ at $N = (0, \dots, 0, 1)$. Hence we have

$$(4.4) \quad q_0(\xi' + i\eta') \neq 0 \text{ when } \xi' \in \mathfrak{E}^{n-1} \text{ and } \eta \in -\Gamma_{\xi_0} - s\vartheta,$$

in particular $\eta \in -\Gamma - s\vartheta$ with s large enough (Note the relation (4.3)).⁶⁾ If we set

$$(4.5) \quad \Gamma_0 = \{\eta' \in \mathfrak{E}^{n-1}; (\eta', 0) \in \Gamma\} = \mathfrak{E}_{\eta'}^{n-1} \cap \Gamma,$$

it follows from (4.1) that the roots of the equation $P(\zeta', \lambda) = 0$ in λ are never real when $\zeta' = \xi' + i\eta' \in \mathfrak{E}^{n-1} - i\Gamma_0 - is\vartheta'$ with s large enough. By (4.4) the roots of $P(\zeta', \lambda) = 0$ in λ are (multivalued) continuous functions of $\zeta' \in \mathfrak{E}^{n-1} - i\Gamma_0 - is\vartheta'$ and therefore the number of roots with positive imaginary part, counted according to multiplicity, is constant when $\zeta' \in \mathfrak{E}^{n-1} - i\Gamma_0 - is\vartheta'$. The roots of $P(\zeta', \lambda) = 0$ in λ with positive imaginary part we denote by $\lambda_1^+(\zeta'), \dots, \lambda_p^+(\zeta')$ and those with negative imaginary part by $\lambda_{p+1}^-(\zeta'), \dots, \lambda_m^-(\zeta')$. Put

$$(4.6) \quad P_+(\zeta', \lambda) = \prod_{j=1}^p (\lambda - \lambda_j^+(\zeta')), \quad \zeta' \in \mathfrak{E}^{n-1} - i\Gamma_0 - is\vartheta'.$$

In this section we consider the case where the number of boundary conditions $B_j(D)$ equals exactly p . In particular, if $p=0$, boundary conditions are not imposed. Therefore, we assume that $p \geq 1$ in this section. We define the Lopatinski determinant of the system $\{P(D); B_j(D), j=1, \dots, p\}$ by

$$(4.7) \quad R(\zeta') = \det \left((2\pi i)^{-1} \int_{\gamma(\zeta')} B_j(\zeta', \lambda) \lambda^{k-1} P_+(\zeta', \lambda)^{-1} d\lambda \right)_{j,k=1, \dots, p},$$

$\zeta' \in \mathfrak{E}^{n-1} - i\Gamma_0 - is\vartheta'$ with s large enough, where $\gamma(\zeta')$ is a positively oriented contour in the complex λ -plane enclosing the roots $\lambda_1^+(\zeta'), \dots, \lambda_p^+(\zeta')$. Denote by $\dot{\Gamma}$ the projection of Γ in \mathfrak{E}^{n-1} , that is,

$$(4.8) \quad \dot{\Gamma} = \{\xi' \in \mathfrak{E}^{n-1}; (\xi', \xi_n) \in \Gamma \text{ for some } \xi_n \in \mathfrak{E}\},$$

which coincides with one defined in Sakamoto [9]. The following two

6) These fact are pointed out by K. Kasahara. He also showed that his discussions in [8] are valid in this case (unpublished).

lemmas are due to Sakamoto [9].

Lemma 4.1 ([9]). $R(\zeta')$ is holomorphic in $\mathfrak{E}^n - i\Gamma - is\vartheta'$ with s large enough.

Lemma 4.2 ([9]). Let t be a complex parameter. Then we have

$$(4.9) \quad R(t\zeta') = t^{h_0} \{R^0(\zeta') + t^{-1}R^1(\zeta') + t^{-2}R^2(\zeta') + \dots\},$$

where the convergence is uniform in $K \times \{\lambda \in C; |\lambda| > T_K\}$ for every compact set K in $\mathfrak{E}^{n-1} - i\dot{\Gamma}$ and for some $T_K > 0$ depending on K . Furthermore

- (I) $R^0(\zeta')$ does not vanish identically in ζ' and h_0 is an integer.
- (II) every $R^j(\zeta')$ is holomorphic in $\dot{\Gamma} = \bigcup_{z \in C - \{0\}} z(\mathfrak{E}^{n-1} - i\dot{\Gamma})$,
- (III) $R^j(t\zeta') = t^{h_0-j}R^j(\zeta')$, $\zeta' \in \dot{\Gamma}$, $t \in C - \{0\}$.

Main purpose of this section is to show the following.

Proposition 4.3. Assume that $P(D)$ is hyperbolic with respect to ϑ and that p which is the number of the roots with positive imaginary part of the equation $P(\xi' - is\vartheta', \lambda) = 0$ in λ for large positive s is greater than 0. If in the mixed problem (1.3)-(1.5) the number of boundary conditions is equal to p and there exist distributions on \mathbf{R}_+^n , $F_j(x)$, $j=1, \dots, p$, which satisfy the equations (1.10) and (1.11) and belong to $C^\infty([0, \delta]; \mathcal{D}'(\mathbf{R}^{n-1}))$ for some positive number δ , whose support is contained in $K \cap \overline{\mathbf{R}_+^n}$, then the Lopatinski determinant of the system $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' , that is,

- (I) $R^0(\vartheta') \neq 0$,
- (II) there exists a real number γ_1 such that

$$R(\xi' + s\vartheta') \neq 0 \text{ when } \xi' \in \mathfrak{E}^{n-1} \text{ and } \text{Im } s < \gamma_1.$$

Proof. Let U and U' be open neighborhoods of the origin in \mathbf{R}^n such that the closure of U is contained in U' , that the closure of U' is contained in $\mathbf{R}^{n-1} \times (-\delta/2, \delta/2)$ and compact in \mathbf{R}^n . Choose

$\phi \in C_0^\infty(U' \times (-\delta/2, \delta/2))$ with $\phi(x) = 1$ in U . Then

$$(4.10) \quad P(D) (\phi F_k) (x) = f_k(x), \quad x \in \mathbf{R}_+^n.$$

$$(4.11) \quad B_j(D) (\phi F_k) (x) |_{x_n=0} = \delta_{jk} \delta(x') + g_{jk}(x'), \quad x' \in \mathbf{R}^{n-1},$$

where $f_k = P(D) (\phi - 1) F_k$ and $g_{jk} = B_j(D) (\phi - 1) F_k |_{x_n=0}$ are distributions with compact support. Taking partial Fourier-Laplace transforms of both sides of (4.10) and (4.11), we obtain

$$(4.12) \quad P(\zeta', D_n) (\widehat{\phi F_k})(\zeta', x_n) = \widehat{f_k}(\zeta', x_n), \quad x_n > 0,$$

$$(4.13) \quad B_j(\zeta', D_n) (\widehat{\phi F_k})(\zeta', x_n) |_{x_n=0} = \delta_{jk} + \widehat{g}_{jk}(\zeta').$$

Now we apply the Euclidean algorithm to the polynomials in λ : $B_j(\zeta', \lambda)$, $P(\zeta', \lambda)$ and $P_+(\zeta', \lambda)$. Then we have

$$(4.14) \quad B_j(\zeta', \lambda) = Q_j(\zeta', \lambda) P(\zeta', \lambda) + T_j(\zeta', \lambda) P_+(\zeta', \lambda) + B_j^*(\zeta', \lambda),$$

where $Q_j(\zeta', \lambda)$, $T_j(\zeta', \lambda)$ and $B_j^*(\zeta', \lambda)$ are polynomials in λ whose coefficients are holomorphic functions of ζ' in $\mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\dot{I}'$ and of polynomial growth in ζ' . Furthermore, $\deg_\lambda Q_j < r_j - m'$, $\deg_\lambda T_j < m' - p$ and $\deg_\lambda B_j^* < p$. Here r_j is the degree of $B_j(\zeta', \lambda)$ in λ . Substituting (4.14) in (4.13), we find

$$(4.15) \quad \begin{aligned} B_j(\zeta', D_n) (\widehat{\phi F_k})(\zeta', x_n) |_{x_n=0} \\ = \delta_{jk} + \widehat{g}_{jk}(\zeta') - Q_j(\zeta', D_n) \widehat{f_k}(\zeta', x_n) |_{x_n=0} \\ - T_j(\zeta', D_n) P_+(\zeta', D_n) (\widehat{\phi F_k})(\zeta', x_n) |_{x_n=0}. \end{aligned}$$

We write $B_j^*(\zeta', D_n)$ in terms of powers of D_n ;

$$(4.16) \quad B_j^*(\zeta', D_n) = \sum_{k=1}^p b_{jk}(\zeta') D_n^{p-k}$$

where all $b_{jk}(\zeta')$ are holomorphic in $\mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\dot{I}'$. Since

$$(4.17) \quad \frac{1}{2\pi i} \int_{r(\zeta')} \lambda^j P_+(\zeta', \lambda)^{-1} d\lambda = \begin{cases} 0 & \text{for } j=0, 1, \dots, p-2, \\ 1 & \text{for } j=p-1, \end{cases}$$

$$\zeta' \in \mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\dot{I}',$$

we have

$$(4.18) \quad R(\zeta') = \det (b_{jk}(\zeta'))_{j,k=1,\dots,p}, \quad \zeta' \in \mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\dot{I}'.$$

From (4.15) and (4.18), we obtain

$$(4.19) \quad R(\zeta') \det (D_n^{j-1} F_k(\zeta', x_n) |_{x_n=0})_{j,k=1,\dots,p} \\ = 1 + S(\zeta'), \zeta' \in \mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}'.$$

We first consider the case where $S(\zeta') \equiv 0$ in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}'$. Then $R(\zeta') \neq 0$ in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - \dot{I}'$. Suppose that $R^0(\vartheta) = 0$. Since $R(\tau\vartheta') = \tau^{j_0}(R^0(\vartheta') + \tau^{-1}R^1(\vartheta') + \dots) \neq 0$ for any $\tau \in C^1$ with $|\tau|$ large enough, there exists an integer $j_0 \geq 1$ such that $R^j(\vartheta') = 0, j=0, \dots, j_0-1$ and $R^{j_0}(\vartheta') \neq 0$. Since $R^0(\zeta') \neq 0$, take a $\zeta^{0'} \in \mathfrak{E}^{n-1} - i\dot{I}'$ such that $R^0(\zeta^{0'}) \neq 0$ and consider the function of (s, z) :

$$(4.20) \quad f(s, z) = R^0(s(\zeta^{0'} - \vartheta') + \vartheta') + zR^1(s(\zeta^{0'} - \vartheta') + \vartheta') + \dots.$$

This function is holomorphic in a neighborhood of the origin of C^2 . Since by assumption $f(0, 0) = \dots = (\partial/\partial z)^{j_0-1} f(0, 0) = 0$ and $(\partial/\partial z)^{j_0} \times f(0, 0) \neq 0$, we can write by applying the Weierstrass preparation theorem (see, for example, [4]) to f

$$(4.21) \quad f(s, z) = (z^{j_0} + \sum_{j=1}^{j_0} a_j(s) z^{j_0-j}) u(s, z),$$

where $u(s, z)$ is holomorphic and $u(s, z) \neq 0$ in a neighborhood of $(s, z) = (0, 0)$ and $a_j(0) = 0, j=1, \dots, j_0$. Then the equation $f(s, z) = 0$ has a solution $z(s) \neq 0$ which is an analytic function of $s^{1/\rho}$ in a neighborhood of the origin and vanishes at the origin:

$$(4.22) \quad z(s) = \sum_{j=1} c_j (s^{1/\rho})^j \quad |s| < d,$$

where d is a positive rational number. Put

$$(4.23) \quad t(s) = 1/z(s).$$

Then we have

$$(4.24) \quad t(s) = as^\nu(1 + o(1)), \text{ as } s \rightarrow 0$$

where ν is a negative rational number and a is a complex number. If we choose the path: $0 < |s| < d$ and $\arg(as^\nu) = -\pi/2$, we have

$$t(s)s(\zeta^{0'} - \vartheta') + t(s)\vartheta' \in \mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}'$$

and

$$R(t(s)(\zeta^{0'} - \vartheta') + t(s)\vartheta') = 0$$

by taking $d > 0$ sufficiently small if necessary. This contradicts that

$R(\zeta') \neq 0$ in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}$. Then we have $R^0(\vartheta') \neq 0$ when $S(\zeta') \equiv 0$.

Next we consider the case where $S(\zeta')$ does not vanish identically in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}$. Put

$$H = \{x'; x = (x', x_n) \in (\text{supp } \phi - U) \cap K \text{ for some } x_n \geq 0\} \subset \mathbf{R}^{n-1}.$$

Then $\text{supp } f_k \subset H$ and $\text{supp } g_{jk} \subset H$. Since the compact set H does not contain the origin in \mathbf{R}^{n-1} and $x_1 = x' \cdot \vartheta' > 0$ on H , we have

$$(4.25) \quad h(-\vartheta') = \text{Max}_{x' \in H} x' \cdot (-\vartheta') < 0.$$

Further we have for some constants C and N

$$(4.26) \quad |\hat{g}_{jk}(\zeta')| \leq C(1 + |\zeta'|)^N \exp h(\text{Im } \zeta')$$

and

$$(4.27) \quad |Q_j(\zeta', D_n) \hat{f}_k(\zeta', x_n)|_{x_n=0} \leq C(1 + |\zeta'|)^N \exp h(\text{Im } \zeta').$$

We shall also denote by C and N some inessential constants which occur in later estimates. Now we estimate the term $T_j(\zeta', D_n) P_+(\zeta', D_n) \times (\hat{\phi} \hat{F}_k)(\zeta', x_n)|_{x_n=0}$ in (4.15). For simplicity we put $v_k(\zeta', x_n) = P_+(\zeta', D_n) (\hat{\phi} \hat{F}_k)(\zeta', x_n)$. Then $\text{supp } v_k(\zeta', x_n) \subset [0, \delta/2]$ and $v_k(\zeta', x_n)$ is a solution of the equation

$$P_-(\zeta', D_n) v_k(\zeta', x_n) = \hat{f}_k(\zeta', x_n), \quad x_n > 0,$$

where $P_-(\zeta', D_n) = P(\zeta', D_n) / P_+(\zeta', D_n)$ is a polynomial in D_n whose coefficients are holomorphic function of ζ' in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}$. Furthermore these functions are of polynomial growth in ζ' . We define $\hat{f}_k(\zeta', x_n) = 0$ for $x_n < 0$. Since the equation $P_-(\zeta, \lambda) = 0$ in λ has only roots $\lambda(\zeta')$ with $\text{Im } \lambda(\zeta') < 0$ when $\zeta' \in \mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}$, we have

$$v_k(\zeta', x_n) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \exp\{i\lambda x_n\} \mathcal{F}_{x_n}[\hat{f}_k(\zeta', x_n)](\lambda) P_-(\zeta', \lambda) d\lambda, \quad x_n \geq 0.$$

Hence we have

$$(4.28) \quad |D_n^l v_k(\zeta', 0)| \leq C \left\{ \int_{-\infty}^{+\infty} |\mathcal{F}_{x_n}[\hat{f}_k(\zeta', x_n)](\lambda)|^2 d\lambda \right\}^{1/2} \times \left\{ \int_{-\infty}^{+\infty} |\lambda^l (P_-(\zeta', \lambda))^{-1}|^2 d\lambda \right\}^{1/2}$$

$$\leq C \left\{ \int_0^{+\infty} |\hat{f}_k(\zeta', x_n)|^2 dx_n \right\}^{1/2}, \quad \text{if } 0 \leq l < m' - p.$$

Now, we know that

$$|\hat{f}_k(\zeta', x_n)| \leq C(1 + |\zeta'|)^N \exp h(\operatorname{Im} \zeta')$$

where the constants C and N are independent of $x_n \geq 0$. Since $\deg T_j(\zeta', \lambda) < m' - p$, we obtain from (4.28)

$$\begin{aligned} (4.29) \quad & |T_j(\zeta', D_n) P_+(\zeta', D_n) (\widehat{\phi F}_k)(\zeta', x_n)|_{x_n=0} \\ & = |T_j(\zeta', D_n) v_k(\zeta', x_n)|_{x_n=0} \\ & \leq C(1 + |\zeta'|)^N \exp \{h(\operatorname{Im} \zeta')\}. \end{aligned}$$

From (4.15), (4.18), (4.19), (4.26), (4.27) and (4.29), we deduce that

$$|S(\zeta')| \leq C(1 + |\zeta'|)^N \exp \{r h(\operatorname{Im} \zeta')\},$$

where r is a positive integer. Then (4.17) implies that

$$1 \leq C(1 + |\zeta'|)^N \exp \{r h(\operatorname{Im} \zeta')\} \quad \text{for any } \zeta' \in \mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\tilde{t}$$

such that $R(\zeta') = 0$. Hence we have

$$(4.30) \quad -h(\operatorname{Im} \zeta') \leq C_1 + C_2 \log(1 + |\zeta'|) \quad \text{for any } \zeta' \text{ with } R(\zeta') = 0.$$

Here C_1 and C_2 are some positive constants independent of ζ' .

Now let us show $R^0(\vartheta') \neq 0$. If $R^j(\vartheta') = 0$ for all $j=0, 1, \dots$, we have $R(-it\vartheta') = 0$ for $t > \gamma_0$ from (4.9). Then we deduce from (4.25) and (4.30) that

$$t \leq C'_1 + C'_2 \log(1 + |t|), \quad t > \gamma_0$$

which is impossible. Consequently there exists an integer $j_0 \geq 0$ such that $R^{j_0}(\vartheta') \neq 0$ and $R^j(\vartheta') = 0$, $j=0, \dots, j_0-1$, if $j_0 \geq 1$. Suppose that $R^0(\vartheta') = 0$. Since $R^0(\zeta') \neq 0$, we take a $\zeta^{0'} \in \mathfrak{E}^{n-1} - i\tilde{t}$ such that $R^0(\zeta^{0'}) \neq 0$ and consider the function $f(s, z)$ defined by (4.20). By the same argument as in the case when $S(\zeta') \equiv 0$, the equation $f(s, z) = 0$ has a solution $z(s) \neq 0$ of the form (4.22). Then $t(s) = z(s)^{-1}$ satisfies (4.24). Set $\zeta'(s) = st(s)(\zeta^{0'} - \vartheta') + t(s)\vartheta'$ and choose the path $\kappa: 0 < |s| < d$ and $\arg(as^v) = -\pi/2$. Then we have

$$(4.31) \quad R(\zeta'(s)) = 0 \quad \text{and} \quad \text{Im } \zeta'(s) = -|a||s|^\nu \vartheta'(1 + o(1)) \quad \text{as } s \rightarrow 0$$

along the path κ . In view of (4.25), we have

$$(4.32) \quad -h(\text{Im } \zeta'(s)) = |a| \cdot |s|^\nu \cdot (1 + o(1)) \quad \text{as } s \rightarrow 0 \text{ along } \kappa.$$

Since $|\zeta'(s)| \leq \text{Const}|s|^\nu$, we deduce from (4.30) and (4.32) that

$$|s|^\nu \leq C'_1 + C'_2 \log(1 + |s|^\nu) \quad \text{as } s \rightarrow 0 \text{ along } \kappa.$$

Here C'_1 and C'_2 are some new positive constants independent of s . Since ν is a negative rational number, this is impossible. Therefore we conclude that $R^0(\vartheta') \neq 0$.

We shall finally prove the assertion (II). We first note that if we consider the polynomial of the variables $\zeta', \lambda_1^+, \dots, \lambda_p^+$:

$$\mathcal{R}(\zeta', \lambda_1^+, \dots, \lambda_p^+) = \det(B_k(\zeta', \lambda_j^+)) / \prod_{\bar{k} < \bar{j}} (\lambda_k^+ - \lambda_j^+),$$

we have

$$R(\zeta') = \mathcal{R}(\zeta', \lambda_1^+(\zeta'), \dots, \lambda_p^+(\zeta')).$$

Put

$$\begin{aligned} M_r = \{ & (\xi', \text{Re } s, \text{Im } s, \text{Re } \lambda_1, \text{Im } \lambda_1, \dots, \text{Re } \lambda_{m'}, \text{Im } \lambda_{m'}); \\ & q_0(\xi' + s\vartheta') \sum_{i=1}^{m'} \lambda_i = -q_1(\xi' + s\vartheta'), \\ & \vdots \\ & q^0(\xi' + s\vartheta') \prod_{i=1}^{m'} \lambda_i = (-1)^{m'} q_{m'}(\xi' + s\vartheta'), \\ & \text{Im } \lambda_1 > 0, \dots, \text{Im } \lambda_p > 0, \quad \text{Im } s < -\gamma_0, \quad |\xi'|^2 = r^2, \quad \xi' \in \mathbb{E}^{n-1}, \\ & \mathcal{R}(\xi', \lambda_1^+, \dots, \lambda_p^+) = 0 \} \end{aligned}$$

and

$$\mu(r) = \sup_{M_r} -\text{Im } s,$$

the supremum being taken over all $(\xi', \text{Re } s, \text{Im } s, \text{Re } \lambda_1, \text{Im } \lambda_1, \dots, \text{Re } \lambda_{m'}, \text{Im } \lambda_{m'}) \in M_r$. Since $R^0(\vartheta') \neq 0$ and $q_0(\xi' + s\vartheta') \neq 0$ when $\text{Im } s < -\gamma_0$, we have $\mu(r) \neq \infty$. Then it follows from the Tarski-Seidenberg theorem that

$$(4.33) \quad \mu(r) = ar^b(1 + o(1)), \quad r \rightarrow +\infty,$$

where a is real and b is rational. Let us denote by $s(\xi')$ the zeros of the equation $R(\xi' + s\vartheta') = 0$ in s . If $\text{Im } s(\xi') < 0$, we have from (4.30)

$$(4.34) \quad -\operatorname{Im} s(\xi') (-h(-\vartheta')) \leq C_1 + C_2 \log(1 + |\xi' + s(\xi')\vartheta'|)$$

because $h(\operatorname{Im}(\xi' + s\vartheta')) = (-\operatorname{Im} s)h(-\vartheta')$. When $\operatorname{Im} s(\xi') \geq 0$, (4.34) is obvious. Using the Tarski-Seidenberg theorem we have also

$$(4.35) \quad \sup_{|\xi'|=r} |s(\xi')| \leq \operatorname{Const} r^d, \quad \text{as } r \rightarrow +\infty.$$

Hence we get from (4.33), (4.34) and (4.35)

$$-h(-\vartheta')\mu(r) \leq C_1 + C_2 \log(1 + r + C_3 r^d) \quad \text{as } r \rightarrow +\infty.$$

Since $-h(-\vartheta') > 0$, this inequality means that $\mu(r)$ remains bounded when $r \rightarrow +\infty$. This implies that there exists a real γ_1 such that $R(\xi' + s\vartheta') \neq 0$ when $\operatorname{Im} s < \gamma_1$ and $\xi' \in \mathfrak{E}^{n-1}$. Thus the proof of Proposition 4.3 is complete.

Assume that the Lopatinski determinant $R(\zeta')$ of $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' , where $P(D)$ is hyperbolic with respect to $\vartheta = (1, 0, \dots, 0)$. Then the principal part $R^0(\zeta')$ is also hyperbolic with respect to ϑ' . Let us denote by $\dot{\Sigma}$ the connected component containing ϑ' of the set $\{\xi' \in \dot{I}; R^0(\xi') \neq 0\}$. Then $\dot{\Sigma}$ is an open convex cone in \mathfrak{E}^{n-1} . (See Wakabayashi [12], Lemma 2.3.) Then we have

Lemma 4.4 ([9]). *There exists a constant γ_1 such that*

$$(4.36) \quad R(\zeta') \neq 0 \quad \text{for } \zeta' \in \mathfrak{E}^{n-1} - i\gamma_1\vartheta' - i\dot{\Sigma}.$$

Moreover there exists locally bounded functions $C(\eta')$ and $N(\eta')$ such that

$$C(t\eta') = C(\eta'), \quad N(t\eta') = N(\eta') \quad \text{when } t \geq 1, \eta' \in -\gamma_1\vartheta' - \dot{\Sigma}$$

and that

$$(4.37) \quad |R(\xi' + i\eta')^{-1}| \leq C(\eta') (1 + |\xi' + i\eta'|)^{N(\eta')},$$

$$\xi' + i\eta' \in \mathfrak{E}^{n-1} - i\gamma_1\vartheta' - i\dot{\Sigma}.$$

§ 5. Proof of the Main Theorem

Let $P(D)$ be a hyperbolic operator with respect to $\vartheta = (1, 0, \dots, 0)$

and let p be the number of the roots with positive imaginary part of the equation $P(\xi' - is\vartheta', \lambda) = 0$ in λ for $\xi' \in \mathbb{E}^{n-1}$ and sufficiently large s . Let $B_j(D), j=1, \dots, q$, be boundary operators with constant coefficients. Then we have the following two propositions.

Proposition 5.1. *Assume that $p=q$ and that the Lopatinski determinant $R(\zeta')$ of the system $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' , if $p \geq 1$. Let $W(x) \in C^\infty([0, \infty); \mathcal{D}'(\mathbb{R}^{n-1}))$ be a solution of the homogeneous problem:*

$$P(D)W(x) = 0, \quad x \in \mathbb{R}_+^n, \quad B_j(D)W(x)|_{x_n=0} = 0, \quad j=1, \dots, p.$$

If $\text{supp } W \subset \{x \in \overline{\mathbb{R}_+^n}; x_1 = x \cdot \vartheta \geq 0\}$, then $W(x) \equiv 0$.

Proof. Put

$$\Sigma = \{\xi \in \Gamma; \xi' \in \dot{\Sigma}\},$$

where $\Gamma = \Gamma(P, \vartheta)$ (see Section 4). Let Γ' and Σ' denote the dual cone of Γ and Σ , respectively. For any positive number b , we put

$$S_b = \{x \in \mathbb{R}^n; b \leq |x| \leq b+1, x_n \geq 0, x_1 \geq 0\},$$

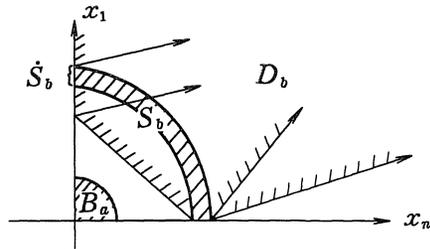
$$\dot{S}_b = \{x' \in \mathbb{R}^{n-1}; (x', 0) \in S_b \text{ (i.e. } b \leq |x'| \leq b+1, x_1 \geq 0)\},$$

$$\Gamma'_{S_b} = \Gamma' + S_b, \quad \dot{\Gamma}'_{S_b} = \{x' \in \mathbb{R}^{n-1}; (x', 0) \in \dot{\Gamma}'_{S_b}\}$$

and

$$D_b = \{[\Sigma' + (\dot{\Gamma}'_{S_b} \cup \dot{S}_b)] \cup \Gamma'_{S_b}\} \cap \overline{\mathbb{R}_+^n}.$$

Since both Γ and Σ are closed proper cone with its vertex at the origin such that $x \cdot \vartheta > 0$ on $\Gamma - \{0\}$, there exists a number $b > 0$ for an arbitrarily given $a > 0$ such that D_b and $B_a = \{x \in \mathbb{R}^n; |x| \leq a, x_1 \geq 0, x_n \geq 0\}$ are disjoint.



Choose such a number $b > 0$. Let $\phi \in C_0^\infty(\{x \in \mathbb{R}^n; |x| \leq b+1\})$ be equal to 1 in $\{x \in \mathbb{R}^n; |x| \leq b\}$. Then we have

$$(5.1) \quad P(D)(\phi W) = f(x), \quad x_n > 0,$$

$$(5.2) \quad B_j(D) (\phi W)|_{x_n=0} = g_j(x'), \quad j=1, \dots, p,$$

where ϕW and f are C^∞ function of x_n in $[0, \infty)$ with values in $\mathcal{E}'(\mathbf{R}^{n-1})$, $g_j(x')$ belongs to $\mathcal{E}'(\mathbf{R}^{n-1})$ and

$$(5.3) \quad \text{supp } f(x) \subset S_b, \quad x_n \geq 0,$$

$$(5.4) \quad \text{supp } g_j(x') \subset \dot{S}_b.$$

Taking partial Fourier-Laplace transforms of both sides of (5.1) and (5.2), we obtain a system of ordinary differential equations with a parameter ζ' :

$$(5.5) \quad P(\zeta', D_n) \widehat{\phi W}(\zeta', x_n) = \widehat{f}(\zeta', x_n), \quad x_n > 0,$$

$$(5.6) \quad B_j(\zeta', D_n) W(\zeta', x_n)|_{x_n=0} = \widehat{g}_j(\zeta'), \quad j=1, \dots, p.$$

$R(\zeta')$ is, by assumption, hyperbolic and so $R(\zeta') \neq 0$ for $\zeta' \in \mathbf{E}^{n-1} - i\gamma_1\vartheta' - i\dot{Z}$. Since $\widehat{\phi W}(\zeta', \cdot), \widehat{f}(\zeta', \cdot) \in C^\infty_{(0)}(\overline{\mathbf{R}}_+)$ for every ζ' , we can write, applying Lemma 3.3,

$$(5.7) \quad \widehat{\phi W}(\zeta', x_n) = v_1(\zeta', x_n) + v_2(\zeta', x_n), \quad x_n \geq 0,$$

where

$$(5.8) \quad v_1(\zeta', x_n) = (2\pi)^{-1} \int \exp(ix_n \xi_n) \mathcal{F}_{x_n}[\widehat{f}_0](\zeta', \xi_n) (P(\zeta', \xi_n))^{-1} d\xi_n$$

$$(5.9) \quad v_2(\zeta', x_n) = (2\pi i)^{-1} \sum_{j,k=1}^p \int_{r(\zeta')} \exp(ix_n \lambda) \lambda^{k-1} (P_+(\zeta', \lambda))^{-1} d\lambda \\ \times R_{jk}(\zeta') [g_j(\zeta') - B_j(\zeta', D_n) v_1(\zeta', x_n)|_{x_n=0}] (R(\zeta'))^{-1}.$$

Here $R_{jk}(\zeta')$ is the (j, k) cofactor of the matrix

$$\left((2\pi i)^{-1} \int_{r(\zeta')} B_j(\zeta', \lambda) \lambda^{k-1} (P_+(\zeta', \lambda))^{-1} d\lambda \right)_{j,k=1, \dots, p},$$

$\widehat{f}_0(\zeta', x_n)$ is a compactly supported C^∞ extension of $\widehat{f}(\zeta', x_n)$ to $x_n < 0$ and

$$(5.10) \quad \mathcal{F}_{x_n}[\widehat{f}_0](\zeta', \xi_n) = \int_{-\infty}^{+\infty} \exp(-ix_n \xi_n) \widehat{f}_0(\zeta', x_n) dx_n.$$

If $p=0$, that is, if the equation $P(\zeta', \lambda) = 0$ in λ has only roots $\lambda(\zeta')$ with $\text{Im } \lambda(\zeta') < 0$ for every $\zeta' \in \mathbf{E}^{n-1} - i\gamma_1\vartheta' - i\dot{Z}$, we have

$$\widehat{\phi W}(\zeta', x_n) = v_1(\zeta', x_n) \quad \text{for } x_n \geq 0.$$

Therefore we define $v_2(\zeta', x_n) \equiv 0$ in this case. Now we have from (5.7)

$$(5.11) \quad (\phi W)(x) = \mathcal{F}_{\zeta'}^{-1}[v_1(\zeta', x_n)] + \mathcal{F}_{\zeta'}^{-1}[v_2(\zeta', x_n)],$$

where $\mathcal{F}_{\zeta'}^{-1}$ denotes the inverse Fourier-Laplace transform in ζ' . From (5.8) and (5.10) we find

$$(5.12) \quad \mathcal{F}_{\zeta'}^{-1}[v_1(\zeta', x_n)] = \mathcal{F}_{\zeta'}^{-1}[(P(\zeta'))^{-1}\hat{f}_0(\zeta)] = \mathcal{F}_{\zeta'}^{-1}[(P(\zeta'))^{-1}] * f_0.$$

Since $\mathcal{F}_{\zeta'}^{-1}[(P(\zeta'))^{-1}]$ is the unique fundamental solution of the Cauchy problem for $P(D)$ such that the support is contained in Γ' , we obtain using (5.3)

$$(5.13) \quad \text{supp } \mathcal{F}_{\zeta'}^{-1}[v_1(\zeta', x_n)] \cap \overline{\mathbf{R}}_+^n \subset \Gamma' + S_b.$$

In particular (5.13) implies that

$$(5.14) \quad \begin{aligned} &\text{supp } (B_j(D) \mathcal{F}_{\zeta'}^{-1}[v_1(\zeta', x_n)]|_{x_n=0}) \\ &\subset \{x' \in \mathbf{R}^{n-1}; (x', 0) \in \Gamma' + S_b\} = \dot{\Gamma}'_{S_b}. \end{aligned}$$

In view of Lemma 4.4, we have applying the Paley-Wiener-Schwartz theorem⁷⁾

$$(5.15) \quad \begin{aligned} &\text{supp } \mathcal{F}_{\zeta'}^{-1} \left[\int_{\Gamma(\zeta')} \exp(ix_n \lambda) \lambda^{k-1} (P_+(\zeta', \lambda))^{-1} d\lambda \right. \\ &\quad \left. \times R_{jk}(\zeta') (R(\zeta'))^{-1} \right] \subset \Sigma'. \end{aligned}$$

From (5.4), (5.9), (5.14) and (5.15), we find

$$(5.16) \quad \begin{aligned} &\text{supp } \mathcal{F}_{\zeta'}^{-1}[v_2(\zeta', x_n)] \cap \overline{\mathbf{R}}_+^n \\ &\subset \{\Sigma' + \cup_{j=1}^p \text{supp} [(g_j - B_j(D)v_1)|_{x_n=0} \otimes \delta(x_n)] \cap \overline{\mathbf{R}}_+^n \\ &\quad \cup \{\Sigma' + (S_b \cup \dot{\Gamma}'_{S_b})\} \cap \overline{\mathbf{R}}_+^n. \end{aligned}$$

From (5.11), (5.13) and (5.16) we conclude that

$$(5.17) \quad \text{supp } \phi W \subset D_b.$$

Since $\phi(x) = 1$ on B_a and $B_a \cap D_b = \emptyset$, (5.17) means that $W(x) \equiv 0$ in B_a . Letting $a \rightarrow +\infty$, we get $W(x) \equiv 0$ in \mathbf{R}_+^n . This completes the proof.

7) For example, see Theorem 2.6 in [1] or Lemma 2.3 in [9].

Under the situation at the beginning of this section we have

Proposition 5.2. *If $p > q$, there exists a non trivial solution $W(x) \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ of the following homogeneous problem whose support is contained in some closed proper cone K_1 with its vertex at the origin such that $x \cdot \vartheta > 0$ on $K_1 - \{0\}$.*

$$(5.18) \quad P(D)W = 0, \quad x_n > 0,$$

$$(5.19) \quad B_j(D)W|_{x_n=0} = 0, \quad j = 1, \dots, q.$$

Proof. Consider the matrix $A(\zeta') = (a_{jk}(\zeta'))_{\substack{j=1, \dots, q \\ k=1, \dots, p}}$ where

$$(5.20) \quad a_{jk}(\zeta') = (2\pi i)^{-1} \int_{r(\zeta')} B_j(\zeta', \lambda) \lambda^{k-1} (P_+(\zeta', \lambda))^{-1} d\lambda.$$

Assume that $\text{rank } A(\zeta') = r$ ($0 \leq r \leq q$), that is, some r -rowed minor $\Delta(\zeta')$ of $A(\zeta')$ is not identically zero in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}'$ when $1 \leq r \leq q$ and every $(r+1), \dots, q$ -rowed minor of $A(\zeta')$ is identically zero in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}'$ when $1 \leq r \leq q-1$. When $r \neq 0$, i.e., $A(\zeta') \neq 0$, we may assume without loss of generality that

$$(5.21) \quad \Delta(\zeta') = \det(a_{jk}(\zeta'))_{j,k=1, \dots, r}.$$

When $1 \leq r \leq q$, we put

$$(5.22) \quad \begin{aligned} \widehat{W}(\zeta', x_n) &= -\sum_{j,k=1}^r \Delta_{jk}(\zeta') a_{jr+1}(\zeta') \\ &\quad \times \left\{ (2\pi i)^{-1} \int_{r(\zeta')} \exp(ix_n \lambda) \lambda^{k-1} (P_+(\zeta', \lambda))^{-1} d\lambda \right\} \\ &\quad + \Delta(\zeta') (2\pi i)^{-1} \int_{r(\zeta')} \exp(ix_n \lambda) \lambda^r (P_+(\zeta', \lambda))^{-1} d\lambda, \end{aligned}$$

where $\Delta_{jk}(\zeta')$ is the (j, k) cofactor of $(a_{jk}(\zeta'))_{j,k=1, \dots, r}$. When $r = 0$, put

$$(5.23) \quad \widehat{W}(\zeta', x_n) = (2\pi i)^{-1} \int_{r(\zeta')} \exp(ix_n \lambda) (P_+(\zeta', \lambda))^{-1} d\lambda.$$

Since

$$(2\pi i)^{-1} \int_{r(\zeta')} \exp(ix_n \lambda) \lambda^{j-1} (P_+(\zeta', \lambda))^{-1} d\lambda, \quad j = 1, \dots, p$$

are linearly independent functions of x_n and $\Delta(\zeta') \neq 0$ in $\mathfrak{E}^{n-1} - i\gamma_0\vartheta' - i\dot{I}'$,

we have $\widehat{W}(\zeta', x_n) \neq 0$. Since $\widehat{W}(\zeta', x_n)$ is analytic in ζ' , we have $\widehat{W}(\zeta', x_n) \neq 0$ in \mathbf{R}_+^n . It is easy to see that $W(x)$ belongs to $C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ and satisfies (5.18) and (5.19). Moreover, using the Paley-Wiener-Schwartz theorem, we see that the support of $W(x)$ is contained in $K_1 \cap \overline{\mathbf{R}_+^n}$ where K_1 is some closed proper cone such that $x \cdot \vartheta > 0$ on $K_1 - \{0\}$. This completes the proof.

Let us now complete our proof of the Main Theorem which is stated in Introduction.

Proof of the Main Theorem. In Proposition 2.1 we proved that the existence of certain fundamental solution $E(x, y)$ of the mixed problem (1.3)-(1.5) for some $y = (0, y'') \in \mathbf{R}_+^n$ (i.e., E satisfies (1.8) and (1.9) and its support is contained in $(K + y) \cap \overline{\mathbf{R}_+^n}$ where K is some closed proper cone such that $x \cdot \vartheta > 0$ on $K - \{0\}$) implies the hyperbolicity of the characteristic polynomial $P(\xi)$ with respect to $\vartheta = (1, 0, \dots, 0)$. The hyperbolicity of $P(\xi)$ determines the number p . Next the uniqueness of the system $\{F_j(x), j=1, \dots, q\}$ implies that the number q of boundary operators $B_j(D)$ must be greater than or equal to p , because if $q < p$ the system $\{F_j(x), j=1, \dots, q\}$ is not unique by Proposition 5.2. Now suppose that $q > p$. When $p=0$, Proposition 5.1 shows that if $P(D)F_1(x) = 0$ and $\text{supp } F_1 \subset \{x \in \overline{\mathbf{R}_+^n}, x_1 \geq 0\}$ then $F_1(x) \equiv 0$. This contradicts the definition of $F_1(x)$. In fact $F_1(x)$ must satisfy the equation $B_1(D)F_1(x)|_{x_n=0} = \delta(x')$. When $p \geq 1$, the existence of $F_j(x), j=1, \dots, p$, implies that the Lopatinski determinant of the system $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' . Since $F_{p+1}(x)$ satisfies the equations:

$$P(D)F_{p+1}(x) = 0 \text{ in } \mathbf{R}_+^n, \quad B_j(D)F_{p+1}(x)|_{x_n=0} = 0, \quad j=1, \dots, p,$$

and $\text{supp } F_{p+1} \subset K \cap \overline{\mathbf{R}_+^n}$, it follows from Proposition 5.1 that $F_{p+1}(x) \equiv 0$ in $\overline{\mathbf{R}_+^n}$. This gives a contradiction because $F_{p+1}(x)$ satisfies by definition the equation: $B_{p+1}(D)F_{p+1}(x)|_{x_n=0} = \delta(x')$. Thus we conclude that the number q of boundary operators $B_j(D)$ must be equal to p . If $p=0$, this means that such boundary conditions cannot be imposed. If $p \geq 1$, we see by Proposition 4.3 that the Lopatinski determinant $R(\zeta')$ of the

system $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' . The proof is complete.

§ 6. Proof of Sufficiency

First we construct a system $\{E(x, y); F_j(x), j=1, \dots, q\}$ of fundamental solutions of the mixed problem (1.3)-(1.5) for any given system $\{P(D); B_j(D), j=1, \dots, q\}$ of differential operators satisfying the conditions:

- (I) $P(\xi)$ is a hyperbolic polynomial with respect to $\vartheta = (1, 0, \dots, 0)$, that is, $P(\xi)$ satisfies (1.1) and (1.2),
- (II) the number q of the boundary operators $B_j(D)$ is equal to the number p of the roots with positive imaginary part of the equation $P(\xi' - is\vartheta', \lambda) = 0$ in λ when $\xi' \in \mathfrak{E}^{n-1}$ and $s > \gamma_0$,
- (III) the Lopatinski determinant $R(\zeta')$ of $\{P(D); B_j(D), j=1, \dots, p\}$ is hyperbolic with respect to ϑ' , that is, $R(\zeta')$ satisfies (1.6) and (1.7).

Let $G(x)$ be the unique fundamental solution of the Cauchy problem for $P(D)$ in the half space $x \cdot \vartheta > 0$. $G(x-y)$ is given in the form

$$(6.1) \quad \langle G(x-y), \check{\phi}(x) \rangle = (2\pi)^{-n} \int_{\mathfrak{E}^n} (P(\xi + i\eta))^{-1} \hat{\phi}(\xi + i\eta) d\xi,$$

$$\eta \in -\gamma_0 \vartheta - \Gamma,$$

where $\check{\phi}(x) = \phi(-x) \in \mathcal{D}(\mathbf{R}^n)$ and $\hat{\phi}(\xi + i\eta)$ is the Fourier-Laplace transform of ϕ . We may assume that ϕ is of the form: $\phi(x) = \psi(x')\rho(x_n)$, $\psi \in \mathcal{D}(\mathbf{R}^{n-1})$ and $\rho \in \mathcal{D}(R^1)$, since the set of all test functions of this form is dense in $\mathcal{D}(\mathbf{R}^n)$. Now we wish to show that $G(x-y) \in C^\infty([0, y_n]; \mathcal{D}'(R_{x'}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(R_{x'}^{n-1}))$.⁸⁾ To do so we write in Section 4

$$(6.2) \quad P(\xi + i\eta) = q_0(\xi' + i\eta')(\xi_n + i\eta_n)^{m'} + \sum_{l=1}^{m'} q_l(\xi' + i\eta')(\xi_n + i\eta_n)^{m'-l}.$$

If $m'=0$, $P(\xi + i\eta) = P(\xi' + i\eta')$. In this case we do not consider any boundary condition and so $E(x, y) = G(x-y)$. Then we have

8) Let I be an interval in \mathbf{R}_+^1 and let $u \in \mathcal{D}'(\mathbf{R}_+^n)$. For simplicity we write $u \in C^\infty(I; \mathcal{D}'(\mathbf{R}_+^{n-1}))$ in place of $u|_{\mathbf{R}_+^{n-1} \times I} \in C^\infty(I; \mathcal{D}'(R_{x'}^{n-1}))$ where $u|_{\mathbf{R}_+^{n-1} \times I}$ is the restriction of u to $\mathbf{R}_+^{n-1} \times I$.

$$\begin{aligned}
 (6.3) \quad & \langle E(x, y), \check{\psi}(x') \rho(x_n) \rangle \\
 &= (2\pi)^{-n+1} \left\{ \int_{\mathbb{R}^n} \exp\{-iy'(\xi' + i\eta')\} (P(\xi' + i\eta'))^{-1} \right. \\
 & \quad \left. \times \widehat{\psi}(\xi' + i\eta') d\xi' \right\} \langle \delta(x_n - y_n), \rho(x_n) \rangle.
 \end{aligned}$$

Thus, in this case it is obvious that $E(\cdot, y) \in C^\infty([0, y_n]; \mathcal{D}'(\mathbb{R}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbb{R}^{n-1}))$.

Let us consider the case where $m' \geq 1$. Take $\chi \in C_0^\infty(\{\xi_n \in \mathbb{R}^1; |\xi_n| < 2\})$ such that $\chi(\xi_n) = 1$ when $|\xi_n| < 1$. Using Lebesgue's dominated convergence theorem and Fubini's theorem we find

$$\begin{aligned}
 (6.4) \quad & \langle G(x - y), \check{\psi}(x') \check{\rho}(x_n) \rangle \\
 &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-n} \int \exp\{-iy(\xi + i\eta)\} \chi(\varepsilon \xi_n) (P(\xi + i\eta))^{-1} \\
 & \quad \times \widehat{\psi}(\xi' + i\eta') \widehat{\rho}(\xi_n + i\eta_n) d\xi \\
 &= \lim_{\varepsilon \downarrow 0} (2\pi)^{-n} \int_{-\infty}^{+\infty} \left[\int_{\mathbb{R}^{n-1}} \exp\{-iy'(\xi' + i\eta')\} \right. \\
 & \quad \left. \times \widehat{\psi}(\xi' + i\eta') I_\varepsilon(x_n, \xi' + i\eta') d\xi' \right] \check{\rho}(x_n) dx_n,
 \end{aligned}$$

where

$$\begin{aligned}
 (6.5) \quad I_\varepsilon(x_n, \xi' + i\eta') &= \int_{-\infty}^{+\infty} \exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \chi(\varepsilon \xi_n) \\
 & \quad \times (P(\xi + i\eta))^{-1} d\xi_n.
 \end{aligned}$$

In order to derive an expression which is valid when $\varepsilon \rightarrow 0$, we shall first establish the following estimate.

$$\begin{aligned}
 (6.6) \quad & |(\partial/\partial \xi_n)^\alpha (P(\xi + i\eta))^{-1}| \leq M_\alpha \frac{(1 + |\xi' + i\eta'|)^{k_\alpha}}{(1 + |\xi_n + i\eta_n|)^{m' + \alpha}}, \\
 & \alpha = 0, 1, 2, \dots,
 \end{aligned}$$

where M_α and k_α are some positive constants independent of $\xi + i\eta \in \mathbb{R}^n - i\gamma_0\partial - i\Gamma$.

In fact, we have for some positive constants C_1 and $C_2 \geq 1$

$$\begin{aligned}
 (6.7) \quad & |q_0(\xi' + i\eta')| \geq C_1 \quad (\text{see (4.4)}) \\
 & \text{and } |q_l(\xi' + i\eta')| < C_2 (1 + |\xi' + i\eta'|)^{m - m' - l}.
 \end{aligned}$$

Then it follows from (6.2) that

$$|P(\xi + i\eta)| \geq C_1 |\xi_n + i\eta_n|^{m'} \left\{ 1 - \sum_{l=1}^{m'} \left| \frac{q_l(\xi' + i\eta')}{q_0(\xi' + i\eta')} \right| |\xi_n + i\eta_n|^{-l} \right\}.$$

Hence if $|\xi_n + i\eta_n| > C_3(1 + |\xi' + i\eta'|)^{m-m'-1}$ with $C_3 = 2m' C_1^{-1} C_2$, we have from (6.7)

$$\sum_{l=1}^{m'} |q_l(\xi' + i\eta') / q_0(\xi' + i\eta')| |\xi_n + i\eta_n|^{-l} \leq 1/2$$

and so

$$(6.8) \quad |P(\xi + i\eta)|^{-1} \leq C_4 (1 + |\xi_n + i\eta_n|)^{-m'}.$$

On the other hand, if $|\xi_n + i\eta_n| < C_3(1 + |\xi' + i\eta'|)^{m-m'-1}$, we have

$$(6.9) \quad |P(\xi + i\eta)|^{-1} \leq C_5 \leq C_6 (1 + |\xi' + i\eta'|)^{m'(m-m'+1)} (1 + |\xi_n + i\eta_n|)^{-m'}.$$

From (6.8) and (6.9) we find (6.6) for $\alpha = 0$ with $M_0 = \text{Max}(C_5(2C_3)^{m'}, C_4)$ and $k_0 = m'(m - m' + 1)$. We show (6.6) for $\alpha \geq 1$ by induction. Now we have

$$\begin{aligned} |(\partial/\partial\xi_n)P(\xi + i\eta)| &= |m'q_0(\xi' + i\eta')(\xi_n + i\eta_n)^{m'-1} \\ &\quad + \sum_{l=1}^{m'} (m' - l - 1)q_l(\xi' + i\eta')(\xi_n + i\eta_n)^{m'-l-1}| \\ &\leq C_7(1 + |\xi' + i\eta'|)^m (1 + |\xi_n + i\eta_n|)^{m'-1}. \end{aligned}$$

Hence

$$\begin{aligned} |(\partial/\partial\xi_n)(P(\xi + i\eta))^{-1}| &= |(\partial/\partial\xi_n)P(\xi + i\eta)| |P(\xi + i\eta)|^{-2} \\ &\leq M_1(1 + |\xi' + i\eta'|)^{k_1} (1 + |\xi_n + i\eta_n|)^{-m'-1}. \end{aligned}$$

Suppose that (6.6) holds for $0 \leq l \leq \alpha - 1$. From the relation

$$0 = (\partial/\partial\xi_n)^\alpha (P^{-1} \cdot P) = \sum_{l=0}^{\alpha} \binom{\alpha}{l} \{(\partial/\partial\xi_n)^l (P^{-1})\} \{(\partial/\partial\xi_n)^{\alpha-l} P\},$$

we get

$$(6.10) \quad (\partial/\partial\xi_n)^\alpha (P^{-1}) = -\sum_{l=0}^{\alpha-1} \binom{\alpha}{l} \{(\partial/\partial\xi_n)^l (P^{-1})\} \{(\partial/\partial\xi_n)^{\alpha-l} P\} P^{-1}.$$

Now we have

$$|(\partial/\partial\xi_n)^{\alpha-l} P(\xi + i\eta)| \leq C(1 + |\xi' + i\eta'|)^m (1 + |\xi_n + i\eta_n|)^{m'-\alpha+l}$$

and by the induction assumption

$$|(\partial/\partial\xi_n)^l (P(\xi + i\eta))^{-1}| \leq M (1 + |\xi' + i\eta'|)^{k_l} (1 + |\xi_n + i\eta_n|)^{-m'-l}$$

$l=0, 1, \dots, \alpha-1$. Applying these estimates to the right-hand side of (6.10), we have (6.6) for α .

We shall now return to the integrals (6.4) and (6.5). Using repeatedly the relation

$$\begin{aligned} & \exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \\ &= -i(x_n - y_n)^{-1}(\partial/\partial\xi_n) \exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \end{aligned}$$

and integrating by parts, we obtain

$$\begin{aligned} (6.11) \quad I_\varepsilon(x_n, \xi' + i\eta') &= (y_n - x_n)^{-\alpha} \left[\int_{-\infty}^{+\infty} \exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \chi(\varepsilon\xi_n) \right. \\ & \quad \left. \times (-i\partial/\partial\xi_n)^\alpha (P(\xi + i\eta))^{-1} d\xi_n + J_\varepsilon(x_n, \xi' + i\eta') \right], \quad x_n \neq y_n, \end{aligned}$$

where

$$\begin{aligned} (6.12) \quad J_\varepsilon(x_n, \xi' + i\eta') &= \sum_{l=1}^{\alpha} \varepsilon^l \int_{-\infty}^{+\infty} \exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \chi^{(l)}(\varepsilon\xi_n) \\ & \quad \times [(-i\partial/\partial\xi_n)^{\alpha-l} (P(\xi + i\eta))^{-1}] d\xi_n, \quad (\chi^{(l)} = (-i\partial/\partial\xi_n)^l \chi). \end{aligned}$$

The estimate (6.6) gives

$$\begin{aligned} (6.13) \quad & |\exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \chi^{(l)}(\varepsilon\xi_n) [(-i\partial/\partial\xi_n)^{\alpha-l} (P(\xi + i\eta))^{-1}]| \\ & \leq M_{\alpha-l} \exp\{(y_n - x_n)\eta_n\} (1 + |\xi' + i\eta'|)^{k_{\alpha-l}} \\ & \quad \times (1 + |\xi_n + i\eta_n|)^{-m'-\alpha+l}. \end{aligned}$$

Using Schwarz inequality and (6.13) we find

$$\begin{aligned} (6.14) \quad & |J_\varepsilon(x_n, \xi' + i\eta')| \leq C\varepsilon (1 + |\xi' + i\eta'|)^k \\ & \quad \times \left\{ \int_{-\infty}^{+\infty} (1 + |\xi_n + i\eta_n|)^{-2} d\xi_n \right\}^{1/2} \\ & \quad \times \left\{ \int_{\varepsilon^{-1} < |\xi_n| < 2\varepsilon^{-1}} d\xi_n \right\}^{1/2} \exp\{(y_n - x_n)\eta_n\} \\ & \leq C\sqrt{\varepsilon} (1 + |\xi' + i\eta'|)^k \exp\{(y_n - x_n)\eta_n\}, \end{aligned}$$

where C and k are some positive constants. Since $\psi \in \mathcal{D}(\mathbf{R}^{n-1})$ we have for any N

$$(6.15) \quad |\hat{\psi}(\xi' + i\eta')| \leq C_N (1 + |\xi' + i\eta'|)^{-N} \exp\{h_\psi(\eta')\},$$

$$h_\psi(\eta') = \text{Max}_{x' \in \text{supp } \psi} x' \cdot \eta' .$$

If we take ρ with support in $(-\infty, y_n) \cup (y_n, \infty)$, we may substitute (6.11) in the right-hand side of (6.4). Since we have (6.13), (6.14) and (6.15), letting ε tend to zero, we get the following expression by Lebesgue's dominated convergence theorem.

$$\begin{aligned} (6.16) \quad & \langle G(x-y), \check{\psi}(x') \check{\rho}(x_n) \rangle \\ &= \int_{-\infty}^{+\infty} (y_n - x_n)^{-\alpha} \int_{\mathbb{R}^{n-1}} \exp\{-iy'(\xi' + i\eta')\} \\ & \quad \times \hat{\psi}(\xi' + i\eta') \left[\int_{-\infty}^{+\infty} \exp\{i(x_n - y_n)(\xi_n + i\eta_n)\} \right. \\ & \quad \left. \times (-i\partial/\partial\xi_n)^\alpha (P(\xi + i\eta))^{-1} d\xi_n \right] \rho(x_n) d\xi' dx_n . \end{aligned}$$

Therefore we have for any sufficiently large α

$$\begin{aligned} (6.17) \quad & \langle G(x-y), \check{\psi}(x') \rangle \\ &= (y_n - x_n)^{-\alpha} \int_{\mathbb{R}^n} \exp\{-iy(\xi + i\eta) - ix_n(\xi_n + i\eta_n)\} \\ & \quad \times \hat{\psi}(\xi' + i\eta') (-i\partial/\partial\xi_n)^\alpha (P(\xi + i\eta))^{-1} d\xi , \end{aligned}$$

when $x_n \neq y_n$ and we see that $G(x-y) \in C^\infty((-\infty, y_n); \mathcal{D}'(\mathbb{R}^{n-1})) \cap C^\infty((y_n, +\infty); \mathcal{D}'(\mathbb{R}^{n-1}))$. This implies that for any integer $l \geq 0$ $D_n^l G(x-y)|_{x_n=0}$ makes sense when $y_n \neq 0$. Moreover the estimates (6.6) and (6.15) enable us to differentiate the right side of (6.17) with respect to x_n under the integral sign and so $D_n^l G(x-y)|_{x_n=0}$ is given in the form

$$\begin{aligned} (6.18) \quad & \langle D_n^l G(x-y)|_{x_n=0}, \check{\psi}(x') \rangle \\ &= \sum_{\nu=0}^l C_{\alpha\nu} y_n^{-\alpha-\nu} \int_{\mathbb{R}^n} \exp\{-iy(\xi + i\eta)\} \hat{\psi}(\xi' + i\eta') \\ & \quad \times (\xi_n + i\eta_n)^{l-\nu} (-i\partial/\partial\xi_n)^\alpha (P(\xi + i\eta))^{-1} d\xi , \quad y_n \neq 0, \end{aligned}$$

where α is chosen sufficiently large and $C_{\alpha\nu} = \alpha(\alpha+1)\cdots(\alpha+\nu-1) \binom{l}{\nu}$. Writing

$$B_j(D) = \sum_{l=0}^{r_j} b_{jl}(D') D_n^l ,$$

we also find

$$\begin{aligned}
 (6.19) \quad & \langle B_j(D)G(x-y)|_{x_n=0}, \check{\psi}(x') \rangle \\
 &= \sum_{i=0}^{r_j} \langle D_n^i G(x-y)|_{x_n=0}, b_{j\iota}(-D') \check{\psi}(x') \rangle \\
 &= \sum_{i=0}^{r_j} \sum_{\nu=0}^i C_{a\nu} y_n^{-\alpha-\nu} \int_{\mathbb{R}^n} \exp\{-iy(\xi+i\eta)\} b_{j\iota}(\xi'+i\eta') \\
 &\quad \times \widehat{\psi}(\xi'+i\eta') (\xi_n+i\eta_n)^{i-\nu} (-i\partial/\partial\xi_n)^\alpha (P(\xi+i\eta))^{-1} d\xi, \\
 &\quad y_n \neq 0.
 \end{aligned}$$

Now we define $F(\cdot, x_n, y) \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ by

$$\begin{aligned}
 (6.20) \quad & \langle F(x, y), \check{\psi}(x') \rangle = (2\pi)^{-n} (-i) \sum_{j,k=1}^p \sum_{i=0}^{r_j} \sum_{\nu=0}^i C_{a\nu} y_n^{-\alpha-\nu} \\
 &\quad \times \int_{\mathbb{R}^n} \left\{ \oint \frac{\exp\{ix_n\lambda\} \lambda^{k-1}}{P_+(\xi'+i\eta', \lambda)} d\lambda \right\} \exp\{-iy(\xi+i\eta)\} \frac{R_{jk}(\xi'+i\eta')}{R(\xi'+i\eta')} \\
 &\quad \times b_{j\iota}(\xi'+i\eta') \widehat{\psi}(\xi'+i\eta') (\xi_n+i\eta_n)^{i-\nu} (-i\partial/\partial\xi_n)^k \\
 &\quad \times (P(\xi+i\eta))^{-1} d\xi,
 \end{aligned}$$

where $R_{jk}(\xi'+i\eta')$ is the (j, k) cofactor of the matrix

$$\begin{aligned}
 & \left((2\pi i)^{-1} \oint B_j(\xi'+i\eta', \lambda) \lambda^{k-1} (P_+(\xi'+i\eta', \lambda))^{-1} d\lambda \right)_{j,k=1,\dots,p} \\
 & \quad \xi' + i\eta' \in \mathfrak{E}^{n-1} - i\gamma_1 \vartheta' - i\dot{S}.
 \end{aligned}$$

If we put

$$E(x, y) = G(x-y) - F(x, y),$$

then $E(x, y)$ is the required fundamental solution, that is, $E(x, y)$ belongs to the space $C^\infty([0, y_n); \mathcal{D}'(\mathbf{R}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ and satisfies the equations

$$P(D_x)E(x, y) = \delta(x-y), x_n > 0, y_n > 0, B_j(D_x)E(x, y)|_{x_n=0} = 0, j=1, \dots, p.$$

From Lemma 4.4 and the Paley-Wiener-Schwartz theorem, it follows that there exists a closed proper cone K with its vertex at the origin such that $x \cdot \vartheta > 0$ on $K - \{0\}$ and $\text{supp } E(x, y) \subset (K + y) \cap \overline{\mathbf{R}}_+^n$.

On the other hand, the fundamental solutions $F_j(x)$ are defined by

$$\begin{aligned}
 (6.21) \quad & \langle F_j(x), \check{\psi}(x') \rangle \\
 &= (2\pi)^{-n} (-i) \sum_{k=1}^p \int_{\mathbb{R}^{n-1}} \left\{ \oint \exp\{ix_n\lambda\} \lambda^{k-1} (P_+(\xi'+i\eta', \lambda))^{-1} d\lambda \right\}
 \end{aligned}$$

$$\times R_{jk}(\xi' + i\eta') (R(\xi' + i\eta'))^{-1} \widehat{\psi}(\xi' + i\eta') d\xi', \quad j=1, \dots, p.$$

Obviously the $F_j(x), j=1, \dots, p$ belong the space $C^\infty([0, \infty), \mathcal{D}'(\mathbf{R}^{n-1}))$ and satisfy the equations:

$$P(D)F_j(x) = 0, x_n > 0, B_j(D)F_k(x)|_{x_n=0} = \delta_{jk}\delta(x'), \quad j, k=1, \dots, p.$$

From Lemma 4.4 and the Paley-Wiener-Schwartz theorem we also see that the support of every $F_j(x)$ is contained in some closed proper cone having no points $\neq 0$ in common with the half-space $x \cdot \vartheta > 0$. Thus we obtained a system $\{E(x, y); F_j(x), j=1, \dots, p\}$ of fundamental solutions of the mixed problem (1.3)-(1.5) for $\{P(D); B_j(D), j=1, \dots, q\}$ which satisfy the required condition on their supports. It only remains to show the uniqueness for such a system.

Proof of the uniqueness of $\{E(x, y); F_j(x), j=1, \dots, p\}$. The uniqueness of the system $\{F_j(x), j=1, \dots, p\}$ follows immediately from Proposition 5.1. In fact, suppose that there exists an another system $\{F'_j(x) \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}_x^{n-1})), j=1, \dots, p\}$ satisfying (1.10), (1.11) and the required condition on support. We may assume that $\text{supp } F'_j \subset K$ replacing K by a larger one if necessary. Then $W(x) = F_j(x) - F'_j(x)$ satisfies (5.1), (5.2) and $\text{supp } W \subset K$. Therefore we have $F_j(x) \equiv F'_j(x)$ by Proposition 5.1.

If the hyperplane $x_n=0$ is non-characteristic for $P(D)$, the uniqueness of $E(x, y)$ also follows immediately from Proposition 3.1 and Proposition 5.1. First we note that if $u \in \mathcal{D}'(\mathbf{R}_+^n)$ belongs to the space $C^\infty([0, \delta), \mathcal{D}'(\mathbf{R}_x^{n-1}))$ for some $\delta > 0$, then $u \in \mathcal{D}'(\overline{\mathbf{R}_+^n})$. In fact, take $\chi(x_n) \in C_0^\infty((-\delta/2, \delta/2))$ with $\chi(x_n) = 1$ on $[-\delta/4, \delta/4]$ and decompose $u = u + (1-\chi)u \equiv u_1 + u_2$. Then $u_1 \in C^\infty([0, \delta); \mathcal{D}'(\mathbf{R}_x^{n-1}))$ and $\text{supp } u_1 \subset \mathbf{R}_x^{n-1} \times [0, \delta/2)$ so $u_1 \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}_x^{n-1})) \subset \mathcal{D}'(\overline{\mathbf{R}_+^n})$. Since $\text{supp } u_2 \subset \mathbf{R}_x^{n-1} \times (\delta/4, \infty)$, we have $u_2 \in \mathcal{D}'(\overline{\mathbf{R}_+^n})$ and so $u \in \mathcal{D}'(\overline{\mathbf{R}_+^n})$. Now suppose that there exists a $E'(x, y) \in \mathcal{D}'(\mathbf{R}_+^n) \cap C^\infty([0, y_n); \mathcal{D}'(\mathbf{R}_x^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbf{R}_x^{n-1}))$ satisfying (1.8), (1.9) and $\text{supp } E'(x, y) \subset K + y$. Then $W(x) = E(x, y) - E'(x, y) \in \mathcal{D}'(\overline{\mathbf{R}_+^n})$ and satisfies (5.1), (5.2). Hence $W \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}_x^{n-1}))$ by Proposition 3.1. Since $\text{supp } W \subset K + y$, it follows from Proposition 5.1 that $W(x) \equiv 0$, that is, $E(x) \equiv E'(x, y)$.

When the hyperplane $x_n=0$ is characteristic for $P(D)$, our argument above does not go through. So we shall prove the uniqueness of $E(x, y)$ by a standard method employing the adjoint system of $\{P(D); B_j(D), j=1, \dots, p\}$. The following lemma is a modification of Green's formula given in Sakamoto [9].

Lemma 6.1. *Suppose that the assumptions (I), (II) and (III) stated at the beginning of this section hold, and that $u(x)$ is an $\mathcal{E}'(\mathbf{R}_x^{n-1})$ -valued C^∞ function of x_n in $[0, \infty)$ with compact support. Then for any $\phi \in C_0^\infty(\mathbf{R}_+^n)$ there exist suitable functions $\psi_j \in C^\infty(\mathbf{R}_x^{n-1})$ ($j=1, \dots, p$), $\rho_j \in C^\infty(\mathbf{R}_x^{n-1})$ ($j=1, \dots, r-m'$) and $\psi \in C^\infty(\overline{\mathbf{R}_+^n})$ such that $\text{supp } \psi \subset \text{supp } \phi + \Lambda$, $\text{supp } \psi_j, \text{supp } \rho_j \subset \{x'; (x', 0) \in \text{supp } \psi\}$ and that*

$$\begin{aligned}
 (6.22) \quad & \int_0^\infty \langle P(D)u(x', x_n), \overline{\psi(x', x_n)} \rangle dx_n \\
 & - \int_0^\infty \langle u(x', x_n), \overline{\phi(x', x_n)} \rangle dx_n \\
 & = \sum_{j=1}^p \langle B_j(D)u|_{x_n=0}, \overline{\psi_j(x')} \rangle + \sum_{k=1}^{r-m'} \langle D_k^k P(D)u|_{x_n=0}, \overline{\rho_k(x')} \rangle,
 \end{aligned}$$

where Λ is a closed proper cone such that $x \cdot (-\vartheta) > 0$ on $\Lambda - \{0\}$, m' is the order of $P(D)$ in D_n , r is the maximum of the orders r_j of $B_j(D)$ and \langle , \rangle denotes the duality between $\mathcal{E}'(\mathbf{R}_x^{n-1})$ and $\mathcal{E}(\mathbf{R}_x^{n-1})$.

Proof. Let $\lambda_j^+(\zeta')$, $j=1, \dots, p$ (resp. $\lambda_j^-(\zeta')$, $j=1, \dots, m'-p$) denote the roots with positive (resp. negative) imaginary part of the equation $P(\zeta', \lambda) = 0$ in λ when $\zeta' \in \mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\dot{I}$, where m' is the order of $P(\zeta', \xi_n)$ in ξ_n (see Section 4). Put

$$P_+(\zeta', \lambda) = \prod_{j=1}^p (\lambda - \lambda_j^+(\zeta')), \quad P_-(\zeta', \lambda) = \prod_{j=1}^{m'-p} (\lambda - \lambda_j^-(\zeta')). \tag{9}$$

Then

$$P(\zeta', \lambda) = \sum_{j=0}^{m'} q_j(\zeta') \lambda^{m'-j} = q_0(\zeta') P_+(\zeta', \lambda) P_-(\zeta', \lambda).$$

As stated in Section 4, we have

$$q_0(\zeta') \neq 0 \quad \text{in} \quad \mathfrak{E}^{n-1} - i\gamma_0 \vartheta' - i\dot{I}.$$

9) This notation is different from one in Section 4.

where tU denotes the transpose of U , then we have

$$(6.23) \quad (P(D_n)u, v) - (u, \bar{P}(D_n)v) = iV^*AU,$$

where V^* denotes the adjoint of V . Put

$$(6.24) \quad B_j(\zeta', \lambda) = Q_j(\zeta', \lambda)P(\zeta', \lambda) + B'_j(\zeta', \lambda), \quad j=1, \dots, m' - p,$$

and

$$(6.25) \quad B'_{j+p}(\zeta', D_n) = D_n^{j-1}P_+(\zeta', D_n), \quad j=1, \dots, m' - p,$$

$$\zeta' \in \mathfrak{E}^{n-1} - i\gamma_1\vartheta' - i\dot{\Sigma},$$

where $\deg Q_j(\zeta', \lambda) \leq r - m'$ (if $r - m' < 0$, $Q_j(\zeta', \lambda) = 0$). When ζ' is fixed in $\mathfrak{E}^{n-1} - i\gamma_1\vartheta' - i\dot{\Sigma}$ we also put

$$B'_j(\lambda) = B'_j(\zeta', \lambda).$$

Let us denote

$$\begin{pmatrix} B'_1(D_n)u(0) \\ \vdots \\ B'_p(D_n)u(0) \\ B'_{p+1}(D_n)u(0) \\ \vdots \\ B'_{m'}(D_n)u(0) \end{pmatrix} = \begin{pmatrix} \beta'_{11}, \dots, \beta'_{1p} & \beta'_{1p+1}, \dots, \beta'_{1m'} \\ \vdots & \vdots \\ \beta'_{p1}, \dots, \beta'_{pp} & \beta'_{pp+1}, \dots, \beta'_{pm'} \\ 0, \dots, 0 & 1 \ 0, \dots, 0 \\ \vdots & 0 \ \ddots \ \vdots \\ 0, \dots, 0 & 0, \dots, 0 \ 1 \end{pmatrix} \begin{pmatrix} u(0) \\ \vdots \\ D_n^{p-1}u(0) \\ P_+(D_n)u(0) \\ \vdots \\ D_n^{m'-p-1}P_+(D_n)u(0) \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{B}_{11} & \vdots & \mathcal{B}_{12} \\ \vdots & \vdots & \vdots \\ 0 & \vdots & I \end{pmatrix} U = \mathcal{B}U,$$

then we have $\det \mathcal{B} = \det \mathcal{B}_{11} = R$, where R is the Lopatinski determinant of the system $\{P(D), B_j(D), j=1, \dots, p\}$. Since $R = R(\zeta') \neq 0$ by Lemma 4.4 for $\zeta' \in \mathfrak{E}^{n-1} - i\gamma_1\vartheta' - i\dot{\Sigma}$, we can put

$$\mathcal{B}'' = (\mathcal{A}\mathcal{B}^{-1})^*,$$

and then we have

$$(P(D_n)u, v) - (u, \bar{P}(D_n)v) = i(\mathcal{B}''V)^*(\mathcal{B}U).$$

Put

$$(6.26) \quad \mathcal{B}''V = \begin{pmatrix} B''_1(\bar{\zeta}', D_n)v(0) \\ \vdots \\ B''_p(\bar{\zeta}', D_n)v(0) \\ B''_{p+1}(\bar{\zeta}', D_n)v(0) \\ \vdots \\ B''_{m'}(\bar{\zeta}', D_n)v(0) \end{pmatrix}$$

$$\begin{aligned}
 &= \left(\begin{array}{cc} b''_{11}(\bar{\zeta}'), \dots, b''_{1p}(\bar{\zeta}') & b''_{1p+1}(\bar{\zeta}'), \dots, b''_{1m'}(\bar{\zeta}') \\ \vdots & \vdots \\ b''_{p1}(\bar{\zeta}'), \dots, b''_{pp}(\bar{\zeta}') & b''_{pp+1}(\bar{\zeta}'), \dots, b''_{pm'}(\bar{\zeta}') \\ b''_{p+11}(\bar{\zeta}'), \dots, b''_{p+1p}(\bar{\zeta}') & b''_{p+1p+1}(\bar{\zeta}'), \dots, b''_{p+1m'}(\bar{\zeta}') \\ \vdots & \vdots \\ b''_{m'1}(\bar{\zeta}'), \dots, b''_{m'p}(\bar{\zeta}') & b''_{m'p+1}(\bar{\zeta}'), \dots, b''_{m'm'}(\bar{\zeta}') \end{array} \right) V \\
 &= \begin{pmatrix} \mathcal{B}''_{11} & \mathcal{B}''_{12} \\ \mathcal{B}''_{21} & \mathcal{B}''_{22} \end{pmatrix} V.
 \end{aligned}$$

Note that if $f(\zeta')$ is holomorphic in $\mathfrak{E}^{n-1} - i\gamma_1\vartheta' - i\dot{S}$, then $\overline{f(\zeta')}$ is holomorphic in $\mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{S}$. The coefficients of $B'_j(\zeta', \lambda)$ are all holomorphic in $\mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{S}$. Hence, it follows from (6.23), (6.24) and (6.26) that

$$\begin{aligned}
 (6.27) \quad & \int_0^\infty P(\zeta', D_n) \hat{u}(\zeta', x_n) \overline{v(x_n)} dx_n \\
 & - \int_0^\infty \hat{u}(\zeta', x_n) \overline{\overline{P(\bar{\zeta}', D_n)} v(x_n)} dx_n \\
 & = i \sum_{j=1}^p B_j(\zeta', D_n) \hat{u}(\zeta', 0) \cdot \overline{B'_j(\bar{\zeta}', D_n) v(0)} \\
 & \quad - i \sum_{j=1}^p \sum_{k=0}^{r-m'} D_n^k P(\zeta', D_n) \hat{u}(\zeta', 0) \cdot \overline{q_j^k(\bar{\zeta}') B'_j(\bar{\zeta}', D_n) v(0)} \\
 & \quad + i \sum_{j=p+1}^{m'} B'_j(\zeta', D_n) \hat{u}(\zeta', 0) \cdot \overline{B'_j(\bar{\zeta}', D_n) v(0)}
 \end{aligned}$$

where $Q_j(\zeta', \lambda) = \sum_{k=0}^{r-m'} q_j^k(\zeta') \lambda^k$ (if $r - m' < 0$ for some j , $q_j^k(\zeta') = 0$) and the coefficients of $B'_j(\bar{\zeta}', D_n)$ and $\overline{q_j^k(\bar{\zeta}')}$ are all holomorphic in $\bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{S}$. Since $\overline{P(\bar{\zeta}', \lambda)} = \overline{q_0(\bar{\zeta}') \cdot P_+(\zeta', \lambda) P_-(\zeta', \lambda)}$ when $\bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{S}$, we denote the roots of the equation $\overline{P(\bar{\zeta}', \lambda)} = 0$ in λ by $\tilde{\lambda}_1^+(\bar{\zeta}'), \dots, \tilde{\lambda}_{m'-p}^+(\bar{\zeta}'), \tilde{\lambda}_1^-(\bar{\zeta}'), \dots, \tilde{\lambda}_p^-(\bar{\zeta}')$, which are enumerated so that $\text{Im } \tilde{\lambda}_k^+(\bar{\zeta}') > 0, k=1, \dots, m'-p, \text{Im } \tilde{\lambda}_k^-(\bar{\zeta}') < 0, k=1, \dots, p, \bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{S}$, and then we have

$$\begin{aligned}
 \overline{P_+(\zeta', \lambda)} &= \lambda^p + \sum_{j=1}^p \overline{a_j^+(\zeta')} \lambda^{p-j} = \prod_{j=1}^p (\lambda - \tilde{\lambda}_j^-(\bar{\zeta}')) \\
 \overline{P_-(\zeta', \lambda)} &= \lambda^{m'-p} + \sum_{j=1}^{m'-p} \overline{a_j^-(\zeta')} \lambda^{m'-p-j} = \prod_{j=1}^{m'-p} (\lambda - \tilde{\lambda}_j^+(\bar{\zeta}')), \\
 \bar{\zeta}' &\in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{S}.
 \end{aligned}$$

Put

$$R^*(\bar{\zeta}') = \det \left((2\pi i)^{-1} \oint_{j,k=1, \dots, m'-p} B''_{j+p}(\bar{\zeta}', \lambda) \lambda^{k-1} (\overline{P_-(\zeta', \lambda)})^{-1} d\lambda \right)$$

$$\bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}.$$

Since

$$B''_{p+j}(\bar{\zeta}', \lambda) = b''_{p+j1}(\bar{\zeta}')\lambda^{p-1}\bar{P}_-(\zeta', \lambda) + \dots + b''_{p+jp}(\bar{\zeta}')\bar{P}_-(\zeta', \lambda) + b''_{p+jp+j}(\bar{\zeta}')\lambda^{m'-p-1} + \dots + b''_{p+jm'}(\bar{\zeta}'),$$

we have

$$R^*(\bar{\zeta}') = \det \mathcal{B}''_{22}, \bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}.$$

On the other hand, since

$$\mathcal{B}^{-1} = \begin{pmatrix} \mathcal{B}_{11}^{-1} & \vdots & \mathcal{B}_{11}^{-1}\mathcal{B}_{12} \\ \dots\dots\dots & \vdots & \dots\dots\dots \\ 0 & \vdots & I \end{pmatrix},$$

we have $\mathcal{B}''_{22} = (\mathcal{A}_-)^*$. And then we have $R^*(\bar{\zeta}') = \overline{(q_0(\zeta'))}^{m'-p}$
 $\bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}$. Put $R^*_{jk}(\bar{\zeta}') =$ the (j, k) cofactor of

$$\left((2\pi i)^{-1} \oint B''_{p+j}(\bar{\zeta}', \lambda) \lambda^{k-1} (\bar{P}_-(\zeta', \lambda))^{-1} d\lambda \right)_{j, k=1, \dots, m'-p},$$

$$\bar{\zeta}' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}.$$

For any $\phi \in C^\infty_0(\mathbf{R}^n_+)$ we put

$$v(\xi' + i\eta', x_n) = v_1(\xi' + i\eta', x_n) - v_2(\xi' + i\eta', x_n),$$

$$\xi' + i\eta' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}$$

where

$$v_1(\xi' + i\eta', x_n) = (2\pi)^{-1} \int \exp\{ix_n \xi_n\} \hat{\phi}(\xi' + i\eta', \xi_n) \times (\bar{P}(\xi' + i\eta', \xi_n))^{-1} d\xi_n,$$

$$v_2(\xi' + i\eta', x_n) = \sum_{j, k=1}^{m'-p} (2\pi i)^{-1} \oint \exp\{ix_n \lambda\} R^*_{jk}(\xi' + i\eta') \lambda^{k-1} (R^*(\xi' + i\eta'))^{-1}$$

$$\times B''_{p+j}(\xi' + i\eta', D_n) v_1(\xi' + i\eta', 0) (\bar{P}_-(\xi' - i\eta', \lambda))^{-1} d\lambda,$$

$$\xi' + i\eta' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}.$$

Put

$$(6.28) \quad \psi(x) = (2\pi)^{-(n-1)} \int_{\gamma' \in \gamma_1\vartheta + \dot{\Sigma}} \exp\{ix'(\xi' + i\eta')\} v(\xi' + i\eta', x_n) d\xi'$$

Since $\overline{P(\zeta)} \neq 0$ when $\zeta \in \mathfrak{E}^n + i\gamma_0\vartheta + i\Gamma$ and the coefficients of $\overline{P_-}(\xi' - i\eta', \lambda)$ and $B''_{p+j}(\xi' + i\eta', \lambda)$ and $R_{jk}^*(\xi' + i\eta')$ is holomorphic in $\xi' + i\eta' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}$, it follows from Paley-Wiener-Schwartz theorem that $\phi(x)$ belongs to $C^\infty(\overline{\mathbf{R}}_+^n)$, whose support is contained in $\{\text{supp } \phi + A\} \cap \{x_n \geq 0\}$, where A is some closed proper cone with its vertex at the origin such that $x \cdot (-\vartheta) > 0$ on $A - \{0\}$. Since $\overline{P}(\xi' + i\eta', D_n)v(\xi' + i\eta', x_n) = \phi(\xi' + i\eta', x_n)$ and $B''_j(\xi' + i\eta', D_n)v(\xi' + i\eta', 0) = 0, j = p+1, \dots, m'$, it follows from (6.27) that

$$\begin{aligned}
 (6.29) \quad & \int_0^\infty P(\xi' - i\eta', D_n)\hat{u}(\xi' - i\eta', x_n) \cdot \overline{v(\xi' + i\eta', x_n)} dx_n \\
 & - \int_0^\infty \hat{u}(\xi' - i\eta', x_n) \cdot \overline{\hat{\phi}(\xi' + i\eta', x_n)} dx_n \\
 & = i \sum_{j=1}^p B_j(\xi' - i\eta', D_n)\hat{u}(\xi' - i\eta', 0) \\
 & \quad \cdot \overline{B''_j(\xi' + i\eta', D_n)v(\xi' + i\eta', 0)} \\
 & - i \sum_{j=1}^p \sum_{k=0}^{r-m'} D_n^k P(\xi' - i\eta', D_n)\hat{u}(\xi' - i\eta', 0) \\
 & \quad \cdot \overline{q_j^k(\xi' - i\eta') B''_j(\xi' + i\eta', D_n)v(\xi' + i\eta', 0)}.
 \end{aligned}$$

Put

$$\begin{aligned}
 (6.30) \quad \psi_j(x') &= (2\pi)^{-(n-1)} \int_{\eta' \in r_1 \vartheta' + \dot{\Sigma}} \exp\{ix'(\xi' + i\eta')\} \\
 & \quad \times B''_j(\xi' + i\eta', D_n)v(\xi' + i\eta', 0) d\xi', \quad j=1, \dots, p
 \end{aligned}$$

$$\begin{aligned}
 (6.31) \quad \rho_j(x') &= (2\pi)^{-(n-1)} \int_{\eta' \in r_1 \vartheta' + \dot{\Sigma}} \exp\{ix'(\xi' + i\eta')\} \{ \sum_{k=1}^p \overline{q_k^j(\xi' - i\eta')} \\
 & \quad B''_j(\xi' + i\eta', D_n)v(\xi' + i\eta', 0) \} d\xi', \quad j=1, \dots, r-m'.
 \end{aligned}$$

Since the coefficients of $B''_j(\xi' + i\eta', D_n)$ and $\overline{q_k^j(\xi' - i\eta')}$ are all holomorphic in $\xi' + i\eta' \in \mathfrak{E}^{n-1} + i\gamma_1\vartheta' + i\dot{\Sigma}$, it follows from the Paley-Wiener-Schwartz theorem that

$$\begin{aligned}
 & \text{supp } \psi_j \subset \{x' \in \mathbf{R}^{n-1}; (x', 0) \in \text{supp } \phi\}, \\
 & \text{supp } \rho_j \subset \{x' \in \mathbf{R}^{n-1}; (x', 0) \in \text{supp } \phi\}.
 \end{aligned}$$

We have

$$(6.32) \quad \langle u, \bar{v} \rangle = \langle u(\xi' - i\eta', x_n), \overline{\mathcal{F}_{x'}[\bar{v}]}(\xi' - i\eta', x_n) \rangle,$$

where

$$\begin{aligned} \overline{\mathcal{F}}_{x'}[\bar{v}](\xi' - i\eta', x_n) &= (2\pi)^{-(n-1)} \int \exp\{ix'(\xi' - i\eta')\} \bar{v}(x', x_n) dx' \\ &= (2\pi)^{-(n-1)} \overline{\int \exp\{-ix'(\xi' + i\eta')\} v(x', x_n) dx'}. \end{aligned}$$

Hence, it follows from (6.28) - (6.32) that

$$\begin{aligned} &\int_0^\infty \langle P(D)u, \bar{\phi} \rangle dx_n \\ &= (2\pi)^{-(n-1)} \left[\int_0^\infty \langle P(\xi' - i\eta', D_n) \hat{u}(\xi' - i\eta', x_n), \overline{\hat{\phi}(\xi' + i\eta', x_n)} \rangle dx_n \right. \\ &= (2\pi)^{-2(n-1)} \int_0^\infty \langle P(\xi' - i\eta', D_n) \hat{u}(\xi' - i\eta', x_n), \\ &\quad \left. \overline{\int_{\eta' \in \tau_1 \theta' + \bar{x}} \exp\{-ix'(\xi' + i\eta')\} \psi(x', x_n) dx'} \rangle dx_n \right. \\ &= (2\pi)^{-(n-1)} \left[\int_0^\infty \langle \hat{u}(\xi' - i\eta', x_n), \overline{\hat{\phi}(\xi' + i\eta', x_n)} \rangle dx_n \right. \\ &\quad + i \sum_{j=1}^p \langle B_j(\xi' - i\eta', D_n) \hat{u}(\xi' - i\eta', 0), \\ &\quad \left. \overline{B'_j(\xi' + i\eta', D_n) v(\xi' + i\eta', 0)} \rangle \right. \\ &\quad - i \sum_{j=1}^p \sum_{k=0}^{r-m'} \langle D_n^k P(\xi' - i\eta', D_n) \hat{u}(\xi' - i\eta', 0), \\ &\quad \left. \overline{q_j^k(\xi' - i\eta') \cdot B'_j(\xi' + i\eta', D_n) v(\xi' + i\eta', 0)} \rangle \right] \\ &= \int_0^\infty \langle u(x), \bar{\phi}(x) \rangle dx_n + i \sum_{j=1}^p \langle B_j(D)u(x', 0), \overline{\psi_j(x')} \rangle \\ &\quad - i \sum_{j=0}^{r-m'} \langle D_n^j P(D)u(x', 0), \overline{\rho_j(x')} \rangle. \end{aligned}$$

Q.E.D.

Proposition 6.2. *Under the hypotheses (I), (II) and (III) we have $W(x) \equiv 0$ provided that $W(x)$ is a solution of the homogeneous boundary value problem in $\overline{\mathbf{R}}_+^n$:*

$$P(D)W(x) = 0 \text{ in } \overline{\mathbf{R}}_+^n, \quad B_j(D)W(x)|_{x_n=0} = 0, \quad j=1, \dots, p,$$

such that $W(x)$ belongs to $C^\infty([0, \delta]; \mathcal{D}'(R_{x'}^{n-1}))$ for some $\delta > 0$ and that $\text{supp } W(x) \subset K \cap \overline{\mathbf{R}}_+^n$.

Proof. It is sufficient to show that

$$(6.33) \quad \langle W, \phi \rangle = 0 \quad \text{for any } \phi \in C_0^\infty(\mathbf{R}_+^n).$$

In fact, if we show (6.33), then we have

$$\langle W(x', x_n), \phi(x') \rangle_{x'} = 0 \quad \text{for any } \phi \in \mathcal{D}'(\mathbf{R}^{n-1})$$

when $0 < x_n < \delta$. $W(x)$ belongs to $C^\infty([0, \delta]; \mathcal{D}'(\mathbf{R}_+^{n-1}))$. Hence

$$\langle W(x', 0), \phi(x') \rangle_{x'} = \lim_{x_n \rightarrow +0} \langle W(x', x_n), \phi(x') \rangle_{x'} = 0.$$

We shall show (6.33). Let ψ, ψ_j ($j=1, \dots, p$) and ρ_j ($j=1, \dots, r-m'$) be functions which are stated in Lemma 6.1 for $\phi \in C_0^\infty(\mathbf{R}_+^n)$. Put

$$\Omega_1 = \{x' \in \mathbf{R}^{n-1}; x_1 \geq 0, (x', x_n) \in \text{supp } \psi \text{ for some } x_n\}$$

$$\Omega_2 = \{x' \in \mathbf{R}^{n-1}; \text{dist}(x', \Omega_1) \leq 1\}.$$

Let $\chi \in C_0^\infty(\Omega_2 \times (-\delta/2, \delta/2))$ be 1 in $\Omega_1 \times (-\delta/4, \delta/4)$. Let $\sigma_1(x_n) \in C_0^\infty((-\delta, \delta))$ be 1 in $(-\delta/2, \delta/2)$ and $\sigma_2(x') \in C^\infty(\mathbf{R}^{n-1})$ be 1 in $x_1 \geq 0$ and 0 in $x_1 \leq -1/2$. Put

$$\begin{aligned} \langle\langle P(D)W, \bar{\psi} \rangle\rangle &= \int_0^\infty \langle \sigma_1(x_n) P(D)W, \sigma_2(x') \bar{\psi} \rangle dx_n \\ &\quad + \langle (1 - \sigma_1(x_n)) P(D)W, \sigma_2(x') \bar{\psi} \rangle \end{aligned}$$

where \langle, \rangle in the first term of the right-hand side denotes the duality between $\mathcal{D}'(\mathbf{R}^{n-1})$ and $\mathcal{D}(\mathbf{R}^{n-1})$ and \langle, \rangle in the second term of the right-hand side denotes the duality between $\mathcal{D}'(\mathbf{R}_+^n)$ and $\mathcal{D}(\mathbf{R}_+^n)$. Since χW is a $C_0^\infty([0, \infty))$ function of x_n with its value in $\mathcal{E}'(\mathbf{R}_+^{n-1})$ and $\sigma_1(x_n)\sigma_2(x') = 1$ in $\text{supp } \chi \cap \{x_1 \geq 0\}$ and $\text{supp } W \subset \{x_1 \geq 0\}$, it follows from Lemma 6.1 that

$$\begin{aligned} \int_0^\infty \langle \sigma_1(x_n) P(D)\chi W, \sigma_2(x_n) \bar{\psi} \rangle dx_n &= \int_0^\infty \langle P(D)\chi W, \bar{\psi} \rangle dx_n = \langle \chi W, \bar{\psi} \rangle \\ &\quad + \sum_{j=1}^p \langle B_j(D)\chi W|_{x_n=0}, \bar{\psi}_j \rangle + \sum_{k=1}^{r-m} \langle D_n^k P(D)\chi W|_{x_n=0}, \bar{\theta}_j \rangle. \end{aligned}$$

Since we have

$$\begin{aligned} B_j(D)\chi W|_{x_n=0} &= \chi(x', 0) B_j(D)W|_{x_n=0} \\ &\quad + \sum_{|\alpha| \geq 1} (D^\alpha \chi / \alpha!) (x', 0) B_j^{(\alpha)}(D)W|_{x_n=0} \\ &= \sum_{|\alpha| \geq 1} (D^\alpha \chi / \alpha!) (x', 0) B_j^{(\alpha)}(D)W|_{x_n=0}, \end{aligned}$$

and $\text{supp}(D^\alpha \chi / \alpha!) (x', 0) \cap \text{supp } \bar{\psi}_j \subset \{x_1 < 0\}$, it follows that

$$\langle B_j(D) \chi W|_{x_n=0}, \bar{\psi}_j \rangle = 0, \quad j=1, \dots, p.$$

Since we have

$$P(D) \chi W|_{x_n=0} = \sum_{|\alpha| \geq 1} (D^\alpha \chi / \alpha!) (x', 0) P^{(\alpha)}(D) W|_{x_n=0}$$

and $\text{supp}(D^\alpha \chi / \alpha!) (x', 0) \cap \text{supp } \bar{\rho}_j \subset \{x_1 < 0\}$, it follows that

$$\langle D_n^k P(D) \chi W|_{x_n=0}, \bar{\rho}_j \rangle = 0, \quad j=1, \dots, r-m'.$$

Hence we have

$$\int_0^\infty \langle \sigma_1(x_n) P(D) \chi W, \sigma_2(x') \bar{\psi} \rangle dx_n = \langle \chi W, \bar{\psi} \rangle.$$

Since $\text{supp}[\sigma_1(x_n) P(D) (1-\chi) W] \cap \text{supp } \bar{\psi} \subset \{x \in \mathbf{R}_+^n; x_1 \geq 0, x_n \geq \delta/4\}$, it follows that

$$\begin{aligned} & \int_0^\infty \langle \sigma_1(x_n) P(D) (1-\chi) W, \sigma_2(x') \bar{\psi} \rangle dx_n \\ &= \int_0^\infty \langle W, \overline{(1-\chi) \bar{P}(D) (\sigma_1(x_n) \sigma_2(x') \psi)} \rangle dx_n. \end{aligned}$$

Since $\text{supp}(1-\sigma_1(x_n)) \sigma_2(x') \bar{\psi} \subset \{x \in \mathbf{R}_+^n; x_1 > -1/2, x_n > \delta/2\}$, it follows that

$$\begin{aligned} & \langle (1-\sigma_1(x_n)) P(D) W, \sigma_2(x') \bar{\psi} \rangle \\ &= \langle P(D) W, (1-\sigma_1(x_n)) \sigma_2(x') \bar{\psi} \rangle \\ &= \langle W, \overline{\bar{P}(D) (1-\sigma_1(x_n)) \sigma_2(x') \psi} \rangle = \langle W, \overline{\bar{P}(D) (1-\sigma_1(x_n)) \sigma_2(x') \psi} \rangle. \end{aligned}$$

Hence we have

$$\langle\langle P(D) W, \psi \rangle\rangle = \langle W, \bar{\psi} \rangle.$$

On the other hand, since

$$\begin{aligned} & \int_0^\infty \langle \sigma_1(x_n) P(D) W, \sigma_2(x') \bar{\psi} \rangle dx_n \\ &= \int_0^\infty \langle P(D) W, \sigma_1(x_n) \sigma_2(x') \bar{\psi} \rangle dx_n = 0, \end{aligned}$$

and since

$$\langle (1-\sigma_1(x_n)) P(D) W, \sigma_2(x') \bar{\psi} \rangle$$

$$= \langle P(D)W, (1 - \sigma_1(x_n))\sigma_2(x')\bar{\psi} \rangle = 0,$$

(6.33) follows.

Q.E.D.

Summing up, we have

Theorem 6.3. *Let $P(D)$ and $B_j(D)$, $j=1, \dots, q$ be differential operators of order m and r_j , $j=1, \dots, q$, respectively, with constant coefficients. The following three assertions are equivalent.*

(I) *There exists a unique system of fundamental solutions $\{E(x, y), F_j(x), j=1, \dots, q\}$ satisfying the following equations:*

$$(6.34) \quad P(D)E(x, y) = \delta(x-y) \text{ in } \mathbf{R}_+^n, y \in \mathbf{R}_+^n, y = (0, y_2, \dots, y_n),$$

$$(6.35) \quad B_j(D)E(x, y)|_{x_n=0} = 0 \text{ in } \mathbf{R}^{n-1}, j=1, \dots, q.$$

$$(6.36) \quad P(D)F_k(x) = 0 \text{ in } \mathbf{R}_+^n, k=1, \dots, q,$$

$$(6.37) \quad B_j(D)F_k(x)|_{x_n=0} = \delta_{jk}\delta(x') \text{ in } \mathbf{R}^{n-1}, j, k=1, \dots, q,$$

such that $E(x, y)$ and $F_j(x)$ belong to $C^\infty([0, \sigma]; \mathcal{D}'(\mathbf{R}^{n-1})) \cap \mathcal{D}'(\mathbf{R}_+^n)$ for some positive number σ and there exists some closed proper cone with its vertex at the origin such that $x \cdot \vartheta > 0$ on $x \in K - \{0\}$ such that $\text{supp } E(x, y)$ contained in $(K + y) \cap \bar{\mathbf{R}}_+^n$ and $\text{supp } F_k(x)$ is contained in $K \cap \bar{\mathbf{R}}_+^n$.

(II) *There exists a unique system of fundamental solutions $\{E(x, y), F_j(x), j=1, \dots, q\}$ satisfying the system of equations (6.34)–(6.37) such that $E(x, y)$ belongs to $C^\infty([0, y_n]; \mathcal{D}'(\mathbf{R}^{n-1})) \cap C^\infty((y_n, \infty); \mathcal{D}'(\mathbf{R}^{n-1})) \cap \mathcal{D}'(\mathbf{R}_+^n)$ and $F_j(x)$ belong to $C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^{n-1}))$ and the support of $E(x, y)$ is contained in $(K + y) \cap \bar{\mathbf{R}}_+^n$ and the support of $F_j(x)$ is contained in $K \cap \bar{\mathbf{R}}_+^n$.*

(III) *$P(\xi)$ is a hyperbolic polynomial with respect to ϑ and q equals the number of roots with positive imaginary part of the equation $P(\xi' - is\vartheta', \lambda) = 0$ in λ for large enough s and $\xi' \in \mathbf{E}^{n-1}$ and the Lopatinski determinant $R(\zeta')$ of the system $\{P(D), B_j(D), j=1, \dots, q\}$ is hyperbolic with respect to ϑ' .*

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