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Cauchy Problem for Non-Strictly Hyperbolic Systems

By

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Introduction

We consider the Cauchy problem for non-strictly hyperbolic systems with characteristic roots of constant multiplicity.

We shall indicate some sufficient condition in order that the Cauchy problem is well posed in C^∞ class. In particular we shall give a necessary and sufficient condition for first order hyperbolic systems.

We introduce the following notation,

$$x=(x_0, x')=(x_0, x_1, \cdots, x_n)\in \boldsymbol{R}^{n+1},$$

$$\xi=(\xi_0, \xi')=(\xi_0, \xi_1, \cdots, \xi_n)\in \boldsymbol{R}^{n+1},$$

$$D=(D_0, D')=(D_0, D_1, \cdots, D_n),$$

$$D_j=-i\frac{\partial}{\partial x_j}.$$

Let G be an open domain in $\boldsymbol{R}^{n+1}$. Let

$$P(x, D)u=\sum_{|\alpha|\leq m} A_\alpha(x)D^\alpha u,$$

where $u={}^t(u_1, \cdots, u_N)$, $A_\alpha(x)$ is a $N\times N$-matrix of elements in $C^\infty(G)$,

$$P(x, \xi)=\sum_{|\alpha|\leq m} A_\alpha(x)\xi^\alpha,\quad P_s(x, \xi)=\sum_{|\alpha|=s} A_\alpha(x)\xi^\alpha,$$

$$\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_n),\quad |\alpha|=\alpha_0+\cdots+\alpha_n,$$

$$D^\alpha=D_0^{\alpha_0}\cdots D_n^{\alpha_n},\quad \xi^\alpha=\xi_0^{\alpha_0}\cdots\xi_n^{\alpha_n}.$$

Let $\hat{x}$ be in G and $G(\hat{x})$ a neighborhood of $\hat{x}$.

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Consider the Cauchy problem

$$\text{(1)} \qquad \begin{cases} Pu=f & \text{in} \quad G(\hat{x}) \cap \{x_0 > \hat{x}_0\}, \\ D_0^j u = g_j & \text{on} \quad G(\hat{x}) \cap \{x_0 = \hat{x}_0\}, \quad j=0, \cdots, m-1, \end{cases}$$

where f and g_j are vector valued functions.

Definition 1. *The Cauchy problem* (1) *for* P *is said to be well posed at* $\hat{x} \in G$, *if the following conditions hold*:

(*E*) *There exists* $G(\hat{x}) \subset G$, *a neighborhood of* $\hat{x}$, *such that for any* $f(x) \in C^\infty(G(\hat{x}))$ *and* $g_j \in C^\infty(G(\hat{x}) \cap \{x_0 = \hat{x}_0\})$, *there is a function* $u(x) \in C^\infty(G(\hat{x}))$ *satisfying* (1).

(*U*) *If for any* $G(\hat{x}) \subset G$, *a neighborhood of* $\hat{x}$, *there exists* $\tilde{G}(\hat{x}) \subset G(\hat{x})$, *a neighborhood of* $\hat{x}$ *such that if* $u \in C^\infty(G(\hat{x}))$ *satisfies* (1) *and* $Pu=0$ *in* $\tilde{G}(\hat{x}) \cap \{x_0 > \hat{x}_0\}$ *and* $\operatorname{supp} u \subset \{x_0 > \hat{x}_0\}$, *then* $u=0$ *in* $\tilde{G}(\hat{x}) \cap \{x_0 > \hat{x}_0\}$.

If for any $\hat{x} \in G$, the Cauchy problem (1) for P is well posed at $\hat{x}$, then P is said to be well posed in G.

We say that P is a hyperbolic system of constant multiplicity, if it holds

$$\text{(2)} \qquad \det P_m(x, \xi) = \prod_{l=1}^{r} (\xi_0 + \lambda^{(l)}(x, \xi'))^{\nu_l},$$

here ν_l is an integer, and $\lambda^{(l)}$ are real valued functions in $C^\infty(G \times \{\boldsymbol{R}^n - 0\})$, $\sum_{l=1}^{r} \nu_l = mN$ and $\lambda^{(l)} \neq \lambda^{(j)}$ for $l \neq j$.

We start with the Cauchy problem for the hyperbolic systems with a diagonal principal part. We say that P is a hyperbolic system with diagonal principal part of constant multiplicity, if the principal part of P has the form,

$$P_m(x, \xi) = \prod_{l=1}^{r} (\xi_0 + \lambda^{(l)}(x, \xi'))^{\nu_l} I,$$

where I is the identity matrix and $\lambda^{(l)}(x, \xi')$ are real valued functions in $C^\infty(G \times \{\boldsymbol{R}^n - 0\})$, $\sum_{l=1}^{r} \nu_l = mN$.

Denote the phase function of P by $\varphi^{(l)}(x)$, that is

$$\varphi^{(l)}_{x_0} + \lambda^{(l)}(x, \varphi^{(l)}_{x'}) = 0, \quad \varphi^{(l)}_{x'} \neq 0.$$

Definition 2. *Let P be a hyperbolic system with diagonal principal part of constant multiplicity. Let P_t^s be the (s, t)-element of P. It is said that P satisfies the Levi's condition in G if there exist integers $n_s^{(l)}$, $l=1, \cdots, r$, $s=1, \cdots, N$ such that for any phase function $\varphi^{(l)}(x)$ and for any scalar function $w \in C_0^\infty(G)$*

$$(L) \qquad e^{-i\rho\varphi^{(l)}} P_t^s (e^{i\rho\varphi^{(l)}} w) = 0(\rho^{m-\nu+n_t{}^{(l)}-n_s{}^{(l)}}), \quad (\rho \to \infty),$$

for $s, t=1, \cdots, N$, $l=1, \cdots, r$.

Remark. This condition was suggested by Leray-Ohya [5]. When $N=1$, this is the usual Levi's condition (c.f. [2], [6]). If we can choose $n_t^{(l)} = n_s^{(l)}$, our condition is same one as Gourdin [3] and Vaillant-Bersin [9].

Theorem 1. *Let P be a hyperbolic system with diagonal principal part of constant multiplicity. If P satisfies the Levi's condition (L) in G then P is well posed in G.*

Remark. The condition (L) is not a necessary condition in order that P is well posed in G. For example

$$P = \begin{pmatrix} D_0^2 & 0 \\ 0 & D_0^2 \end{pmatrix} + \begin{pmatrix} D_1 & D_1 \\ -D_1 & -D_1 \end{pmatrix}$$

is well posed in $\mathbb{R}^2$. But we can never choose integers satisfying (L). We shall investigate the necessary condition in the section 3 (Theorem 3. 2).

We next consider a hyperbolic system of constant multiplicity. We say that Q is a cofactor system of P, if the principal symbol of Q is the cofactor matrix of $P_m(x, \xi)$. Then if P is a hyperbolic system of constant multiplicity, PQ and QP are both hyperbolic systems with diagonal principal part of constant multiplicity.

We obtain the following theorem as a corollary of Theorem 1.

Theorem 2. *Let P be a hyperbolic system of constant multiplicity. If there exist two cofactor systems Q, R of P such that PQ*

and RP satisfy the Levi's condition in G, then P is well posed in G.

Example (Petkov). Let

$$P=ID_0+\begin{pmatrix}0 & 1 & 0\\ & 0 & 0\\ & & 0\end{pmatrix}D_1+\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}.$$

If $b_{21}(x)=b_{31}(x)=0$ or $b_{21}(x)=b_{23}(x)=0$ is valid, then P is well posed. Set

$$Q=ID_0^2-\begin{pmatrix}0 & 1 & 0\\ & 0 & 0\\ 0 & & 0\end{pmatrix}D_0D_1\,.$$

Then PQ satisfies the Levi's condition (L), when we choose

$$(n_1, n_2, n_3)=(0, 1, 1)\,, \quad \text{if} \quad b_{21}=b_{31}=0$$
$$=(0, 1, 0)\,, \quad \text{if} \quad b_{21}=b_{23}=0\,.$$

We shall examine more precisely a first order hyperbolic system of constant multiplicity, that is, $m=1$,

$$P(x, D)=ID_0+\sum_{j=1}^{n}A_j(x)D_j+B(x)\,,$$

where $A_j(x)$ and $B(x)$ are $N\times N$ matrices of elements in $C^\infty(G)$. Moreover we assume

$$\text{(R)} \qquad \operatorname{rank}(\lambda^{(l)}(x, \xi')I-\sum_{j=1}^{n}A_j(x)\xi_j)=N-1,\ l=1, \cdots, r\,.$$

Then we can construct a pseudo-differential operator $N(x, D')$ of order zero with respect to D', which transforms microlocally P into $\tilde{P}$ such that

$$P(x, D)N(x, D')=N(x, D')\tilde{P}(x, D)\,,$$

where

$$\tilde{P}(x, D)=\begin{pmatrix}\tilde{P}^{(1)}(x, D) & & 0\\ & \ddots & \\ 0 & & \tilde{P}^{(r)}(x, D)\end{pmatrix},$$

$$\tilde{P}^{(l)}(x, D) = I(D_0 + \lambda^{(l)}(x, D')) + \begin{pmatrix} 0 & |D'| & & 0 \\ & & \ddots & \\ 0 & & & |D'| \\ \gamma_1^{(l)}(x, D') & \cdots & & \gamma_{\nu_l}^{(l)}(x, D') \end{pmatrix},$$

$$l = 1, \cdots, r,$$

$\gamma_k^{(l)}(x, D')$ is a pseudo-differential operator of order zero with respect to D' (c.f. Proposition 2. 2).

Then we obtain,

Theorem 3. *Let P be a first order hyperbolic system of constant multiplicity. Assume that the condition (R) is valid. Then P is well posed in G, if and only if*

$$(L_1) \qquad \text{order } \gamma_k^{(l)}(x, D') \leq k - \nu_l,$$

for $k = 1, \cdots, \nu_l - 1$, $l = 1, \cdots, r$.

Remark. We note that our theorem holds, if we assume instead of (R),

$$\operatorname{rank}\,(\lambda^{(l)}(x, \xi')I - \sum A_j(x)\xi_j) = \begin{cases} N-1 \\ \text{or} \\ N-\nu_l. \end{cases}$$

In [7] Petkov has given the Levi's condition as follows; P satisfies the condition (L_2), if for any $w(x) \in C_0^\infty(G)$ and for any phase function $\varphi^{(l)}(x)$, there exist vector valued functions $V_k^{(l)}(x, \varphi^{(l)}, w)$, $(k = 1, \cdots, \nu_l - 1)$, such that

$$(L_2) \qquad e^{-i\rho\varphi^{(l)}} P[e^{i\rho\varphi^{(l)}}\{w(x)N_1^{(l)}(x, \varphi_{x'}^{(l)}) + \sum_{k=1}^{\nu_l - 1} \rho^{-k} V_k(x, \varphi^{(l)}, w)\}]$$
$$= 0(\rho^{1-\nu_l}), \quad (\rho \to \infty),$$

where $N_1^{(l)}(x, \xi')$ is a right null vector of the matrix $(\lambda^{(l)}(x, \xi')I - \sum A_j(x)\xi_j)$.

Theorem 4. *Let P be a first order hyperbolic system of constant multiplicity. Assume that (R) is valid. Then our condition (L_1) is equivalent to (L_2) given by Petkov.*

In [8] Petkov constructs the parametrix of P under the condition (L_2). When $\nu_l=2$, Yamahara has derived the condition (L_1) in [10].

§ 1. Systems with Dagonal Principal Part

We shall construct a parametrix of the Cauchy problem for P, a hyperbolic system with diagonal principal part of constant multiplicity satisfying the condition (L). It follows from the existence of the parametrix that the Cauchy problem for P is well posed. For, if P satisfies (L), then $P^{(*)}$ the adjoint operator of P does so and since the condition (L) is invariant under a transform of coordinate variables, a solution of the Cauchy problem for P has a finite propagation speed and therefore the Cauchy problem for P has the local uniqueness (c.f. [1]).

Denote by $L^m(G)$ the class of pseudo-differential operators of which symbol is developed asymptotically,

$$a(x, \xi)=\sum_{j=0}^{\infty} a_j(x, \xi),$$

where $a_j(x, \xi)$ are homogeneous degree $m-j$ in ξ and polynomials with respect to ξ_0.

Let $a(x, D)$ be in $L^m(G)$ and w in $C^\infty(G)$. Then we can develop asymptotically

$$e^{-i\rho\varphi}a(x, D)(e^{i\rho\varphi}w)=\sum_{j=0}^{\infty}\rho^{m-j}\sigma_j(\varphi, a),$$

where $\sigma_j(\varphi, a)$ are the differential operators of which principal part is given by

$$\sum_{|\alpha|=j}\left(\left(\frac{\partial}{\partial\xi}\right)^{\alpha}a_0\right)(x, \varphi_x)D^\alpha/\alpha!,$$

in particular, we have

$$\sigma_0(\varphi, a)=a_0(x, \varphi_x)$$

$$\sigma_1(\varphi, a)=\sum_{j=0}^{n}\left(\frac{\partial}{\partial\xi_j}a_0\right)(x, \varphi_x)D_j+a_1(x, \varphi_x)$$

$$+i\sum_{|\alpha|=2}\left(\left(\frac{\partial}{\partial\xi}\right)^{\alpha}a_0\right)(x, \varphi_x)D^\alpha/\alpha!.$$

Let us consider P, a hyperbolic system with diagonal principal part,

$$a(x, D)=\prod_{l=1}^{r} q^{(l)}(x, D)^{\nu_l},$$

where $q^{(l)}(x, \xi)=\xi_0+\lambda^{(l)}(x, \xi')$, $\sum_{l=1}^{r} \nu_l=m$. We note that $a(x, D)$ satisfies,

$$e^{-i\rho\varphi^{(l)}} a(x, D)(e^{i\rho\varphi^{(l)}} w)=0(\rho^{m-\nu_l}) \tag{1.1}$$

for $l=1, \cdots, r$, where $\varphi^{(l)}(x)$ is a phase function corresponding to $q^{(l)}$. Let $b_t^s(x, D)$ be in $L^{m-1}(G)$, $s, t=1, \cdots, N$.

Consider the following Cauchy problem;

$$\begin{gathered} a(x, D) u^s(x)+\sum_{t=1}^{N} b_t^s(x, D) u^t(x)=0, \quad s=1, \cdots, N, \\ D_0^j u^s|_{x_0=0}=e^{i\langle x', \xi'\rangle} g_j^s(x, \xi'), \quad j=0,1, \cdots, m-1, \end{gathered} \tag{1.2}$$

where

$$g_j^s(x, \xi')=\sum_{p=0}^{\infty} g_{jp}^s(x, \xi')|\xi'|^{\gamma_0-p},$$

$g_{jp}^s(x, \xi')$ are homogeneous degree zero in ξ'. Let us choose a phase function $\varphi^{(l)}(x, \xi')$ such that

$$\begin{gathered} q^{(l)}(x, \varphi_{x'}^{(l)})=0 \\ \varphi^{(l)}|_{x_0=0}=\langle x', \xi'\rangle/|\xi'|. \end{gathered} \tag{1.3}$$

Then we have

Theorem 1.1. *Let $b_s^t(x, D)$ be in $L^{m-1}(G)$. Assume that $b_s^t(x, D)$ satisfies the condition (L), that is, for any $w \in C^{\infty}(G)$,*

$$e^{-i\rho\varphi^{(l)}} b_t^s(x, D)(e^{i\rho\varphi^{(l)}} w)=0(\rho^{m-\nu_l+n_t(l)-n_s(l)}), \tag{1.4}$$

then the Cauchy problem (1.2) has an asymptotic solution (u^s), $s=1, \cdots, N$ such that,

$$u^s(x, \xi')=\sum_{l=1}^{r} \sum_{j=0}^{\infty} e^{i\rho\varphi^{(l)}}(x, \xi') \rho^{m_0-\nu_l-n_s(l)-j} u_{lj}^s(x, \xi'), \tag{1.5}$$

where $\rho=|\xi'|$, $m_0=\gamma_0+\max_{l,t} n_t^{(l)}$, and $u_{lj}^s(x, \xi')$ are homogeneous degree zero in ξ'.

Proof. We can write by virtue of (1.1) and (1.4),

$$e^{-i\rho\varphi^{(l)}}a(x, D)(e^{i\rho\varphi^{(l)}}w)$$

$$=\sum_{p=0}^{\infty}\sigma_p(\varphi^{(l)}, a)\rho^{m-p}w$$

$$=\sum_{p=0}^{\infty}\sigma_{p+\nu_l}(\varphi^{(l)}, a)\rho^{m-\nu_l-p}w\ ,$$

and

$$e^{-i\rho\varphi^{(l)}}b_t^s(x, D)(e^{i\rho\varphi^{(l)}}w)$$

$$=\sum_{p=0}^{\infty}\sigma_p(\varphi^{(l)}, b_t^s)\rho^{m-1-p}w$$

$$=\sum_{p=0}^{\infty}\sigma_{p+\nu_l-1+n_s(l)-n_t(l)}(\varphi^{(l)}, b_t^s)\rho^{m-\nu_l+n_t(l)-n_s(l)-p}w\ .$$

Inserting $u^s(t, \xi')$ of (1.5) into (1.2), we obtain

$$\sum_{l=1}^{r}\sum_{j,p=0}^{\infty}e^{i\rho\varphi(l)}\rho^{m+m_0-n_s(l)-p-j}\{\sigma_{p+\nu_l}(\varphi^{(l)}, a)u_{lj}^s$$

$$+\sum_{t=1}^{N}\sigma_{p+\nu_l-1+n_s(l)-n_t(l)}(\varphi^{(l)}, b_t^s)u_{lj}^t\}=0\ ,$$

for $s=1, \cdots, N$. Hence we have

$$\sum_{j+p=k}\{\sigma_{p+\nu_l}(\varphi^{(l)}, a)u_{lj}^s+\sum_{t=1}^{N}\sigma_{p+\nu_l-1+n_s(l)-n_t(l)}(\varphi^{(l)}, b_t^s)u_{lj}^t\}=0$$

for $l=1, \cdots, r$, $s=1, \cdots, N$, $k=0, 1, \cdots$, that is,

$$\sigma_{\nu_l}(\varphi^{(l)}, a)u_{lk}^s+\sum_{t=1}^{N}\sigma_{\nu_l-1+n_s(l)-n_t(l)}(\varphi^{(l)}, b_t^s)u_{lk}^t=f_{lk}^s\ , \tag{1.6}$$

where

$$f_{lk}^s=-\sum_{\substack{j+p=k\\ j<k}}\sigma_{p+\nu_l}(\varphi^{(l)}, a)u_{lj}^s+\sum_{t=1}^{N}\sigma_{p+\nu_l-1+n_s(l)-n_t(l)}(\varphi^{(l)}, b_t^s)u_{lj}^t\ .$$

From the initial condition of (1.2) it follows that

$$D_0^q u^s|_{x_0=0}=\sum_{l=1}^{r}e^{i\langle x', \xi'\rangle}\sum_{j,p=0}^{\infty}\sigma_p(\varphi^{(l)}, D_0^q)\rho^{q+m_0-n_s(l)-p-j}u_{lj}^s|_{x_0=0}$$

$$=e^{i\langle x', \xi'\rangle}\sum_{k=0}^{\infty}g_{qk}^s(x, \xi')\rho^{\gamma_0-k}\ .$$

Put $m_s^{(l)}=m_0-\gamma_0-n_s^{(l)}$. Then we have

$$\sum_{l=1}^{r} \sum_{p+j=k+m_s(l)+q} \sigma_p(\varphi^{(l)}, D_0^q)\, u_{lj}^s|_{x_0=0} = g_{qk}^s .$$

Noting that the principal part of $\sigma_q(\varphi^{(l)}, D_0^q)$ is $\binom{q}{p}(\varphi_{x_0})^{q-p}D_0^p$, we have

$$\sum_{l=1}^{r} \sum_{p=0}^{q} \left\{\binom{q}{p}(\varphi_{x_0}^{(l)})^{q-p} D_0^p u_{lk+m_s(l)-p}^s + B_{qp}^{(l)}(x, D_0)\, u_{lk+m_s(l)-p}^s\right\}_{x_0=0} \tag{1.7}$$

$$= g_{qk-q}^s ,$$

here $B_{qp}^{(l)}(x, D_0)$ are of order $p-1$. Since $\varphi_{x_0}^{(l)}|_{x_0=0} = -\lambda^{(l)}(0, x, \xi'/|\xi'|)$, $l=1, \cdots, r$, are distinct, the determinant of Van der Monde $\left\{\binom{q}{p}(\varphi_{x_0}^{(l)})^{q-p}, q=0, 1, \cdots, m-1, p=0, 1, \cdots, \nu_l-1, l=1, \cdots, r,\right\}$ is not zero. Hence we can solve (1.7) with respect to $\{D_0^p u_{lk+m_s(l)-p}^s\}$, $p=0, 1, \cdots, \nu_l-1$, $l=1, \cdots, r$, for any k, where $u_{lk}^s=0$ if $k<0$. Therefore we can solve (1.6) and (1.7) successively by use of the following lemma. For, we have

$$\sigma_{\nu_l}(\varphi^{(l)}, a) = \sum_{k=0}^{\nu_l} H_l(x, D)^{\nu_l-k} a_k^{(l)}(x), \quad a_0^{(l)}(x) \neq 0 ,$$

and

$$\sigma_{\nu_l-1+n_s(l)-n_t(l)}(\varphi^{(l)}, b_t^s) = \sum_{k=0}^{\nu_l-1+n_s(l)-n_t(l)} B_{tk}^s(x) H_l(x, D)^k ,$$

where $H_l(x, D) = D_0 + \sum_{j=1}^{n} \lambda_{\xi_j}^{(l)}(x, \varphi_{x'}^{(l)}) D_j$.

Lemma 1.2. (*c.f.* [1]). *Let $b(x, D)$ be in $L^{m-1}(G)$ and $\varphi^{(l)}$ be a phase function satisfying* (1.3). *Assume that for any phase function $\varphi^{(l)}$ and for any $w \in C_0(G)$, $\rho = |\xi'|$,*

$$e^{-i\rho\varphi^{(l)}} b(x, D)\, (e^{i\rho\varphi^{(l)}} w) = 0(\rho^{m-\nu}),$$

then we obtain

$$\sigma_{\nu-1}(\varphi^{(l)}, b) = \sum_{k=0}^{\nu-1} b_{\nu k}^{(l)}(x) H_l(x, D)^k , \tag{1.8}$$

where

$$H_l(x, D) = D_0 + \sum_{j=1}^{n} \lambda_{\xi_j}^{(l)}(x, \varphi_{x'}^{(l)}) D_j .$$

Proof. We transform coodinate variables $x' = x'(t, z)$, $x_0 = t$, such that

$$\varphi^{(l)}(t, x'(t, z'), \xi') = \langle z', \xi' \rangle / |\xi'|$$

that is, $x'(t, z')$ is a solution;

$$\frac{d}{dt} x'(t, z') = \lambda^{(l)}_{\xi'}(t, x'(t, z'), \varphi^{(l)}_{x'})$$

$$x'(0, z') = z' .$$

Then

$$(1.9) \qquad D_t(w(t, x'(t, z'))) = (D_t + \sum \lambda^{(l)}_{\xi_j} D_j) w |_{x=(t\ x'(t,z'))}$$

and

$$e^{-i\rho\varphi^{(l)}} b(x, D) (e^{i\rho\varphi^{(l)}} w) |_{x=(t,x'(t,z'))}$$

$$= e^{-i\langle z', \xi' \rangle} \tilde{b}(t, z, D_t, D_z) (e^{i\langle z, \xi' \rangle} w(t, x'(t, z'))) = 0(\rho^{m-\nu}), \quad \rho = |\xi'|.$$

Hence we have

$$\sigma_{\nu-1}(\varphi^{(l)}, b) |_{x=(t,x'(t,z'))} = \sigma_{\nu-1}(\langle z', \xi'/|\xi'| \rangle, \tilde{b})$$

$$= \sum_{k=0}^{\nu-1} \tilde{b}^{(l)}_{\nu k}(t, z') D_t^k ,$$

which implies (1. 8) with (1. 9).

Thus we have proved Theorem 1. 1 which implies Theorem 1 and Theorem 2 in the introduction.

§ 2. First Order Systems

Consider the first order system,

$$P = ID_0 + \sum_{j=1}^{n} A_j(x) D_j + B(x),$$

here $A_j(x)$ and $B(x)$ are $N \times N$ matrices of $C^\infty(G)$-elements. Set

$$A(x, \xi') = \sum_{j=1}^{n} A_j(x) \xi_j .$$

and

$$M^{(l)}(x, \xi') = A(x, \xi') - \lambda^{(l)}(x, \xi') I .$$

We suppose that

$$(2.1)\qquad \begin{cases} \det(\xi_0 + A(x,\xi')) = \prod_{l=1}^{r} (\xi_0 + \lambda^{(l)}(x,\xi'))^{\nu_l}, \\ (\nu_l;\ \text{positive integers}), \\ \operatorname{rank} M^{(l)}(x,\xi') = N-1, \quad (l=1,\cdots,N). \end{cases}$$

Lemma 2.1 ([4]). *Under the assumption* (2.1), *for any* $(\hat{x}, \hat{\xi}')$ $\in G\times\{R^n - 0\}$, *there exists a conic neighborhood* $V(\hat{x}, \xi')$ *and a matrix* $N_0(x,\xi') \in C^\infty(V(\hat{x},\xi'))$ *such that*

$$A(x,\xi')N_0(x,\xi') = N_0(x,\xi')A_0(x,\xi'),$$

$$A_0(x,\xi') = \begin{pmatrix} \lambda^{(l)}(x,\xi')I + C^{(l)}|\xi'| & & 0 \\ & \ddots & \\ 0 & & \lambda^{(l)}(x,\xi')I + C^{(l)}|\xi'| \end{pmatrix},$$

$$C^{(l)} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & 0 \end{pmatrix},$$

and $N_0(x,\xi')$ *is homogeneous of degree zero and its determinant does not vanish for* $(x,\xi') \in V(\hat{x},\xi')$.

Proposition 2.2. *Under the assumption* (2.1), *for any* $(\hat{x}, \hat{\xi}')$ $\in G\times\{R^{n+1}\backslash 0\}$, *there exists a pseudo-differential operator* $N(x, D')$ *of order zero such that*

$$P(x,D)N(x,D') = N(x,D')\tilde{P}(x,D), \ (\text{mod } L^{-\infty}(G)),$$

$$(2.2)\qquad \tilde{P}(x,D) = ID_0 + \begin{pmatrix} \Lambda^{(1)}(x,D') & & \\ 0 & \ddots & 0 \\ & & \Lambda^{(r)}(x,D') \end{pmatrix},$$

symbol of $\Lambda^{(l)} = \lambda^{(l)}(x,\xi')I + C^{(l)}|\xi'|$

$$+ \begin{pmatrix} & 0 & \\ \gamma_1^{(l)}(x,\xi') & \cdots & \gamma_{\nu_l}^{(l)}(x,\xi') \end{pmatrix}$$

for $(x,\xi') \in V(\hat{x},\hat{\xi}')$, *a conic neighborhood of* $(\hat{x},\hat{\xi}')$, *here* $\gamma_k^{(l)}(x,\xi')$ $\in C^\infty(V(\hat{x},\xi'))$ *and* $C^{(l)}$ *is* $\nu_l\times\nu_l$*-Jordan's matrix of rank* $\nu_l - 1$.

Proof. We shall seek $N(x,D')$ such that,

$$N(x, D')=\sum_{j=0}^{\infty} N_j(x, D')$$

here $N_j(x, \xi')$, the symbol of $N_j(x, D')$ is homogeneous degree $-j$ in ξ'. Then we can write the symbol of PN and NP,

$$(PN)(x, \xi')=\sum_{j=0}^{\infty}(\xi_0+A(x, \xi'))N_j(x, \xi')$$
$$+P(x, D)N_{j-1}(x, \xi')$$

and

$$(NP)(x, \xi)=\sum_{\alpha}\left(\frac{\partial}{\partial\xi'}\right)^{\alpha} N(x, \xi')D^{\alpha}\tilde{P}(x, \xi)/\alpha!$$
$$=N(x, \xi')\xi_0+\sum_{j,\alpha}\left(\frac{\partial}{\partial\xi}\right)^{\alpha} N_j(x, \xi')D^{\alpha}\Lambda_k(x, \xi')/\alpha!\,.$$

Hence we have

$$(2.3)\qquad A(x, \xi')N_p(x, \xi')+P(x, D)N_{p-1}(x, \xi')\Lambda_0(x, \xi'),$$
$$=\sum_{j+k+|\alpha|=p}\left(\frac{\partial}{\partial\xi}\right)^{\alpha} N_j(x, \xi')D_x^{\alpha}\Lambda_k(x, \xi')/\alpha!$$

for $p=0, 1, \cdots$. For $p=0$, we have

$$A(x, \xi')N_0(x, \xi')=N_0(x, \xi')\Lambda_0(x, \xi)$$

where $N_0(x, \xi')$ is given in Lemma 2.1. Set

$$F_p(x, \xi')=\sum_{j+|\alpha|+k=p}\left(\frac{\partial}{\partial\xi}\right)^{\alpha} N_j(x, \xi')D_x^{\alpha}\Lambda_k(x, \xi')/\alpha!$$
$$-P(x, D)N_{p-1}(x, \xi').$$

Then for $p\geqq 1$, we have from (2.3),

$$(2.4)\qquad A(x, \xi')N_p(x, \xi')-N_p(x, \xi')\Lambda_0(x, \xi')$$
$$=F_p(x, \xi')+N_0(x, \xi')\Lambda_p(x, \xi').$$

Set

$$\tilde{N}_p=N_0^{-1}N_p\,,\quad \tilde{F}_p=N_0^{-1}F_p\,,$$

and

$$\tilde{N}_p=\begin{pmatrix} N_p^{(11)} & \cdots & N_p^{(1r)} \\ & \cdots & \\ & \cdots & \\ N_p^{(r1)} & \cdots & N_p^{(rr)} \end{pmatrix}, \quad \tilde{F}_p=\begin{pmatrix} F_p^{(11)} & \cdots & N_p^{(1r)} \\ & \cdots & \\ & \cdots & \\ F_p^{(r1)} & \cdots & F_p^{(rr)} \end{pmatrix}$$

$$\Lambda_p=\begin{pmatrix} \Lambda_p^{(1)} & & 0 \\ & \ddots & \\ 0 & & \Lambda_p^{(r)} \end{pmatrix}, \qquad \Lambda_p^{(l)}=\begin{pmatrix} 0 & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & 0 \\ \gamma_{p1}^{(l)} & \cdots & \gamma_{p\nu_l}^{(l)} \end{pmatrix}, \quad (p\geq 1),$$

where $N_p^{(ij)}$ and $F_p^{(ij)}$ are $\nu_i\times\nu_j$ matrices. Then we can write from (2. 4) for $p\geq 1$,

$$\Lambda_0^{(i)}N_p^{(ii)}-N_p^{(ii)}\Lambda_0^{(i)}=F_p^{(ii)}+\Lambda_p^{(i)} \tag{2.5}$$

$$\Lambda_0^{(i)}N_p^{(ij)}-N_p^{(ij)}\Lambda_0^{(j)}=F_p^{(ij)}, \quad (i\neq j), \tag{2.6}$$

where

$$\Lambda_0^{(i)}=\lambda^{(i)}I+C^{(i)}|\xi'|.$$

For (2. 6), we can solve $N_p^{(ij)}$ as follows

$$N_p^{(ij)}=\sum_{l=0}^{\nu_j-1}(\lambda^{(i)}-\lambda^{(j)}+C^{(i)}|\xi'|)^{-l-1}F_p^{(ij)}(C^{(j)}|\xi'|)^l$$

for $i\neq j$, $p\geq 1$. We can also solve (2. 5), if we choose $\Lambda_p^{(i)}$ suitably.

Lemma 2. 3. *Let N and F be $m\times m$ matrices and let C be a Jordan's matrix of the form,*

$$C=\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & 0 & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

We consider the following linear equations,

$$CN-NC=F. \tag{2.7}$$

Then we can seek for a solution N of (2. 7), *if and only if the elements $\{f_{i,j}\}$ of F satisfy,*

$$\sum_{l=0}^{k}f_{m-l,k+1-l}=0, \quad k=0,1,\cdots,m-1. \tag{2.8}$$

Proof. It follows from (2. 7) that the elements $\{n_{i,j}\}$ of N satisfy

$$n_{j+1,1}=f_{j,1}\ ,\quad j=1,\cdots,m-1\ ,$$

(2.9) $$n_{j+1,1}=f_{j,1}\ ,\quad j=1,\cdots,m-1\ ,$$

(2.10) $$n_{j+1,k}-n_{j,k-1}=f_{j,k}\ ,\quad k=2,\cdots,m,\ j=1,\cdots,m-1\ ,$$

(2.11) $$-n_{m,k}=f_{m,k+1}\ ,\quad k=1,\cdots,m-1\ ,$$

$$0=f_{m,1}\ .$$

Hence we have from (2.10),

$$n_{j+1,k}=\sum_{l=0}^{\min(j,k)-1}f_{j-l,k-l}\ ,$$

$$j=1,\cdots,m-1\ ,\ k=2,\cdots,m\ .$$

In particular,

$$n_{m,k}=\sum_{l=1}^{k}f_{m-l,k+1-l}\ ,\quad k=2,\cdots,m\ .$$

On the other hand, $n_{m,k}$ satisfy (2.11). Hence we have the relation (2.7). We can choose $n_{1,k}$, $(k=1,\cdots,m)$ arbitrarily.

In order to apply this lemma to (2.5), we put

(2.12) $$\gamma_{p,k}^{(i)}=-\sum_{l=1}^{k}(F_p^{(ii)})_{\nu_i-k+l,l}\ ,$$

for $k=1,\cdots,\nu_i$, $p=1,2,\cdots$, where $(F_p^{(ii)})_{s,t}$ stands for the (s,t) element of $F_p^{(ii)}$. Then $F_p^{(ii)}+\Lambda_p^{(i)}$ satisfies (2.7). Hence we can solve (2.5).

Proposition 2.4. *Let $\tilde{P}$ be the operator given by* (2.2). *We assume that*

(L_1) order $\gamma_k^{(l)}(x,D')\leq k-\nu_l\ ,\quad k=1,\cdots,\nu_l-1,\ l=1,\cdots,r\ .$

Then there exists Q, a cofactor system of P such that PQ satisfies the condition (L) *for* $(x,\varphi_{x'}^{(l)})\in V(\hat{x},\hat{\xi}')$.

Proof. Set

$$Q=\begin{pmatrix}Q^{(1)} & & 0\\ & \ddots & \\ 0 & & Q^{(r)}\end{pmatrix},$$

here

$$Q^{(l)}=\prod_{k\neq l} q^{(k)}(x, D)^{\nu_k}\sum_{k=1}^{\nu_l-1} q^{(l)}(x, D)^{\nu_l-k}(-C^{(l)}|D'|)^{k-1},$$

$$q^{(l)}(x, \xi)=\xi_0+\lambda^{(l)}(x, \xi'),$$

and set

$$n_s^{(l)}=s, \quad s=\{\sum_{k=1}^{l=1}\nu_k\}+1, \cdots, \sum_{k=1}^{l}\nu_k,$$

$$=0, \quad \text{otherwise}$$

Then noting that the condition (L_1) implies

$$e^{-i\rho\varphi^{(l)}}\sum_{k=1}^{\nu_l-1}\gamma_k^{(l)}(x, D')q^{(l)}(x, D)^{k-1}(-C^{(l)}|D'|)^{\nu_l-k}(e^{i\rho\varphi^{(l)}}w)=0(1),$$

for $(x, \varphi_{x'}^{(l)})\in V(\hat{x}, \hat{\xi}')$, we can verify easily that $\tilde{P}Q$ satisfies the condition (L).

Thus applying Theorem 1. 1, we can construct a parametrix of the Cauchy problem for P in some neighborhood $G(\hat{x})$ of $\hat{x}$, which implies the existence of the solution of the Cauchy problem (1) for P in $G(x)$, (c.f. [1]). Concerning with the local uniqueness, we must prove that the condition (L_1) is satisfied for $P^{(*)}$, the adjoint operator of P, and that (L_1) is invariant under the transform of coordinate variables. To do so, we shall prove that our condition (L_1) is equivalent to the condition (L_2), given by Petkov. In [7] he has proved that $P^{(*)}$ satisfies (L_2), if P does so, and that (L_2) is invariant under the transform of coordinate variables.

We need the following preliminary. The proof is easy.

Proposition 2. 5. *Let $T(x, D')$ be an elliptic operator in $L^0(G)$ and $S(x, D')$ be the inverse of $T(x, D')$, that is, $S(x, D')T(x, D')\equiv I$,* mod $L^{-\infty}(G)$. *If P satisfies (L_2), then SPT does so.*

It follows from this proposition that $\tilde{P}$ given by (2. 2) satisfies (L_2) blookwisely, that is, for any scalar function $f(x)\in C_0^\infty(G(\hat{x}))$ and for $(x, \varphi_{x'}^{(l)})\in V(\hat{x}, \hat{\xi}')$, there exist ν_l-vector valued functions $v_k^{(l)}(x, \varphi^{(l)}, f)\in C^\infty(G(\hat{x}))$ such that

$$(\tilde{\mathrm{L}}_2)\qquad e^{-i\rho\varphi^{(l)}}\tilde{P}^{(l)}[\{f(x)e_1^{(l)}+\sum_{k=1}^{\nu_l-1}v_k^{(l)}(x,\varphi^{(l)},f)\rho^{-k}\}e^{i\rho\varphi^{(l)}}]$$
$$=0(\rho^{-\nu_l+1}),\quad(\rho\to\infty),\ l=1,\cdots,r,$$

where $e_k^{(l)}$ is a ν_l-vector of which k th-component is 1 and otherwise zero.

Proposition 2. 6. *The condition* (L_1) *is equivalent to* $(\tilde{L}_2)$.

Proof. We can expand asymptotically

$$\gamma_k^{(l)}(x,\xi')=\sum_{p=1}^{\infty}\gamma_{kp}^{(l)}(x,\xi'),$$

where functions $\gamma_{kp}^{(l)}(x,\xi')$ are homogeneous degree $1-p$ in ξ'. Hence (L_1) is equivalent to

$$(\tilde{\mathrm{L}}_1)\qquad \gamma_{kp}^{(l)}(x,\xi')=0,\quad p=1,\cdots,\nu_l-k,\quad k=1,\cdots,\nu_l-1.$$

Therefore it suffices to prove that $(\tilde{\mathrm{L}}_1)$ is equivalent to $(\tilde{\mathrm{L}}_2)$. We write asymptotically

$$\tilde{P}^{(l)}=\sum_{j=0}^{\infty}\tilde{P}_j^{(l)}.$$

Then we have

$$e^{-\rho\varphi^{(l)}}\tilde{P}^{(l)}[e^{i\rho\varphi^{(l)}}\{f(x)e_1^{(l)}+\sum_{k=1}^{\nu_l-1}v_k^{(l)}\rho^{-k}\}]$$
$$=\sum_{p=0}^{\infty}\rho^{1-p}\{\sum_{j+k=p}\sigma_j(\varphi^{(l)},\tilde{P}_k^{(l)})f(x)e_1^{(l)}+\sum_{j+k+s=p}\sigma_j(\varphi^{(l)},\tilde{P}_k^{(l)})v_s^{(l)}\}.$$

Hence $(\tilde{\mathrm{L}}_2)$ is equivalent to

$$(2.13)_p\qquad \sum_{j+k=p}\sigma_j(\varphi^{(l)},\tilde{P}_k^{(l)})f(x)e_1^{(l)}+\sum_{j+k+s=p}\sigma_j(\varphi^{(l)},\tilde{P}_k^{(l)})v_s^{(l)}=0,$$

for $p=1,\cdots,\nu_l-1$. When $p=0$,

$$\sigma_0(\varphi^{(l)},\tilde{P}_0^{(l)})e_1^{(l)}f=C^{(l)}|\varphi_{x'}^{(l)}|e_1^{(l)}f=0.$$

Put $h^{(l)}(x)=|\varphi_{x'}^{(l)}|$. For $p=1$, we have

$$h^{(l)}C^{(l)}v_1^{(l)}+(\sigma_0(\varphi^{(l)},\tilde{P}_1^{(l)})+\sigma_1(\varphi^{(l)},\tilde{P}_0^{(l)}))e_1^{(l)}f=0.$$

Hence there exists $v_1^{(l)}$ if and only if

$${}^te_{\nu_l}^{(l)}\{\sigma_0(\varphi^{(l)},\tilde{P}_1^{(l)})+\sigma_1(\varphi^{(l)},\tilde{P}_0^{(l)}\}e_1^{(l)}f=\gamma_{11}^{(l)}(x,\varphi_{x'}^{(l)})f=0,$$

which implies

$$\gamma_{11}^{(l)}(x, \hat{\xi}') = 0 .$$

Then we can find $v_1^{(l)}$ of the form

$$v_1^{(l)} = -\frac{1}{h^{(l)}(x)} H_l(x, D) f(x) e_2^{(l)},$$

noting,

$$\sigma_1(\varphi^{(l)}, \tilde{P}_0^{(l)}) = H_l(x, D) + \frac{1}{h^{(l)}} C^{(l)} \langle \varphi_{x'}^{(l)}, D' \rangle ,$$

where $H_l(x, D) = D_0 + \sum_{j=1}^{n} \lambda_{\xi_j}^{(l)}(x, \varphi_{x'}^{(l)}) D_j$.

In general we shall prove our statement by induction. Assume that there exist $v_q^{(l)}$ satisfying $(2.13)_q$, $q=1, \cdots, p-1$, of the form

$$(2.14)_q \qquad v_q^{(l)} = \sum_{s=2}^{q+1} a_s^{(q)}(x, D) f(x) e_s^{(l)}, \quad q=1, \cdots, p-1 ,$$

here,

$$(2.15)_q \qquad a_{q+1}^{(q)}(x, D) = \left(-\frac{1}{h^{(l)}} H_l(x, D)\right)^q ,$$

if and only if

$$(2.16)_p \qquad \gamma_{sq}^{(l)}(x, \varphi_{x'}^{(l)}) = 0 \qquad \text{for} \quad s+q \leq p .$$

Then we shall prove that we can find $v_p^{(l)}$ satisfying $(2.13)_p$ of the form $(2.14)_{p+1}$, if and only if $(2.16)_{p+1}$ holds. We have from $(2.13)_p$ and $(2.14)_q$, $q \leq p-1$,

$$(2.17) \quad h^{(l)} C^{(l)} v_p^{(l)} + \sum_{\substack{j+k+q=p \\ q \leq p-1}} \sigma_j(\varphi^{(l)}, \tilde{P}_k^{(l)}) v_q^{(l)} + \sum_{j+k=p} \sigma_j(\varphi^{(l)}, \tilde{P}_k^{(l)}) e_1^{(l)} f$$

$$= h^{(l)} C^{(l)} v_p^{(l)} + \sum_{\substack{j+k+q=p \\ q \leq p-1}} \sum_{s=2}^{q+1} \sigma_j(\varphi^{(l)}, \tilde{P}_k^{(l)}) a_s^{(q)} f e_s^{(l)}$$

$$+ \sum_{j+k=p} \sigma_j(\varphi^{(l)}, \tilde{P}_k^{(l)}) f e_1^{(l)} = 0 .$$

Hence we can find $v_p^{(l)}$ if and only if

$$(2.18) \qquad {}^t e_{\nu_l}^{(l)} \{ \sum_{\substack{j+k+q=p \\ q \leq p-1}} \sigma_j(\varphi^{(l)}, \tilde{P}_k^{(l)}) \sum_{s=2}^{q+1} a_s^{(q)}(x, D) f e_s^{(l)}$$

$$+\sum_{j+k=p}\sigma_j(\varphi^{(l)},\tilde{P}_k^{(l)})fe_1^{(l)}\}=0\,.$$

It follows from $(2.16)_p$ that

$${}^te_{\nu_l}^{(l)}\sigma_j(\varphi^{(l)},\tilde{P}_k^{(l)})e_s^{(l)}=\sigma_j(\varphi^{(l)},\gamma_{sk}^{(l)})=0\,,\quad s+k\leq p\,,$$

for any j. Hence we have from (2.18)

$$\sum_{\substack{j+k+q=p\\ q\leq p-1}}\sigma_j(\varphi^{(l)},\gamma_{sk}^{(l)})\sum_{s+k=p+1}a_s^{(q)}(x,D)f+\sigma_0(\varphi^{(l)},\gamma_{1p}^{(l)})f$$

$$=\sum_{k+q=p}\sigma_0(\varphi^{(l)},\gamma_{q+1\,k}^{(l)})\,a_{q+1}^{(q)}(x,D)f+\sigma_0(\varphi^{(l)},\gamma_{1p}^{(l)})f=0\,.$$

Since $\{1,a_{q+1}^{(q)}(x,D),q=1,\cdots,p-1\}$ is linearly independent from $(2.15)_q$, $q\leq p-1$, we obtain $(2.16)_{p+1}$. Inserting $(2.14)_p$ into (2.17), we obtain by virtue of $(2.16)_{p+1}$,

$$(2.19)\qquad \sum_{j+q=p}\sigma_j(\varphi^{(l)},\tilde{P}_0^{(l)})\,a_s^{(q)}(x,D)fe_s^{(l)}+\sigma_p(\varphi^{(l)}),\tilde{P}_0^{(l)})fe_1^{(l)}=0\,.$$

Hence, noting that

$$\sigma_j(\varphi^{(l)},\tilde{P}_0^{(l)})=I\sigma_j(\varphi^{(l)},q^{(l)})+\sigma_j(\varphi^{(l)},|D'|)C^{(l)}$$

we obtain from (2.19)

$$a_{q+1}^{(p)}(x,D)=-\frac{1}{h^{(l)}}\{\sum_{s=q-1}^{p-1}\sigma_{p-s}(\varphi^{(l)},q^{(l)})\,a_q^{(s)}+\sum_{s=q}^{p-1}\sigma_{p-s}(\varphi^{(l)},|D'|)\,a_{q+1}^{(s)}\}$$

for $q=1,\cdots,p$, where $a_1^{(0)}=1$. In particular for $q=p$, we obtain $(2.15)_p$.

§ 3. Proof of the Necessity Part of Theorem 3

In this section we shall show that the condition (L_1) is necessary in order that the Cauchy problem for P is well posed. Assume that the Cauchy problem for P is well posed at $\hat{x}\in G$. Then for any neighborhood $U(\hat{x})$ of $\hat{x}$, there exists a neighborhood $G(\hat{x})\subset U(\hat{x})$ and a positive integer s_0 such that

$$(3.1)\qquad |u|_{0,G^+(\hat{x})}\leq C(x)\{|Pu|_{s_0,G^+(\hat{x})}+|u|_{s_0,G_0(\hat{x})}\}$$

for $u\in C^\infty(U(\hat{x}))$, where $G^+(\hat{x})=\{x\in G(\hat{x}),x_0>\hat{x}_0\}$ and $G_0(\hat{x})=\{x\in G(\hat{x}),x_0=\hat{x}_0\}$. This inequality is derived by the closed graph theorem. We shall prove the necessity of Theorem 3 by contradiction, that is, we shall construct an asymptotic solution which does not satisfy the inequality (3.1).

By virtue of Proposition 2.2 we can transform P to $\tilde{P}$ by N, where $\tilde{P}$ has the form (2.2). Then we assume that the property (L_1) is not valid for some l_0 and some k_0. We introduce a cofactor system $Q^{(l_0)}$ of $P^{(l_0)}$ such that

$$Q^{(l_0)}=\sum_{k=0}^{\nu_{l_0}-1} q^{(l_0)}(x, D)^{\nu_{l_0}-1-k}(-C^{(l_0)}|D'|)^k ,$$

where $q^{(l_0)}(x, D)=D_0+\lambda^{(l_0)}(x, D')$. Then we have

$$P^{(l_0)}(x, D)Q^{(l_0)}(x, D)=I_{\nu_{l_0}}(q^{(l_0)}(x, D))^{\nu_{l_0}}+\{b_t^s(x, D)\}_{s,t=1,\cdots,\nu_{l_0}} ,$$

where

$$\begin{aligned} &b_t^s(x, D)=0 , \quad t\leq s\leq \nu_{l_0}-1 ,\\ (3.2)\qquad &b_{s+k}^s(x, D)=[q^{(l_0)}(x, D)^{\nu_{l_0}-k}, |D'|](-|D'|)^{k-1} ,\\ &\qquad s=1, \cdots, \nu_{l_0}-k , \quad k=1, \cdots, \nu_{l_0}-1 ,\\ &b_t^{\nu_{l_0}}(x, D)=\sum_{k=1}^{t}\gamma_k^{(l_0)}(x, D')q^{(l_0)}(x, D)^{k-1}(-|D'|)^{\nu_{l_0}-1-t+k} ,\\ &\qquad t=1, \cdots, \nu_{l_0} . \end{aligned}$$

We shall construct the asymptotic solutions of the following equations (put $\nu_{l_0}=\nu$ for simplicity),

$$(3.3)\qquad q^{(l_0)}(x, D)^{\nu}v^s+\sum_{t=1}^{\nu} b_t^s(x, D)v^t=0 \quad s=1, \cdots, \nu .$$

We seek $v^s=v^s(x, \rho)$ of the form,

$$\begin{aligned} &v^s(x, \rho)=\rho^{-n_s}e^{iE(\rho,x)}\sum_{j=0}^{\infty}\rho^{-j\varepsilon}v_j^s(x),\\ &E(\rho, x)=\sum_{i=0}^{d}\rho^{\sigma_i}\varphi^{(i)}(x),\\ &1=\sigma^{(0)}>\sigma^{(1)}>\cdots>\sigma^{(d)}>0 , \end{aligned}$$

and ε^{-1} is the common denominator of the rational numbers $\sigma^{(j)}$ $(j=1, \cdots, d)$. We shall determine $(\sigma^{(j)}, \varphi^{(j)})$ inductively.

At first we define $\varphi^{(0)}$ as follows

$$\begin{aligned} &\varphi_{x_0}^{(0)}+\lambda^{(l_0)}(x, \varphi_{x'}^{(0)})=0\\ &\varphi^{(0)}|_{x_0=\hat{x}_0}=\langle x', \omega^{(0)}\rangle, \ w^{(0)}\in \mathbb{R}^n\backslash 0 . \end{aligned}$$

We put

$$L = q^{(l_0)}(x, D)^{\nu} + b_{\nu}^{\nu}(x, D)$$
$$= \sum_{k=0}^{\nu} L_k(x, D') q^{(l_0)}(x, D)^{k-1},$$

where $L_0 = 1$ and L_k are pseudo-differential operators in x'. We denote by $d_k^{(0)}$ the order of L_k. Then from (3.2) we have

(3.5) $$\nu_{l_0} - k \geq d_k^{(0)} \geq \text{order } \gamma_k^{(l_0)} + \nu_{l_0} - k,\ k = 1, \cdots, \nu_{l_0}.$$

We define

$$L_\rho^{(0)} w = e^{-i\rho\varphi^{(0)}} L(e^{i\rho\varphi^{(0)}} w)$$
$$= \sum_{l \geq 0} \rho^{d_k(0) - l} \sigma_l(L_k, \varphi^{(0)}) \sum_{j \geq 0} \rho^{-k} \sigma_{k-1+j}(q^{(l_0)(k-1)}, \varphi^{(0)})$$
$$= \sum_{k=0}^{\nu} \sum_{s \geq 0} \rho^{d_k(0) - s} L_{k,s}^{(0)}(x, D) w,$$

where

(3.6) $$L_{k,s}^{(0)} = \sum_{l+j=s} \sigma_l(L_k, \varphi^{(0)}) \sigma_{k-1+j}(q^{(l_0)(k-1)}, \varphi^{(0)}).$$

Then from Lemma 1.2 it follows that the principal part of $\sigma_{k-1}(q^{(l_0)(k-1)}, \varphi^{(0)})$ is given by

(3.7) $$H(x, \xi)^{(k-1)}$$

where $H(x, \xi) = \xi_0 + \sum_{i=j}^{n} \lambda_{\xi_i}^{(l_0)}(x, \varphi_{x'}^{(0)}) \xi_i$. We note that the order of $L_{k,s}^{(0)} \leq k - 1 + s$. We put

$$\sigma^{(1)} = \max_{1 \leq k \leq \nu - 1} \frac{d_k^{(0)}}{\nu - k + 1},$$

$$\not\equiv^{(1)} = \left\{k, \frac{d_k^{(0)}}{\nu - k + 1} = \sigma^{(1)}\right\}.$$

Then if (L_1) is not valid, we have from (3.5)

$$d_k^{(0)} > 0,$$

for some k. Therefore we have

(3.8) $$0 < \sigma^{(1)} < 1.$$

We define

$$L_\rho^{(1)}=\exp\{-i\rho^{\sigma^{(1)}}\varphi^{(1)}\}\,L_\rho^{(0)}\,\exp\{i\rho^{\sigma^{(1)}}\varphi^{(1)}\}$$
$$=\sum \rho^{d_k^{(0)}-s+\sigma^{(1)}(k-1+s-j)}\sigma_j(L_{k,0}^{(0)},\varphi^{(1)})$$
$$=\rho^{\sigma^{(1)}\nu}\{(H(x,D)\varphi^{(1)})^\nu+\sum_{k\in\#^{(1)}}\sigma_0(L_{k,0}^{(0)},\varphi^{(1)})+o(1)\}.$$

We put

$$h^{(1)}(x,\varphi_x^{(1)})=H(x,\varphi_x^{(1)})^\nu+\sum_{k\in\#^{(1)}}\sigma_0(L_{k,0}^{(0)},\varphi^{(1)}),$$

which is a polynomial in $\varphi_x^{(1)}$. But from (3. 6) and (3. 7) it follows that the principal part of $L_{k,0}^{(0)}$ is given by $\sigma_0(L_k,\varphi^{(0)})H(x,\xi)^{\nu-k}$. Hence $h^{(1)}(x,\varphi_x^{(1)})$ is a polynomial in $H(x,\varphi_x^{(1)})$ and is decomposed

$$h^{(1)}(x,\varphi_x^{(1)})=(H(x,\varphi_x^{(1)})-C^{(1)}(x))^{m^{(1)}}Q^{(1)}(x,H)$$

for $x\in U^{(1)}$, an open set, where $Q^{(1)}(x,H)$ is a polynomial in H, $Q^{(1)}(x,C^{(1)})\neq 0$ and $C^{(1)}(x)$ is a C^∞-function in $U^{(1)}$. We note that we can choose $C^{(1)}(x)$ such that

$$\text{(3. 9)}\qquad \operatorname{Im} C^{(1)}(x)<0 \quad\text{in}\quad U^{(1)}.$$

For, we have

$$h^{(1)}(x,\varphi_x^{(1)})=H(x,\varphi_x^{(1)})^\nu+\sum_{k\in\#^{(1)}}\hat{L}(x,\varphi_x^{(0)})H(x,\varphi_x^{(1)})^{\nu-k}$$
$$=\hat{h}^{(1)}(x,H,\varphi_x^{(0)})$$

where $\hat{L}_k(x,\varphi_x^{(0)})$, the principal part L_k, is a polynomial of order $\sigma^{(1)}k$ for $k\in\#^{(1)}$. Hence we have

$$\hat{h}^{(1)}(x,H,-\varphi_x^{(0)})=H^\nu+\sum_{k\in\#^{(1)}}(-1)^{\sigma^{(1)}k}L_k(x,\varphi_x^{(0)})H^{\nu-k}$$
$$=(-1)^{\nu\sigma^{(1)}}\hat{h}^{(1)}(x,(-1)^{\sigma^{(1)}}H,\varphi_x^{(0)}).$$

Therefore $(-1)^{-\sigma^{(1)}}C^{(1)}(x)$ is a root of $h^{(1)}(x,H,-\varphi_x^{(0)})=0$. Since $C^{(1)}(x)\neq 0$ and $0<\sigma^{(1)}<1$, we can choose a branch of $(-1)^{\sigma^{(1)}}$ such that $\operatorname{Im}(-1)^{-\sigma^{(1)}}C^{(1)}(x)<0$.

We choose $\varphi^{(1)}$ as a solution

$$\text{(3. 10)}\qquad \begin{cases} H(x,\varphi_x^{(1)})=C^{(1)}(x) \\ \varphi^{(1)}|_{x_0=\hat{x}_0}=\langle x',\omega^{(1)}\rangle,\ \omega^{(1)}\in R^n\backslash 0\,. \end{cases}$$

Then (3. 9) implies

$$\text{(3. 11)} \qquad \operatorname{Im}\varphi^{(1)}<0\,, \quad x\in U^{(1)}, \quad x_0>\hat{x}_0\,.$$

We define $L^{(j)}, \varphi^{(j)}, \sigma^{(j)}$ and $h^{(j)}(x, H)$ inductively, for $j\geq 2$,

$$L_\rho^{(j)}=\exp\{-i\rho^{\sigma^{(j)}}\varphi^{(j)}\}L_\rho^{(j-1)}\exp\{i\rho^{\sigma^{(j)}}\varphi^{(j)}\},$$
$$=\rho^{M^{(j)}}\sum_{k\geq 0}\rho^{-k\xi^{(j)}}L_k^{(j)},$$

$$L_0^{(j)}=h^{(j)}(x, H(x, \varphi_x^{(j)}))$$
$$=(H(x, \varphi_x^{(j)})-C^{(j)}(x))^{m^{(j)}}Q^{(j)}(x, H) \quad \text{in } U^{(j)}\subset U^{(j-1)},$$

$$d_k^{(j)}=\text{order } L_k^{(j)}\,, \quad (d_k^{(j)}=-\infty, \quad \text{if} \quad L_k^{(j)}\equiv 0),$$

$$\sigma^{(j)}=\max_{0<k<m^{(j-1)}\sigma^{(j-1)}/\varepsilon^{(j-1)}}\frac{m^{(j-1)}\sigma^{(j-1)}-k\varepsilon^{(j-1)}}{m^{(j-1)}-d_k^{(j-1)}},$$

$\varepsilon^{(j)-1}$; the common denominator of the rational numbers $\sigma^{(1)}, \cdots, \sigma^{(j)}$,

$$\xi^{(j)}=\left\{0<k<m^{(j-1)}\sigma^{(j-1)}/\varepsilon^{(j-1)}, \frac{m^{(j-1)}\sigma^{(j-1)}-k\varepsilon^{(j-1)}}{m^{(j-1)}-d_k^{(j-1)}}=\sigma^{(j)}\right\}\cup\{0\},$$

$$M^{(j)}=M^{(j-1)}+m^{(j-1)}(\sigma^{(j)}-\sigma^{(j-1)}), \quad M^{(1)}=\nu\sigma^{(1)}\,,$$

$$\nu>m^{(1)}>\cdots>m^{(j)}, \quad 1>\sigma^{(1)}>\cdots>\sigma^{(j)},$$

and $\varphi^{(j)}$ a solution as

$$H(x, \varphi_x^{(j)})=C^{(j)}(x)$$
$$\varphi^{(j)}|_{x_0=\hat{x}_0}=\langle x', \omega^{(j)}\rangle\,,$$

where $H(x, \xi)=\xi_0+\sum_{i=1}^{n}\lambda_{\xi_i}^{(l_0)}(x, \varphi_{x'}^{(0)})\xi_i$.

We must prove that $L_0^{(j)}$, the coefficient of the leading power $\rho^{M^{(j)}}$ in $L_\rho^{(j)}$ is a polynomial in $H(x, \varphi_x^{(j)})$. To do so, we decompose $L_k^{(j-2)}$ such that

$$L_k^{(j-2)}=\sum_{s=0}^{d_k(j-2)}L_{k,s}^{(j-2)}(x, D')H(x, D)^{d_k(j-2)-s}$$

where $L_{k,s}^{(j-2)}$ is a differential operator in x' of order s. We rewrite $H(x, D)^k$ as

$$H(x, D)^k=(H(x, D)-\rho C^{(j-1)}(x)+\rho C^{(j-1)}(x))^k$$
$$=\sum_{p=0}^{k}C_{k,p}^{(j-1)}(x)\rho^p(H-\rho C^{(j-1)}(x))^{k-p}\,,$$

where $C_{k,0}^{(j-1)}=1$. Then we have

$$L_{\rho}^{(j-2)}=\rho^{M(j-2)}\sum\rho^{-k\varepsilon^{(j-2)}+p\sigma^{(j-1)}}L_{k,s}^{(j-2)}C_{d_k(j-1)-s,p}^{(j-1)}(H-\rho^{\sigma^{(j-1)}}C^{(j-1)})^{d_{k-s-p}^{(j-2)}}.$$

Hence we obtain

$$\begin{aligned}L_{\rho}^{(j-1)}&=\exp\{-i\rho^{\sigma^{(j-1)}}\varphi^{(j-1)}\}L_{\rho}^{(j-2)}\exp\{i\rho^{\sigma^{(j-1)}}\varphi^{(j-1)}\}\\&=\rho^{M(j-2)}\sum\rho^{-k\varepsilon^{(j-2)}+(p+s-l)\sigma^{(j-1)}}\sigma_l(L_{k,s}^{(j-2)},\varphi^{(j-1)})C_{d_k(j-2)-s,p}^{(j-1)}(x)\\&\qquad\times H(x,D)^{d_k(j-2)-s-p},\\&=\rho^{M(j-1)}\sum_t\rho^{-t\varepsilon^{(j-1)}}L_t^{(j-1)}.\end{aligned}$$

Therefore we have

$$L_t^{(j-1)}=\sum\sigma_l(L_{k,s}^{(j-2)},\varphi^{(j-1)})C_{d_k(j-2)-s,p}^{(j-1)}(x)H(x,D)^{d_k(j-2)-s-p},$$

where the summation is

$$k\varepsilon^{(j-2)}-(p+s-l)\sigma^{(j-1)}+m^{(j-2)}(\sigma^{(j-1)}-\sigma^{(j-2)}=t\varepsilon^{(j-1)}.$$

On the other hand, we have

$$\begin{aligned}L_{\rho}^{(j)}&=\exp\{-i\rho^{\sigma(j)}\varphi^{(j)}\}L_{\rho}^{(j-1)}\exp\{i\rho^{\sigma(j)}\varphi^{(j)}\}\\&=\rho^{M(j-1)}\sum\rho^{-t\varepsilon(j-1)+\sigma^{(j)}(d_t^{(j)}-q)}\sigma_q(L_t^{(j-1)},\varphi^{(j)})\\&=\rho^{M(j)}(\sum_{t\in\sharp(j)}\sigma_0(L_t^{(j-1)},\varphi^{(j)})+o(1)).\end{aligned}$$

Hence we have

$$L_0^{(j)}=\sum_{t\in\sharp(j)}\sigma_0(L_t^{(j-1)},\varphi^{(j)}).$$

In order to prove that $L_0^{(j)}$ is a polynomial in $H(x,\varphi_x^{(j)})$, it suffices to know that the principal part of $L_t^{(j-1)}$ for $t\in\sharp^{(j)}$ is a polynomial in $H(x,\varphi_x^{(j)})$, that is, in the expression of $L_t^{(j-1)}$ for $t\in\sharp^{(j)}$ the terms $\sigma_l(L_{k,s}^{(j-1)},\varphi^{(j-1)})$ become zero for $l>0$. The principal part of $L_t^{(j-1)}$ is given by

$$\sum\hat{\sigma}_l(L_{k,s}^{(j-2)},\varphi^{(j-1)})H(x,\xi)^{d_k(j-2)-s},$$

where the summation is

$$l+d_k^{(j-2)}-s=d_t^{(j-1)},$$

for, the order of $L_t^{(j-1)}$ is equal to $d_t^{(j-1)}$, where $\hat{\sigma}_l(L_{k,s}^{(j-2)},\varphi^{(j-1)})$ stands for the principal part of $\sigma_l(L_{k,s}^{(j-2)},\varphi^{(j-1)})$. Assume that for some $\hat{l}\neq 0$

and $\hat{s}$,

$$\sum_{\hat{l}+d_{k,s}^{(j-2)}-\hat{s}=d_t^{(j-1)}} \hat{\sigma}_{\hat{l}}(L_{k,\hat{s}}^{(j-2)}, \varphi^{(j-1)}) \neq 0 . \tag{3.12}$$

Since the principal part $\hat{L}_{k,\hat{s}}^{(j-2)}(x, \xi)$ of $L_{k,\hat{s}}^{(j-2)}$ is a homogeneous polynomial in ξ of order $\hat{s}$ and $\hat{\sigma}_{\hat{l}}(L_{k,s}^{(j-2)}, \varphi^{(j-1)})$ is given by

$$\sum_{|\alpha|=\hat{l}} \frac{1}{\alpha!}\left(\frac{\partial}{\partial \xi'}\right)^{\alpha} \hat{L}_{k,s}^{(j-2)}(x, \varphi_{x'}^{(j-1)}) \xi'^{\alpha} ,$$

which homogeneous in $\varphi_{x'}^{(j-1)}$. Hence if (3.12) is valid,

$$\sum_{\hat{l}+d_k{}^{(j-2)}-\hat{s}=d_t{}^{(j-1)}} \hat{\sigma}_{\hat{l}-1}(L_{k,\hat{s}}^{(j-2)}, \varphi^{(j-1)}) \tag{3.13}$$

does not vanish, if we choose a suitable $\omega^{(j-1)}$, the direction of the initial data of $\varphi^{(j-1)}$. On the other hand

$$\sum_{\hat{l}+d_k{}^{(j-2)}-\hat{s}=d_t{}^{(j-1)}} \hat{\sigma}_{\hat{l}-1}(L_{k,\hat{s}}^{(j-2)}, \varphi^{(j-1)}) H(x, \xi)^{d_k{}^{(j-2)}-\hat{s}}$$

is involved in the terms of $L_{t-\sigma^{(j-1)}/\varepsilon^{(j-1)}}^{(j-1)}$. Hence we have

$$d_{t-\sigma^{(j-1)}/\varepsilon^{(j-1)}}^{(j-1)} \geq d_t^{(j-1)} - 1$$

which contradicts to the definition of $\sigma^{(j)}$. For,

$$\sigma^{(j)} = \frac{m^{(j-1)}\sigma^{(j-1)} - t\varepsilon^{(j-1)}}{m^{(j-1)} - d_t^{(j-1)}}$$

and on the other hand, $\sigma^{(j-1)} > \sigma^{(j)}$ implies

$$\frac{m^{(j-1)}\sigma^{(j-1)} - (t-\sigma^{(j-1)}/\varepsilon^{(j-1)})}{m^{(j-1)} - d_{t-\sigma^{(j-1)}/\varepsilon^{(j-1)}}^{(j-1)}} > \frac{m^{(j-1)}\sigma^{(j-1)} - t\varepsilon^{(j-1)} + \sigma^{(j-1)}}{m^{(j-1)} - d_t^{(j-1)} + 1} > \sigma^{(j)} .$$

Thus we have proved that $L_0^{(j)}$ is a polynomial in $H(x, \varphi_x^{(j)})$, that is,

$$h^{(j)}(x, H(x, \varphi_x^{(j)}) = L_0^{(j)} = \sum_{t \in \#^{(j)}} \sigma_0(L_t^{(j-1)}, \varphi^{(j)}) . \tag{3.14}$$

Then the coefficient of the leading power $H^{m^{(j-1)}}$ in $L_0^{(j)}$ is $Q^{(j-1)}(x, C^{(j-1)}) \neq 0$. Hence there exists an open set $U^{(j)} \subset U^{(j-1)}$ such that we can decompose

$$h^{(j)}(x, H) = (H - C^{(j)}(x))^{m^{(j)}} Q^{(j)}(x, H) \quad \text{in} \quad U^{(j)} ,$$

where $C^{(j)}(x) \in C^{\infty}(U^{(j)})$ and $Q^{(j)}(x, C^{(j)}) \neq 0$. Moreover we note that

$$m^{(j)} < m^{(j-1)} , \tag{3.15}$$

if $\sigma^{(j)}>0$, that is, $L_k^{(j-1)}\neq 0$ for some $k<m^{(j-1)}\sigma^{(j-1)}/\varepsilon^{(j-1)}$.

Next we shall prove that

$$d_k^{(j)}<k\varepsilon^{(j)}/\sigma^{(j)} \qquad \text{for} \quad 0<k<m^{(j)}\sigma^{(j)}/\varepsilon^{(j)}\,. \tag{3.16}$$

We have

$$\begin{aligned} L_\rho^{(j)} &= \exp\{-i\rho^{\sigma^{(j)}}\varphi^{(j)}\}\, L_\rho^{(j-1)} \exp\{i\rho^{\sigma^{(j)}}\varphi^{(j)} \\ &= \rho^{M^{(j-1)}}\sum \rho^{-\varepsilon^{(j-1)}p+\sigma^{(j)}(d_p{}^{(j-1)}-s)}\sigma_s(L_p^{(j-1)}, \varphi^{(j)}) \\ &= \rho^{M^{(j)}}\sum \rho^{-\varepsilon^{(j)}k} L_k^{(j)}\,. \end{aligned}$$

Hence we obtain

$$L_k^{(j)} = \sum \sigma_s(L_p^{(j-1)}, \varphi^{(j)}) \tag{3.17}$$

where the summation is,

$$\varepsilon^{(j-1)}p-\sigma^{(j)}(d_p^{(j-1)}-s)+m^{(j-1)}(\sigma^{(j)}-\sigma^{(j-1)})=k\varepsilon^{(j)}. \tag{3.18}$$

If $k\varepsilon^{(j)}/\sigma^{(j)}$ is not an integer, it is evident that (3.18) implies (3.16). When $k\varepsilon^{(j)}/\sigma^{(j)}$ is an integer, it follows from (3.14) that the term of order $k\varepsilon^{(j)}/\sigma^{(j)}$ in $L_k^{(j)}$ is given by

$$\begin{aligned} &\sum_{p\in\#^{(j)}}{}' \hat{\sigma}_s(L_p^{(j-1)}, \varphi^{(j)}) \\ &\qquad = \frac{1}{s!}\left(\frac{\partial}{\partial H}\right)^s h^{(j)}(x, H(x, \varphi_x^{(j)}))=0\,, \end{aligned}$$

for $s=k\varepsilon^{(j)}/\sigma^{(j)}<m^{(j)}$, if $k<m^{(j)}\sigma^{(j)}/\varepsilon^{(j)}$. Thus we have proved (3.16). It is evident that (3.16) implies that $\sigma^{(j+1)}<\sigma^{(j)}$, if $\sigma^{(j)}\neq 0$. Moreover from (3.17) we have

$$L_k^{(j)}=\sum_{s=0}^{m^{(j)}} L_{k,s}^{(j)}(x)\, H(x, D)^{m^{(j)}-s} \tag{3.19}$$

for $k=m^{(j)}\sigma^{(j)}/\varepsilon^{(j)}$, where $L_{k,0}^{(j)}=Q^{(j)}(x, C^{(j)}(x))\neq 0$ for $k=m^{(j)}\sigma^{(j)}/\varepsilon^{(j)}$. Thus it follows from (3.15) that in the finite step we have

$$L_p^{(d)}=\rho^{M^{(d)}}\sum_{k\geq k(d)}\rho^{-\varepsilon^{(d)}k}L_k^{(d)}\,, \tag{3.20}$$

where $k(d)=m^{(d)}\sigma^{(d)}/\varepsilon^{(d)}$ and $L_{k(d)}^{(d)}$ is given by (3.19) with $j=d$.

Now we return to construct asymptotic solutions of the equations (3.3). Noting that from (3.2) we have, $(n_s=(1-\sigma^{(1)})s)$,

$$\rho^{-n_s}e^{-iE(\rho,x)}q^{(l_0)}(x, D)^{\nu}e^{iE(\rho,x)}$$
$$=\rho^{\nu\sigma(1)-n_s}(H(x, \varphi_x^{(1)})^{\nu}+0(\rho^{-\sigma(1)})),$$
$$\rho^{-n_t}e^{-iE(\rho,x)}b_t^s(x, D)e^{iE(\rho,x)}=0(\rho^{-n_s+\sigma(1)(\nu-1)}),$$
$$s=1, \cdots, \nu-1, \ t=1, \cdots, \nu,$$
$$\rho^{-n_t}e^{-iE}b_t^{\nu}(x, D)e^{iE}=0(\rho^{M(d)}-n_\nu),$$
$$t=1, \cdots, \nu-1,$$

and inserting $v^s(x, \rho)$ into (3. 3), we obtain by virtue of (3. 20),

$$q^{(l_0)}(x, D)^{\nu}v^s+\sum_{t=1}^{\nu}b_t^s(x, D)v^t$$
$$=e^{iE(\rho,x)}\rho^{\nu\sigma(1)}\sum_{j\geq 0}\rho^{-j\sigma}(H(x, \varphi_x^{(1)})^{\nu}v_j^s+g_t^s),$$
$$s=1, \cdots, \nu-1,$$
$$q^{(l_0)}(x, D)^{\nu}v^{\nu}+\sum_{t=1}^{\nu}b_t^{\nu}(x, D)v^t$$
$$=L(x, D)v^{\nu}+\sum_{t=1}^{\nu-1}b_t^{\nu}(x, D)v^t$$
$$=e^{iE(\rho,x)}\rho^{M(d)}\sum_{j\geq 0}\rho^{-j\sigma}(L_{k(d)}^{(d)}v_j^{\nu}+g_j^{\nu}),$$

where we put $\nu=\nu_{l_0}$ and $\sigma=\varepsilon^{(d)}$, and g_t^s are functions of $(x, v_0^1, \cdots, v_{j-1}^1, \cdots, v_1^{\nu}, \cdots, v_{j-1}^{\nu})$ and g_j^{ν} functions of $(x, v_0^1, \cdots, v_j^1, \cdots, v_0^{\nu-1}, \cdots, v_j^{\nu-1}, v_0^{\nu}, \cdots, v_{j-1}^{\nu})$, and in particular $g_0^s=0, s=1, \cdots, \nu$, and $L_0^{(d)}$ is given by (3. 19).

Thus we have the following equations,

$$\text{(3. 21)} \qquad \begin{cases} H(x, \varphi_x^{(1)})^{\nu}v_j^s+g_j^s=0, & s=1, \cdots, \nu-1, \\ L_{k(d)}^{(d)}(x, D)v_j^{\nu}+g_j^{\nu}=0, \end{cases}$$

for $j=0, 1, 2, \cdots$. Since $H(x, \varphi_x^{(1)})=C^{(1)}(x)\neq 0$ and $L_{k(d)}^{(d)}$ involved only the differential operator $H(x, D)$, and $g_0=0$, we can solve (3. 21) successively. Since $g_0^s=0$, $s=1, \cdots, \nu-1$ and $g_0^{\nu}=0$ for $v_0^s=0$, $s=1, \cdots, v-1$,

$$\text{(3. 22)} \qquad v_0^s=0, \ s=1, \cdots, \nu-1,$$
$$L_{k(d)}^{(d)}v_0^{\nu}=0.$$

Hence we can seek v_0^{ν} and an open set $U\subset U^{(d)}$ such that

(3. 23) $$v_0^\nu \neq 0 \quad \text{in } U .$$

Decompose $N(x, D') = (N^{(1)}(x, D'), \cdots, N^{(r)}(x, D'))$ which is given by Proposition 2. 2, where $N_0^{(l)}(x, \xi')$ the principal part of $N^{(l)}(x, D')$ is generated by eigen vectors of $A(x, \xi')$ corresponding to $\lambda^{(l)}(x, \xi')$. Put

$$u(x, \rho) = \sum_{j=0}^{M} \rho^{-j\sigma} N^{(l_0)}(x, D') Q^{(l_0)}(x, D) \begin{pmatrix} \rho^{-n_1} & & 0 \\ & \ddots & \\ 0 & & \rho^{-n_\nu} \end{pmatrix} e^{iE(\rho,x)} \begin{pmatrix} v_j^1 \\ \vdots \\ v_j^\nu \end{pmatrix}.$$

Then by virtue of (3. 22) and (3. 23) we have

$$u(x, \rho) = e^{iE(\rho,x)} \{N_{0\nu}^{(l_0)}(x, \varphi_{x'}^{(0)}) (-|\varphi_{x'}^{(0)}|)^{\nu-1} v_0^\nu(x) + 0(\rho^{-\sigma})\},$$

where $N_{0\nu}^{(l_0)}(x, \xi')$ is the ν-th eigen vector of $A(x, \xi')$ corresponding to $\lambda^{(l_0)}$. Therefore it follows from (3. 11) that $u(x, \rho)$ violates (3. 1), if M is sufficiently large and ρ tends to ∞. Thus we have completed the proof of Theorem 3. 1.

We shall here give a necessary condition in order that the Cauchy (1) for P, a hyperbolic system with diagonal principal part of constant multiplicity, is well posed. It seems that our condition is deeply connected with that given by Mizohata in [11].

We consider

$$P_t^s(x, D) = \delta_t^s a(x, D) + B_t^s(x, D), \quad s, t = 1, \cdots, N,$$

where $a(x, D) = Q(x, D) q(x, D)^\nu$, $q(x, D) = D_0 + \lambda(x, D')$ and $\hat{Q}(x, \lambda \times (x, \xi'), \xi') \neq 0$ ($\hat{Q}$ the principal part of Q) and order $a(x, D) = m$, order $B_t^s \leq m-1$.

We decompose

$$B_t^s(x, D) = \sum_{j=0}^{m-1} B_{tj}^s(x, D') C_{tj}^s(x, D),$$

where B_{tj}^s is a pseudo-differential operator in $x' \in R^n$ of order $m-1-j$ and

$$C_{tj}^s(x, D) = \sum_{k=0}^{d_{tj}^s} C_{tjk}^s(x) q(x, D)^{d_{tj}^s - k}.$$

We note that $m-j-1+d_{tj}^s \leq m-1$, that is

(3. 24) $$d_{tj}^s \leq j .$$

We develop for a phase function φ corresponding to λ,

$$e^{-i\rho\varphi}B_t^s(x, D)\, e^{i\rho\varphi}=\sum \sigma_k(B_{tj}^s, \varphi)\,\sigma_{d_{tj}^s+l}(C_{tj}^s, \varphi)\,\rho^{m-j-1-k-l}$$
$$=\sum \rho^{m-j-1-i}B_{tji}^s(x, D),$$

where

$$B_{tji}^s=\sum_{k+l=i}\sigma_k(B_{tj}^s, \varphi)\,\sigma_{d_{tj}^s+l}(C_{tj}^s, \varphi),$$

$$\text{order } B_{tji}^s\leq d_{tj}^s+i\,.$$

In particular, the principal part of B_{tj0}^s is given by

$$\hat{B}_{tj0}^s(x, \xi)=\hat{B}_{tj}^s(x, \varphi_x)\,C_{tj0}^s(x)\,H(x.\,\xi)^{d_{tj}^s}, \tag{3.25}$$

where

$$H(x, \xi)=\xi_0+\sum_{i=1}^n \lambda_{\xi_i}(x, \varphi_{x'})\,\xi_i\,.$$

We put

$$E(\rho, x)=\rho\varphi(x)+\rho^\sigma\psi(x), \quad 0<\sigma<1\,.$$

Then we have

$$e^{-iE(\rho,x)}B_t^s(e^{iE(\rho,x)})$$
$$=\sum \rho^{m-j-1-i+\sigma(d_{tj}^s+i-k)}\sigma_k(B_{tji}^s, \psi)$$
$$=\sum \rho^{m-j-1+\sigma d_{tj}^s}(\sigma_0(B_{tj0}^s, \psi)+0(1)).$$

We set

$$d_j=\max_\pi \sum_{s=1}^N d_{\pi(s)j}^s/N, \tag{3.26}$$

where π stands for a permutation of $[1, \cdots, N]$. We choose the rational number σ such that

$$m-j-1+\sigma d_j\leq m-\nu+\sigma\nu$$

for any j. To do so, we put

$$\sigma=\max_{0\leq j<\nu}\frac{\nu-j-1}{\nu-d_j}\,. \tag{3.27}$$

By (3.24) we have $\sigma<1$. We put

$$M_t^s=\max_{0\leq j<\nu}(m-j-1+d_{tj}^s),$$

$$\sharp_t^s = \{0 \leq j < \nu,\ m-j-1+d_{tj}^s = M_t^s\}.$$

Then we obtain by (3. 26)

$$\max_{\pi} \sum_{s=1}^{N} M_{\pi(s)}^s / N = m - \nu + \sigma\nu .$$

Volevich's lemma (c.f. [11], [12]) implies that there exist the rational numbers m_s, $s=1, \cdots, N$, such that

$$M_t^s \leq m - \nu + \sigma\nu + m_t - m_s , \quad s, t = 1, \cdots, N.$$

We define $\hat{B}_t^s(x, H)$ such that

$$\hat{B}_t^s(x, H) = \begin{cases} \sum_{j \in \sharp_t^s} B_{tj}^s(x, \varphi_x) C_{tj0}^s(x) H^{d_{tj}^s} & \text{if } M_t^s = m - \nu + \sigma\nu + m_t - m_s , \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $A(x, H)$ the matrix of which (s, t)-element is given by

$$\delta_t^s Q(x, \varphi_x) H^\nu + \hat{B}_t^s(x, H).$$

Then it is evident that all elements of $A(x, H)$ are polynomials in H of order ν. Then we have the following theorem whose proof is analogous to that of Theorem 3. 1.

Theorem 3. 2. *Let P be a hyperbolic system with diagonal principal part of constant multiplicity. If the Cauchy problem for P is well posed in G and σ given by* (3. 27) *is not zero, then all the roots with respect to H of the determinant of the characteristic matrix $A(x, H)$ are zero in G.*

Example 1. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_0^2 + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} D_1 + \begin{pmatrix} a & b \\ c & d \end{pmatrix} D_0 .$$

Then we have

$$A(x, H) = \begin{pmatrix} H^2 + \alpha & \beta \\ \gamma & H^2 + \delta \end{pmatrix} .$$

Hence if P is well posed in $\boldsymbol{R}^2$, we obtain

$$\alpha + \delta = 0 \tag{3. 28}$$

$$\alpha\delta-\beta\gamma=0 .$$

Assume $\gamma(x)\neq 0$. We put

$$N=\begin{pmatrix}\alpha & 1\\ \gamma & 0\end{pmatrix}.$$

Then

$$\tilde{P}=N^{-1}PN=D_0^2+\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}D_1+\frac{1}{\gamma}\begin{pmatrix}* & *\\ B_1 & *\end{pmatrix}D_0+\frac{1}{\gamma}\begin{pmatrix}* & *\\ B_2 & *\end{pmatrix}$$

where

$$B_1=\gamma D_0\alpha-\alpha D_0\gamma+\gamma(a\alpha+b\gamma)-\alpha(\alpha c+d\gamma)$$

$$B_2=\gamma D_0^2\alpha-\alpha D_0^2\gamma+\gamma(aD_0\alpha+bD_0\gamma)-\alpha(cD_0\alpha+dD_0\gamma).$$

Since $\tilde{P}$ is also well posed at $\hat{x}$, and the characteristic matrix is given by

$$A(x, H)=\begin{pmatrix}H^2 & 1\\ B_1H & H^2\end{pmatrix},$$

we have $B_1\equiv 0$. Moreover when $B_1\equiv 0$, we have

$$A(x, H)=\begin{pmatrix}H^2 & 1\\ B_2 & H^2\end{pmatrix}.$$

Hence we obtain $B_2\equiv 0$. Thus we have (2. 28) and $B_1\equiv B_2\equiv 0$ as the necessary conditions. If the rank of $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is constant, then (2. 28) and $B_1\equiv B_2\equiv 0$ is also sufficient.

Example 2. Let

$$P=\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}D_0+\begin{pmatrix}0 & & \\ & 0^{\,1} & \\ & & 0_{\,0}\end{pmatrix}D_1+\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}.$$

Put

$$Q=\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}D_0-\begin{pmatrix}0 & & \\ & 0^{\,1} & \\ & & 0_{\,0}\end{pmatrix}D_1,$$

which is evidently well posed. If P is well posed,

$$PQ=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}D_0^2-\begin{pmatrix}0&b_{11}&0\\0&b_{21}&0\\0&b_{31}&0\end{pmatrix}D_1+BD_0$$

is also well posed. Then the characteristic matrix for PQ

$$A(x,H)=\begin{pmatrix}H^2&b_{11}&0\\0&H^2+b_{21}&0\\0&b_{31}&H^2\end{pmatrix}.$$

Hence $b_{21}=0$ is necessary. Moreover when $b_{21}=0$, we have

$$A(x,H)=\begin{pmatrix}H^2&b_{11}&0\\0&H^2&b_{23}H\\0&b_{31}&H^2\end{pmatrix}.$$

Therefore $b_{31}b_{23}=0$ is necessary. Thus we obtain as the necessary condition

$$\begin{cases}b_{21}=0\\b_{31}b_{23}=0\,,\end{cases}$$

which Petkov has already derived by a different method from ours in [7]. In general, let

$$P=\begin{pmatrix}1&&\\&\ddots&\\&&1\end{pmatrix}D_0+\begin{pmatrix}0&1&&&\\&0&\ddots&&\\&&\ddots&\ddots&\\&&&\ddots&0\\&&&&0\end{pmatrix}D_1+\begin{pmatrix}b_{11}&\cdots&b_{1N}\\&\cdots&\\b_{N1}&\cdots&b_{NN}\end{pmatrix}.$$

If P is well posed in $\mathbf{R}^2$, it is necessary,

$$\begin{cases}b_{21}=0\,,\\\sum_{k=3}^{N}b_{k1}b_{2k}=0\,.\end{cases}$$

Example 3. Let

$$P=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}D_0+\begin{pmatrix}0&1&&\\&0&0&\\&&0&1\\&&&0\end{pmatrix}D_1+\begin{pmatrix}0&0&0&0\\a&b&c&d\\0&0&0&0\\\alpha&\beta&\gamma&\delta\end{pmatrix}$$

If P is well posed,

$$a+\delta=0,$$

$$a\delta-\alpha c=0,$$

$$\alpha(ab+\alpha d)-a(a\beta+\alpha\delta)+2(\alpha D_0 a-aD_0\alpha)=0,$$

$$\alpha(bD_0 a+dD_0\alpha)-a(\beta D_0 a+\delta D_0\alpha)+\alpha D_0^2 a-aD_0^2\alpha=0,$$

are necessary conditions. If the rank of $\begin{pmatrix} a & c \\ \alpha & \delta \end{pmatrix}$ is constant, these conditions are sufficients.

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