Asymptotic Behavior and Stability of Solutions of Semilinear Diffusion Equations

By

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§ 1. Introduction

This paper is divided into two chapters. In the first chapter (§ 2 ∼ § 4) we study the asymptotic behavior of solutions of semilinear diffusion equations—in particular their behavior around unstable equilibrium solutions, so as to get a clearer understanding of the situation where those unstable equilibrium solutions are placed. The results will then be applied to making some stability criteria as well as to establishing theorems on the existence of stable equilibrium solutions and on the structure of multiple equilibrium solutions. In the second chapter (§ 5, § 6) we are confined to Neumann problems and discuss the possibility of the existence of non-constant stable equilibrium solutions.

Let D denote a bounded domain in $\mathbf{R}^n$ with a smooth boundary ∂D. What we shall deal with first are initial-boundary value problems of the type

(1. 1a) $\qquad \dfrac{\partial u}{\partial t} = Lu + f(x, u) \quad \text{in} \quad D \times (0, T)$,

(1. 1b) $\qquad u(x, 0) = u_0(x) \quad \text{on} \quad D$,

(1. 1c) $\qquad \alpha(x) u + \{1 - \alpha(x)\} \dfrac{\partial u}{\partial \nu} = \beta(x) \quad \text{on} \quad \partial D \times (0, T')$,

where L is a second order uniformly elliptic operator of the form

$$Lu = \sum_{i, j=1}^{n} \frac{\partial}{\partial x_i} \left\{ a_{ij}(x) \frac{\partial u}{\partial x_j} \right\}$$

and $\partial/\partial \nu$ denotes the outward conormal derivative. It is always assumed

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that $0 \leq \alpha \leq 1$ everywhere on ∂D. A function $v = v(x)$ is said to be an equilibrium solution, or a steady-state solution, of (1.1) if it satisfies the following elliptic boundary value problem:

$$(1.2a) \qquad\qquad Lv + f(x, v) = 0 \quad \text{in} \quad D,$$

$$(1.2b) \qquad\qquad \alpha v + (1-\alpha)\frac{\partial v}{\partial \nu} = \beta \quad \text{on} \quad \partial D.$$

Throughout this paper only real-valued classical solutions are considered.

Our main result in Chap. I is related to the question of how a solution of (1.1) will behave in the long run when starting from nearby an unstable equilibrium solution. In short, we shall show that above an unstable solution v of (1.2) there exists a minimal solution v^+ in a certain sense and that for any continuous function ψ satisfying $v < \psi \leq v^+$ the solution of (1.1) with initial data $u_0 = \psi$ converges to v^+ as t increases. Here, roughly speaking, v^+ must often be replaced by $+\infty$ according to circumstances. A similar assertion holds for a maximal solution below v.

The main result in Chapter II is on the stability of solutions of the following Neumann problem:

$$(1.3a) \qquad\qquad \varDelta v + f(v) = 0 \quad \text{in} \quad D,$$

$$(1.3b) \qquad\qquad \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial D,$$

where $\partial/\partial n$ denotes the outer normal derivative. It shall be shown that if D is a convex domain or one of other certain kinds of domains then any non-constant solution of (1.3) is, if it exists, unstable.

As regards the stability of equilibrium solutions, D. H. Sattinger has shown in [12] how stability and instability are characterized in terms of upper and lower solutions. The characterization of stability in this direction presents us a clear understanding of the global behavior of solutions around stable and unstable equilibrium solutions. But so far, it seems that stability—including instability—of solutions of (1.2) has not yet been completely characterized by means of those upper and lower solutions, or supersolutions and subsolutions in [3]. In this present paper, taking advantage of the self-adjointness of the elliptic

operator L, we shall make quite another approach to the understanding of the global behavior of solutions which will provide us with a complete knowledge of "the extent of instability". The fundamental technique for the discussion of the asymptotic behavior of solutions is given in the next section. One of our contributions here is to reveal the extent of instability in pretty general situations, and another is to emphasize the importance of this result.

In § 2, we prove, as is mentioned just above, some fundamental theorems on the asymptotic behavior of solutions which are based upon estimates of fundamental solutions, Schauder's estimates, and the so-called energy methods. For convenience' sake, we employ the idea of ω-limiting sets in the discussion. What is important is the fact that the ω-limiting set of a solution of (1. 1) consists only of the solutions of (1. 2) and that in many cases it is not empty (Theorem 2. 8). We usually discuss the behavior of solutions simply in the topology of $C^1(\overline{D}) \cap C^2(D)$ and never in a stronger one even if possible, since it is sufficient for the essential part of the following argument.

In § 3, the main result in this chapter is presented (Theorem 3. 1). It should be noted that no further assumption than a smoothness condition is made on the nonlinear term f. Applications of this theorem are found mainly in § 4, but also the results in § 6 are essentially (if not directly) based upon this theorem.

In § 4, theorems on the existence and on the structure of solutions of (1. 2) are given. We first prove that a subset of $C^1(\overline{D}) \cap C^2(D)$ which is stable as a set in a certain sense contains at least one stable solution (Theorem 4. 2). Next we show that between two unstable solutions below and above there exists at least one stable solution (Theorem 4. 3) and that between two stable solutions exists at least one solution (Theorem 4. 4). It is H. Amann who has proved the existence of minimal and maximal solutions between a lower solution below and an upper solution above (see [1]). Our theorems in § 3, § 4 are partly analogous to [1] as well as to [12], but include stronger results that probably can hardly be obtained by their methods, namely the monotone iteration methods. From a practical point of view we must note Corollary 4. 5 and Remark 4. 6, which are sometimes useful for investigating the stability of

solutions. For example, given a maximal solution v of (1. 2) (i.e., a solution above which there lies no other solution), one can see, from Remark 4. 6, that it is quite sufficient to conclude the instability of v if he only finds a continuous function ψ which makes the solution of (1. 1) with initial data ψ blow up to $+\infty$ in a finite time or diverge to $+\infty$ as $t \to \infty$ (here we assume $n=1$ and that f is independent of x).

In § 5, we consider (1. 3), which is a special case of (1. 2), and present one of the main theorems in Chap. II (Theorem 5. 1). This theorem is a kind of generalization of N. Chafee's result [4; Section 6] on the instability of solutions of (1. 3). He has proved that f being of class C^2 any isolated non-constant solution of (1. 3) is unstable provided that $n=1$. Our theorem here, which contains the same kind of result for convex domains in $\boldsymbol{R}^n$, of course includes his result and, moreover, is stronger than his even if restricted to one-dimensional cases, for it is all the same applicable to non-isolated solutions. One also finds other two types of domains in which any non-constant solution is unstable, namely a body of rotation with a convex section and a domain bounded by two concentric spheres. (∂D is always assumed to be smooth.)

In § 6, it is shown that when $n \geq 2$ there are actually cases where (1. 3) has a non-constant stable solution (Theorem 6. 2). We give sufficient conditions on the domain D and the function f for the existence of such a solution. This result is obtained by applying Theorem 4. 2 to a certain invariant subset of $C^1(\overline{D}) \cap C^2(D)$ which contains no constant function.

Chapter I. General Theory

§ 2. Preliminaries

Our assumptions concerning the equations (1.1) and (1.2) are all put together as follows:

(A)   1) D is bounded and its boundary is sufficiently smooth, say of class C^3.

        2)   Each $a_{ij}(x)$ is smooth, say all its second derivatives are uniformly Hölder continuous in $\bar{D}$ (of exponent γ, $0 < \gamma < 1$), and the matrix (a_{ij}) is uniformly positive definite and symmetric.

        3)   Both $\alpha(x)$ and $\beta(x)$ belong to $C^2(\partial D)$, and $0 \leq \alpha \leq 1$ everywhere on ∂D.

        4) $f(x, u)$ is of class C^2 in u. Moreover, f and $\partial f / \partial u$ are Hölder continuous (of exponent γ) in $\bar{D} \times R$.

We make no further assumption throughout this section, while some of the results in next two sections require slightly stronger boundary conditions. Smoothness conditions in (A) may be relaxed somewhat, for which we shall not care in this paper.

For the purpose of convenience in making a priori estimates and in discussing the asymptotic behavior of solutions, let us introduce some function spaces.

First, for description of the asymptotic behavior: Given an interval I in $\boldsymbol{R}^1$ and a metric space X, we denote by $C(I; X)$ the space of all X-valued continuous functions on I with the topology of uniform convergence on compact subintervals.

Secondly, for a priori estimates: Given a closed interval I, a compact set K in $\bar{D}$ and a positive number $\gamma < 1$, we introduce the notation

$$|g|_{\gamma, I, K} = \sup_{\substack{x \in K \\ t \in I}} |g(x, t)| + \sup_{\substack{x, x' \in K \\ t, t' \in I}} \frac{|g(x, t) - g(x', t')|}{(|x - x'|^2 + |t - t'|)^{\gamma/2}},$$

where g is a function defined on $K \times I$. We also need the notation

$$|g|_{2+r,I,K} = \sup_{\substack{x \in K \\ t \in I}} |g| + \sum_{i,j=1}^{n} \left|\frac{\partial^2 g}{\partial x_i \partial x_j}\right|_{r,I,K} + \left|\frac{\partial g}{\partial t}\right|_{r,I,K}.$$

The space of all functions with finite norm $|g|_{r,I,\bar{D}}$ is denoted by $C^r(\bar{D} \times I)$. And $C^{2+r}(D \times (t_1, t_2])$ denotes the Fréchet space defined by the system of seminorms $|g|_{2+r,[s,t_2],K}$ where K varies over all the compact sets in D and s over $(t_1, t_2]$. It is not difficult to see that $C^1(\bar{D} \times I)$ is embedded continuously both into $C^r(\bar{D} \times I)$ and into $C(I; C^1(\bar{D}))$, and that $C^{2+r}(D \times (t_1, t_2])$ completely continuously into $C((t_1, t_2]; C^2(D))$.

We are now ready to formulate our a priori estimates—a pretty weakened form of what is known as the interior estimates of the Schauder type:

Lemma 2.1. *Let* $g(x, t)$ *belong to* $C^r(\bar{D} \times [t_1, t_2])$ *and let* $u(x, t)$ *be a continuous function on* $\bar{D} \times [t_1, t_2]$ *satisfying*

$$\frac{\partial u}{\partial t} = Lu + g(x, t) \quad in \quad D \times (t_1, t_2].$$

Then u *belongs to* $C^{2+r}(D \times (t_1, t_2])$. *Moreover, given any compact set* $K \subset D$ *and any* $s \in (t_1, t_2]$, *there exists a constant* $C > 0$ *such that*

$$|u|_{2+r,[s,t_2],K} \leq C(\|u\|_{L^\infty(D \times (t_1, t_2))} + |g|_{r,[t_1,t_2],\bar{D}})$$

for any continuous function u *satisfying the above equation.*

Our assumption on D and L is quite enough for the estimates of this type. See A. Friedman [5; Theorems 5 and 9 in Chapter 3] for details.

Let $U = U(x, y, t)$, defined on $\bar{D} \times \bar{D} \times (0, \infty)$, be the fundamental solution of the equation

$$\frac{\partial u}{\partial t} = Lu \quad in \quad D \times (0, \infty)$$

with the boundary condition

$$\alpha u + (1 - \alpha) \frac{\partial u}{\partial \nu} = 0 \quad on \quad \partial D \times (0, \infty),$$

where L and α are as in (**A**). We owe the following lemma to S. Itô [private communication]. (This result can naturally be obtained in the

course of constructing the fundamental solution in a bounded domain; see [8; § 3, § 4] for the construction of fundamental solutions.)

Lemma 2.2. U, $\nabla_x U$ and $\partial U/\partial t$ are continuous in $\bar{D} \times \bar{D} \times (0, \infty)$, and $U(x, y, t) = U(y, x, t) > 0$ on $D \times D \times (0, \infty)$, where $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_n)$. Moreover, given any $t_0 > 0$, there exist positive constants M_i $(i = 1, 2, 3)$ such that

$$(2.1) \qquad \sup_{x \in \bar{D}} \int_D U(x, y, t) \, dy \leq M_1,$$

$$(2.2) \qquad \sup_{x \in \bar{D}} \int_D \left| \frac{\partial U(x, y, t)}{\partial t} \right| dy \leq M_2 t^{-1},$$

for $0 < t \leq t_0$ and that

$$(2.3) \qquad \sup_{x \in \bar{D}} \int_0^{t_0} d\tau \int_D |\nabla_x U(x, y, \tau)| \, dy \equiv M_3 < \infty .$$

We introduce linear operators T and S with parameter t: For each $t \geq 0$, $T(t)$ is an operator on $L^\infty(D)$ defined by

$$[T(t) w](x) = \int_D U(x, y, t) w(y) \, dy ,$$

where we understand $T(0) w = w$. $S(t)$ is defined by

$$[S(t) g](x) = \int_0^t d\tau \int_D U(x, y, t - \tau) g(y, \tau) \, dy ,$$

and here we understand $S(0) g = 0$. These integrals are easily seen to induce such operators as

(2.4a) $\quad T: L^\infty(D) \to C^\gamma(\bar{D} \times [\delta, t_0]) \cap C([\delta, t_0]; C^1(\bar{D}))$

(2.4b) $\quad S: L^\infty(D \times (0, t_0)) \to C^\gamma(\bar{D} \times [0, t_0]) \cap C([0, t_0]; C^1(\bar{D}))$

for any given $0 < \delta < t_0$ and $0 < \gamma < 1$. Moreover;

Lemma 2.3. *The operators in* (2.4) *are both compact.*

Proof. By virtue of the continuity of $\partial U/\partial t$ and $\nabla_x U$ on the compact set $\bar{D} \times \bar{D} \times [\delta, t_0]$, T is compact if regarded as a mapping from $L^\infty(D)$ into $C^1(\bar{D} \times [\delta, t_0])$. Since $C^1(\bar{D} \times [\delta, t_0])$ is embedded

continuously into $C^r(\overline{D} \times [\delta, t_0]) \cap C([\delta, t_0]; C^1(\overline{D}))$, the compactness of the mapping in (2. 4a) follows.

Well, let

$$A = \{g \in L^\infty (D \times (0, t_0)) ; \|g\|_\infty \leq 1\}.$$

Using (2. 1) and (2. 3) we get

(2. 5) $\qquad \sup_{\substack{x \in \overline{D} \\ t \in [0, t_0]}} \{|[S(t)g](x)| + |\nabla_x[S(t)g](x)|\} \leq M_1 t_0 + M_3,$

for all $g \in A$. As $\overline{D}$ is compact and as $\nabla_x U$ is continuous on $\overline{D} \times (0, t_0]$, it also follows from (2. 3) that given any $\varepsilon > 0$ there exists a $d > 0$ such that

(2. 6) $\qquad |\nabla_x[S(t)g](x) - \nabla_x[S(t)g](x')| < \varepsilon$

for all $g \in A$, $t \in [0, t_0]$ and $x, x' \in \overline{D}$ satisfying $|x - x'| < d$. (Note that $|\nabla_x S(t)g|$ itself is sufficiently small if t is sufficiently small.) On the other hand, for any $0 \leq t_1 < t_2 \leq t_0$ and any $g \in A$

$$|[S(t_2)g](x) - [S(t_1)g](x)|$$

$$= \left| \int_0^{t_2} d\tau \int_D U(x, y, t_2 - \tau) g(y, \tau) dy \right.$$

$$\left. - \int_0^{t_1} d\tau \int_D U(x, y, t_1 - \tau) g(y, \tau) dy \right|$$

$$\leq \int_{\max(t_1 - \Delta t, 0)}^{t_2} d\tau \int_D |U(x, y, t_2 - \tau) g(y, \tau)| dy$$

$$+ \int_{\max(t_1 - \Delta t, 0)}^{t_1} d\tau \int_D |U(x, y, t_1 - \tau) g(y, \tau)| dy$$

$$+ \int_0^{\max(t_1 - \Delta t, 0)} d\tau \int_D |U(x, y, t_2 - \tau) - U(x, y, t_1 - \tau)| |g(y, \tau)| dy$$

$$\leq 3M_1 \Delta t + \int_{\min(t_1, \Delta t)}^{t_1} d\tau \int_D |U(x, y, \tau + \Delta t) - U(x, y, \tau)| dy$$

$$\leq 3M_1 \Delta t + \int_{\min(t_1, \Delta t)}^{t_1} d\tau \int_D dy \int_0^{\Delta t} \left| \frac{\partial U(x, y, \tau + s)}{\partial t} \right| ds,$$

where $\Delta t = t_2 - t_1$. Using (2. 2), we get

$$\int_D dy \int_0^{\Delta t} \left| \frac{\partial U(x, y, \tau + s)}{\partial t} \right| ds = \int_0^{\Delta t} ds \int_D \left| \frac{\partial U(x, y, \tau + s)}{\partial t} \right| dy$$

$$\leq M_2 \int_0^{\Delta t} \frac{ds}{\tau + s} < M_2 \Delta t \tau^{-1}.$$

Hence, an elementary calculation yields

$$(2.7) \qquad \sup_{x \in \overline{D}} | [S(t_2)g](x) - [S(t_1)g](x) |$$

$$\leq \left(3M_1 + M_2 \log \frac{t_1}{\min(t_1, \Delta t)} \right) \Delta t$$

$$\leqq \{ 3M_1 + M_2 (|\log t_0| + |\log(t_2 - t_1)|) \} (t_2 - t_1)$$

for all $g \in A$ and $0 \leq t_1 < t_2 \leq t_0$. Seeing (2.5) and (2.7), we easily find that $\{Sg\}_{g \in A}$ is relatively compact in $C^r(\overline{D} \times [0, t_0])$ for any $0 < \gamma < 1$. It remains to show the relative compactness of $\{Sg\}_{g \in A}$ in $C([0, t_0]; C^1(\overline{D}))$. For each fixed $t \in [0, t_0]$, $S(t)g$ is a function of $x \in \overline{D}$, and it remains in a compact set of $C^1(\overline{D})$ as t varies in $[0, t_0]$ and g in A, by virtue of (2.5), (2.6) and Ascoli-Arzelà's theorem. So what we are to prove is the equicontinuity of $\{Sg\}_{g \in A}$ in t with respect to the topology of $C^1(\overline{D})$. Since the norm of $C^1(\overline{D})$ and that of $C(\overline{D})$ are equivalent on the set $\{S(t)g | 0 \leq t \leq t_0, g \in A\}$ as it is relatively compact in $C^1(\overline{D})$, we have only to prove the equicontinuity in the topology of $C(\overline{D})$, which follows immediately from (2.7). Thus the relative compactness of $\{Sg\}_{g \in A}$ in $C^r(\overline{D} \times [0, t_0]) \cap C([0, t_0]; C^1(\overline{D}))$ is established. Hence the compactness of the operator S in (2.4b).   Q.E.D.

Now we turn to the initial-boundary value problem (1.1). Converting (1.1) to an integral equation and then using the successive approximation method, one easily finds that for any bounded continuous initial data the unique existence of the solution of (1.1) is guaranteed so long as t remains small. Therefore, for each $t \geq 0$, an operator $Q(t)$ on $\mathscr{B}^0(D) \equiv L^\infty(D) \cap C(D)$ can be defined as follows:

$$(2.8) \qquad [Q(t)w](x) = u_w(x, t) \quad \text{for} \quad w \in \mathscr{B}^0(D),$$

where u_w is the solution of (1.1) with initial data $u_0 = w$. For each fixed $t' \geq 0$, $Q(t')$ is generally a nonlinear operator and its domain — denoted by $\text{Dom}(Q(t'))$ — is not necessarily the whole space $\mathscr{B}^0(D)$. A function w belongs to $\text{Dom}(Q(t'))$ if and only if u_w can be extended farther than $t = t'$ as a solution of (1.1); in other words, if and only if u_w does not blow up in the time interval $[0, t']$. So we have $\text{Dom}(Q(0)) = \mathscr{B}^0(D)$, and one can easily see

$$\mathrm{Dom}\,(Q\,(t_1))\supset\mathrm{Dom}\,(Q\,(t_2)),$$

$$Q\,(t_2)=Q\,(t_2-t_1)\,Q\,(t_1),$$

for any $0\leqq t_1\leqq t_2$.

Theorem 2. 4. *Let* (**A**) *hold and let* B *be a bounded set in* $\mathscr{B}^0(D)$. *Then*

i) *there exists a positive number* t_0 *such that* $B\subset\mathrm{Dom}\,(Q\,(t_0))$ *and that* $\{Qw\}_{w\in B}$ *is bounded in* $L^\infty(D\times(0,t_0))$, *where* Qw *denotes a function of* x,t *expressed by* $[Q\,(t)\,w]\,(x)$;

ii) *if* $\{Qw\}_{w\in B}$ *is bounded in* $L^\infty(D\times(0,t_1))$ *for some* $t_1>0$, *it is relatively compact in* $C\,((0,t_1]\,;\,C^1(\overline{D})\cap C^2(D))$.

We prove this theorem only in the case where $\beta\equiv0$, for otherwise (1. 1) can be reduced to this case by subtracting from u a function $w_0\in C^1(\overline{D})\cap C^2(D)$ which satisfies the same boundary condition as (1. 2b) and makes Lw_0 uniformly Hölder continuous in $\overline{D}$. (We can indeed find such a w_0, since given any uniformly Hölder continuous data φ the elliptic equation $Lw_0=\varphi$ with the boundary condition (1. 2b) is solvable except that when $\alpha\equiv0$, φ must be so chosen as to satisfy

$$\int_D\varphi\,dx=\int_{\partial D}a\,(\tilde{x})\,\beta\,(\tilde{x})\,d\tilde{x}\,,$$

where

(2. 9) $$a\,(\tilde{x})=[\sum_i\{\sum_j a_{ij}\,(\tilde{x})\,n_j\,(\tilde{x})\}^2]^{1/2}$$

and here $(n_j\,(\tilde{x}))$ denotes the outer normal to ∂D at $\tilde{x}$.) Note that if the assumption (**A**) holds for (1. 1) it is still the same with the reduced problem

$$\begin{cases}\dfrac{\partial\overline{u}}{\partial t}=L\overline{u}+\bar{f}(x,\overline{u})\\[2mm]\overline{u}\,(x,0)=\overline{u}_0(x)\\[2mm]\alpha\overline{u}+(1-\alpha)\dfrac{\partial\overline{u}}{\partial\nu}=0\end{cases}$$

where

$$\bar{f}(x, \bar{u}) = f(x, \bar{u} + w_0(x)) + Lw_0(x).$$

For Lw_0 is uniformly Hölder continuous.

Proof of Theorem 2.4. As is mentioned above, we can assume $\beta \equiv 0$ without any loss of generality.

i) Let $f_1(u)$, $f_2(u)$ be locally Lipschitz continuous functions satisfying

(2.10a) $$f_1(u) \leq \min_{x \in \bar{D}} (0, f(x, u)),$$

(2.10b) $$f_2(u) \geq \max_{x \in \bar{D}} (0, f(x, u)),$$

for all $u \in R,$ and let us then consider the following ordinary differential equations:

(2.11) $$\begin{cases} \dfrac{du_1}{dt} = f_1(u_1), \quad t \geq 0 \\ u_1(0) = -M, \end{cases}$$

(2.12) $$\begin{cases} \dfrac{du_2}{dt} = f_2(u_2), \quad t \geq 0 \\ u_2(0) = M, \end{cases}$$

where $M = \sup\limits_{w \in B} \|w\|_{L^\infty(D)}$. Either of these equations has a solution at least on some interval $0 \leq t \leq t_0$ $(t_0 > 0)$. Seeing (2.10), we have

$$u_1(t) \leq 0 \leq u_2(t)$$

for $0 \leq t \leq t_0$. Besides, regarded as functions of x, t, these solutions obviously satisfy $Lu_i = 0$ and $\partial u_i / \partial \nu = 0$ $(i = 1, 2)$. Considering these facts and using (2.10) again, we get, instead of (2.11) and (2.12), the following partial differential inequalities:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} \leq Lu_1 + f(x, u_1) \quad \text{in} \quad D \times (0, t_0] \\ u_1(x, 0) \leq w(x) \quad \text{on} \quad D \\ \alpha u_1 + (1 - \alpha) \dfrac{\partial u_1}{\partial \nu} \leq 0 \quad \text{on} \quad \partial D \times (0, t_0], \end{cases}$$

$$\begin{cases} \dfrac{\partial u_2}{\partial t} \geqq Lu_2 + f(x, u_2) \quad \text{in} \quad D \times (0, t_0] \\[2mm] u_2(x, 0) \geqq w(x) \quad \text{on} \quad D \\[2mm] \alpha u_2 + (1-\alpha) \dfrac{\partial u_2}{\partial \nu} \geqq 0 \quad \text{on} \quad \partial D \times (0, t_0], \end{cases}$$

here w is any element of B. Since $0 \leqq \alpha \leqq 1$, it follows from a standard comparison theorem that

$$u_1(t) \leqq [Q(t)w](x) \leqq u_2(t)$$

for all $x \in \overline{D}$ and $t \in (0, t_0]$. (See, for instance, M. H. Protter- H. F. Weinberger [11; Theorem 8 in Chapter 3], where the proof of the comparison theorem of this type is implicitly suggested.) Strictly speaking, this inequality should be understood to hold so long as $Q(t)w$ is definable, i.e., $w \in \mathrm{Dom}(Q(t))$, but as a matter of fact it is the case because any solution of (1.1) can be extended until it blows up or to $t = \infty$. Hence this completes the proof of i)

ii) Converting (1.1) into an integral equation in the usual manner, we get

(2.13) $Q(t)w = T(t)w + S(t)\tilde{f}_w,$

where

$$\tilde{f}_w(x, t) = f(x, [Q(t)w](x)).$$

(Here the boundary integral term does not appear, for we have assumed that $\beta \equiv 0$.) As $\{Qw\}_{w \in B}$ is bounded in $L^\infty(D \times (0, t_1))$, so is $\{\tilde{f}_w\}_{w \in B}$ because of the continuity of f. Therefore Lemma 2.3 can be applied to get the relative compactness of $\{Qw\}_{w \in B}$ in $C^\tau(\overline{D} \times [\delta, t_1]) \cap C([\delta, t_1]; C^1(\overline{D}))$ for any given $\delta \in (0, t_1]$. Hence $\{Qw\}_{w \in B}$ is relatively compact in $C([\delta, t_1]; C^1(\overline{D}))$. On the other hand, as f is uniformly Hölder continuous of exponent γ in x and Lipschitz continuous in u, $\{\tilde{f}_w\}_{w \in B}$ is bounded in $C^\tau(\overline{D} \times [\delta, t_1])$. So it follows from Lemma 2.1 that $\{Qw\}_{w \in B}$ is bounded in $C^{2+\tau}(D \times (\delta, t_1])$, hence relatively compact in $C((\delta, t_1]; C^2(D))$. Combining these facts, and considering that δ is any number in $(0, t_1]$, we now obtain what is expected.   Q.E.D.

Corollary 2. 5. *Let* **(A)** *hold and let* $\{v_\lambda\}_{\lambda \in A}$ *be a set of solutions of* (1. 2). *Suppose* $\{v_\lambda\}_{\lambda \in A}$ *is bounded in* $\mathscr{B}^0(D)$. *Then it is relatively compact in* $C^1(\bar{D}) \cap C^2(D)$.

The proof is immediate, since $\{v_\lambda\}_{\lambda \in A} = \{Q(t) v_\lambda\}_{\lambda \in A}$ for $t > 0$.

Lemma 2. 6. *Let* **(A)** *hold.*

i) *For each fixed* $t > 0$, *the following mappings are continuous linear operators* (cf. (2. 4)):

$$T(t) : L^2(D) \to L^2(D),$$

$$S(t) : L^\infty(0, t; L^2(D)) \to L^2(D).$$

ii) *Let* B, t_0 *be as in Theorem* 2. 4. *Then there exists a constant* $C > 0$ *such that for all* $w_1, w_2 \in B$,

$$\sup_{0 \leq t \leq t_0} \|Q(t) w_1 - Q(t) w_2\|_{L^2(D)} \leq C \|w_1 - w_2\|_{L^2(D)}.$$

We omit the proof of this lemma, for such kinds of results are so popular. Suffice it to say that ii) is an immediate consequence of a usual successive approximation in the integral equation (2. 13).

Our estimates obtained so far are not yet sufficient for the discussion of the asymptotic behavior of solutions; it is because they are merely local estimates (with respect to t). We need some more information on solutions which is in a sense global in t. For this purpose we introduce an energy form $J(u)$ associated with (1. 1):

$$J(u) = \int_D \left\{ \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - F(x, u) \right\} dx$$

$$+ \int_{\partial D} a \left\{ \frac{\alpha}{2} u^2 + \frac{1-\alpha}{2} \left(\frac{\partial u}{\partial \nu} \right)^2 - \beta u \right\} d\tilde{x},$$

where

$$F(x, u) = \int_0^u f(x, u) du$$

and $a(\tilde{x})$ is as in (2. 9). For any $t > 0$ and any $w \in \mathrm{Dom}(Q(t))$ the

form $J(Q(t)w)$ is definable.

Lemma 2. 7. *Let* **(A)** *hold and let* t_0 *be any positive number. Then for any* $w \in \mathrm{Dom}(Q(t_0))$

$$\frac{d}{dt}J(Q(t)w) = -\int_D \left\{\frac{\partial}{\partial t}Q(t)w\right\}^2 dx, \quad 0 < t \leq t_0.$$

Proof. For brevity's sake we write $u(x, t)$ instead of $[Q(t)w](x)$. First let us show $\partial u/\partial t$ belongs to $C((0, t_0]; C^1(\overline{D}))$. Differentiating (1.1) by t, we formally obtain the linear equation

(2. 14a) $$\frac{\partial p}{\partial t} = Lp + f_u(x, u(x, t))p$$

and the boundary condition

(2. 14b) $$\alpha p + (1-\alpha)\frac{\partial p}{\partial \nu} = 0 .$$

As $f_u(x, u(x, t))$ is Hölder continuous in $\overline{D} \times (0, t_0]$, (2.14) has a unique classical solution for any bounded continuous initial data at $t = t_1$, where t_1 is any given number in $(0, t_0]$. Set

$$p(x, t_1) = \frac{\partial u(x, t_1)}{\partial t}$$

(note that $\partial u/\partial t$ is bounded by virtue of the Hölder continuity of f). We claim that $p = \partial u/\partial t$ on $\overline{D} \times [t_1, t_0]$. To see this, we put

(2. 15) $$\tilde{u}(x, t) = u(x, t_1) + \int_{t_1}^t p(x, s)\,ds$$

and prove $\tilde{u} = u$. Since

$$L\tilde{u} = Lu(x, t_1) + \int_{t_1}^t Lp(x, s)\,ds$$

$$= Lu(x, t_1) + p - \frac{\partial u(x, t_1)}{\partial t} - \int_{t_1}^t f_u(x, u)p\,ds$$

$$= p - f(x, u(x, t_1)) - [f_u(x, u)\tilde{u}]_{t_1}^t + \int_{t_1}^t f_{uu}(x, u)\frac{\partial u}{\partial t}\tilde{u}\,ds ,$$

it is not difficult to see that $w \equiv \tilde{u} - u$ satisfies the equation

$$\frac{\partial w}{\partial t} = Lw + f_u(x, u) w - \int_{t_1}^t f_{uu}(x, u) \frac{\partial u}{\partial t} w\, ds\,,$$

as well as the boundary condition

$$\alpha w + (1-\alpha) \frac{\partial w}{\partial \nu} = 0\,.$$

Putting $\widetilde{w} = e^{-kt} w$, we get

$$\begin{cases} \dfrac{\partial \widetilde{w}}{\partial t} = L\widetilde{w} - \{k - f_u(x, u)\} \widetilde{w} - \displaystyle\int_{t_1}^t e^{-k(t-s)} f_{uu} \frac{\partial u}{\partial t} \widetilde{w}\, ds & \text{in} \quad D \times [t_1, t_0] \\[2ex] \widetilde{w}(x, t_1) = 0 \quad \text{on} \quad D \\[1ex] \alpha \widetilde{w} + (1-\alpha) \dfrac{\partial \widetilde{w}}{\partial \nu} = 0 \quad \text{on} \quad \partial D \times [t_1, t_0]\,, \end{cases}$$

where k is a positive number so large as to satisfy the inequality

$$(2.16) \qquad k - f_u(x, u(x, t)) > \int_{t_1}^{t_0} \left| f_{uu}(x, u) \frac{\partial u}{\partial t} \right| ds$$

on $\overline{D} \times [t_1, t_0]$. By the boundary condition, $\widetilde{w}$ cannot take its positive maximum nor negative minimum anywhere on $\partial D \times [t_1, t_0]$. Now suppose $\widetilde{w} \not\equiv 0$, and let (x', t') be the point where $|\widetilde{w}|$ takes its maximum. We can assume without loss of generality that $\widetilde{w}(x', t') > 0$. Since (x', t') $\in D \times (t_1, t_0]$, it follows that $\frac{\partial \widetilde{w}}{\partial t} \geqq 0$, $L\widetilde{w} \leqq 0$ and $\widetilde{w} > 0$ at (x', t'). So

$$0 \leqq - \{k - f_u(x, u)\} \widetilde{w}(x', t') + \int_{t_1}^{t'} \left| f_{uu} \frac{\partial u}{\partial t} \right| \widetilde{w}\, ds$$

$$\leqq - \{k - f_u(x, u)\} \widetilde{w}(x', t') + \widetilde{w}(x', t') \int_{t_1}^{t_0} \left| f_{uu} \frac{\partial u}{\partial t} \right| ds\,,$$

which clealy contradicts (2.16). Therefore $\widetilde{w}$ must vanish everywhere, hence the equality $u = \widetilde{u}$ is established. As t_1 is any number in $(0, t_0]$, we get $\partial u / \partial t = p \in C((0, t_0]; C^1(\overline{D}))$.

Now we differentiate the form $J(u)$ by t. The following formal calculation is well-founded because $\partial u / \partial t$ belongs to $C((0, t_0]; C^1(\overline{D}))$ and $Lu (= \partial u / \partial t - f(x, u))$ is bounded. By Green's formula,

$$\frac{d}{dt} \int_D \left\{ \frac{1}{2} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - F(x, u) \right\} dx$$

$$= \int_D \left\{ \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial t \partial x_i} \frac{\partial u}{\partial x_j} - f(x, u) \frac{\partial u}{\partial t} \right\} dx$$

$$= \int_{\partial D} a \frac{\partial u}{\partial t} \frac{\partial u}{\partial \nu} d\tilde{x} - \int_D \{Lu + f(x, u)\} \frac{\partial u}{\partial t} dx .$$

From the boundary condition of u it follows that

$$\frac{\partial u}{\partial t} \frac{\partial u}{\partial \nu} = \left(\alpha \frac{\partial u}{\partial t}\right) \frac{\partial u}{\partial \nu} + \frac{\partial u}{\partial t} \left\{(1-\alpha) \frac{\partial u}{\partial \nu}\right\}$$

$$= -(1-\alpha) \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial \nu}\right) \frac{\partial u}{\partial \nu} + \frac{\partial u}{\partial t} (-\alpha u + \beta)$$

$$= -\frac{\partial}{\partial t} \left\{\frac{1-\alpha}{2} \left(\frac{\partial u}{\partial \nu}\right)^2 + \frac{\alpha}{2} u^2 - \beta u\right\} .$$

Therefore

$$\frac{d}{dt} J(u) = - \int_D \{Lu + f(x, u)\} \frac{\partial u}{\partial t} dx = - \int_D \left(\frac{\partial u}{\partial t}\right)^2 dx . \qquad \text{Q.E.D.}$$

Let ψ be an element of $\mathrm{Dom}(Q(\infty))$ $(\equiv \bigcap_{t>0} \mathrm{Dom}(Q(t)))$. Looking upon the set $\{Q(t)\psi\}_{0<t<\infty}$ as an orbit described in some function space, we can define its ω-limiting set in the usual way—as is usual in the theory of dynamical systems:

Definition 1. For any $\psi \in \mathrm{Dom}(Q(\infty))$ the *ω-limiting set* of ψ in a function space X is defined as follows:

$$\Omega(X|\psi) = \bigcap_{t>0} cl_X(\{Q(\tau)\psi\}_{\tau \geq t}),$$

where $cl_X(\cdot)$ denotes the closure of a set in X. (We understand that $\Omega(X|\psi) = \emptyset$ if ψ belongs to $\mathscr{B}^0(D) \backslash \mathrm{Dom}(Q(\infty))$, i.e., if the solution of (1.1) with initial data ψ blows up in some finite time interval.)

Of importance to us is the case where $X = C^1(\overline{D}) \cap C^2(D)$, but we also consider the case where $X = L^\infty(D)$ with the weak∗ topology. As $L^1(D)$ is separable, $L^\infty(D)$-weak∗ satisfies the first countability axiom. Therefore, in either of these cases a function $\varphi(x)$ belongs to $\Omega(X|\psi)$ if and only if there exists a sequence $0 < t_1 < t_2 \cdots \to \infty$ such that

$$\lim_{m \to \infty} Q(t_m)\psi = \varphi \quad \text{in} \quad X .$$

The continuous embeddings $C^1(\overline{D}) \cap C^2(D) \subset C(\overline{D}) \subset L^\infty(D) - w.*$

yield

$$\Omega(C^1(\overline{D}) \cap C^2(D) \,|\psi) \subset \Omega(C(\overline{D}) \,|\psi) \subset \Omega(L^\infty(D) - w.* \,|\psi).$$

As a matter of fact, these apparently different ω-limiting sets coincide with each other as is seen in the following theorem:

Theorem 2.8. *Let* (A) *hold and let* $\psi \in \mathrm{Dom}(Q(\infty))$. *Then*

   i) $\Omega(C^1(\overline{D}) \cap C^2(D) \,|\psi) = \Omega(L^\infty(D) - w.* \,|\psi)$;

   ii) *it consists of solutions of* (1.2);

   iii) *it is not empty if there exists a sequence* $0 < t_1 < t_2 < \cdots \to \infty$ *such that* $Q(t_m)\psi$ *remains bounded in* $L^\infty(D)$ *as* $m \to \infty$.

Proof. We first show iii). For simplicity let $u_m(x)$ denote $[Q(t_m)\psi](x)$ for $m = 1, 2, \cdots$. As $\{u_m\}_{m \in \mathbf{N}}$ is bounded in $\mathscr{B}^0(D)$, it follows from Theorem 2.4 that $\{Qu_m\}_{m \in \mathbf{N}}$ is relatively compact in $C((0, t_0]; C^1(\overline{D}) \cap C^2(D))$ for some $t_0 > 0$. So $\{Q(t_0)u_m\}_{m \in \mathbf{N}} \equiv \{Q(t_m + t_0)\psi\}_{m \in \mathbf{N}}$ is relatively compact in $C^1(\overline{D}) \cap C^2(D)$, hence the conclusion.

Secondly we prove i) and ii) at one time. Suppose $\Omega(L^\infty(D) - w.* \,|\psi)$ is not empty and let φ be any element of it. What we have to show is that φ belongs to $\Omega(C^1(\overline{D}) \cap C^2(D) \,|\psi)$ and that it is a solution of (1.2). By definition, there exists a sequence $0 < t_1 < t_2 < \cdots \to \infty$ such that

$$\lim_{m \to \infty} Q(t_m)\psi = \varphi \quad \text{in} \quad L^\infty(D)\text{-weak}*.$$

Using Banach-Steinhaus' theorem, we see that $u_m \equiv Q(t_m)\psi$ remains bounded in $L^\infty(D)$ as $m \to \infty$. So, with a suitable positive number t_0 as above, $\{Qu_m\}_{m \in \mathbf{N}}$ is bounded in $L^\infty(D \times (0, t_0))$ and relatively compact in $C((0, t_0]; C^1(\overline{D}) \cap C^2(D))$. Since $J(Q(t)\psi)$ is monotone decreasing in t by virtue of Lemma 2.7,

$$J(u_1) \geqq J(u_m),$$

hence

$$\int_D \sum_{i,j} a_{ij} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} dx + \int_{\partial D} a(1-\alpha) \left(\frac{\partial u_m}{\partial \nu}\right)^2 d\tilde{x}$$

$$\leqq 2J(u_1) + 2 \int_D F(x, u_m) dx + \int_{\partial D} a(2\beta u_m - \alpha u_m^2) d\tilde{x},$$

for $m = 1, 2, \cdots$. The right-hand side of this inequality is bounded with respect to m because of the boundedness of $\{u_1, u_2, \cdots\}$ in the maximum norm. Considering that the matrix $(a_{ij}(x))$ is uniformly positive definite and taking into account the non-negativity of $a(1-\alpha)(\partial u_m/\partial\nu)^2$, we find that u_m remains bounded in $H^1(D)$[1] as $m \to \infty$. So $\{u_m\}_{m \in N}$ is relatively compact in $L^2(D)$, since D is a bounded domain. Recall that u_m converges to φ as $m \to \infty$ in $L^\infty(D) - w.*$. It follows that this convergence also occurs in $L^2(D)$, hence, by Lemma 2.6 ii), there exists a function $h(x, t)$ such that

$$(2.17) \qquad \lim_{m \to \infty} \sup_{0 < t \le t_0} \| Q(t) u_m - h(x, t) \|_{L^2(D)} = 0 .$$

Combining (2.17) and the fact that $\{Qu_m\}_{m \in N}$ is relatively compact in $C((0, t_0] ; C^1(\overline{D}) \cap C^2(D))$, we see, under a suitable modification of h on a null set, that given any $\delta \in (0, t_0]$ and any compact set $K \subset D$

$$(2.18) \qquad \lim_{m \to \infty} Q(t) u_m = h(x, t) \quad \text{in} \quad C^2(K),$$

$$(2.19) \qquad \lim_{m \to \infty} Q(t) u_m = h(x, t) \quad \text{in} \quad C^1(\overline{D}),$$

uniformly in $\delta \le t \le t_0$.

On the other hand, using Lemma 2.7 again, we get

$$\int_{t_1 + t_0}^{t_m + t_0} d\tau \int_D \left\{ \frac{\partial}{\partial t} [Q(\tau) \psi] (x) \right\}^2 dx = J(Q(t_0) u_1) - J(Q(t_0) u_m).$$

Since $\{Q(t_0) u_m\}$ is a bounded set in $C^1(\overline{D})$, $J(Q(t_0) u_m)$ remains bounded as $m \to \infty$. Therefore

$$(2.20) \qquad \int_{t_1 + t_0}^{\infty} d\tau \int_D \left\{ \frac{\partial}{\partial t} Q(\tau) \psi \right\}^2 dx < \infty$$

Using (2.18) and (2.20) we obtain

$$0 = \lim_{m \to \infty} \int_{t_m + \delta}^{t_m + t_0} d\tau \int_D \left\{ \frac{\partial}{\partial t} Q(\tau) \psi \right\}^2 dx$$

$$= \lim_{m \to \infty} \int_{\delta}^{t_0} d\tau \int_D \left\{ \frac{\partial}{\partial t} Q(\tau) u_m \right\}^2 dx$$

---

[1] $H^1(D)$ denotes the Banach space of all functions with finite norm
$\left\{ \int_D w^2 dx + \int_D |\nabla w|^2 dx \right\}^{1/2}$.

$$\geq \lim_{m \to \infty} \int_{\delta}^{t_0} d\tau \int_K \left\{ \frac{\partial}{\partial t} Q(\tau) u_m \right\}^2 dx$$

$$= \lim_{m \to \infty} \int_{\delta}^{t_0} d\tau \int_K \{LQ(\tau) u_m + f(x, Q(\tau) u_m)\}^2 dx$$

$$= \int_{\delta}^{t_0} d\tau \int_K \{Lh(x, \tau) + f(x, h(x, \tau))\}^2 dx,$$

for any $\delta \in (0, t_0]$ and any compact set $K \subset D$, and hence

$$(2.21) \qquad Lh(x, t) + f(x, h(x, t)) = 0 \quad \text{in} \quad D \times (0, t_0].$$

It also follows that

$$(2 \cdot 22) \qquad \qquad \frac{\partial h}{\partial t}(x, t) = 0 \quad \text{in} \quad D \times (0, t_0],$$

since $u_m \to h$ and $\partial u_m / \partial t \to 0$ (as $m \to \infty$) uniformly on any $K \times [\delta, t_0]$. Meanwhile it follows from (2.19) and the boundary condition of $Q(t) u_m$ that

$$(2 \cdot 23) \qquad \alpha h(x, t) + (1 - \alpha) \frac{\partial h}{\partial \nu}(x, t) = 0 \quad \text{on} \quad \partial D \times (0, t_0].$$

Combining (2.21), (2.22) and (2.23), we conclude that h is independent of t and is a solution of (1.2) regarded as a function of x. Moreover it belongs to $\Omega(C^1(\overline{D}) \cap C^2(D) | \psi)$ because of (2.18) and (2.19). Passing to the limit in the equality $Q(t) u_m = T(t) u_m + S(t) f(x, Q(t) u_m)$ and using (2.17) and Lemma 2.6 i), we get

$$h(x, t) = [T(t)\varphi](x) + [S(t) f(x, h)](x).$$

Consequently

$$\lim_{t \to 0} h = \lim_{t \to 0} T(t)\varphi = \varphi \quad \text{in} \quad L^2(D)$$

(see [8; Theorem 6] for the latter equality). It implies $\varphi(x) = h(x, t_0)$ for a.e. $x \in D$, hence φ is a solution of (1.2) and belongs to $\Omega(C^1(\overline{D}) \cap C^2(D) | \psi)$. Q.E.D.

Owing to this theorem, there is no need to distinguish $\Omega(X | \psi)$ from $\Omega(C^1(\overline{D}) \cap C^2(D) | \psi)$ so long as we have the continuous embeddings

$$C^1(\overline{D}) \cap C^2(D) \subset X \subset L^\infty(D) - \text{weak}^*.$$

So hereafter we adopt the abridged notation

(2·24) $\Omega(\psi) \equiv \Omega(X|\psi)$,

where X is any space satisfying the above condition.

Theorem 2. 9. *Let* (A) *hold, let* $\psi \in \mathrm{Dom}(Q(\infty))$, *and let* Y *be a nonempty subset of* $\Omega(\psi)$ *such that* Y *and* $\Omega(\psi) \backslash Y$ *are closed in* $C^1(\overline{D}) \cap C^2(D)$. *Suppose* Y *is bounded in* $L^\infty(D)$. *Then* $\Omega(\psi) = Y$ *and, given any neighborhood* W *of* Y *in* $C^1(\overline{D}) \cap C^2(D)$, *there exists a finite number* $T \geqq 0$ *such that*

$$\{Q(t)\psi\}_{t \leq T} \subset W.$$

Proof. We have only to prove the latter part, for the former follows immediately from the latter. Let

$$V(\varepsilon) = \{w \in \mathscr{B}^0(D) \,; \, \|w - \varphi\|_{L^\infty(D)} < \varepsilon \quad \text{for some} \quad \varphi \in Y\}.$$

As Y is bounded in $L^\infty(D)$, so is $V(\varepsilon)$ for each fixed $\varepsilon > 0$. Hence, by virtue of Corollary 2.5, $V(1) \cap \Omega(\psi)$ is relatively compact in $C^1(\overline{D})$ $\cap C^2(D)$. So the topology induced from $C^1(\overline{D}) \cap C^2(D)$ and the one from $L^\infty(D)$ are equivalent on this set. Considering that Y is compact and is separated by an open set in $C^1(\overline{D}) \cap C^2(D)$ from $\Omega(\psi) \backslash Y$, we find that there exists an $\varepsilon_0 \in (0, 1]$ such that

(2. 25) $V(\varepsilon_0) \cap \Omega(\psi) = Y$.

Now we claim that for any $0 < \varepsilon < \varepsilon_0$ there exists a $T' > 0$ such that

(2. 26) $\{Q(t)\psi\}_{t \geq T'} \subset V(\varepsilon)$.

Suppose (2. 26) does not hold. Then there exist an $\varepsilon_1 \in (0, \varepsilon_0)$ and a sequence $0 < t_1 < t_2 < \cdots \to \infty$ such that

(2. 27a) $Q(t_{2m-1})\psi \in V\left(\dfrac{\varepsilon_1}{2}\right)$,

(2. 27b) $Q(t_{2m})\psi \notin V(\varepsilon_1)$,

for $m = 1, 2, \cdots$. By the way, for any $\varphi \in Y$ and $w \in V(\varepsilon_1/2)$, $\bar{u} = Q(t)w$ $- \varphi$ satisfies the reduced equation

$$\frac{\partial \overline{u}}{\partial t} = L\overline{u} + \{f(x, \overline{u} + \varphi(x)) - f(x, \varphi(x))\},$$

as well as the boundary condition

$$\alpha \overline{u} + (1-\alpha)\frac{\partial \overline{u}}{\partial \nu} = 0.$$

So it is not difficult to see that there exists a $t_0 > 0$ such that

(2·28) $Q(t)w \in V(\varepsilon_1), \quad 0 \leqq t \leqq t_0,$

for all $w \in V(\varepsilon_1/2)$ (see the proof of Theorem 2. 4 i)). Combining (2. 27) and (2. 28), we find that for each $m = 1, 2, \cdots$ there exists a $t_m' \in [t_{2m-1}, t_{2m})$ such that

$$Q(t_m' + t)\psi \in V(\varepsilon_1) \setminus V\left(\frac{\varepsilon_1}{2}\right), \quad 0 \leqq t \leqq t_0.$$

Consequently, by virtue of Theorem 2. 4 ii), $\{Q(t_m' + t_0)\psi\}_{m=1,2,\cdots}$ is relatively compact in $C^1(\overline{D}) \cap C^2(D)$. It implies that $\Omega(\psi)$ meets with the closure of $V(\varepsilon_1) \setminus V(\varepsilon_1/2)$, hence $\Omega(\psi) \setminus Y$ meets with $V(\varepsilon_0)$, a contradiction to (2. 25). Thus (2. 26) is established.

As a result of (2. 26), $\{Q(t)\psi\}_{t \geqq \delta}$ is relatively compact in $C^1(\overline{D}) \cap C^2(D)$ for any $\delta > 0$. It can be seen from the boundedness of $\{Q(t)\psi\}_{t \geqq 0}$ in $L^\infty(D)$, Theorem 2. 4 ii) and the fact that

$$\{Q(t)\psi\}_{t \geqq \delta} = \{Q(\delta)w; w \in \{Q(t)\psi\}_{t \geqq 0}\}.$$

Accordingly the topology of $C^1(\overline{D}) \cap C^2(D)$ and that of $L^\infty(D)$ are equivalent on the closure of $\{Q(t)\psi\}_{t \geqq \delta}$, hence, combining this fact and (2. 26), we obtain the conclusion.   Q.E.D.

Corollary 2. 10. *Let* (A) *hold and let* $\psi \in \mathrm{Dom}(Q(\infty))$. *Suppose* $Q(t)\psi$ *remains bounded in the* $L^\infty(D)$ *norm as* $t \to \infty$. *Then* $\Omega(\psi)$ *is a non-empty connected set in* $C^1(\overline{D}) \cap C^2(D)$.

Corollary 2. 11. *Let* (A) *hold and let* $\psi \in \mathrm{Dom}(Q(\infty))$. *Suppose* $\Omega(\psi)$ *contains an isolated point* v *(with respect to the topology of* $C^1(\overline{D}) \cap C^2(D)$). *Then* $\Omega(\psi) = \{v\}$ *and*

$$\lim_{t \to \infty} Q(t)\psi = v \quad in \quad C^1(\overline{D}) \cap C^2(D).$$

These corollaries are direct consequences of Theorems 2. 9 and 2. 8 iii).

§ 3. Strong Instability of Equilibrium Solutions

A solution v of (1. 2) is said to be *stable* (in the maximum norm) if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|Q(t)\psi - v\|_{L^\infty(D)} < \varepsilon, \quad 0 < t < \infty,$$

for any $\psi \in \mathscr{B}^0(D)$ satisfying $\|\psi - v\|_{L^\infty(D)} < \delta$, where Q is what is introduced in (2. 8). We say v is unstable if v is not stable. As we are concerned with real-valued solutions, it is possible, and convenient as well, to discriminate between upward instability and downward one. It is also convenient to introduce the notion of instability in a stronger sense.

Definition 2. A solution v of (1. 2) is *unstable upward* if there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists a function $\psi \in \mathscr{B}^0(D)$ which satisfies $v(x) \leqq \psi(x) \leqq v(x) + \delta$ everywhere on D and also the inequality

(3. 1) $[Q(t_0)\psi](x_0) \geqq v(x_0) + \varepsilon_0$

at some (x_0, t_0). We say v is *strongly unstable upward* if there exists an $\varepsilon_0 > 0$ such that given any $\psi \in \mathscr{B}^0(D)$ satisfying $\psi \geqq v$ and $\psi \not\equiv v$ (3. 1) holds at some (x_0, t_0). By reversing the inequalities, *downward instability* in either sense is defined likewise.

It is easy to see, by virtue of the comparison theorem, that v is stable in the maximum norm if and only if it is unstable neither upward nor downward.

Before presenting our results we introduce the following notation:

$$X^+(v) = \{w \in \mathscr{B}^0(D) ; w \geqq v, w \not\equiv v\},$$

$$X^-(v) = \{w \in \mathscr{B}^0(D) ; w \leqq v, w \not\equiv v\},$$

$$S^+(v) = S \cap X^+(v),$$

$$S^-(v) = S \cap X^-(v),$$

where S is the set of all the solutions of (1. 2).

Here we are ready to state the most important theorem in this chapter:

Theorem 3. 1. *Let* **(A)** *hold (see* §2) *and let* v *be a solution of* (1.2) *which is strongly unstable upward.*

i) *Suppose* $S^+(v) \neq \emptyset$. *Then it has the minimum of it, i.e., there exists a solution* $v^+ \in S^+(v)$ *such that*

$$v^+(x) \leqq \tilde{v}(x) \quad on \quad \bar{D}$$

for all $\tilde{v} \in S^+(v)$. *Moreover, for any* $\psi \in X^+(v) \cap X^-(v^+)$,

$$\lim_{t \to \infty} Q(t)\psi = v^+$$

in the topology of $C^1(\bar{D}) \cap C^2(D)$.

ii) *Suppose* $S^+(v) = \emptyset$. *Then, given any* $\psi \in X^+(v)$, *there exists a* T, $0 < T \leqq \infty$, *such that*

$$\lim_{t \to T} \max_{x \in \bar{D}} [Q(t)\psi](x) = +\infty .$$

Remark 3. 2. A similar proposition holds true for $S^-(v)$ and its maximum v^- if v is strongly unstable downward.

The proof shall be given later in this section. This theorem will find its wider application when coupled with one of the following theorems which are more or less familiar to us.

Theorem 3. 3. *Let* **(A)** *hold, let* v *be a solution of* (1.2), *and let* λ_1 *be the least eigenvalue of the following eigenvalue problem*:

(3. 2a) $$\tilde{L}\varphi + \lambda\varphi = 0 \quad in \quad D,$$

(3. 2b) $$\alpha\varphi + (1-\alpha)\frac{\partial\varphi}{\partial\nu} = 0 \quad on \quad \partial D,$$

where

$$\tilde{L}\varphi \equiv L\varphi + f_u(x, v(x))\varphi .$$

If $\lambda_1 < 0$, v *is strongly unstable both upward and downward.*

Next theorem requires a further assumption on the boundary condition.

(B) Either 1) or 2) holds:

   1) $\alpha \equiv 1$ on ∂D.

   2) $1-\alpha$ never vanishes on ∂D.

Theorem 3.4. *Let* (A), (B) *hold and let v be a solution of* (1.2). *If v is unstable upward (resp. downward), it is strongly unstable upward (resp. downward).*

As a consequence of Theorems 3.1 and 3.4 we have

Corollary 3.5. *Let* (A), (B) *hold and let v be a solution of* (1.2). *Suppose one of the following conditions holds. Then v is stable from above, i.e., not unstable upward.*

   1)   *There exists a function $\psi \in X^+(v)$ such that $Q(t)\psi$ converges to v as $t \to \infty$ (for instance in the maximum norm).*

   2)
$$\inf_{\tilde{v} \in S^+(v)} \tilde{v}(x) = v(x) \quad on \quad \bar{D}.$$

Of course an analogous assertion holds good for the stability from below.

Some applications of Theorem 3.1 in combination with Theorem 3.4 will be found in § 4 and § 6, while in § 5 are those of Theorem 3.1 in combination with Theorem 3.3.

Now let us begin the proof of these results.

Lemma 3.6. *Let* (A) *hold and let v be a solution of* (1.2). *Suppose v is strongly unstable upward. Then*

   i)   *v is isolated from $S^+(v)$ in the maximum norm;*

   ii) *for any $\psi \in X^+(v)$,*

$$\Omega(\psi) \subset S^+(v).$$

(See (2.24).)

Proof. i) Let ε_0 be as in Definition 2 and let $V = \{w \in \mathscr{B}^0(D) ; \|w - v\|_\infty < \varepsilon_0\}$. It follows from the definition of ε_0 that $V \cap X^+(v)$ does not contain any equilibrium solutions. So $V \cap S^+(v) = \emptyset$.

ii) We assume $\Omega(\psi) \neq \emptyset$, for otherwise the statement is trivial. By virtue of the comparison Theorem and Theorem 2.8 ii), $\Omega(\psi)$ is contained in $S^+(v) \cup \{v\}$. Since v is an isolated point of $S^+(v) \cup \{v\}$, it follows from Corollary 2.11 that either $\Omega(\psi) = \{v\}$ or $\Omega(\psi) \subset S^+(v)$. Suppose that the former holds. Then we see, from Corollary 2.11 again, that $Q(t)\psi$ converges to v in $C^1(\overline{D}) \cap C^2(D)$, which contradicts the strong upward instability of v. Therefore the former cannot hold, hence the conclusion. Q.E.D.

We introduce into $S^+(v)$ (resp. $S^-(v)$) an order relation defined as follows:

$v_1 \geqq v_2$ means $v_1(x) \geqq v_2(x)$ everywhere on $\overline{D}$. $S^+(v)$ (resp. $S^-(v)$) is then regarded as a partially ordered set with respect to this relation.

Lemma 3.7. *Let* (**A**) *hold and let* v *be a solution of* (1.2). *Suppose* v *is strongly unstable upward and that* $S^+(v) \neq \emptyset$. *Then for any* v_1, $v_2 \in S^+(v)$ *there exists the greatest lower bound of* $\{v_1, v_2\}$ *in the ordered set* $S^+(v)$.

Proof. Put

$$h(x) = \min\{v_1(x), v_2(x)\}.$$

By virtue of the strong maximum principle, $h(x) > v(x)$ on D (cf. Lemmas 3.8 and 3.9). Hence $h \in X^+(v)$ and, by the comparison theorem,

$$v \leqq Q(t)h \leqq v_i, \quad i = 1, 2,$$

for all $t \geqq 0$. Applying Corollary 2.10 and Lemma 3.6 ii) to h, we find that $\Omega(h)$ is a non-empty subset of $S^+(v)$. The above inequalities thus imply that $\{v_1, v_2\}$ has at least one lower bound in $S^+(v)$. Let $\tilde{v} \in S^+(v)$ be any lower bound of $\{v_1, v_2\}$. Since $\tilde{v} \leqq h$,

$$\tilde{v} \leqq Q(t)h \leqq v_i, \quad i = 1, 2,$$

for all $t \geqq 0$. So any element of $\Omega(h)$ is an upper bound of $\{\tilde{v}\}$. It is now easy to see that $\Omega(h)$ contains only one element, namely the greatest lower bound of $\{v_1, v_2\}$. Q.E.D.

It follows from this lemma that any finite subset of $S^+(v)$ has the greatest lower bound of it in $S^+(v)$. We denote by $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ the greatest lower bound of $\{v_1, v_2, \cdots, v_m\}$.

Proof of Theorem 3.1. i) We merely show the existence of the minimum v^+. The rest of i) then follows immediately, if the comparison theorem, Theorem 2.8 iii), Lemma 3.6 ii) and Corollary 2.11 are taken into account.

Put

$$\Psi(x) = \inf_{\widetilde{v} \in S^+(v)} \widetilde{v}(x), \quad x \in \overline{D}.$$

What we have to show now is $\Psi \in S^+(v)$. Take an element v_0 of $S^+(v)$ and fix it. Then

$$\Psi(x) = \inf_{\widetilde{v} \in S^+ \wedge v_0} \widetilde{v}(x),$$

where $S^+ \wedge v_0$ denotes $\{\widetilde{v} \wedge v_0; \widetilde{v} \in S^+(v)\}$. So, given any $\varepsilon > 0$ and any finite number of points $x_1, \cdots, x_k \in \overline{D}$, there exist $v_1, \cdots, v_k \in S^+ \wedge v_0$ such that

$$\Psi(x_i) \leq v_i(x_i) \leq \Psi(x_i) + \varepsilon$$

for $i = 1, \cdots, k$. Hence

$$\Psi(x_i) \leq v_1 \wedge \cdots \wedge v_k(x_i) \leq \Psi(x_i) + \varepsilon$$

for $i = 1, \cdots, k$. It follows that Ψ is an accumulation point of $S^+ \wedge v_0$ in the topology of pointwise convergence on $\overline{D}$. On the other hand, since $S^+ \wedge v_0$ is bounded by v from below and by v_0 from above, $S^+ \wedge v_0$ is relatively compact in $C^1(\overline{D}) \cap C^2(D)$ by virtue of Corollary 2.5. Consequently Ψ is also an accumulation point of $S^+ \wedge v_0$ in $C^1(\overline{D}) \cap C^2(D)$, therefore Ψ is a solution of (1.2). This means that Ψ belongs to $S^+(v) \cup \{v\}$, and as v is isolated from $S^+(v)$ by Lemma 3.6 i), Ψ must belong to $S^+(v)$. Thus the statement i) is established.

ii) Let $T = \sup\{t > 0; \psi \in \mathrm{Dom}(Q(t))\}$, and suppose there exist a constant $M > 0$ and a sequence $0 < t_1 < t_2 < \cdots \to T$ such that

(3.3) $\displaystyle\sup_{x \in \overline{D}} [Q(t_m)\psi](x) \leq M$

for $m = 1, 2, \cdots$. As $Q(t)\psi$ is bounded by v from below, it then follows

that $\{Q(t_m)\psi\}_{m\in N}$ is bounded in $L^\infty(D)$. Here we have two cases, i.e., the case where T is a finite number and the case where $T=\infty$. In the first case, on applying Theorem 2.4 i) to $B=\{Q(t_m)\psi\}_{m\in N}$, we are led to a contradiction. In the second case, an application of Theorem 2.8 iii) yields $\Omega(\psi)\neq\emptyset$. So it follows from Lemma 3.6 ii) that $S^+(v)$ $\neq\emptyset$, which contradicts the presupposition. Thus, in either case, our supposition (3.3) has turned out false.   QED

Proof of Theorem 3.3. Let φ_1 be one of the eigenfunctions of (3.2) belonging to λ_1. It is well-known that φ_1 never changes sign in D (a consequence of Krein-Rutman Theory). So we can assume in the sequel that $\varphi_1>0$ in D and that $\|\varphi_1\|_{L^2(D)}=1$. Since $\lambda_1<0$, there exists an $\varepsilon_0>0$ such that

$$(3.4) \qquad f(x, v(x)+\varepsilon)-f(x, v(x))\geqq\varepsilon\left\{f_u(x, v(x))+\frac{\lambda_1}{2}\right\}$$

for all $0\leqq\varepsilon\leqq\varepsilon_0$ and $x\in\overline{D}$.

Now suppose that some function $\psi\in X^+(v)$ satisfies the inequality

$$(3.5) \qquad\qquad\qquad \|Q(t)\psi-v\|_{L^\infty(D)}\leqq\varepsilon_0$$

for all $t\geqq0$. By Green's formula,

$$\frac{d}{dt}\langle Q(t)\psi-v, \varphi_1\rangle=\left\langle\frac{\partial}{\partial t}Q(t)\psi, \varphi_1\right\rangle$$

$$=\langle LQ(t)\psi+f(x, Q(t)\psi), \varphi_1\rangle$$

$$=\langle L\{Q(t)\psi-v\}+\{f(x, Q(t)\psi)-f(x, v)\}, \varphi_1\rangle,$$

$$=\langle Q(t)\psi-v, L\varphi_1\rangle+\langle f(x, Q(t)\psi)-f(x, v), \varphi_1\rangle,$$

where $\langle,\rangle$ denotes the inner product on $L^2(D)$. Using (3.4) and (3.5), we get

$$\frac{d}{dt}\langle Q(t)\psi-v, \varphi_1\rangle\geqq-\frac{\lambda_1}{2}\langle Q(t)\psi-v, \varphi_1\rangle,$$

therefore

$$\langle Q(t)\psi-v, \varphi_1\rangle\geqq\exp\left\{-\frac{\lambda_1}{2}t\right\}\langle\psi-v, \varphi_1\rangle.$$

As $\lambda_1<0$ and $\langle\psi-v, \varphi_1\rangle>0$, it follows that

$$\lim_{t \to \infty} \langle Q(t)\psi - v, \varphi_1 \rangle = \infty \, ,$$

which contradicts (3.5) because D is bounded. So there exists no such a function $\psi \in X^+(v)$ as to satisfy (3.5). Hence the strong upward instability of v. The strong downward instability can be shown likewise.

<div align="right">Q.E.D.</div>

Lemma 3.8. *Let* (A), (B1) *hold and let* v *be a solution of* (1.2). *Given any* $\psi \in X^+(v)$ *and any* $t_1 > 0$ *such as* $\psi \in \mathrm{Dom}(Q(t_1))$, *there exists a constant* $c > 0$ *such that*

(3.6) $Q(t_1)\psi \geqq v + c\rho \, ,$

where $\rho(x)$ *is the distance between* x *and* ∂D.

Proof. Put

$$M = \sup_{0 \leqq t \leqq t_1} \| f_u(x, Q(t)\psi) \|_{L^\infty(D)}$$

and let us consider the initial-boundary value problem

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} = L\bar{u} - M\bar{u} \\[2mm] \bar{u}(x, 0) = \psi(x) - v(x) \\[2mm] \alpha\bar{u} + (1-\alpha)\dfrac{\partial \bar{u}}{\partial \nu} = 0 \, . \end{cases}$$

By virtue of the comparison theorem we have

(3.7) $Q(t_1)\psi - v \geqq \bar{u}(x, t_1) = e^{-Mt_1} \displaystyle\int_D U(x, y, t_1)\{\psi(y) - v(y)\}\, dy \, ,$

where U is the fundamental solution of the equation $\partial u/\partial t = Lu$ with the same boundary condition as above. Since

$$U(x, y, t_1) > 0 \quad \text{for } (x, y) \in D \times D \, ,$$

$$\frac{\partial U}{\partial \nu}(x, y, t_1) < 0 \quad \text{for } (x, y) \in \partial D \times D$$

(see [8], §9), and as $\psi - v$ is non-negative and is not identical to zero, it follows that

$$\bar{u}(x, t_1) > 0 \quad \text{in} \quad D,$$

$$\frac{\partial \bar{u}}{\partial \nu}(x, t_1) < 0 \quad \text{on} \quad \partial D.$$

So we can easily find that there exists a positive constant c such that

$$\bar{u}(x, t_1) \geqq c\rho,$$

hence we obtain the conclusion. Q.E.D.

Lemma 3.9. *Let* (A), (B2) *hold and let* v *be a solution of* (1.2). *Given any* $\psi \in X^+(v)$ *and any* $t_1 > 0$ *such as* $\psi \in \mathrm{Dom}(Q(t_1))$, *there exists a constant* $c > 0$ *such that*

$$Q(t_1)\psi \geqq v + c$$

everywhere on $\bar{D}$.

The proof is just similar to the previous one except that we use here the following property of U:

$$U(x, y, t_1) > 0 \quad \text{on} \quad \bar{D} \times \bar{D}.$$

Proof of Theorem 3.4. First we prove this theorem in the case where (A), (B1) hold. Let ε_0 be as in the definition of the upward instability of v (see Definition 2) and let ψ be any element of $X^+(v)$. Further we put

$$M = \max_{x \in \bar{D}, 0 \leqq \varepsilon \leqq \varepsilon_0} f_u(x, \varepsilon).$$

It is easily seen, from the proof of Theorem 2.4 i), that there exists a positive number t_1 such that $\psi \in \mathrm{Dom}(Q(t_1))$ and that

(3.8) $$0 \leqq Q(t)w - v < \varepsilon_0, \quad 0 \leqq t \leqq t_1$$

for any continuous function w satisfying $0 \leqq w - v \leqq \varepsilon_0/2$. Because $f(x, Q(t)w) - f(x, v) \leqq M\{Q(t)w - v\}$ so long as $0 \leqq t \leqq t_1$, we get

$$Q(t_1)w - v \leqq e^{Mt_1} \int_D U(x, y, t_1)\{w(y) - v(y)\}\,dy$$

by the aid of the comparison theorem (cf. (3.7)). As

$$\left|\nabla_x \int_D U(x, y, t_1)\{w(y) - v(y)\}dy\right|$$

$$\leqq \|w - v\|_{L^\infty(D)} \int_D |\nabla_x U(x, y, t_1)|dy,$$

and since $Q(t_1)w - v$ vanishes on ∂D, it follows that

(3. 9) $Q(t_1)w - v \leqq c_1\|w - v\|_{L^\infty(D)}\rho,$

where

$$c_1 = e^{Mt_1} \sup_{x \in \bar{D}} \int_D |\nabla_x U(x, y, t_1)|dy.$$

Let c be a small positive constant such that (3. 6) in Lemma 3. 8 holds for the present ψ and t_1 (recall $\psi \in \text{Dom}(Q(t_1))$). As v is unstable upward, there exists a continuous function $\psi_1(x)$ which satisfies the inequalities

$$0 \leqq \psi_1 - v \leqq \min\left(\frac{c}{c_1}, \frac{\varepsilon_0}{2}\right)$$

on $\bar{D}$ and also the inequality

(3. 10) $[Q(t_0)\psi_1](x_0) - v(x_0) \geqq \varepsilon_0$

at some (x_0, t_0). Here $t_0 > t_1$ because of (3. 8). Combining (3. 6) and (3. 9), we get

$$Q(t_1)\psi_1 \leqq v + c\rho \leqq Q(t_1)\psi,$$

so it follows from the comparison theorem and (3. 10) that

$$[Q(t_0)\psi](x_0) - v(x_0) \geqq \varepsilon_0,$$

since $t_0 > t_1$. As ψ is any element of $X^+(v)$, the strong upward instability of v is established.

Secondly, the case where (A), (B2) hold. Using Lemma 3. 9 instead of Lemma 3. 8, we obtain the conclusion more easily.   Q.E.D.

§4. Multiple Existence of Equilibrium Solutions

In this section we consider problems under both of the hypotheses (A) and (B) (see §2 and §3), so the boundary condition of (1. 1) can be expressed in one of the forms

$$u = \beta(x),$$

or

$$\frac{\partial u}{\partial \nu} + \sigma(x) u = \gamma(x).$$

Definition 3. A closed set Y in $C^1(\bar{D}) \cap C^2(D)$ has *the property* (S) if there exists a family of sets $\{Y_\alpha\}_{\alpha \in A}$ in $C^1(\bar{D}) \cap C^2(D)$ which satisfies the following conditions:

1) $\bigcap_{\alpha \in A} Y_\alpha = Y$;

2) for any $\alpha_1, \alpha_2 \in A$ there exists an $\alpha_3 \in A$ with $Y_{\alpha_3} \subset Y_{\alpha_1} \cap Y_{\alpha_2}$;

3) each Y_α is closed in $C^1(\bar{D}) \cap C^2(D)$ and bounded in $L^\infty(D)$;

4) given any element w of Y, any $\alpha \in A$ and any $\delta > 0$, there exist continuous functions w_1, w_2 and a finite number $T \geq 0$ such that

$$w - \delta \leq w_1 \leq w \leq w_2 \leq w + \delta, \quad w_1, w_2 \not\equiv w,$$

and that

$$Q(t) w_i \in Y_\alpha, \quad i = 1, 2,$$

for $T \leq t \leq \infty$, where Q is the operator introduced in (2.8).

Remark 4.1. If a closed plus invariant set Y is plus stable, i.e., if given any neighborhood U of Y there exists a neighborhood $V \supset Y$ such that $Q(t) V \subset U$ for $t \geq 0$, then Y has the property (S).

Now we present our main results in this section, which consist of the following three theorems. The first one may seem rather different from the others because of its abstract form, but one will find the second theorem to be an immediate consequence of it, and in §6 will be found a more concrete and more interesting application of it.

Theorem 4.2. *Let* (A), (B) *hold and let* Y *be a nonempty closed set in* $C^1(\bar{D}) \cap C^2(D)$ *which has the property* (S). *Then* Y *contains at least one stable solution of* (1.2).

Theorem 4.3. *Let* (A), (B) *hold and let* v_1, v_2 *be distinct*

solutions of (1.2). *Suppose* $v_1 \leqq v_2$ *on* $\bar{D}$ *and that* v_1 *is unstable upward while* v_2 *downward. Then*

   i)   *there exist a minimal solution* v_3 *and a maximal solution* v_4 *in such a sense that*

$$v_1 \leqq v_3 \leqq v \leqq v_4 \leqq v_2, \quad v_1 \not\equiv v_3, \quad v_4 \not\equiv v_2,$$

for any solution v *satisfying* $v_1 \leqq v \leqq v_2$, $v \not\equiv v_1, v_2$;

   ii)   *there exists at least one stable solution* v *satisfying* $v_1 \leqq v \leqq v_2$.

Theorem 4.4. *Let* (A), (B) *hold and let* v_1, v_2 *be distinct solutions of* (1.2) *satisfying* $v_1 \leqq v_2$. *Suppose* v_1 *is stable from above and* v_2 *from below. Then there exists at least one other solution* v *which satisfies* $v_1 \leqq v \leqq v_2$.

Here we say v is *stable from above* (resp. *below*) if it is not unstable upward (resp. downward). As a result of Theorems 4.3 and 4.4 we have the following corollary:

Corollary 4.5. *Let* (A), (B) *hold and let* v_1, v_2 *be distinct solutions of* (1.2). *Suppose that* $v_1 \leqq v_2$ *and that there exists no other solution lying between* v_1 *and* v_2. *Then, either*

$$\lim_{t \to \infty} Q(t)w = v_1 \quad in \quad C^1(\bar{D}) \cap C^2(D)$$

for all $w \in X^+(v_1) \cap X^-(v_2)$ *or*

$$\lim_{t \to \infty} Q(t)w = v_2 \quad in \quad C^1(\bar{D}) \cap C^2(D)$$

for all $w \in X^+(v_1) \cap X^-(v_2)$.

The proof is immediate if Theorem 3.1 i) and Theorem 3.4 are taken into account.

Remark 4.6.   1)   One may expect that a similar assertion still holds when v_2 is replaced by $+\infty$ or v_1 by $-\infty$. But, so far, any satisfactory result has not been obtained except for the case where $n = 1$

and the function f is independent of x. For this special case we state the result without proof.

Let n and f be just as above and let v be a solution of (1.2) satisfying $S^+(v) = \emptyset$ (resp. $S^-(v) = \emptyset$). Then we have the following alternatives:

$$\lim_{t \to \infty} Q(t)w = v \quad \text{in} \quad C^1(\bar{D}) \cap C^2(D)$$

for all $w \in X^+(v)$ (resp. $X^-(v)$) or

$$\lim_{t \to T} \max_{x \in \bar{D}} [Q(t)w](x) = +\infty$$

$$(\text{resp. } \lim_{t \to T} \min_{x \in \bar{D}} [Q(t)w](x) = -\infty)$$

for all $w \in X^+(v)$ (resp. $X^-(v)$), where T is a positive number or infinity depending on w.

2) The same is true in the case where $n = 1$ and $f(x, u)$ is bounded from below (resp. above) in the domain $D \times (0, +\infty)$ (resp. $D \times (-\infty, 0)$). We can prove these results by using Corollary 4.5.

From this remark, we get rather a simple sufficient (and necessary) condition for the instability of a solution. Given such a solution as in the above remark, suppose we could find a continuous function w such that $Q(t)w$ blows up to $-\infty$ (resp. $-\infty$) in a finite time or diverges to $+\infty$ (resp. $-\infty$) as $t \to \infty$; then the upward (resp. downward) instability of v follows. In fact, the former alternative in Remark 4.6. 1) cannot hold in this case by virtue of the comparison theorem.

In finding such a function w the idea of upper and lower solutions is often useful. For, if w is a lower (resp. an upper) solution satisfying $w(x_0) > v(x_0)$ (resp. $w(x_0) < v(x_0)$) at some $x_0 \in D$, it is easily verified that $Q(t)w$ is monotone increasing (resp. decreasing) in t to finally blow up or diverge to $+\infty$ (resp. $-\infty$) (see [12; Remark in § 4]). So the question is merely how to find such a lower or an upper solution. This method can be applied, for example, to the boundary value problem $v'' + e^v = 0$, $v|_{\partial D} = 0$. In fact, given any positive number M, a lower solution w satisfying $w(x_0) > M$ at some $x_0 \in D$ is always found, and so we can a priori conclude that any solution of this problem above

which there lies no other solution is unstable upward in the sense that
the second alternative in Remark 4. 6 1) holds. (Instability of such
solutions has already been shown in H. Fujita [6] under the hypothesis
that the problem has at least two solutions.)

Now, let us begin the proof of the theorems.

Proof of Theorem 4. 2.   Suppose Y does not contain any stable
solution of (1. 2).   By definition, there exists a family of sets $\{Y_\alpha\}_{\alpha \in A}$
which satisfies the conditions stated in Definition 3.   So, for any $\alpha \in A$,
we can find a continuous function w_α such that $Q(t)w_\alpha \in Y_\alpha$ for all
$t \geq T_\alpha$, where T_α is a finite number depending on α and w_α.   Since Y_α
is closed in $C^1(\overline{D}) \cap C^2(D)$ and bounded in $L^\infty(D)$, it follows from
Theorem 2. 8 that $\varOmega(w_\alpha)$ is a nonempty subset of Y_α.   Set $K = $closure
$\bigcup_{\alpha \in A} \varOmega(w_\alpha)$, where the closure is with respect to $C^1(\overline{D}) \cap C^2(D)$. K is
a set of solutions of (1. 2) and, by Corollary 2. 5, $Y_\alpha \cap K$ is nonempty
and compact.   Considering the conditions 1) and 2), we see $\bigcap_{\alpha \in A}(Y_\alpha \cap K)$
$= Y \cap K \neq \varnothing$, which implies that Y contains at least one solution of (1 .2).
Let $v_0 \in Y \cap K$.   As it is supposed that Y contains no stable solution,
v_0 must be unstable upward or unstable downward.   Without loss of
generality we can assume the upward instability of it.   Then, as is seen
in Theorem 3. 4, it is strongly unstable upward.

Now let Z be the set of all the solutions of (1. 2) that are unstable
upward and contained in Y.   Naturally Z is regarded as a partially
ordered set (the relation $v_1 \leq v_2$ means $v_1(x) \leq v_2(x)$ everywhere in $\overline{D}$).
Denote by M the set of all the well-ordered subset of Z.   Again can
M be regarded as a partially ordered set with respect to the following
order relation: $W_1 \leq W_2$ if W_1 coincides with a segment of W_2.   Clearly,
M is an inductively ordered set and is not empty as it contains $\{v_0\}$.
Therefore, by Zorn's lemma, there exists a maximal element W of M.
Here we have two cases.

Case 1.   Suppose that W has its greatest element $\tilde{v}$, in other words,
consider the case where W is isomorphic to the segment of an isolated
ordinal number.   In this case the minimum of $S^+(\tilde{v})$ exists and belongs
to Z.   To see this, we first note that there exists a function $w \in X^+(\tilde{v})$
such that $Q(t)w$ remains bounded in $L^\infty(D)$ as $t \to \infty$ (see Definition 3

and recall that each Y_α is bounded in $L^\infty(D)$). So, considering that $\tilde{v}$ is unstable upward, we find from Theorem 3.1 that $S^+(\tilde{v})$ is not empty and has its minimum $\tilde{v}^+$. Again by definition, given any $\delta > 0$ and any $\alpha \in A$, there exists a function $\psi \in X^+(\tilde{v}) \cap X^-(\tilde{v} + \delta)$ satisfying $Q(t)\psi \in Y_\alpha$ on some interval $T \leq t < \infty$. Here, taking a sufficiently small δ, we can make $\tilde{v} \leq Q(t_1)\psi \leq \tilde{v}^+$ for some $t_1 \geq 0$. (In the case where (**B2**) holds, $\tilde{v}^+ \geq \tilde{v} + c$ for some $c > 0$ by virtue of Lemma 3.9. So the above statement is trivial if we put $\delta = c$ and $t_1 = 0$. In the case where (**B1**) holds, $\tilde{v}^+ \geq \tilde{v} + c\rho$ for some $c > 0$ by virtue of Lemma 3.8. Therefore, as is seen in the proof of Theorem 3.4, we can find such a $\delta > 0$ and a $t_1 \geq 0$ as above.) It means that $Q(t)\psi$ converges to $\tilde{v}^+$ in $C^1(\overline{D}) \cap C^2(D)$ as $t \to \infty$ (see Theorem 3.1 i)). Hence $\tilde{v}^+$ is contained in Y_α. As α is any element of A, $\tilde{v}^+$ is also contained in Y. Seeing that $Q(t)\psi$ converges to $\tilde{v}^+$ from below, we find, by the aid of Corollary 3.5, that $\tilde{v}^+$ is stable from below. So it must be unstable upward since Y contains no stable solution. Consequently $\tilde{v}^+$ belongs to Z and hence $W \cup \{\tilde{v}^+\}$ to M.

Case 2. Suppose that W does not have the greatest element, in other words, consider the case where W is isomorphic to the segment of a limit ordinal number. In this case, putting

$$\hat{v}(x) = \sup_{v \in W} v(x),$$

we claim that $\hat{v}$ belongs to Z. In fact, since Y is bounded in $L^\infty(D)$, $\hat{v}$ is an accumulation point of W in the topology of $C^1(\overline{D}) \cap C^2(D)$. So $\hat{v}$ is a solution of (1.2). The relation $\hat{v} \in Y$ and the stability of $\hat{v}$ from below follow from the closedness of Y and Corollary 3.5 respectively. So $\hat{v}$ belongs to Z.

Thus, in either case, the existence of an element that violates the maximality of W has been shown. So our supposition that Y contains no stable solution must be false.   Q.E.D.

Proof of Theorem 4.3. i) As $v_2 \in S^+(v_1)$, $S^+(v_1)$ is not empty, and so it follows from Theorems 3.1 and 3.4 that the minimum of $S^+(v_1)$ exists. We denote it by v_3. Not only $v_1 \leq v_3 \leq v_2$ and $v_1 \not\equiv v_3$, but also $v_3 \not\equiv v_2$ since v_3 is stable from below while v_2 is not. Considering

that $v_3 \in S^-(v_2)$, we can similarly show the existence of $v_4 = \max S^-(v_2)$, which satisfies

$$v_3 \leqq v_4 \leqq v_2 , \quad v_4 \not\equiv v_2 .$$

The rest of i) follows immediately.

ii) Let v_3, v_4 be as above and put

$$Y = \{w \in C^1(\overline{D}) \cap C^2(D) ; v_3 - \overline{v}/4 \leqq w \leqq v_4 + \overline{\overline{v}}/4\},$$

where $\overline{v} = v_3 - v_1$ and $\overline{\overline{v}} = v_2 - v_4$. Note that in the case where (**A**), (**B1**) (resp. (**A**), (**B2**)) hold there exists a constant $c > 0$ such that $\overline{v}$, $\overline{\overline{v}} \geqq c\rho$ (resp. $\geqq c$) thanks to Lemma 3.8 (resp. 3.9). So there exists a finite number $T \geqq 0$ such that

$$Q(t)(v_3 - \overline{v}/2), \quad Q(t)(v_4 + \overline{\overline{v}}/2) \in Y$$

for all $t \geqq T$, because these two functions converge to v_3, v_4, respectively in $C^1(\overline{D})$ as $t \to \infty$. It follows from the comparison theorem that $Q(t)w \in Y$ for all $t \geqq T$ and any function $w \in C^1(\overline{D}) \cap C^2(D)$ satisfying $v_3 - \overline{v}/2 \leqq w \leqq v_4 + \overline{\overline{v}}/2$. Putting $Y_\alpha = Y$ we easily find that Y has the property (S). The conclusion now follows from Theorem 4.2.

$$\text{Q.E.D.}$$

Proof of Theorem 4.4. First we prove this theorem under the assumptions (**A**), (**B1**). Suppose $X^+(v_1) \cap X^-(v_2)$ contains no solution of (1.2). Then it follows from the comparison theorem, Theorem 2.8 ii) and Corollary 2.11 that for any $w \in X^+(v_1) \cap X^-(v_2)$ $Q(t)w$ converges either to v_1 or to v_2 as $t \to \infty$ in $C^1(\overline{D}) \cap C^2(D)$. Set

$$w_\theta = v_1 + \theta(v_2 - v_1)$$

for $0 \leqq \theta \leqq 1$. By virtue of the comparison theorem there exists a number θ_0 such that

(4.1a) $Q(t)w_\theta \to v_1$ (as $t \to \infty$) if $0 \leqq \theta < \theta_0$,

(4.1b) $Q(t)w_\theta \to v_2$ (as $t \to \infty$) if $\theta_0 < \theta \leqq 1$.

Neither $\theta_0 = 0$ nor $\theta_0 = 1$ because of the stability of v_1 and v_2. Without loss of generality we can assume $Q(t)w_{\theta_0} \to v_1$ as $t \to \infty$. Since this convergence occurs in $C^1(\overline{D})$, given any $c > 0$ there exists a $t' \geqq 0$ such

that

$$v_1 \leqq Q(t') w_{\theta_0} \leqq v_1 + c\rho ,$$

where $\rho(x) = \mathrm{dist}(x, \partial D)$. Considering

(4. 2) $$v_2 - v_1 \geqq c_0 \rho$$

holds for some $c_0 > 0$ by virtue of Lemma 3. 8, we then find that

(4. 3) $$v_1 \leqq Q(t_1) w_{\theta_0} \leqq w_{\theta_0/4}$$

for some $t_1 > 0$. By the way there exists a constant $c_1 > 0$ such that for any $w \in X^+(v_1) \cap X^-(v_2)$

(4. 4) $$\| Q(t_1) w_{\theta_0} - Q(t_1) w \|_{L^\infty(D)} \leqq c_1 \| w_{\theta_0} - w \|_{L^\infty(D)} \rho .$$

This inequality can be shown just similarly to (3. 9). Combining (4. 2), (4. 3) and (4. 4), we obtain by a simple calculation the inequality

(4. 5) $$Q(t_1) w_\theta \leqq v_1 + \left\{ \frac{\theta_0}{4} + c_0^{-1} c_1 (\theta - \theta_0) \| v_2 - v_1 \|_{L^\infty(D)} \right\} (v_2 - v_1)$$

for any θ satisfying $\theta_0 \leqq \theta \leqq 1$. Let θ_1 satisfy

$$0 < \theta_1 - \theta_0 < \frac{1}{4} \theta_0 c_0 c_1^{-1} (\| v_2 - v_1 \|_{L^\infty(D)})^{-1} .$$

It then follows from (4. 5) that $Q(t_1) w_{\theta_1} \leqq w_{\theta_0/2}$. So we can see from (4. 1a) that $Q(t+t_1) w_{\theta_1}$ converges to v_1 as $t \to \infty$. On the other hand, since $\theta_1 > \theta_0$, $Q(t) w_{\theta_1}$ must converge to v_2 by virtue of (4. 1b). This contradiction proves the existence of at least one solution of (1. 2) between v_1 and v_2.

Secondly, under the assumptions (A), (B2). The proof is just similar to the above one, except that we use Lemma 3 9 instead of Lemma 3. 8.   Q.E.D.

Chapter II. Homogeneous Neumann Problems

§ 5. Instability of Non-constant Equilibrium Solutions

In this section, and also in next section, we study a special form of (1. 1):

(5. 1a) $\dfrac{\partial u}{\partial t} = \varDelta u + f(u),$

(5. 1b) $u(x, 0) = u_0(x),$

(5. 1c) $\dfrac{\partial u}{\partial n} = 0 .$

The stationary equation corresponding to (5.1) is (1.3). Note that f is assumed to be independent of x. So assumptions (A), (B) are put into the reduced form

(**A**′)   1) D is bounded and ∂D is smooth;

      2) f is of class C^2.

Our aim in this section is to pick out a few types of domains in $\boldsymbol{R}^n$ in which any stable solution of the boundary value problem (1.3) is necessarily a trivial solution, i.e., a zero of $f(u)$. We mention here three types; convex domains, bodies of rotation with convex sections and domains bounded by concentric spheres. From the standpoint of practial applications, it is important to determine, for a given function f, in what domain (1.3) admits non-constant stable solutions and in what domain not. Our results here form part of the answer to the latter question, and part of the answer to the former is given in § 6.

To define the above-mentioned second type of domains exactly, let us introduce the following notation:

$$G(U) = \{U^{-1}T_\theta U\}_{0 \leq \theta < 2\pi} ,$$

where

$$T_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

and U denotes an element of $SO(n)$. Here it is assumed that $n \geq 2$. We consider $G(U)$ a one-parameter group of transformations. Obviously $G(U)$ is a subgroup of $SO(n)$.

Definition 4. A subset Y of R^n is $G(U)$-*invariant* if

$$U^{-1}T_\theta U x \in Y$$

for all $x \in Y$ and $0 \leq \theta < 2\pi$. A function $w(x)$ is $G(U)$-*invariant* if its domain is $G(U)$-invariant and

$$w(U^{-1}T_\theta U x) = w(x)$$

for all x, θ.

It is easy to see that Y is $G(U)$-invariant if and only if $UY(=\{Ux; x \in Y\})$ is $G(E)$-invariant, where E is the unit matrix. When $n = 3$, $G(E)$ is nothing but a group of rotations around x_3-axis.

In a $G(E)$-invariant domain it is convenient to introduce a new co-ordinate system $(\theta, r, x_3, \cdots, x_n)$ instead of the usual orthogonal coordinate system $(x_1, \cdots, x_n)$, where

$$x_1 = r \cos \theta ,$$

$$x_2 = r \sin \theta ,$$

for $r \geq 0$, $0 \leq \theta < 2\pi$. A domain D is $G(E)$-invariant if and only if there exists an $(n-1)$-dimensional domain D_0 in $(r, x_3, \cdots, x_n)$-space such that

(5. 2) $$D = [0, 2\pi) \times D_0 .$$

(Note that for each $x_3, \cdots, x_n$, the set $\{(\theta, 0, x_3, \cdots, x_n); 0 \leq \theta < 2\pi\}$ is identified to one point.) Similarly, a function $w = w(\theta, r, x_3, \cdots, x_n)$ is $G(E)$-invariant if and only if w is independent of θ.

Definition 5. Let D be a $G(E)$-invariant domain. The *E-section* of D is a domain in $(r, x_3, \cdots, x_n)$-space defined by D_0 in (5. 2). We denote it by $S_E(D)$. For any $U \in SO(n)$, U-section of a $G(U)$-invariant domain D is defined by $S_U(D) = S_E(UD)$.

Now we are ready to form a clear definition of the three types of domains.

(**T1**): D is convex.

(**T2**): D is $G(U)$-invariant for some $U \in SO(n)$, $\overline{UD}$ does not meet

with the 2-codimensional subspace $x_1 = x_2 = 0$ (i.e. $r = 0$) and $S_U(D)$ is convex.   (Here $n \geq 2$.)

(T3): $D = \{x \in \mathbf{R}^n; r_1 < |x| < r_2\}$ for some $0 < r_1 < r_2$. $(n \geq 2.)$

These conditions are incompatible with each other, except that (T2) and (T3) are equivalent conditions when $n = 2$. A typical example of domain of type (T2) for $n = 3$ is the inside of a torus.

Theorem 5. 1.   *Let* (A′) *hold, let D be of one of the types* (T1) $\sim$ (T3) *and let v be a solution of* (1.3). *Suppose v is not a constant function. Then it is unstable in the following sense:*

   i)   *If* $A^+ \equiv \{\alpha \in \mathbf{R}; f(\alpha) = 0, \alpha \geq \max\limits_{x \in \bar{D}} v(x)\}$ *is not empty, then any continuous function ψ satisfying $v \leq \psi \leq \alpha^+$ $(= \min A^+)$, $\psi \not\equiv v$ belongs to* $\mathrm{Dom}(Q(\infty))$ *and*

$$\lim_{t \to \infty} Q(t)\psi = \alpha^+$$

in the topology of $C^1(\bar{D}) \cap C^2(D)$, where Q is the operator introduced in (2.8).

   ii)   *If $A^+ = \emptyset$, then given any continuous function ψ satisfying $\psi \geq v$ and $\psi \not\equiv v$ there exists a $T \in (0, \infty]$ such that*

$$\lim_{t \to T} \max_{x \in \bar{D}} Q(t)\psi = +\infty .$$

   i)′, ii)′ *similar propositions hold good for* $A^- = \{\alpha \in R; f(\alpha) = 0, \alpha \leq \min\limits_{x \in \bar{D}} v(x)\}$.

This is our main theorem in Chapter II. The following theorem, though not so strong a statement as above, is also interesting.

Theorem 5. 2.   *Let* (A′) *hold, let D be $G(U)$-invariant for some $U \in SO(n)$ and let v be a solution of* (1.3). *If v is not $G(U)$-invariant, it is unstable both upward and downward.*

To see the above results by the use of Theorems 3. 1 and 3. 3, we have to show the negativity of the first (i.e., least) eigenvalue λ_1 of the following eigenvalue problem:

$$(5.3) \quad \begin{cases} \widetilde{L}\varphi + \lambda\varphi = 0 \quad \text{in} \quad D, \\ \dfrac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \partial D, \end{cases}$$

where

$$\widetilde{L}\varphi = \Delta\varphi + f'(v(x))\varphi.$$

As is well-known,

$$\lambda_1 = \min_{\varphi \in H^1(D),\, \|\varphi\|_{L^2(D)} = 1} \mathcal{H}(\varphi),$$

where

$$\mathcal{H}(\varphi) \equiv \int_D \{|\nabla\varphi|^2 - f'(v(x))\varphi^2\}\, dx,$$

and the eigenfunctions corresponding to λ_1 are characterized as to mini-mize the form

$$\mathcal{H}(\varphi) / \|\varphi\|_{L^2(D)}.$$

As a consequence of Krein's theorem, λ_1 is a simple eigenvalue and any eigenfunction subject to λ_1 does not change sign in D. Therefore, it is sufficient for concluding $\lambda_1 < 0$ if we find such a function $\varphi \in H^1(D)$ as to make $\mathcal{H}(\varphi) < 0$ or as to make $\mathcal{H}(\varphi) = 0$ and change sign in D.

Next lemma will play an important role in the sequel.

Lemma 5.3. *Let D_1 be a convex domain in $\mathbf{R}^n$ with a smooth boundary. Suppose w belongs to $C^3(\overline{D}_1)$ and satisfies $\partial w/\partial n = 0$ on ∂D_1. Then*

$$\frac{\partial}{\partial n}|\nabla w|^2 \leq 0 \quad on \quad \partial D_1.$$

Proof. Let ρ be a function of class C^2 defined in a neighborhood of $\overline{D}_1$ such that

$$(5.4) \quad \begin{cases} \rho < 0 \quad \text{in} \quad D_1 \\ \rho = 0 \quad \text{on} \quad \partial D_1 \\ \rho > 0 \quad \text{outside} \quad \overline{D}_1, \end{cases}$$

and that $\nabla\rho$ never vanishes on ∂D_1. Since $\nabla\rho$ is outer normal to ∂D_1

at each point of ∂D_1, we have

$$\frac{\partial w}{\partial n} = \frac{1}{|\nabla \rho|} \nabla \rho \cdot \nabla w|_{\partial D_1},$$

where $\cdot$ denotes the usual Euclidean inner product. So it follows from the boundary condition $\partial w/\partial n = 0$ that there exists a function $g \in C^1(\overline{D}_1)$ such that

(5.5) $$\nabla \rho \cdot \nabla w = \rho g \quad \text{on} \quad \overline{D}_1.$$

Differentiating (5.5) by x_i yields

$$\nabla \rho \cdot \nabla \left(\frac{\partial w}{\partial x_i} \right) + \nabla \left(\frac{\partial \rho}{\partial x_i} \right) \cdot \nabla w = \frac{\partial \rho}{\partial x_i} g + \rho \frac{\partial g}{\partial x_i},$$

for $i = 1, \cdots, n$. Multiplying this equality by $\partial w/\partial x_i$ and then summing over $i = 1, \cdots, n$, we get

$$\frac{1}{2} \nabla \rho \cdot \nabla (|\nabla w|^2) + \sum_{i,j} \frac{\partial^2 \rho}{\partial x_i \partial x_j} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j}$$

$$= g \nabla \rho \cdot \nabla w + \rho \nabla g \cdot \nabla w = \rho g^2 + \rho \nabla g \cdot \nabla w.$$

Therefore

$$\frac{1}{2} \nabla \rho \cdot \nabla (|\nabla w|^2) + \sum_{i,j} \frac{\partial^2 \rho}{\partial x_i \partial x_j} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} = 0 \quad \text{on} \quad \partial D_1,$$

hence

(5.6) $$\frac{\partial}{\partial n} |\nabla w|^2 = -\frac{2}{|\nabla \rho|} \sum_{i,j} \frac{\partial^2 \rho}{\partial x_i \partial x_j} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \quad \text{on} \quad \partial D_1.$$

As is well-known, for each $\tilde{x} \in \partial D_1$ the Hessian of ρ at $\tilde{x}$ induces a positive semi-definite quadratic form on the tangent hyperplane to ∂D_1 at $\tilde{x}$, because D_1 is convex. Since the boundary condition $\partial w/\partial n = 0$ implies that the vector $\nabla w(\tilde{x})$ belongs to this tangent hyperplane, it follows that the right-hand side of (5.6) is nonpositive.   Q.E.D.

Lemma 5.4. *Let* (A′) *hold and let* v *be a solution of* (1.3). *Suppose* v *is not a constant and that*

$$\sum_{i=1}^{n} \mathcal{H} \left(\frac{\partial v}{\partial x_i} \right) \leq 0.$$

Then $\lambda_1 < 0$.

Proof. The statement is trivial if $\sum_i \mathcal{H}(\partial v/\partial x_i) < 0$, for in this case there exists at least one integer k, $1 \leq k \leq n$, such that $\mathcal{H}(\partial v/\partial x_k)$ < 0. So we assume $\sum_i \mathcal{H}(\partial v/\partial x_i) = 0$.

Now suppose that $\lambda_1 \geq 0$. Then it follows from the above equality that

$$\mathcal{H}(\partial v/\partial x_i) = 0, \quad i = 1, \cdots, n,$$

and that $\lambda_1 = 0$. This means each $\partial v/\partial x_i$ is an eigenfunction subject to λ_1 or is identically zero. Since λ_1 is simple, we can find a function $\varphi_1(x)$ and constants c_i $(i = 1, \cdots, n)$ such that

$$\frac{\partial v}{\partial x_i} = c_i \varphi_1, \quad i = 1, \cdots, n.$$

It follows that there exists a function $\mathit{\Phi}$ of one variable satisfying

(5. 7) $$v(x_1, \cdots, x_n) = \mathit{\Phi}(c_1 x_1 + \cdots + c_n x_n)$$

for any $(x_1, \cdots, x_n) \in \overline{D}$. Here $c \equiv \sqrt{c_1^2 + \cdots + c_n^2} \neq 0$, because v is not a constant. Put

$$a = \min_{x \in \overline{D}} (c_1 x_1 + \cdots + c_n x_n),$$

$$b = \max_{x \in \overline{D}} (c_1 x_1 + \cdots + c_n x_n)$$

and let $\tilde{x}$ be the point of ∂D where the above minimum a is attained. It is easy to see that the vector $(c_1, \cdots, c_n)$ is inner normal to ∂D at $\tilde{x}$. Combining (5. 7) and (1. 3), and considering the above-mentioned fact, we get an ordinary differential equation of the form

(5. 8a) $$\mathit{\Phi}''(y) + c^{-2} f(\mathit{\Phi}(y)) = 0, \quad a < y < b,$$

with the initial condition

(5. 8b) $$\mathit{\Phi}'(a) = 0.$$

Recall that each $\partial v/\partial x_i$ $(i = 1, \cdots, n)$ is an eigenfunction of (5. 3) or is identically zero. So it satisfies the Neumann boundary condition on ∂D. Consequently,

$$c_i \mathit{\Phi}''(a) = 0, \quad i = 1, \cdots, n,$$

hence $\varPhi''(a) = 0$. It then follows from (5.8a) that

(5.9) $f(\varPhi(a)) = 0$.

Combining (5.8) and (5.9), we find, by the aid of the uniqueness theorem for solutions of ordinary differential equations, that

$$\varPhi(y) = \varPhi(a), \quad a \leqq y \leqq b .$$

It implies that v is a constant, which contradicts the presumption. Therefore our supposition that $\lambda_1 \geqq 0$ must be false, thus the negativity of λ_1 is established.   Q.E.D.

To consider (1.3) and (5.1) in a $G(E)$-invariant domain, it is convenient to express $\varDelta \equiv \sum_i \partial^2/\partial x_i^2$ in the coordinates θ, r, x_3, $\cdots$, x_n:

(5.10) $\varDelta = \tilde{\varDelta} + \dfrac{1}{r} \dfrac{\partial}{\partial r} + \dfrac{1}{r^2} \dfrac{\partial^2}{\partial \theta^2}, \quad r > 0$,

where

$$\tilde{\varDelta} = \dfrac{\partial^2}{\partial r^2} + \sum_{i=3}^{n} \dfrac{\partial^2}{\partial x_i^2} .$$

We also use the following formula:

(5.11) $\varGamma w_1 \cdot \varGamma w_2 = \tilde{\varGamma} w_1 \cdot \tilde{\varGamma} w_2 + \dfrac{1}{r^2} \dfrac{\partial w_1}{\partial \theta} \dfrac{\partial w_2}{\partial \theta}, \quad r > 0$,

where

$$\tilde{\varGamma} = \left(\dfrac{\partial}{\partial r}, \dfrac{\partial}{\partial x_3}, \cdots, \dfrac{\partial}{\partial x_n} \right).$$

It is easily seen from (5.10) that $\varDelta$ and $\partial/\partial\theta$ are commutative. Consequently, differentiating (1.3a) by θ yields

(5·12) $\varDelta \left(\dfrac{\partial v}{\partial \theta} \right) + f'(v) \dfrac{\partial v}{\partial \theta} = 0$.

This equality should be understood to hold as long as $r > 0$, since (5.10) is valid for $r > 0$. However, putting $\partial/\partial\theta = 0$ when $r = 0$ and then regarding $\partial/\partial\theta$ as a smooth vector field defined everywhere, we can see (5.12) hold everywhere in D.

Lemma 5. 5. *Let D_1 be a $G(E)$-invariant domain with a smooth boundary and let w be a function of class $C^3(\overline{D}_1)$ satisfying the boundary condition $\partial w/\partial n = 0$ on ∂D_1. Then*

$$\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial \theta}\right) = 0 \quad on \quad \partial D_1 \, .$$

(As is mentioned above, we understand $\partial w/\partial \theta = 0$ if $r = 0$.)

Proof. Let ρ be a $G(E)$-invariant function of class C^2 satisfying (5. 4). (It is not difficult to see that we can actually take such a function ρ.) Combining (5. 5) and (5. 11). We get

$$\rho g = \tilde{\nabla}\rho \cdot \tilde{\nabla}w \, ,$$

where g is a function of $C^1(\overline{D}_1)$. Differentiating this equality by θ, we then obtain

$$\rho\frac{\partial g}{\partial \theta} = \tilde{\nabla}\rho \cdot \tilde{\nabla}\left(\frac{\partial w}{\partial \theta}\right) = \nabla\rho \cdot \nabla\left(\frac{\partial w}{\partial \theta}\right).$$

So $\partial(\partial w/\partial \theta)/\partial n$ vanishes on ∂D_1.   Q.E.D.

Proof of Theorem 5. 2. First of all, note that if v is a solution of (1. 3) then it belongs to $C^3(\overline{D})$ because of the smoothness of f. It should also be noted that we have only to prove this theorem in the case $U = E$ since UD is $G(E)$-invariant and (1. 3) does not change its form under any orthogonal transformation of coordinates.

Using (5. 12) and Green's formula, we get

$$\mathcal{H}\left(\frac{\partial v}{\partial \theta}\right) \equiv \int_D \left\{\left|\nabla\left(\frac{\partial v}{\partial \theta}\right)\right|^2 - f'(v)\left(\frac{\partial v}{\partial \theta}\right)^2\right\}dx$$

$$= \int_D \left\{\left|\nabla\left(\frac{\partial v}{\partial \theta}\right)\right|^2 + \Delta\left(\frac{\partial v}{\partial \theta}\right)\frac{\partial v}{\partial \theta}\right\}dx$$

$$= \int_{\partial D} \frac{\partial v}{\partial \theta}\,\frac{\partial}{\partial n}\left(\frac{\partial v}{\partial \theta}\right)d\tilde{x} \, .$$

Consequently, by Lemma 5. 5,

(5. 13) $$\mathcal{H}\left(\frac{\partial v}{\partial \theta}\right) = 0 \, .$$

As v is not $G(E)$-invariant, $\partial v/\partial \theta$ is not identically zero. So (5.13) implies that $\lambda_1 \leqq 0$. If $\lambda_1 = 0$, it follows from (5.13) that $\partial v/\partial \theta$ is an eigenfunction of (5.3) subject to λ_1, hence $\partial v/\partial \theta$ does not change sign in D. But this is impossible because

$$\int_0^{2\pi} \frac{\partial v}{\partial \theta}(\theta, r, x_3, \cdots, x_n)\,d\theta = 0\,.$$

Therefore λ_1 must be negative. On applying Theorem 3.3 we get to the completion of the proof.   Q.E.D.

Proof of Theorem 5.1. We first show the negativity of λ_1.

Case (T1): By Green's formula,

$$(5.14) \qquad \sum_{i=1}^n \mathcal{H}\left(\frac{\partial v}{\partial x_i}\right) \equiv \sum_i \int_D \left\{\left|\mathcal{V}\left(\frac{\partial v}{\partial x_i}\right)\right|^2 - f'(v)\left(\frac{\partial v}{\partial x_i}\right)^2\right\}dx$$

$$= \sum_i \int_D \left\{\left|\mathcal{V}\left(\frac{\partial v}{\partial x_i}\right)\right|^2 + \varDelta\left(\frac{\partial v}{\partial x_i}\right)\frac{\partial v}{\partial x_i}\right\}dx$$

$$= \sum_i \int_{\partial D} \frac{\partial v}{\partial x_i}\frac{\partial}{\partial n}\left(\frac{\partial v}{\partial x_i}\right)d\tilde{x}$$

$$= \frac{1}{2}\int_{\partial D} \frac{\partial}{\partial n}|\mathcal{V}v|^2 d\tilde{x}\,.$$

Considering that $v \in C^3(\overline{D})$ and applying Lemmas 5.3 and 5.4, we get $\lambda_1 < 0$.

Case (T2): As is mentioned in the proof of Lemma 5.5, we can assume $U = E$. Further we assume that v is $G(E)$-invariant, for otherwise the negativity of λ_1 is already shown in the proof of Theorem 5.2.

Let ρ be as in the proof of Lemma 5.5. Then

$$(5.15) \qquad\qquad \frac{\partial}{\partial n}|\mathcal{V}v|^2 = \frac{1}{|\mathcal{V}\rho|}\mathcal{V}\rho \cdot \mathcal{V}(|\mathcal{V}v|^2)$$

$$= \frac{1}{|\tilde{\mathcal{V}}\rho|}\tilde{\mathcal{V}}\rho \cdot \tilde{\mathcal{V}}(|\tilde{\mathcal{V}}v|^2)\,.$$

Here we have used the fact that $|\tilde{\mathcal{V}}v| = |\mathcal{V}v|$ and that $\partial\rho/\partial\theta = 0$. The former follows from the assumption that $\partial v/\partial\theta = 0$. Since $S_E(D)$ is convex and $\partial D = [0, 2\pi) \times \partial S_E(D)$, applying Lemma 5.3 to $D_1 = S_E(D)$ yields the nonpositivity of the right-hand side of (5.15). Combining this

fact and (5.14), and then applying Lemma 5.4, we find $\lambda_1 < 0$.

Case (T3): Since D is $G(U)$-invariant for any $U \in SO(n)$, we can assume that v is $G(U)$-invariant for any U, for otherwise the statement is already established in the proof of Theorem 5.2. In this case, as is easy to see, v is a function of $|x|$. So we put $v(x) = \Phi(|x|)$. Then

$$|\nabla v(x)| = |\Phi'(|x|)| \, |\nabla|x|| = |\Phi'(|x|)|,$$

since $|x| \neq 0$ as long as $x \in D$. Hence on the boundary $|x| = r_i$ ($i = 1, 2$) we have

$$(5.16) \qquad \frac{\partial}{\partial n}|\nabla v(x)|^2 = \frac{\partial}{\partial n}\{\Phi'(|x|)\}^2$$

$$= (-1)^i 2\Phi'(|x|)\Phi''(|x|).$$

Considering that

$$\Phi'(|x|) = (-1)^i \frac{\partial v}{\partial n} = 0, \quad |x| = r_i,$$

for $i = 1, 2$, we find, from (5.16), that

$$(5.17) \qquad \frac{\partial}{\partial n}|\nabla v|^2 = 0 \quad \text{on} \quad \partial D.$$

Combining (5.17), (5.14) and Lemma 5.4, we thus get $\lambda_1 < 0$.

Now that λ_1 is shown to be negative in each case, the rest of the proof is not difficult. For instance we prove i) and ii). Suppose $S^+(v) \neq \emptyset$. Since v is strongly unstable upward by virtue of Theorem 3.3, it follows from Theorem 3.1 that $S^+(v)$ has its minimum v^+. Here, by Cororally 3.5, v^+ is stable from below. And so it is a constant function, for, as is just shown above, any non-constant solution of (1.3) is in these cases unstable both upward and downward. Consequently v^+ is a zero of $f(u)$, and from the minimality of v^+ in $S^+(v)$ follows $v^+ = \alpha^+$. This means that $S^+(v) \neq \emptyset$ and $A^+ \neq \emptyset$ are equivalent, hence the assertions i), ii) follow immediately from Theorem 3.1 i), ii), respectively. Q.E.D.

§ 6. Domains Admitting Non-constant Stable Solutions

As is seen in Theorem 5.1 for case (T1), any non-constant solution

of (1.3) is unstable when $n=1$. But it must be noted that when $n\geq2$ (1.3) sometimes has non-constant stable solutions. To show this, we present sufficient conditions on D and f for the existence of such solutions, by which various examples can easily be made.

Let the nonlinear term f be of the form

$$f(u)=kg(u),$$

where k is a positive constant and g is a function of class C^2 satisfying

(6.1a) $g(a)=g(0)=g(b)=0$

for some $a<0<b$ and

(6.1b) $u\leq g(u)<0, \quad a<u<0$

(6.1c) $0<g(u)\leq u, \quad 0<u<b.$

From (6.1b) and (6.1c) it follows that $G(u)\equiv\int_0^u g(w)dw$ is nonnegative in $a\leq u\leq b$ and takes its maximum at $u=a$ or at $u=b$. Without loss of generality we can assume that $G(b)$ is the maximum.

Secondly, let D be a bounded domain in $\boldsymbol{R}^n$ with a sufficiently smooth boundary and let D_1, D_2 be subdomains of D in each of which Poincaré's second inequality holds, i.e., there exist positive constants $\lambda_2(D_1)$ and $\lambda_2(D_2)$ such that for any function $w\in H^1(D_i)$ the following inequality holds:

(6.2) $\lambda_2(D_i)^{-1}\int_{D_i}|\nabla w|^2 dx+\dfrac{(\int_{D_i}w\,dx)^2}{\int_{D_i}dx}\geq\int_{D_i}w^2 dx \quad (i=1,2).$

(Here these constants are assumed to be the best possible ones. So they are nothing but the second eigenvalues of $-\Delta$ under the Neumann boundary conditions if ∂D_i $(i=1,2)$ are smooth.) We need some more notation: Let μ be the Lebesgue measure in $\boldsymbol{R}^n$ and let $J(u)$ be the energy form

$$J(u)=\int_D\left\{\frac{1}{2}|\nabla u|^2-F(u)\right\}dx,$$

where $F(u)=kG(u)$. Finally, put

$$R[-,+]=\left\{w\in C^1(\bar{D})\cap C^2(D); \ a\leq w\leq b \ \text{ on } \ \bar{D},\right.$$

$$\left.\int_{D_1}w(x)dx<0, \ \int_{D_2}w(x)dx>0\right\}.$$

We are now going to show that under suitable conditions (1.3) has stable solutions belonging to $R[-, +]$. Of course any function belonging this set is not a constant function. The essential part of the result in this section is based on Theorem 4.2. To apply this theorem, we must find a subset Y of $R[-, +]$ having the property (S) given in Definition 3. Let us begin with the following lemma:

Lemma 6.1. *Let f, D, D_1 and D_2 be as above and put*

$$\varepsilon_0 = G(b) \min \{\mu(D_1) \min \{k, \lambda_2(D_1)\}, \quad \mu(D_2) \min \{k, \lambda_2(D_2)\}\}.$$

Suppose a function $w \in R[-, +]$ satisfies the inequality

$$J(w) < \varepsilon_0 - kG(b)\mu(D).$$

Then, $Q(t)w$ belongs to $R[-, +]$ for all $t \geq 0$.

Proof. Since (6.1a) implies that both $v = a$ and $v = b$ are solutions of (1.3), it follows from the comparison theorem that $a \leq Q(t)w \leq b$ for all $t \geq 0$. So what must be proved are the inequalities

(6.3a) $$\int_{D_1} Q(t)w \, dx < 0, \quad t \geq 0,$$

(6.3b) $$\int_{D_2} Q(t)w \, dx > 0, \quad t \geq 0.$$

Suppose, for instance, that (6.3a) does not hold. Then, because of the continuity of $Q(t)w$ in t, there exists a positive number t_1 such that

$$\int_{D_1} Q(t_1)w \, dx = 0.$$

Therefore, seeing (6.2), we get

(6.4) $$\int_{D_1} |\nabla w_1|^2 dx \geq \lambda_2(D_1) \int_{D_1} w_1^2 \, dx$$

$$\geq 2\lambda_2(D_1) \int_{D_1} G(w_1) \, dx,$$

where, for simplicity, the function $Q(t_1)w$ is denoted by w_1. The latter inequality follows from (6.1b) and (6.1c). On the other hand, by virtue of Lemma 2.7, $J(Q(t)w)$ is monotone decreasing with respect to

t in the interval $(0, \infty)$. As a matter of fact, this monotonicity still holds in $[0, \infty)$, since w belongs to $C^1(\overline{D})$ (note that w is the limit of a sequence of functions satisfying the Neumann boundary conditions in the topology of $H^1(D)$). Consequently

$$J(w_1) \leqq J(w)$$

$$< \varepsilon_0 - kG(b)\mu(D).$$

So, considering that the inequality

$$\int_{D \backslash D_1} \left\{ \frac{1}{2} |\nabla w_1|^2 - F(w_1) \right\} dx \geqq -kG(b)\mu(D \backslash D_1)$$

holds because of the maximality of $G(b)$, we find

(6.5) $$\int_{D_1} \left\{ \frac{1}{2} |\nabla w_1|^2 - F(w_1) \right\} dx < \varepsilon_0 - kG(b)\mu(D_1).$$

It follows from (6.4) and (6.5) that

$$\{\lambda_2(D_1) - k\} \int_{D_1} G(w_1) dx < \varepsilon_0 - kG(b)\mu(D_1).$$

Hence, using the inequalities

$$0 \leqq \int_{D_1} G(w_1) dx \leqq G(b)\mu(D_1),$$

we obtain

$$\varepsilon_0 > \{\lambda_2(D_1) - k\} \int_{D_1} G(w_1) dx + kG(b)\mu(D_1)$$

$$\geqq G(b)\mu(D_1) \min\{k, \lambda_2(D_1)\},$$

which, however, is impossible by the definition of ε_0. This contradiction proves (6.3a). Similarly (6.3b) can be verified.   Q.E.D.

Now we are ready to present the main theorem in this section.

Theorem 6.2. *Let f, D, D_1 and D_2 be as above, and suppose that there exists a function $w_0 \in R[-, +]$ satisfying*

(6.6) $$J(w_0) < \varepsilon_0 - kG(b)\mu(D),$$

where ε_0 is the number introduced in Lemma 6.1. Then, (1.3)

has at least one stable solution belonging to $R[-, +]$.

Proof. Let J_0 be the right-hand side of (6. 6) and set

$$Y_m = \left\{ w \in R[-, +]\,;\, J(w) \leq \frac{mJ(w_0) + J_0}{m+1} \right\}$$

for $m = 1, 2, \cdots$. It is clear from Lemma 6. 1 and from the monotone decreasing property of $J(Q(t)w)$ that each Y_m is invariant under $Q(t)$, i.e.,

$$Q(t)\,w \in Y_m, \quad t \geq 0$$

for all $w \in Y_m$. Moreover, each Y_m is bounded in $L^\infty(D)$ and closed in $C^1(\bar{D}) \cap C^2(D)$. So it is not difficult to see that $Y = \bigcap_{m=1}^{\infty} Y_m$ has the property (S), for Y is contained in the interior of each Y_m in the topology of $C^1(\bar{D}) \cap C^2(D)$. The nonemptiness of Y follows from the fact that $w_0 \in Y$. Therefore, by Theorem 4. 2, Y contains at least one stable solution of (1. 3). Considering $Y \subset R[-, +]$, we get to the completion of the proof.   Q.E.D.

It may not be immediately clear that the sufficient conditions given in the above theorem are really possible conditions. So we next make a concrete description of a domain with which above conditions can be realized.

(P)   1) D is a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary.

2) D_1 and D_2 are subdomains of D in each of which Poincaré's second inequality holds (such as convex domains, domains with piecewise smooth boundaries, and the like).

3)   There exists a connected component (or a union of connected components) of $D \cap \left\{ x\,;\, -\dfrac{l}{2} \leq x_1 \leq \dfrac{l}{2} \right\}$, denoted by D_3, such that $D \backslash D_3$ is divided into disjoint open sets O_1 and O_2 containing D_1 and D_2 respectively and satisfying $\partial D_3 \cap \partial O_1 \subset \left\{ x\,;\, x_1 = -\dfrac{l}{2} \right\}$ and $\partial D_3 \cap \partial O_2 \subset \left\{ x\,;\, x_1 = \dfrac{l}{2} \right\}$ (see Figure).

4)   The $(n-1)$-dimensional measure of the intersection of D_3 and the hyperplane $\{x\,;\, x_1 = \xi\}$ does not exceed S for $-\dfrac{l}{2} \leq \xi \leq \dfrac{l}{2}$.
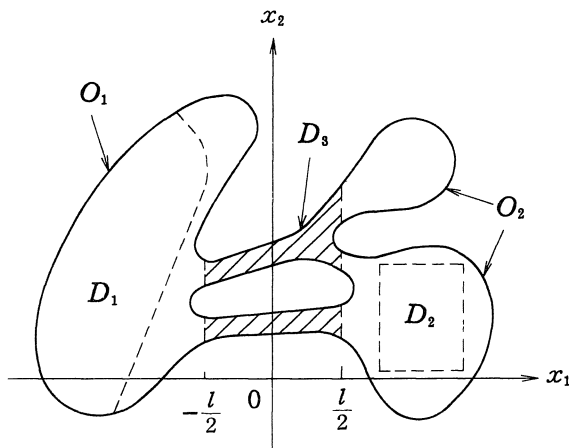
Figure   Example of domain D for $n=2$.

In terms of the above notation we can derive from Theorem 6.2 some more simplified sufficient conditions:

Corollary 6.3.   *Let* **(P)** *hold and let* $f(u) = kg(u)$ *satisfy* (6.1). *Suppose*

$$\left\{\frac{(b-a)^2}{2l} + kG(b)l\right\} S \leq \varepsilon_0 - k\{G(b) - G(a)\}\mu(O_1),$$

where ε_0 *is as in Lemma* 6.1. *Then* (1.3) *has at least one stable solution belonging to* $R[-, +]$.

Proof.   To apply the previous theorem, we must show the existence of such a function $w_0 \in R[-, +]$ as to satisfy (6.6).   Let $\tilde{w}$ be a function on D such that

$$\tilde{w} = \begin{cases} a & (x \in O_1) \\ \dfrac{a+b}{2} + \dfrac{(b-a)x_1}{l} & (x \in D_3) \\ b & (x \in O_2). \end{cases}$$

Obviously $\tilde{w}$ belongs to $H^1(D)$ and satisfies

$$\int_{D_1} \tilde{w}(x)\,dx < 0, \quad \int_{D_2} \tilde{w}(x)\,dx > 0.$$

In addition,

$$J(\widetilde{w}) = -kG(a)\mu(O_1) - kG(b)\{\mu(D_3) + \mu(O_2)\}$$

$$+ \int_{D_3} \left\{\frac{1}{2}|\nabla\widetilde{w}|^2 - kG(\widetilde{w}) + kG(b)\right\}dx$$

$$< -kG(a)\mu(O_1) - kG(b)\{\mu(D_3) + \mu(O_2)\}$$

$$+ \frac{(b-a)^2}{2l}S + kG(b)lS$$

$$\leq \varepsilon_0 - kG(b)\mu(D).$$

Therefore, as $C^1(\overline{D}) \cap C^2(D)$ is dense in $H^1(D)$, there exists a function $w_0 \in R[-,+]$ satisfying (6.6). Hence, by Theorem 6.2, $R[-,+]$ contains at least one stable solution of (1.3). Q.E.D.

Remark 6.4. From the above corollary it immediately follows that given any function f satisfying (6.1) there exists a domain D such that (1.3) has a non-constant stable solution. To see this, put $\mu(D_1) = \mu(O_1) \leq \mu(D_2)$ and $k \leq \lambda_2(D_i)$ $(i=1,2)$, and then take a sufficiently small S.

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Note added after submission: After completing this work, the author got acquainted with the following paper: Casten, R. G., and Holland, C. J., Instability results for reaction diffusion equations with Neumann boundary conditions, *J. Differential Equations* **27** (1978), 266-273.

It contains the same results as in our Theorem 5.1. Further it is shown that any nonconstant solution of (1. 3) is unstable if $f(u)$ is convex or concave. But the problem of finding f and D with which there exists a nonconstant stable solution of (1. 3) is still unsolved there, the answer to which can be found in our Theorem 6. 2 and Corollary 6. 3.