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Holonomic Quantum Fields III

By

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Introduction

In this article we shall study an analogue of the Riemann-Hilbert problem and the monodromy preserving deformation for solutions of the 2-dimensional Euclidean Klein-Gordon and Dirac equations.

The topic discussed here has its own interest from a purely mathematical viewpoint; to the authors' knowledge, it will provide one of relatively few examples of deformation theory of linear differential equations in which calculation of the deformation equations is explicitly carried out to the end. However, the most salient feature of this theory consists in its close connection with quantum field operators. In fact, as has been exemplified in the case of the 1-dimensional Riemann-Hilbert problem (Chapter II [2]), we shall see that the whole theory is most naturally and effectively described in terms of a class of field operators "belonging to the Clifford group" ([1]). This will be done in the forthcoming Chapter IV. The purpose of Chapter III included here is to clarify the mathematical aspects and to prepare necessary ingredients so as to serve a sound basis for the operator theory.

In the first § 3. 1, local (multi-valued) solutions of the Euclidean Klein-Gordon and Dirac equations are examined. We introduce here a series of special solutions which will play a rôle of power functions in the 1-dimensional case.

The monodromy problem is formulated in the next § 3. 2. For given branch points $\{(a_\nu, \bar{a}_\nu)\}_{\nu=1,\cdots,n}$ and exponents $l_\nu \in \boldsymbol{R}$ $(\nu=1, \cdots, n)$, we consider the space $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ (resp. $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$) consisting of solutions v

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(resp. $w={}^t(w_+, w_-)$) of the Euclidean Klein-Gordon (resp. Dirac) equations satisfying the monodromic property

$$(3.0.1)\qquad v(a_\nu+e^{2\pi i}(z-a_\nu),\ \bar{a}_\nu+e^{-2\pi i}(\bar{z}-\bar{a}_\nu))=e^{-2\pi i l_\nu}(z,\bar{z})$$

$$(\text{resp. } w_\pm(a_\nu+e^{2\pi i}(z-a_\nu),\ \bar{a}_\nu+e^{-2\pi i}(\bar{z}-\bar{a}_\nu))=-e^{-2\pi i l_\nu}w_\pm(z,\bar{z}))$$

$$0<|z-a_\nu|\ll 1,\quad \nu=1,\cdots,n,$$

as well as the growth order conditions

$$(3.0.2)\qquad |v|,\ |\bar{\partial}v|=O(|z-a_\nu|^{-[l_\nu]-1})$$

$$(\text{resp. } |w_\pm|=O(|z-a_\nu|^{-[l_\nu+(1/2)]-1})$$

$$|z-a_\nu|\to 0,\ \nu=1,\cdots,n$$

$$(3.0.3)\qquad |v|\ (\text{resp. } |w_\pm|)=O(e^{-2m|z|}),\quad |z|\to\infty.$$

Assuming $0<l_1,\cdots,l_n<1$ $\left(\text{resp.}-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}\right)$ we shall establish the n-dimensionality of the space $W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}$. We also show that, for general $l'_\nu\equiv l_\nu \bmod \boldsymbol{Z}$, an element $W^{l_1',\cdots,l_n'}_{*,a_1,\cdots,a_n}$ is obtained by applying differential operators with constant coefficients to a basis of $W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}$. Analogous results are obtained by specifying in place of (3.0.1) a class of n-dimensional monodromy representation parametrized by a symmetric matrix Λ. The case (3.0.1) is shown to be a degenerating limit of the latter.

In § 3.3 we shall derive a holonomic system of linear differential equations in $(z,\bar{z})$ satisfied by a basis of $W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}$. We show further that a canonical basis constructed in § 3.2 should satisfy a linear system of Pfaffian equations in the total set of variables $(z,\bar{z},a_1,\bar{a}_1,\cdots,a_n,\bar{a}_n)$, and that the coefficient matrices appearing in this system obey a nonlinear completely integrable system of Pfaffian equations (the deformation equations). Concerning the latter we note that, for $n=2$, there arises the Painlevé equation of the fifth kind [7], which in the case of equal exponents $l_1=l_2$ reduces to the third kind of restricted type ($\nu=0$ in [8]). Finally we introduce a closed 1-form associated with the deformation equations. It will be shown to coincide with the logarithmic derivative of the τ-function later in Chapter IV.

Main results of this chapter has been announced in [3], [6].

Chapter III. Monodromy Preserving Deformation in 2-Dimensional Euclidean Space.

§ 3. 1. The Euclidean Klein-Gordon and Dirac Equations

Let $z=(x^1+ix^2)/2$, $\bar{z}=(x^1-ix^2)/2$ denote the complex coordinate of the Euclidean 2-space $X^{\mathrm{Euc}}=\boldsymbol{R}^2$. We set $\partial=\partial_z=\dfrac{\partial}{\partial z}$, $\bar{\partial}=\partial_{\bar{z}}=\dfrac{\partial}{\partial\bar{z}}$, $M_B=z\partial-\bar{z}\bar{\partial}$ and $M_F=z\partial-\bar{z}\bar{\partial}+\dfrac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}$. Then we have the commutation relation

$$[\partial,\bar{\partial}]=0,\ [M_*,\partial]=-\partial,\ [M_*,\bar{\partial}]=\bar{\partial} \qquad (M_*=M_B \text{ or } M_F) \tag{3.1.1}$$

and $\{\partial,\bar{\partial},M_*\}$ spans the Lie algebra of the Euclidean motion group $E(2)$.

Consider the Euclidean Klein-Gordon equation

$$(m^2-\partial\bar{\partial})v=0 \tag{3.1.2}$$

and the Euclidean Dirac equation

$$(m-\Gamma)w=0 \tag{3.1.3}$$

$$\Gamma=\begin{pmatrix} & \partial \\ \bar{\partial} & \end{pmatrix},\quad w=\begin{pmatrix} w_+ \\ w_- \end{pmatrix}$$

with positive mass $m>0$. In this paragraph we shall study the local solutions of (3.1.2) and (3.1.3).

In general let $\mathcal{D}$ denote the sheaf of differential operators on a real analytic manifold X. By abuse of notation the sheaf of $N\times N$ matrices of differential operators will also be denoted by $\mathcal{D}$. Let $\mathcal{M}$: $Pu=0$, $P\in\mathcal{J}$, be a system of differential equations, where $\mathcal{J}$ denotes a coherent left Ideal of $\mathcal{D}$. We set $\mathcal{D}_0=\mathcal{D}_{0,\mathcal{J}}=\{P\in\mathcal{D}\,|\,\mathcal{J}\cdot P\subset\mathcal{J}\}$. Then $\mathcal{D}_0$ is the unique maximal subring of $\mathcal{D}$ containing $\mathcal{J}$ as its bi-Ideal, and consists of operators with the following property: for any $P\in\mathcal{D}_0$ and any solution u of $\mathcal{M}$, Pu is again a solution of $\mathcal{M}$.

Let $\mathcal{J}_B=\mathcal{D}(m^2-\partial_z\partial_{\bar{z}})$ (resp. $\mathcal{J}_F=\mathcal{D}(m-\Gamma)$) and set $\mathcal{D}_{0,B}=\mathcal{D}_{0,\mathcal{J}_B}$ (resp. $\mathcal{D}_{0,F}=\mathcal{D}_{0,\mathcal{J}_F}$). Then we have

Proposition 3. 1. 1.

(i) $\mathcal{D}_{0,B}=\mathcal{J}_B+\boldsymbol{C}[\partial,\bar{\partial},M_B]$, *and the set* $\{P\in\mathcal{D}\,|\,[P,m^2-\partial\bar{\partial}]=0\}$ *coincides with* $\boldsymbol{C}[\partial,\bar{\partial},M_B]$.

(ii) $\mathcal{D}_{0,F}=\mathcal{J}_F+\boldsymbol{C}[\partial,\bar{\partial},M_F]$, *and* $\{P\in\mathcal{D}\,|\,[P,m-\Gamma]=0\}=\boldsymbol{C}[\partial,\bar{\partial},M_F]\oplus\boldsymbol{C}[\partial,\bar{\partial},M_F]\Gamma$.

Proof. Denote the set $\mathcal{J}_*+\boldsymbol{C}[\partial,\bar{\partial},M_*]$ by $\mathcal{D}'_{0,*}$. The inclusion $\mathcal{D}'_{0,*}\subset\mathcal{D}_{0,*}$ being obvious, we prove the converse.

First we show (i). Let $P\in\mathcal{D}_{0,B}$ and set $P\equiv\sum_{j=-l}^{l}p_j(z,\bar{z})\,(m^{-1}\partial)^j$ mod $\mathcal{J}_B$, where $(m^{-1}\partial)^{-1}$ stands for $m^{-1}\bar{\partial}$. For any $u\neq0$, $Pe^{m(uz+u^{-1}\bar{z})}$ is a solution of (3. 1. 2), so that $0=e^{-m(uz+u^{-1}\bar{z})}(m^2-\partial\bar{\partial})Pe^{m(uz+u^{-1}\bar{z})}=-(\partial\bar{\partial}+mu^{-1}\partial+mu\bar{\partial})P(z,\bar{z},u)$ holds with $P(z,\bar{z},u)=\sum_{j=-l}^{l}p_j(z,\bar{z})\,u^j$. Setting $p_j=0$ for $|j|>l$, we obtain $0=m\bar{\partial}p_{j-1}+\partial\bar{\partial}p_j+m\partial p_{j+1}$ for any j. In particular $(m^{-1}\bar{\partial})^{[j/2]+1}p_{l-j}=0$ and $(m^{-1}\partial)^{[j/2]+1}p_{-l+j}=0$, hence $p_j(z,\bar{z})$ are polynomials. Since the operator $\partial\bar{\partial}+mu^{-1}\partial+mu\bar{\partial}$ commutes with $z\partial-\bar{z}\bar{\partial}-u\partial_u$, we may assume that $P(z,\bar{z},u)$ is homogeneous in $(z,\bar{z},u)$, i.e. that $P(z,\bar{z},u)=u^kP_1(uz,u^{-1}\bar{z})$, where $k\in\boldsymbol{Z}$ and $P_1(x,y)$ is a polynomial. We may assume further $k=0$, since $P\in\mathcal{D}'_{0,B}$ is equivalent to $P(m^{-1}\partial_z)^{-k}\in\mathcal{D}'_{0,B}$. We have then $(\partial_x\partial_y+m\partial_x+m\partial_y)P_1(x,y)=0$. This implies in particular that the highest order term $P_{1,l}$ of P_1 satisfies $(\partial_x+\partial_y)P_{1,l}=0$, and hence $P_{1,l}(x,y)=c_l(x-y)^l$ $(c_l\in\boldsymbol{C})$. Summing up, we have shown $P-c_lM_B^l\equiv\sum_{|j|\leq l-1}p_j(m^{-1}\partial)^j$ mod $\mathcal{J}_B$. By induction on l we have thus $\mathcal{D}'_{0,B}=\mathcal{D}_{0,B}$. Using this one verifies easily the rest of (i) by induction on the order of P.

Next let $P\in\mathcal{D}_{0,F}$. Noting $m^2-\partial\bar{\partial}=(m+\Gamma)(m-\Gamma)$, we see that each entry of $P(m+\Gamma)$ belongs to $\mathcal{D}_{0,B}$. It follows from (i) that $2mP\equiv P(m+\Gamma)\equiv Q$ mod $\mathcal{J}_F$ for some $Q\in M_2(\boldsymbol{C}[\partial,\bar{\partial},M_B])$, where $M_2(\boldsymbol{C}[\partial,\bar{\partial},M_B])$ denotes the set of 2×2 matrices with entries in $\boldsymbol{C}[\partial,\bar{\partial},M_B]$. Hence Q is uniquely written as $\sum_{j=0}^{k}Q'_j(\partial,\bar{\partial})M_B^j$, or, by rearranging terms, as $\sum_{j=0}^{k}Q_j(\partial,\bar{\partial})M_F^j$, where $Q_j,Q'_j\in M_2(\boldsymbol{C}[\partial,\bar{\partial}])$. On the other hand, by the definition of $\mathcal{D}_{0,F}$, there exists an $R\in\mathcal{D}$ such that $(m-\Gamma)Q=R(m-\Gamma)\in M_2(\boldsymbol{C}[\partial,\bar{\partial},M_B])$. Then R commutes with $m^2-\partial\bar{\partial}$, so that R also has the form $\sum_{j=0}^{l}R_j(\partial,\bar{\partial})M_F^j$. Since M_F commutes with Γ, we see that $(m-\Gamma)Q_j=R_j(m-\Gamma)$ by the uniqueness of representation. Set

$Q_j \equiv \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \bmod \mathcal{D}(m^2 - \partial_z\partial_{\bar{z}})$, $q_\nu = q_\nu(m^{-1}\partial_z) = \sum_{k=-l}^{l} q_{\nu k}(m^{-1}\partial)^k$ $(\nu = 1, 2, 3, 4)$. From $0 = (m - \Gamma)Q_j \begin{pmatrix} u \\ 1 \end{pmatrix} e^{m(uz + u^{-1}\bar{z})}$ we get $u(q_1(u) - uq_3(u)) + (q_2(u) - uq_4(u)) = 0$. This implies $Q_j \underset{\bmod \mathcal{I}_F}{\equiv} \begin{pmatrix} q_1 - (m^{-1}\partial q_3 + q_4) & \\ & -m^{-1}\partial q_3 \end{pmatrix} \cdot (1 - m^{-1}\Gamma) + (m^{-1}\partial q_3 + q_4) \in \mathcal{D}'_{0,F}$. Similar argument proves the rest of (ii).

We now introduce a series of multi-valued special solutions of (3.1.2) and (3.1.3). For $l \in \mathbb{C}$ let $I_l(x)$ and $K_l(x)$ denote the modified Bessel functions of the first and second kind, respectively. We set

$$v_l(z, \bar{z}) = e^{il\theta} I_l(mr),\ v_l^*(z, \bar{z}) = e^{-il\theta} I_l(mr) \tag{3.1.4}$$

$$\tilde{v}_l(z, \bar{z}) = \tilde{v}^*_{-l}(z, \bar{z}) = e^{il(\theta+\pi)} K_l(mr),$$

where $z = \frac{1}{2} re^{i\theta}$, $\bar{z} = \frac{1}{2} re^{-i\theta}$ $(r \geqq 0, \theta \in \mathbb{R})$, and

$$w_l(z, \bar{z}) = \begin{pmatrix} v_{l-1/2}(z, \bar{z}) \\ v_{l+1/2}(z, \bar{z}) \end{pmatrix},\quad w_l^*(z, \bar{z}) = \begin{pmatrix} v^*_{l+1/2}(z, \bar{z}) \\ v^*_{l-1/2}(z, \bar{z}) \end{pmatrix} \tag{3.1.5}$$

$$\tilde{w}_l(z, \bar{z}) = \tilde{w}^*_{-l}(z, \bar{z}) = \begin{pmatrix} \tilde{v}_{l-1/2}(z, \bar{z}) \\ \tilde{v}_{l+1/2}(z, \bar{z}) \end{pmatrix}.$$

For $l \not\equiv 0 \bmod \mathbb{Z}$, we have $\tilde{v}_l = \frac{\pi}{2} \frac{e^{\pi i l}}{\sin \pi l}(-v_l + v^*_{-l})$. For $l \equiv 0 \bmod \mathbb{Z}$, $v_l = v^*_{-l}$ holds and $v_l, \tilde{v}_l$ are linearly independent. Likewise we have $\tilde{w}_l = \frac{\pi}{2} \frac{ie^{\pi i l}}{\cos \pi l}(-w_l + w^*_{-l})$ for $l \not\equiv \frac{1}{2} \bmod \mathbb{Z}$, and $w_l = w^*_{-l}$ for $l \equiv \frac{1}{2} \bmod \mathbb{Z}$. These functions are multi-valued solution (outside the origin) of (3.1.2) and (3.1.3) respectively, having the following local behavior as $|z| \to 0$:

$$v_l(z, \bar{z}) = (mz)^l \left(\frac{1}{l!} + \frac{m^2 z\bar{z}}{(l+1)!} + \cdots \right) \tag{3.1.6}$$

$$v_l^*(z, \bar{z}) = (m\bar{z})^l \left(\frac{1}{l!} + \frac{m^2 z\bar{z}}{(l+1)!} + \cdots \right)$$

$$w_l(z, \bar{z}) = \begin{pmatrix} (mz)^{l-1/2} \left(\dfrac{1}{(l-\frac{1}{2})!} + \dfrac{m^2 z\bar{z}}{(l+\frac{1}{2})!} + \cdots \right) \\ (mz)^{l+1/2} \left(\dfrac{1}{(l+\frac{1}{2})!} + \dfrac{m^2 z\bar{z}}{(l+\frac{3}{2})!} + \cdots \right) \end{pmatrix} \tag{3.1.7}$$

$$w_l^*(z,\bar z)=\begin{pmatrix}(m\bar z)^{l+1/2}\left(\dfrac{1}{(l+\frac12)!}+\dfrac{m^2z\bar z}{(l+\frac32)!}+\cdots\right)\\ (m\bar z)^{l-1/2}\left(\dfrac{1}{(l-\frac12)!}+\dfrac{m^2z\bar z}{(l+\frac12)!}+\cdots\right)\end{pmatrix}$$

where $l!=\Gamma(1+l)$.

For integral $l=0, 1, 2, \cdots$ we have

$$(3.1.8)\qquad \tilde v_l(z,\bar z)=\frac12(-)^l\frac{(l-1)!}{(m\bar z)^l}\sum_{j=0}^{l-1}(-)^j\frac{(l-j-1)!}{j!(l-1)!}(m^2z\bar z)^j+\cdots$$
$$-(\gamma+\log m\sqrt{z\bar z})\cdot(mz)^l\left(\frac{1}{l!}+\frac{m^2z\bar z}{(l+1)!}+\cdots\right),$$
$$\tilde v_l^*(z,\bar z)=\overline{\tilde v_l(z,\bar z)},$$

and for half integral $l=1/2, 3/2, \cdots$ we have

$$(3.1.9)\qquad \tilde w_l(z,\bar z)=\frac12(-)^{l-1/2}$$
$$\times\begin{pmatrix}\dfrac{(l-3/2)!}{(m\bar z)^{l-1/2}}\displaystyle\sum_{j=0}^{l-3/2}(-)^j\frac{(l-j-3/2)!}{j!(l-3/2)!}(m^2z\bar z)^j+\cdots\\ -\dfrac{(l-1/2)!}{(m\bar z)^{l+1/2}}\displaystyle\sum_{j=0}^{l-1/2}(-)^j\frac{(l-j-1/2)!}{j!(l-1/2)!}(m^2z\bar z)^j+\cdots\end{pmatrix}$$
$$-(\gamma+\log m\sqrt{z\bar z})\cdot$$
$$\times\begin{pmatrix}(mz)^{l-1/2}\left(\dfrac{1}{(l-1/2)!}+\dfrac{m^2z\bar z}{(l+1/2)!}+\cdots\right)\\ (mz)^{l+1/2}\left(\dfrac{1}{(l+1/2)!}+\dfrac{m^2z\bar z}{(l+3/2)!}+\cdots\right)\end{pmatrix}$$
$$\tilde w_l^*(z,\bar z)=\begin{pmatrix}\tilde w_{l-}(z,\bar z)\\ \tilde w_{l+}(z,\bar z)\end{pmatrix}.$$

In particular $\tilde v_0$ and $\tilde w_{\pm1/2}$ are fundamental solutions of (3.1.2) and (3.1.3), respectively:

$$(3.1.10)\qquad (m^2-\partial\bar\partial)\,\tilde v_0(z,\bar z)=2\pi\delta(x^1)\,\delta(x^2)$$

$$(3.1.11)\qquad (m-\Gamma)\,(\tilde w_{1/2}(z,\bar z),\ \tilde w_{1/2}^*(z,\bar z))=\frac1m I_2\cdot 2\pi\delta(x^1)\,\delta(x^2)$$

where $z=(x^1+ix^2)/2$, $\bar z=(x^1-ix^2)/2$ and $I_2=\begin{pmatrix}1&\\&1\end{pmatrix}$.

Remark. (3.1.4) or (3.1.5) reduce to elementary functions if $l \equiv \frac{1}{2} \bmod \boldsymbol{Z}$ or $l \equiv 0 \bmod \boldsymbol{Z}$, respectively. For example

$$v_{-1/2}(z, \bar{z}) = \frac{1}{\sqrt{m\pi}} \frac{1}{\sqrt{z}} \cosh(2m|z|),$$

$$v^*_{-1/2}(z, \bar{z}) = \frac{1}{\sqrt{m\pi}} \frac{1}{\sqrt{\bar{z}}} \cosh(2m|z|),$$

$$\tilde{v}_{-1/2}(z, \bar{z}) = \frac{1}{2} \sqrt{\frac{\pi}{m}} \frac{-i}{\sqrt{z}} e^{-2m|z|}.$$

$$w_0(z, \bar{z}) = \frac{1}{\sqrt{m\pi}} \begin{pmatrix} \frac{1}{\sqrt{z}} \cosh(2m|z|) \\ \sqrt{z} \frac{\sinh(2m|z|)}{|z|} \end{pmatrix},$$

$$w^*_0(z, \bar{z}) = \frac{1}{\sqrt{m\pi}} \begin{pmatrix} \sqrt{\bar{z}} \frac{\sinh(2m|z|)}{|z|} \\ \frac{1}{\sqrt{\bar{z}}} \cosh(2m|z|) \end{pmatrix},$$

$$\tilde{w}_0(z, \bar{z}) = \frac{1}{2} \sqrt{\frac{\pi}{m}} \begin{pmatrix} \frac{-i}{\sqrt{z}} e^{-2m|z|} \\ \frac{i}{\sqrt{\bar{z}}} e^{-2m|z|} \end{pmatrix}.$$

Proposition 3.1.2. *We have the following recursion relations:*

$$\partial v_l = m v_{l-1}, \quad \bar{\partial} v_l = m v_{l+1}, \quad M_B v_l = l v_l, \tag{3.1.12}$$

$$\partial v^*_l = m v^*_{l+1}, \quad \bar{\partial} v^*_l = m v^*_{l-1}, \quad M_B v^*_l = -l v^*_l.$$

The same relations hold if we replace v_l by $\tilde{v}_l$ and v^*_l by $\tilde{v}^*_l$. Likewise we have

$$\partial w_l = m w_{l-1}, \quad \bar{\partial} w_l = m w_{l+1}, \quad M_F w_l = l w_l, \tag{3.1.13}$$

$$\partial w^*_l = m w^*_{l+1}, \quad \bar{\partial} w^*_l = m w^*_{l-1}, \quad M_F w^*_l = -l w^*_l.$$

The same relations hold if we replace w_l by $\tilde{w}_l$ and w^*_l by $\tilde{w}^*_l$.

Proof. Using $\partial = e^{-i\theta}\left(\frac{\partial}{\partial r} + \frac{1}{ir}\frac{\partial}{\partial \theta}\right)$, $\bar{\partial} = e^{i\theta}\left(\frac{\partial}{\partial r} - \frac{1}{ir}\frac{\partial}{\partial \theta}\right)$ and $M_B =$

$-i\dfrac{\partial}{\partial\theta}$, we obtain (3. 1. 12) $\sim$ (3. 1. 13) immediately from the formula

$$r\frac{d}{dr}I_l(r)\pm lI_l(r)=rI_{l\mp1}(r)$$

$$r\frac{d}{dr}K_l(r)\pm lK_l(r)=-rK_{l\mp1}(r)\,.$$

Proposition 3. 1. 3. *Let v be a multi-valued local solution of* (3. 1. 2) *at* $(z,\bar z)=(a,\bar a)$ *such that, for some* $l_0\in\boldsymbol{C}$,

$$v(a+e^{2\pi i}(z-a),\ \bar a+e^{-2\pi i}(\bar z-\bar a))=e^{2\pi i l_0}v(z,\bar z) \tag{3. 1. 14}$$

$$|v(z,\bar z)|=\begin{cases} O(|z-a|^{\mathrm{Re}\,l_0}) & (l_0\neq0)\\ O(|\log|z-a||) & (l_0=0)\end{cases} \tag{3. 1. 15}$$

as $|z-a|\to0$ *and* $|\mathrm{Arg}(z-a)|<C$ *for any* $C>0$. *Here the left hand side of* (3. 1. 14) *indicates the analytic continuation of* $v(z,\bar z)$ *along the path* $z=a+\frac{1}{2}re^{i\theta}$, $\bar z=\bar a+\frac{1}{2}re^{-i\theta}$ $(r>0,\theta:0\to2\pi)$. *Then v is uniquely expanded in the form*

$$v(z,\bar z)=\begin{cases}\displaystyle\sum_{\substack{l\equiv l_0 \bmod \boldsymbol{Z}\\ \mathrm{Re}\,l\geqq\mathrm{Re}\,l_0}}c_lv_l[a]+\sum_{\substack{l\equiv -l_0 \bmod \boldsymbol{Z}\\ \mathrm{Re}\,l\geqq\mathrm{Re}\,l_0}}c_l^*v_l^*[a] & (l_0\not\equiv0 \bmod \boldsymbol{Z})\\ \displaystyle\sum_{\substack{l\equiv0 \bmod \boldsymbol{Z}\\ -l_0\geqq l\geqq0}}(\tilde c_l\tilde v_l[a]+\tilde c_l^*\tilde v_l^*[a])\\ \qquad\displaystyle+\sum_{\substack{l\equiv0 \bmod \boldsymbol{Z}\\ l\geqq\max(l_0,0)}}(c_lv_l[a]+c_l^*v_l^*[a]) & (l_0\equiv0 \bmod \boldsymbol{Z}).\end{cases} \tag{3. 1. 16}$$

Here $c_l, c_l^*, \tilde c_l, \tilde c_l^*\in\boldsymbol{C}$,(*) *and we have set* $v_l[a]=v_l(z-a,\bar z-\bar a)$, *etc.*

Proof. Set $z-a=\frac{1}{2}re^{i\theta}$, $\bar z-\bar a=\frac{1}{2}re^{-i\theta}$, $r\geqq0$ and $\theta\in\boldsymbol{R}$. By the definition v is a function of (r,θ) defined on $(0,\varepsilon)\times\boldsymbol{R}$ $(\varepsilon>0)$ and is real analytic there by virtue of the ellipticity of $m^2-\partial\bar\partial$. Expanding the single-valued function $e^{-il_0\theta}v$ into a Fourier series in θ, we have

$$v\Big(a+\frac{1}{2}re^{i\theta},\ \bar a+\frac{1}{2}re^{-i\theta}\Big)=\sum_{l\equiv l_0 \bmod \boldsymbol{Z}}e^{il\theta}f_l(r), \tag{3. 1. 17}$$

which converges absolutely and uniformly on any compact subset of

(*) Regarding $v_0=v_0^*$ and $\tilde v_0=\tilde v_0^*$, we set $c_0=c_0^*$, $\tilde c_0=\tilde c_0^*$ to assure the uniqueness of representation.

$(0, \varepsilon) \times \boldsymbol{R}$. Substitution of (3. 1. 17) into (3. 1. 2) yields

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\left(m^2+\frac{l^2}{r^2}\right)\right)f_l(r)=0\,,$$

which implies

$$(3.1.18)\qquad f_l(r)=\begin{cases} c_l I_l(mr)+c_{-l}^* I_{-l}(mr) & (l\not\equiv 0 \bmod \boldsymbol{Z}) \\ c_l' I_l(mr)+\tilde{c}_l' K_l(mr) & (l\equiv 0 \bmod \boldsymbol{Z}). \end{cases}$$

On the other hand, assumption (3. 1. 15) gives

$$(3.1.19)\qquad |f_l(r)|\leqq \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\left|e^{-il\theta}v\left(a+\frac{1}{2}re^{i\theta},\ \bar{a}+\frac{1}{2}re^{-i\theta}\right)\right|$$

$$\leqq \text{const.}\begin{cases} r^{\operatorname{Re} l_0} & (l_0\neq 0) \\ |\log r| & (l_0=0) \end{cases}$$

as $r\to 0$. Combining (3. 1. 17), (3. 1. 18) and (3. 1. 19) we obtain (3. 1. 16). Uniqueness of the expansion follows from that of the Fourier expansion.

As an immediate consequence of Proposition 3. 1. 3 we obtain

Proposition 3. 1. 4. *A multi-valued local solution of* (3. 1. 3) *at* $(a, \bar{a})$ *satisfying*

$$(3.1.20)\qquad w(a+e^{2\pi i}(z-a),\ \bar{a}+e^{-2\pi i}(\bar{z}-\bar{a}))=e^{2\pi i(l_0-1/2)}w(z,\bar{z})$$

$$(3.1.21)\qquad |w(z,\bar{z})|=O(|z-a|^{\operatorname{Re} l_0-1/2})\quad \textit{as}\quad |z-a|\to 0\,,$$

$$|\operatorname{Arg}(z-a)|<C\quad \textit{for any } C>0$$

is uniquely expanded as follows:

$$(3.1.22)\qquad w(z,\bar{z})=\begin{cases} \displaystyle\sum_{\substack{l\equiv l_0 \bmod \boldsymbol{Z}\\ \operatorname{Re} l\geqq \operatorname{Re} l_0}} c_l w_l[a]+\sum_{\substack{l\equiv -l_0 \bmod \boldsymbol{Z}\\ \operatorname{Re} l\geqq \operatorname{Re} l_0}} c_l^* w_l^*[a] & \left(l_0\not\equiv\frac{1}{2} \bmod \boldsymbol{Z}\right) \\ \displaystyle\sum_{\substack{l\equiv 1/2 \bmod \boldsymbol{Z}\\ -l_0\geqq l\geqq 1/2}} (\tilde{c}_l\tilde{w}_l[a]+\tilde{c}_l^*\tilde{w}_l^*[a]) & \\ \quad+\displaystyle\sum_{\substack{l\equiv 1/2 \bmod \boldsymbol{Z}\\ l\geqq \max(l_0,\,1/2)}} (c_l w_l[a]+c_l^* w_l^*[a]) & \left(l_0\equiv\frac{1}{2} \bmod \boldsymbol{Z}\right) \end{cases}$$

where $w_l[a]=w_l(z-a,\bar{z}-\bar{a})$, *etc.*

In the terminology of [3], w is said to be of *Fermi-type* at $(a, \bar{a})$ if $l_0 \in \boldsymbol{Z}$ in (3.1.22), and of *strict Fermi-type* if in addition $l_0 \geqq 0$.

From Proposition 3.1.2 we see that the form of the local expansion (3.1.16) (resp. (3.1.22)) is not changed by applying operators in $\boldsymbol{C}[\partial, \bar{\partial}, M_B]$ (resp. $\boldsymbol{C}[\partial, \bar{\partial}, M_F]$) except for the growth order at the branch point $(a, \bar{a})$. The following Proposition shows that asymptotic behavior as $|z| \to \infty$ is preserved as well.

Proposition 3.1.5. *Let $v(z, \bar{z})$ be a multi-valued solution of (3.1.2) in a neighborhood of $\{2|z| \geqq R\}$ having the monodromy property $v(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = e^{2\pi i l_0} v(z, \bar{z})$ (i.e. $e^{-il_0\theta}v$ is single-valued) with some $l_0 \in \boldsymbol{C}$. Then the following are equivalent:*

(i) $|e^{-il_0\theta}v| \in L^2(\{2|z| \geqq R\})$

(ii) $\sup_{2|z| \geqq R} |e^{-il_0\theta}v| < \infty$

(iii) $v(z, \bar{z}) = \sum_{l \equiv l_0 \bmod \boldsymbol{Z}} \tilde{c}_l e^{il\theta} K_l(mr) \quad \left(z = \frac{1}{2} re^{i\theta},\ r \geqq R\right)$

(iv) $|e^{-il_0\theta}(pv)(z, \bar{z})| = O\left(\frac{e^{-2m|z|}}{\sqrt{|z|}}\right)$ $(|z| \to \infty)$ *for any* $p \in \boldsymbol{C}[\partial, \bar{\partial}, M_B]$.

Proof. Implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are obvious. We shall prove (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

Consider the Fourier expansion

$$v\left(\frac{1}{2} re^{i\theta}, \frac{1}{2} re^{-i\theta}\right) = \sum_{l \equiv l_0 \bmod \boldsymbol{Z}} e^{il\theta} f_l(r) \quad (r \geqq R)$$

where $f_l(r) = c_l I_l(mr) + \tilde{c}_l K_l(mr)$ with some $c_l, \tilde{c}_l \in \boldsymbol{C}$. Now (ii) implies $\sup_{r \geqq R} |f_l(r)| < \infty$, hence $c_l = 0$ for all l. This proves (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (iv), it suffices to prove (iv) for $p = 1$, since pv satisfies the same condition (iii) if $p \in \boldsymbol{C}[\partial, \bar{\partial}, M_B]$. From the integral representation $\left(l > \frac{1}{2},\ r > 0\right)$

$$K_l(r) = K_{-l}(r) = \sqrt{\frac{\pi}{2r}} e^{-r} \frac{1}{\Gamma(l + \frac{1}{2})} \int_0^\infty e^{-t} t^{l-1/2} \left(1 + \frac{t}{2r}\right)^{l-1/2} dt$$

$$= \sqrt{\frac{\pi}{2r}} e^{-r} \times (\text{monotone decreasing function of } r)$$

it follows that

$$0<K_l(mr)/K_l(mR)\leqq\sqrt{R}e^{mR}\frac{1}{\sqrt{r}}e^{-mr}\quad\left(|l|>\frac{1}{2}\right).$$

Consequently we have

$$\sum_{\substack{l\equiv l_0 \bmod \boldsymbol{Z}\\ |l|>1/2}}|\tilde{c}_lK_l(mr)|=\sum_{\substack{l\equiv l_0 \bmod \boldsymbol{Z}\\ |l|>1/2}}|\tilde{c}_lK_l(mR)||K_l(mr)/K_l(mR)|$$

$$\leqq\{\sqrt{R}e^{mR}\sum_{\substack{l\equiv l_0 \bmod \boldsymbol{Z}\\ |l|>1/2}}|\tilde{c}_lK_l(mR)|\}\cdot\frac{1}{\sqrt{r}}e^{-mr}.$$

Terms with $|l|\leqq\frac{1}{2}$ are estimated directly as $|K_l(mr)|=O\left(\frac{1}{\sqrt{mr}}e^{-mr}\right)$. This proves Proposition 3.1.5.

§ 3.2. Wave Functions

In this paragraph we shall consider a 2-dimensional analogue of the Riemann-Hilbert problem(*) for solutions of the Euclidean Klein-Gordon and Dirac equations.

Let $(a_\nu, \bar{a}_\nu)$ $(\nu=1,\cdots,n)$ be distinct n-points of X^{Euc}. Denote by $\tilde{X}'=\tilde{X}'_{a_1,\cdots,a_n}$ the universal covering manifold of $X'=X'_{a_1,\cdots,a_n}=X^{\mathrm{Euc}}-\{(a_\nu,\bar{a}_\nu)\}_{\nu=1,\cdots,n}$ with the covering projection $\pi:\tilde{X}'\to X'$. We fix base points $\tilde{x}_0\in\tilde{X}'$, $x_0\in X'$ so that $\pi(\tilde{x}_0)=x_0$, and denote by $\pi_1(X';x_0)$ the fundamental group of X'. We use the following convention: A closed path $\gamma=\gamma(t)$ $(0\leqq t\leqq1,\ \gamma(0)=\gamma(1)=x_0)$ is confused with its homotopy class in $\pi_1(X';x_0)$. Product of $\gamma,\gamma'\in\pi_1(X';x_0)$ is defined in the order $(\gamma\gamma')(t)=\gamma'(2t)$ $(0\leqq t\leqq1/2)$, $=\gamma(2t-1)$ $(1/2\leqq t\leqq1)$. An element $\gamma\in\pi_1(X';x_0)$ is identified with the covering transformation it induces on $\tilde{X}'$. For a function u on $\tilde{X}'$ we set $(\gamma u)(\tilde{x})=u(\gamma^{-1}(\tilde{x}))$.(**) Finally we fix a set of generators $\gamma_1,\cdots,\gamma_n\in\pi_1(X';x_0)$, where γ_ν is a closed path encircling $(a_\nu,\bar{a}_\nu)$ clockwise (Fig. 3.2.1):

First consider the case of the Euclidean Klein-Gordon equation (3.1.2). For $l_1,\cdots,l_n\in\boldsymbol{R}$, let

(*) For a technical reason we shall exclusively deal with the case of unitary monodromy in this Chapter III.

(**) We have changed the notation from the one in II. In the notation here the analytic continuation of u along γ is denoted by $\gamma^{-1}u$.

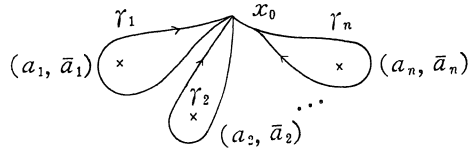

Fig. 3.2.1.

(3.2.1) $\rho_{l_1,\cdots,l_n}: \pi_1(X'; x_0) \to U(1)$, $\gamma_\nu \mapsto e^{-2\pi i l_\nu}$ $(\nu=1, \cdots, n)$

be a unitary representation of the fundamental group.

Definition 3.2.1. *For $l_\nu \notin \boldsymbol{Z}$ $(\nu=1, \cdots, n)$, $W_{B,a_1,\cdots,a_n}^{l_1,\cdots,l_n}$ will denote the space consisting of complex-valued real analytic function v on $\widetilde{X}'$ with the following properties:*

(3.2.2) $$(m^2-\partial\bar{\partial})\, v=0 \quad \text{on } \widetilde{X}'.$$

(3.2.3) $$\gamma v = v \cdot \rho_{l_1,\cdots,l_n}(\gamma) \quad \text{for any } \gamma \in \pi_1(X'; x_0),$$

$$\text{i.e.} \quad (\gamma_\nu v)(z, \bar{z}) = v(a_\nu + e^{2\pi i}(z-a_\nu),\ \bar{a}_\nu + e^{-2\pi i}(\bar{z}-\bar{a}_\nu))$$

$$= e^{-2\pi i l_\nu} v(z, \bar{z})$$

$$(0<|z-a_\nu| \ll 1;\ \nu=1, \cdots, n).$$

(3.2.4)$_\nu$ $$|v| = O(|z-a_\nu|^{-[l_\nu]-1}),\quad |\bar{\partial} v| = O(|z-a_\nu|^{-[l_\nu]-1})\ ^{*)}$$

$$as \quad |z-a_\nu| \to 0 \quad (\nu=1, \cdots, n),$$

(3.2.4)$_\infty$ $$|v| = O(e^{-2m|z|}) \qquad as \quad |z| \to \infty.$$

Since $\rho(\gamma)$ is unitary, (3.2.3) guarantees that $|v|$ is a single-valued function on X'. From Proposition 3.1.3, we see that, under (3.2.2) and (3.2.3), the growth order condition (3.2.4)$_\nu$ is equivalent to the expansion of the form

(3.2.4)$'_\nu$ $$v = \sum_{j=0}^{\infty} c_{-l_\nu+j}^{(\nu)}(v) \cdot v_{-l_\nu+j}[a_\nu] + \sum_{j=0}^{\infty} c_{l_\nu^*+j}^{*(\nu)}(v) \cdot v_{l_\nu^*+j}^*[a_\nu]$$

where $c_{-l_\nu+j}^{(\nu)}(v)$, $c_{l_\nu^*+j}^{*(\nu)}(v) \in \boldsymbol{C}$ and $l_\nu^* = l_\nu - 2[l_\nu]$. For definiteness, in (3.2.4)$'_\nu$ we choose the branch of v on the "first sheet", i.e. the value

(*) For $l \in \boldsymbol{R}$, $[l]$ denotes the integer satisfying $[l] \leqq l < [l]+1$.

$v|_{\tilde{\gamma}_\nu}$ on the lifting $\tilde{\gamma}_\nu$ of γ_ν to the base point $\tilde{x}_0 \in \tilde{X}'$.

Likewise $W^{l_1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_n}$ will denote the space of v's satisfying (3. 2. 2), (3. 2. 3), (3. 2. 4)$_\infty$ and

$$(3.2.4)^{\text{strict}}_\nu \qquad v=\sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(v)\cdot v_{-l_\nu+j}[a_\nu]+\sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu+j}(v)\cdot v^*_{l_\nu+j}[a_\nu]$$

for $\nu=1,\cdots,n$. (for $0<l_1,\cdots,l_n<1$ this coincides with $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$).

We set also $W^{*l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}=\{\bar{v}\,|\,v\in W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}\}$, $W^{*l_1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_n}=\{\bar{v}\,|\,v \in W^{l_1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_n}\}$.

For the moment we consider the case $0<l_\nu<1$ $(\nu=1,\cdots,n)$.

Example $(n=1)$. For $0<l<1$, $W^l_{B,a}$ is a 1-dimensional space spanned by $\tilde{v}_{-l}[a]=\dfrac{\pi}{2}\dfrac{e^{-\pi i l}}{\sin \pi l}(v_{-l}[a]-v^*_l[a])$. It is shown below (Theorem 3. 2. 4) that $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ $(0<l_1,\cdots,l_n<1)$ is exactly n-dimensional. We note that the exponential fall-off condition (3. 2. 4)$_\infty$ is crucial for the finite dimensionality. For instance, $v_{-l+j}[a]$ and $v^*_{l+j}[a]$ $(j=0,1,2,\cdots)$ all satisfy (3. 2. 2)$\sim$(3. 2. 4) with $n=1$ except for (3. 2. 4)$_\infty$.

Now we set

$$(3.2.5) \qquad I_B(v,v')=\frac{1}{2}\int\!\!\int_{X^{\text{Euc}}} i dz\wedge d\bar{z}\,(\bar{\partial}v\cdot\partial\bar{v}'+m^2 v\bar{v})=\overline{I_B(v',v)}.$$

If v, v' both satisfy the monodromy condition (3. 2. 3), the integrand is single-valued on X' and hence (3. 2. 5) makes sense.

Proposition 3. 2. 2. *Assume* $0<l_\nu<1$ *for* $\nu=1,\cdots,n$. *Under the conditions* (3. 2. 2) *and* (3. 2. 3), (3. 2. 4) *holds if and only if* $I_B(v,v)<\infty$. *In this case we have*

$$(3.2.6) \qquad I_B(v,v')=-\sum_{\nu=1}^{n} c^{(\nu)}_{-l_\nu}(v)\cdot\overline{c^{*(\nu)}_{l_\nu}(v')}\cdot\sin\pi l_\nu .$$

Proof. In view of Propositions 3. 1. 3 and 3. 1. 5, conditions (3. 2. 4)$_\nu$ and (3. 2. 4)$_\infty$ are equivalent to

$$(3.2.4)''_\nu \qquad |v|,\ |\bar{\partial}v|\in L^2(D_{\nu,\varepsilon})$$

$$(3.2.4)''_\infty \qquad |v|,\ |\bar{\partial}v|\in L^2(D_{\infty,R})$$

respectively, where $D_{\nu,\varepsilon}=\{(z,\bar{z})\in X^{\mathrm{Euc}}||z-a_\nu|\leqq\varepsilon\}$ $(0<\varepsilon\ll 1)$ and $D_{\infty,R}=\{(z,\bar{z})\in X^{\mathrm{Euc}}||z|\geqq R\}$ $(R\gg 1)$. Hence (3.2.4) holds if and only if $|v|$, $|\bar{\partial}v|\in L^2(X^{\mathrm{Euc}})$, i.e. $I_B(v,v)<\infty$. Now assume that v, v' satisfies this condition. Notinng

$$(\bar{\partial}v\cdot\partial\bar{v}'+m^2v\bar{v}')\,idz\wedge d\bar{z}=-id(v\partial\bar{v}'dz)$$

for $(z,\bar{z})\neq(a_\nu,\bar{a}_\nu)$ $(\nu=1,\cdots,n)$ we have

$$I_B(v,v')=\lim_{\varepsilon\downarrow 0}\frac{1}{2}\int\!\!\int_{X^{\mathrm{Euc}}-\bigcup_{\nu=1}^{n}D_{\varepsilon,\nu}}-id(v\partial\bar{v}'dz)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{2}\sum_{\nu=1}^{n}\oint_{\partial D_{\nu,\varepsilon}}iv\partial\bar{v}'dz.$$

Substitution of the expansion $(3.2.4)'_\nu$ yields

$$\oint_{\partial D_{\nu,\varepsilon}}iv\partial\bar{v}'dz$$

$$=\oint_{|z-a_\nu|=\varepsilon}idz\left(c^{(\nu)}_{-l_\nu}(v)\cdot\frac{(m(z-a_\nu))^{-l_\nu}}{(-l_\nu)!}+\cdots\right)$$

$$\times m\left(\overline{c^{*(\nu)}_{l_\nu}(v')}\cdot\frac{(m(z-a_\nu))^{l_\nu-1}}{(l_\nu-1)!}+\cdots\right)$$

$$=c^{(\nu)}_{-l_\nu}(v)\cdot\overline{c^{*(\nu)}_{l_\nu}(v')}\cdot\frac{\sin\pi(-l_\nu+1)}{\pi}\cdot i\cdot 2\pi i+O(\varepsilon).$$

This proves (3.2.6).

Proposition 3.2.2 shows that the spaces $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ and $W^{*l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ are mutually dual through the $\boldsymbol{C}$-bilinear inner product $I_B(v,\bar{v}')$ $(v\in W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$, $v'\in W^{*l_1,\cdots,l_n}_{B,a_1,\cdots,a_n})$. For $v\in W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$, we set $\boldsymbol{c}(v)=(c^{(1)}_{-l_1}(v),\cdots,c^{(n)}_{-l_n}(v))$. Since I_B is positive definite, (3.2.6) shows that the linear map

$$\boldsymbol{c}: W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}\to\boldsymbol{C}^n,\quad v\mapsto\boldsymbol{c}(v)$$

is injective for $0<l_1,\cdots,l_n<1$. Therefore we have the following result.

Corollary 3.2.3. *We have* $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}=0$ *if* $l_1,\cdots,l_n<0$, *and* $\dim_{\boldsymbol{C}}W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}\leqq n$ *if* $0<l_1,\cdots,l_n<1$.

Remark. Injectivity of the map $\boldsymbol{c}$ is also shown by the following argument. Let $v \in W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ $(0<l_1,\cdots,l_n<1)$ satisfy $\boldsymbol{c}(v)=0$, and consider the real analytic function $f=v\bar{v}\geqq 0$ on $X'_{a_1,\cdots,a_n}$. Since $f\to 0$ at the boundary $|z-a_\nu|\to 0$ $(\nu=1,\cdots,n)$ or $|z|\to\infty$, f attains its maximum on X'. On the other hand we have $(-\partial\bar{\partial}+m^2)f=-(|\bar{\partial}v|^2+|\partial v|^2+m^2|v|^2)\leqq 0$, hence by the maximum principle we obtain $\max f\leqq 0$. This implies $f\equiv 0$.

We shall now show that the equality $\dim_{\boldsymbol{C}} W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}=n$ holds for $0<l_1,\cdots,l_n<1$, by making use of some functional analysis. In the forthcoming Chapter IV, we shall explicitly construct a basis of $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ in terms of field operators.

Theorem 3.2.4. *For $0<l_1,\cdots,l_n<1$, there exists a unique basis $v_1,\cdots,v_n$ of $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ such that $c^{(\nu)}_{-l_\nu}(v_\mu)=\delta_{\mu\nu}$ $(\mu,\nu=1,\cdots,n)$. In particular* $\dim_{\boldsymbol{C}} W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}=n$.

Proof. Uniqueness follows from Corollary 3.2.3. We are only to prove the existence.

Let $\mathcal{H}'_\rho$ denote the space of C^∞-functions v on $\tilde{X}'$ such that v satisfies (3.2.3) and $\mathrm{supp}|v|$ is compact in X'. On $\mathcal{H}'_\rho$ we introduce a positive definite hermitian inner product by

$$2I_B(v,v')=\frac{1}{2}\int\!\!\int_{X^{\mathrm{Euc}}} idz\wedge d\bar{z}\,(\bar{\partial}v\cdot\partial\bar{v}'+\partial v\cdot\bar{\partial}\bar{v}'+2m^2v\bar{v}')$$

$$=\int\!\!\int_{X^{\mathrm{Euc}}} idz\wedge d\bar{z}\; v\cdot(-\partial\bar{\partial}+m^2)\bar{v}',\quad v,v'\in\mathcal{H}'_\rho.$$

Denote by $\mathcal{H}_\rho$ the Hilbert space completion of $\mathcal{H}'_\rho$ with respect to $2I_B$. Note that $\mathcal{H}_\rho$ is nothing but the usual Sobolev space when restricted to a simply connected open subset $U\subset X'$ (more precisely, if we consider the restriction $v|_{\tilde{U}_1}$ to a connected component $\tilde{U}_1$ of $\pi^{-1}(U)\subset\tilde{X}'$). Hence an element v of $\mathcal{H}_\rho$ is identified with a measurable function on $\tilde{X}'$, satisfying (3.2.3) and $|v|,|\partial v|,|\bar{\partial}v|\in L^2(X^{\mathrm{Euc}})$.

Now let φ_μ be a C^∞-function on X^{Euc} such that $\varphi_\mu\equiv 1$ $(|z-a_\mu|\leqq\varepsilon/2)$, $\varphi_\mu\equiv 0$ $(|z-a_\mu|\geqq\varepsilon)$ with $0<\varepsilon\ll 1$. Then it is possible to extend

the function $\varphi_\mu v_{-l_\mu}[a_\mu]$ as a C^∞-function $\widetilde{\varphi_\mu v_{-l_\mu}[a_\mu]}$ on $\widetilde{X}'$ satisfying (3. 2. 3). Setting $g_\mu=(m^2-\partial\bar{\partial})\,(\widetilde{\varphi_\mu v_{-l_\mu}[a_\mu]})$ we see that $g_\mu\in\mathcal{H}'_\rho$.

Consider the skew-linear map

$$G_\mu\colon \mathcal{H}_\rho\to\boldsymbol{C},\quad v\mapsto\int\!\!\int_{X^{\mathrm{Euc}}} idz\wedge d\bar{z}\cdot g_\mu\bar{v}\ .$$

By the Schwarz' inequality G_μ is continuous in the topology of $\mathcal{H}_\rho$. Hence by the Riesz' representation theorem there exists a unique $v'_\mu\in\mathcal{H}_\rho$ such that $2I_B(v'_\mu, v)=G_\mu(v)$ for any $v\in\mathcal{H}_\rho$. This implies $(m^2-\partial\bar{\partial})\,v'_\mu=g_\mu=(m^2-\partial\bar{\partial})\,(\widetilde{\varphi_\mu v_{-l_\mu}[a_\mu]})$ in the sense of hyperfunctions. We set $v_\mu=v'_\mu-\widetilde{\varphi_\mu v_{-l_\mu}[a_\mu]}$. It is clear that v_μ satisfies (3. 2. 2) (and hence is real analytic on $\widetilde{X}'$), (3. 2. 3) and $(3.2.4)_\infty$. Since $|\partial v'_\mu|$, $|\bar{\partial}v'_\mu|$ both belong to $L^2(X^{\mathrm{Euc}})$, the local expansion of v_μ at $(a_\nu, \bar{a}_\nu)$ has the form

$$v_\mu=\delta_{\mu\nu}v_{-l_\nu}[a_\nu]+\alpha_{\mu\nu}v_{-l_\nu+1}[a_\nu]+\cdots \\ +\beta_{\mu\nu}v^*_{l_\nu}[a_\nu]+\cdots \tag{3.2.7}$$

This proves Theorem 3. 2. 4.

In the sequel we shall sometimes refer to the above basis as the canonical basis of $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$. More generally for any $l_1,\cdots,l_n\in\boldsymbol{R}-\boldsymbol{Z}$, we use the notation $v_\mu=v_\mu(L)=v_\mu(z,\bar{z};L)$ to indicate the dependence on parameters $L=\begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_n\end{pmatrix}$ of the functions satisfying (3. 2. 2), (3. 2. 3), $(3.2.4)_\infty$ and (3. 2. 7). We also write the coefficients as $\alpha_{\mu\nu}=\alpha_{\mu\nu}(L)$, $\beta_{\mu\nu}=\beta_{\mu\nu}(L)$ and so forth. Theorem 3. 2. 4 guarantees the existence of $v_\mu(L)$ in the case $k<l_1,\cdots,l_n<k+1$ with some $k\in\boldsymbol{Z}$, for we are only to set $v_\mu(L+k)=(m^{-1}\partial)^k v_\mu(L)$ $((m^{-1}\partial)^{-1}=m^{-1}\bar{\partial})$. In particular $\alpha_{\mu\nu}(L+k)=\alpha_{\mu\nu}(L)$, $\beta_{\mu\nu}(L+k)=\beta_{\mu\nu}(L)$.

In the coming § 4. 4 and § 4. 5 we shall explicitly construct n independent canonical elements in $W^{l_1,\cdots,l_n,\mathrm{strict}}_{B,a_1,\cdots,a_n}$ for $l_1,\cdots,l_n\in\boldsymbol{C}-\boldsymbol{Z}$ under certain convergence condition. We can then prove that $W^{l_1,\cdots,l_n,\mathrm{strict}}_{B,a_1,\cdots,a_n}$ is exactly n-dimensional for arbitrary $l_1,\cdots,l_n\in\boldsymbol{R}-\boldsymbol{Z}$. In fact, it is sufficient to show that, if v satisfies besides (3. 2. 2), (3. 2. 3), $(3.2.4)^{\mathrm{strict}}_\nu$ and $(3.2.4)_\infty$ the conditions

$$c^{(\nu)}_{-l_\nu}(v)\cdot c^{*(\nu)}_{l_\nu}(v)=0\quad(\nu=1,\cdots,n),$$

then $v\equiv 0$. Let $\nu_1\cup\nu_2=\{1,\cdots,n\}$ be a partition such that $c^{(\nu)}_{-l_\nu}(v)=0$ for $\nu\in\nu_1$ and $c^{*(\nu)}_{l_\nu}(v)=0$ for $\nu\in\nu_2$. Assume that $\nu_0\in\nu_2$. There exists an element v_{ν_0} (resp. $v^*_{\nu_0}$) in $W^{l_1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_n}$ (resp. $W^{l_1,\cdots,l_{\nu_0}+1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_{\nu_0},\cdots,a_n}$) such that $c^{(\nu)}_{-l_\nu}(v_{\nu_0})=0$ for $v\in\nu_1$, $c^{*(\nu)}_{l_\nu}(v_{\nu_0})=0$ for $\nu(\neq\nu_0)\in\nu_2$ and $c^{(\nu_0)}_{-l_{\nu_0}}(v_{\nu_0})\neq 0$ (resp. $c^{(\nu)}_{-l_\nu}(v^*_{\nu_0})=0$ for $\nu\in\nu_1$, $c^{*(\nu)}_{l_\nu}(v^*_{\nu_0})=0$ for $\nu(\neq\nu_0)\in\nu_2$ and $c^{*(\nu_0)}_{l_{\nu_0}+1}(v^*_{\nu_0})\neq 0$). Subtracting some constant multiple of v_{ν_0} or $v^*_{\nu_0}$, we can reduce the problem to the case of exponents $l_1,\cdots,l_{\nu_0}\pm 1,\cdots,l_n$. By repeating this argument reduction to the case $0<l_1,\cdots,l_n<1$ is possible, for which our assertion is proved.

Now we return to the case $0<l_1,\cdots,l_n<1$. Comparing the local expansion of $\bar\partial v_\mu(L)$ with that of $\bar v_\nu(-L+1)$, we find that $\bar\partial v_\mu(L)-m\sum_{\nu=1}^n\beta_{\mu\nu}(L)\bar v_\nu(-L+1)$ belongs to the kernel of the map c. Hence we have

$$\bar\partial v_\mu(L)=m\sum_{\nu=1}^n\beta_{\mu\nu}(L)\bar v_\nu(-L+1). \tag{3.2.8}$$

Proposition 3.2.5. *Set* $\alpha(L)=(\alpha_{\mu\nu}(L))_{\mu,\nu=1,\cdots,n}$, $\beta(L)=(\beta_{\mu\nu}(L))_{\mu,\nu=1,\cdots,n}$. *Then*

(i) $\alpha_{\mu\nu}(L)\cdot\sin\pi l_\nu=\sin\pi l_\mu\cdot\alpha_{\nu\mu}(-L+1)\quad(\mu\neq\nu)$.

(ii) $\beta(L)\cdot\overline{\beta(-L+1)}=1$, *and there exists a unique* $H(L)={}^t\overline{H(L)}$ *such that* $\beta(L)\sin\pi L=-e^{2H(L)}$.

In particular, if $l_\nu=\dfrac{1}{2}$ $(\nu=1,\cdots,n)$, *we have* ${}^t\alpha=\alpha$ *and* $\beta=-e^{2H}$, $H={}^t\bar H=-\bar H$.

Proof. Assuming $\mu\neq\nu$, we compute $I_B(v_\mu(L),\overline{v_\nu(-L+1)})$ in two ways and obtain

$$\lim_{\varepsilon\downarrow 0}\sum_{\lambda=1}^n\oint_{\partial D_{\lambda,\varepsilon}}iv_\mu(L)\cdot\partial v_\nu(-L+1)\,dz$$
$$=\lim_{\varepsilon\downarrow 0}\sum_{\lambda=1}^n\oint_{\partial D_{\lambda,\varepsilon}}-i\bar\partial v_\mu(L)\cdot v_\nu(-L+1)\,d\bar z$$

or

$$2(\alpha_{\mu\nu}(L)\cdot\sin\pi l_\nu-\alpha_{\nu\mu}(-L+1)\cdot\sin\pi l_\mu)=0.$$

This proves (i). Comparing the coefficient of $v_{-l_\nu+1}[a_\nu]$ of the local expansion of (3. 2. 8), we obtain $\delta_{\mu\nu}=\sum_{\lambda=1}^{n}\beta_{\mu\lambda}(L)\overline{\beta_{\lambda\nu}(-L+1)}$. Finally let $k=(k_1, \cdots, k_n)\in \boldsymbol{C}^n$ and set $v=\sum_{\mu=1}^{n}k_\mu v_\mu$. Then (3. 2. 6) yields

$$0\leqq I_B(v, v)=-\sum_{\mu,\nu=1}^{n}k_\nu\bar{k}_\mu\overline{\beta_{\mu\nu}(L)}\sin\pi l_\nu .$$

This implies that $-\beta(L)\sin\pi L$ is a positive definite hermitian matrix, and hence is uniquely written as $e^{2H(L)}$, $H(L)=\overline{{}^tH(L)}$. This proves (ii).

Remark. From (3. 2. 8) the local expansion of $v_\mu(L)$ has the form

$$v_\mu(L)=\delta_{\mu\nu}v_{-l_\nu}[a_\nu]+\alpha_{\mu\nu}(L)\cdot v_{-l_\nu+1}[a_\nu]+\cdots$$
$$+\sum_{\sigma=1}^{n}\beta_{\mu\sigma}(L)\cdot(\delta_{\sigma\nu}v^*_{l_\nu}[a_\nu]+\overline{\alpha_{\sigma\nu}(-L-1)}\cdot v^*_{l_\nu+1}[a_\nu]+\cdots).$$

As long as solutions of (3. 2. 2) are concerned, we may identify the operator $\partial\bar{\partial}$ with m^2. The quotient ring $\boldsymbol{C}[\partial, \bar{\partial}]/(m^2-\partial\bar{\partial})$ is isomorphic to the polynomial ring $\boldsymbol{C}[u, u^{-1}]$ in one variable $u=m^{-1}\partial$ admitting its inverse $u^{-1}=m^{-1}\bar{\partial}$. Thus any element of $\boldsymbol{C}[\partial, \bar{\partial}]/(m^2-\partial\bar{\partial})$ is a residue class of a unique element of the form $p(\partial, \bar{\partial})=\sum_{k=k_1}^{k_2}p_k(m^{-1}\partial)^k$, where $(m^{-1}\partial)^{-1}$ stands for $m^{-1}\bar{\partial}$. We set $(\boldsymbol{C}[\partial, \bar{\partial}]/(m^2-\partial\bar{\partial}))_j=\{p(\partial, \bar{\partial})=\sum_{|k|\leqq j}p_k(m^{-1}\partial)^k\}$. The following theorem tells the structure of $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$ for general $l_\nu\in \boldsymbol{R}-\boldsymbol{Z}$.

Theorem 3. 2. 6. *Let* $0<l_\nu<1$ $(\nu=1, \cdots, n)$. *Then the space* $\bigcup_{j=0}^{\infty}W^{l_1+j,\cdots,l_n+j}_{B,a_1,\cdots,a_n}$ *is a left* $\boldsymbol{C}[\partial, \bar{\partial}, M_B]$*-module. Moreover we have an isomorphism*

$$(3.\ 2.\ 9)\qquad (\boldsymbol{C}[\partial, \bar{\partial}]/(m^2-\partial\bar{\partial}))_j\underset{\boldsymbol{C}}{\otimes}W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}\xrightarrow{\sim}W^{l_1+j,\cdots,l_n+j}_{B,a_1,\cdots,a_n}$$

given by $p(\partial, \bar{\partial})\otimes v\mapsto p(\partial, \bar{\partial})v$.

Proof. Let $v\in W^{l_1+j,\cdots,l_n+j}_{B,a_1,\cdots,a_n}$, $p\in\boldsymbol{C}[\partial, \bar{\partial}, M_B]$. Since p belongs to $\mathcal{D}_{0,B}$ (Proposition 3. 1. 1), pv satisfies (3. 2. 2). Clearly (3. 2. 3) remains valid, and $(3.\ 2.\ 4)_\infty$ is also preserved by Proposition 3. 1. 5. Finally if $\operatorname{ord}p=k$, $(3.\ 2.\ 4)_\nu$ is fulfilled with l_ν replaced by $l_\nu+j+k$. Hence we

have shown $pW^{l_1+j,\cdots,l_n+j}_{B,a_1,\cdots,a_n}\subset W^{l_1+j+k,\cdots,l_n+j+k}_{B,a_1,\cdots,a_n}$.

To prove (3. 2. 9), let $v_1,\cdots,v_n$ be the canonical basis of $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$, and take $p^{(\mu)}=\sum_{|k|\leqq j}p^{(\mu)}_k(m^{-1}\partial)^k\in(\mathbb{C}[\partial,\bar\partial]/(m^2-\partial\bar\partial))_j$, $\mu=1,\cdots,n$, arbitrarily. Then we find

$$(3.2.10)\quad \sum_{\mu=1}^n p^{(\mu)}v_\mu=p^{(\nu)}_j v_{-l_\nu-j}[a_\nu]+\cdots+(\sum_{\mu=1}^n p^{(\mu)}_{-j}\beta_{\mu\nu})\cdot v^*_{l_\nu-j}[a_\nu]+\cdots$$

at $(a_\nu,\bar a_\nu)$ $(\nu=1,\cdots n,)$. Here $\beta=(\beta_{\mu\nu})$ is an invertible matrix by Proposition 3. 2. 5. Hence (3. 2. 10) implies that, for any $v\in W^{l_1+j,\cdots,l_n+j}_{B,a_1,\cdots,a_n}$. there exist unique constants $p^{(\mu)}_j$, $p^{(\mu)}_{-j}$ $(\mu=1,\cdots,n)$ such that $v-\sum_{\mu=1}^n(p^{(\mu)}_j(m^{-1}\partial)^j+p^{(\mu)}_{-j}(m^{-1}\bar\partial)^j)v_\mu$ belongs to $W^{l_1+j-1,\cdots,l_n+j-1}_{B,a_1,\cdots,a_n}$. Repeating this argument we see that $v-\sum_{\mu=1}^n p^{(\mu)}(\partial,\bar\partial)v_\mu\in W^{l_1-1,\cdots,l_n-1}_{B,a_1,\cdots,a_n}=0$ for unique $p^{(\mu)}\in(\mathbb{C}[\partial,\bar\partial]/(m^2-\partial\bar\partial))_j$. This completes the proof of Theorem 3. 2. 6.

We remark here that most of the above results are valid if we admit some of l_ν's to be integral. In this case we impose in place of $(3.2.4)_\nu$ the condition

$$|v|=\begin{cases}O(|z-a_\nu|^{-l_\nu}) & (l_\nu\in\mathbb{Z}-\{0\})\\ O(|\log|z-a_\nu||) & (l_\nu=0)\end{cases}$$

to define the space $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$, or what amounts to the same thing, assume the expansion

$$v=\sum_{l=0}^{l_\nu}(\tilde c^{(\nu)}_l(v)\cdot\tilde v_l[a_\nu]+\tilde c^{*(\nu)}_l(v)\cdot\tilde v^*_l[a_\nu])$$

$$+\sum_{l=\max(-l_\nu,0)}^{\infty}(c^{(\nu)}_l(v)\cdot v_l[a_\nu]+c^{*(\nu)}_l(v)\cdot v^*_l[a_\nu])$$

$$(\tilde c^{(\nu)}_0(v)=\tilde c^{*(\nu)}_0(v),\ c^{(\nu)}_0(v)=c^{*(\nu)}_0(v))$$

to hold at $(a_\nu,\bar a_\nu)$. We have then $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}\subset W^{l_1,\cdots,l_n,0,\cdots,0}_{B,a_1,\cdots,a_n,a_{n+1},\cdots,a_{n+k}}$, and $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}=W^{l_1}_{B,a_1}\oplus\cdots\oplus W^{l_n}_{B,a_n}$ if $l_1,\cdots,l_n\in\mathbb{Z}$. Although the inner product I_B diverges in general if $l_\nu=0,1,2,\cdots$ for some ν, Theorems 3. 2. 4(*) and 3. 2. 6 are valid for $0\leqq l_\nu<1$ $(\nu=1,\cdots,n)$. Let $W^{0,l_1,\cdots,l_n,\mathrm{strict}}_{B,z^*,a_1,\cdots,a_n}$ denote the space consisting of v's satisfying $|v|=O(|\log|z-z^*||)$ at $(z^*,\bar z^*)$ besides the conditions for $W^{l_1,\cdots,l_n,\mathrm{strict}}_{B,a_1,\cdots,a_n}$. It is spanned by the

(*) We replace $c^{(\nu)}_{l_\nu}$ by $\tilde c^{(\nu)}_0$ if $l_\nu=0$.

canonical basis $v_\mu(L)=v_\mu(z;L)$ (*) of $W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}$, and a unique $v_0(L)=v_0(z^*,z;L)$ with the characteristic properties (3.2.2), (3.2.3), $(3.2.4)_\infty$ and

$$(3.2.11)\qquad v_0(z^*,z;L)=\begin{cases}\tilde{v}_0[z^*]+\text{regular function}\\ \qquad\text{at}\quad (z,\bar{z})=(z^*,\bar{z}^*)\\ \alpha_{0\nu}(z^*;L)\cdot v_{-l_\nu+1}[a_\nu]+\cdots+\beta_{0\nu}(z^*;L)\cdot v^*_{l_\nu}[a_\nu]+\cdots\\ \qquad\text{at}\quad (z,\bar{z})=(a_\nu,\bar{a}_\nu)\quad(\nu=1,\cdots,n).\end{cases}$$

In view of the formula (3.1.10), this $v_0(z^*,z;L)$ is an analogue of the Green's function. Existence of $v_0(L)$ is proved by a similar method for $0<l_1,\cdots,l_n<1$ (for general $l_\nu\in\boldsymbol{R}-\boldsymbol{Z}$, see Chapter IV). The coefficients $\alpha_{0\nu},\beta_{0\nu}$ in (3.2.11) are obtained by calculating the inner product $I_B(v_0(z^*,z;L),\overline{v_\nu(z;1-L)})$ and $I_B(v_0(z^*,z;L),v_\nu(z;L))$ respectively, noting $I_B(v,v')=\overline{I_B(v',v)}$. We find

$$(3.2.12)\qquad \alpha_{0\nu}(z^*;L)=\frac{\pi}{2\sin\pi l_\nu}v_\nu(z^*;1-L)$$

$$\beta_{0\nu}(z^*;L)=\frac{\pi}{2\sin\pi l_\nu}\overline{v_\nu(z^*;L)}.$$

That is, the canonical basis of $W^{l_1,\cdots,l_n,\mathrm{strict}}_{B,a_1,\cdots,a_n}$ appears as the leading coefficient of the local expansion for the "Green's function" $v_0(z^*,z;L)$. Likewise we find the following symmetry property by evaluating $I_B(v_0(z,z';L),v_0(z^*,z';L))$ with respect to z':

$$(3.2.13)\qquad v_0(z^*,z;L)=\overline{v_0(z,z^*;L)}.$$

Now we proceed to the case of Euclidean Dirac equation (3.1.3). Let $l_1,\cdots,l_n\in\boldsymbol{R}$ be such that $l_\nu\not\equiv\frac{1}{2}\bmod\boldsymbol{Z}$ $(\nu=1,\cdots,n)$.

Definition 3.2.7. *We denote by* $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$ *the space consisting of 2-component real analytic functions* $w={}^t(w_+,w_-)$ *on* $\tilde{X}'$, *satisfying*

$$(3.2.14)\qquad (m-\Gamma)w=0\qquad \text{on}\quad \tilde{X}'.$$

$$(3.2.15)\qquad \gamma\cdot w=w\cdot\rho_{l_1-1/2,\cdots,l_n-1/2}(\gamma)\qquad \text{for}\quad \gamma\in\pi_1(X';x_0),$$

(*) For brevity we write the variables $(z,\bar{z})$ simply as z.

$$i.e. \quad (\gamma_\nu w)(z,\bar{z}) = w(a_\nu + e^{2\pi i}(z-a_\nu),\ \bar{a}_\nu + e^{-2\pi i}(\bar{z}-\bar{a}_\nu))$$
$$= -e^{-2\pi i l_\nu} w(z,\bar{z}) \quad (0<|z-a_\nu|\ll 1;\ \nu=1,\cdots,n)$$

$$(3.2.16)_\nu \quad |w_\pm| = O(|z-a_\nu|^{-[l_\nu+1/2]-1}) \quad as\ |z-a_\nu|\to 0 \quad (\nu=1,\cdots,n),$$

$$(3.2.16)_\infty \quad |w_\pm| = O(e^{-2m|z|}) \quad as\ |z|\to\infty.$$

Likewise we define $W^{l_1,\cdots,l_n,\mathrm{strict}}_{F,a_1,\cdots,a_n}$, *replacing* $(3.2.16)_\nu$ *by*

$$(3.2.16)^{\mathrm{strict}}_\nu \quad w = \sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(w)\cdot w_{-l_\nu+j}[a_\nu] + \sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu+j}(w)\cdot w^*_{l_\nu+j}[a_\nu]$$

for $\nu=1,\cdots,n$.

Condition $(3.2.16)_\nu$ is equivalent (under (3.2.14) and (3.2.15)) to

$$(3.2.16)'_\nu \quad w = \sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(w)\cdot w_{-l_\nu+j}[a_\nu] + \sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu{}^*+j}(w)\cdot w^*_{l_\nu{}^*+j}[a_\nu]$$

where $l^*_\nu = l_\nu - 2\left[l_\nu+\frac{1}{2}\right]$. From the definition we have the following isomorphism

$$(3.2.17) \quad W^{l_1+1/2,\cdots,l_n-1/2}_{B,a_1,\cdots,a_n} \xrightarrow{\sim} W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n},\ v\mapsto \begin{pmatrix} v \\ m^{-1}\bar{\partial}v \end{pmatrix}.$$

Hence the results for $W^{l_1+1/2,\cdots,l_n+1/2}_{B,a_1,\cdots,a_n}$ are straightly translated into those for $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$.

The case $n=1$, $-\frac{1}{2}<l<\frac{1}{2}$.

$$W^l_{F,a} = \mathbb{C}\tilde{w}_{-l}[a],\quad \tilde{w}_{-l}[a] = \frac{\pi}{2}\frac{ie^{-\pi i l}}{\cos\pi l}(-w_{-l}[a]+w^*_l[a]).$$

Inner product

$$(3.2.18) \quad I_F(w,w') = \frac{m^2}{2}\int\!\!\int_{X^{\mathrm{Euc}}} idz\wedge d\bar{z}\,(w_+\bar{w}'_+ + w_-\bar{w}'_-)$$
$$= \overline{I_F(w',w)}.$$

Assuming $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$, we find for $w, w'\in W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$

$$(3.2.18)' \quad I_F(w,w') = -\sum_{\nu=1}^{n} c^{(\nu)}_{-l_\nu}(w)\cdot\overline{c^{*(\nu)}_{l_\nu}(w')}\cdot\cos\pi l_\nu.$$

Theorem 3.2.8. *We have* $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}=0$ *for* $l_1,\cdots,l_n<-\frac{1}{2}$, *and* $\dim_{\boldsymbol{C}} W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}=n$ *if* $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$. *In the latter case there exists a unique canonical basis* $w_\mu(L)$ $(\mu=1,\cdots,n)$ *such that*

$$(3.2.19)\quad w_\mu(L)=\delta_{\mu\nu}w_{-l_\nu}[a_\nu]+\alpha_{\mu\nu}\Big(L+\frac{1}{2}\Big)\cdot w_{-l_\nu+1}[a_\nu]+\cdots$$
$$+\sum_{\sigma=1}^{n}\beta_{\mu\sigma}\Big(L+\frac{1}{2}\Big)\cdot\Big(\delta_{\sigma\nu}\cdot w^*_{l_\nu}[a_\nu]+\overline{\alpha_{\sigma\nu}\Big(-L+\frac{1}{2}\Big)}$$
$$\times w^*_{l_\nu+1}[a_\nu]+\cdots\Big).$$

at $(a_\nu,\bar a_\nu)$ $(\mu,\nu=1,\cdots,n)$. *Here* $\alpha_{\mu\nu}\Big(L+\frac{1}{2}\Big)$, $\beta_{\mu\nu}\Big(L+\frac{1}{2}\Big)$ *are those in* (3.2.7) *corresponding to exponents* $l_1+\frac{1}{2},\cdots,l_n+\frac{1}{2}$.

Theorem 3.2.9. *Assume* $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$. *Then* $\bigcup_{j=0}^{\infty}W^{l_1+j,\cdots,l_n-j}_{F,a_1,\cdots,a_n}$ *is a left* $\boldsymbol{C}[\partial,\bar\partial,M_F]$*-module. We have*

$$(3.2.20)\quad (\boldsymbol{C}[\partial,\bar\partial]/(m^2-\partial\bar\partial))^{\otimes_{\boldsymbol{C}}}_{j}W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}\xrightarrow{\sim}W^{l_1+j,\cdots,l_n+j}_{F,a_1,\cdots,a_n}$$

by the map $p(\partial,\bar\partial)\otimes w\mapsto p(\partial,\bar\partial)w$.

These are immediate consequences of Theorems 3.2.4 and 3.2.6.

Extension to the case of half integral l_ν is also possible. We merely note here the existence of solutions $w_0^{(\pm)}(L)=w_0^{(\pm)}(z^*,z;L)$ of (3.2.14) satisfying (3.2.15), $(3.2.16)_\infty$ and the local behavior

$$(3.2.21)_0^{(\pm)}\quad w_0^{(\pm)}(z^*,z;L)=\begin{cases}-\tilde w^*_{1/2}[z^*]\\ \tilde w_{1/2}[z^*]\end{cases}+\text{regular function}$$
$$\text{at}\quad (z,\bar z)=(z^*,\bar z^*)$$

$$(3.2.21)_\nu^{(\pm)}\quad w_0^{(\pm)}(z^*,z;L)=\alpha_{0\nu}^{(\pm)}(z^*;L)\cdot w_{-l_\nu+1}[a_\nu]+\cdots$$
$$+\beta_{0\nu}^{(\pm)}(z^*;L)\cdot w^*_{l_\nu}[a_\nu]+\cdots$$
$$\text{at}\quad (z,\bar z)=(a_\nu,\bar a_\nu)\quad(\nu=1,\cdots,n),$$

where $\alpha_{0\nu}={}^t(\alpha_{0\nu}^{(+)},\alpha_{0\nu}^{(-)})$, $\beta_{0\nu}={}^t(\beta_{0\nu}^{(+)},\beta_{0\nu}^{(-)})$ are obtained by calculating the inner product of $w_0^{(\pm)}(z^*,z;L)$ and $w^*_\nu(z;1-L)$ or $w_\nu(z;L)$, as

follows.

$$\alpha_{0\nu}(z^*;\ L)=\frac{\pi}{2\cos\pi l_\nu}w_\nu(z^*;\ 1-L) \tag{3.2.22}$$

$$\beta_{0\nu}(z^*;\ L)=\frac{\pi}{2\cos\pi l_\nu}w_\nu^*(z^*;\ L).$$

We also see that (cf. (3.2.13))

$$w_+^{(\mp)}(z^*,\ z;\ L)=\bar{w}_-^{(-)}(z,\ \bar{z}^*;\ L) \tag{3.2.23}$$

$$w_\mp^{(\pm)}(z^*,\ z;\ L)=w_\mp^{(\pm)}(z,\ z^*;\ L).$$

In terms of $v_0(L)=v_0(z^*,\ z;\ L)$, $w_0^{(\pm)}(L)=w_0^{(\pm)}(z^*,\ z;\ L)$ are given by

$$(w_0^{(+)}(L),\ w_0^{(-)}(L)) \tag{3.2.24}$$

$$=\begin{pmatrix}-m^{-1}\partial_z v_0(L-1/2) & v_0(L+1/2)\\ -v_0(L-1/2) & m^{-1}\bar{\partial}_z v_0(L+1/2)\end{pmatrix}.$$

We denote by $W_{F,z^*,a_1,\cdots,a_n}^{(\pm)0,l_1,\cdots,l_n,\mathrm{strict}}$ the space of w's which, besides the conditions for $W_{F,a_1,\cdots,a_n}^{l_1,\cdots,l_n,\mathrm{strict}}$, admit of singularities of the form $(3.2.21)_0^{(\pm)}$ at $(z^*,\ \bar{z}^*)$.

We remark a few words concerning the special case $l_1=0,\cdots,l_n=0$. For a solution $w=\begin{pmatrix}w_+\\ w_-\end{pmatrix}$ of the Euclidean Dirac equation (3.2.14), define its *-*conjugate* by $w^*=\begin{pmatrix}\bar{w}_-\\ \bar{w}_+\end{pmatrix}$. Then assuming $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$ we have the skew-linear isomorphism $*: W_{F,a_1,\cdots,a_n}^{l_1,\cdots,l_n}\tilde{\to}W_{F,a_1,\cdots,a_n}^{-l_1,\cdots,-l_n}$. These two are mutually dual spaces through the $\boldsymbol{C}$-bilinear form $I_F(w,\ w'^*)$ $(w\in W_{F,a_1,\cdots,a_n}^{l_1,\cdots,l_n},\ w'\in W_{F,a_1,\cdots,a_n}^{-l_1,\cdots,-l_n})$. Now if $l_1=0,\cdots,l_n=0$, the space $W_{F,a_1,\cdots,a_n}^{0,\cdots,0}(=W_{F,a_1,\cdots,a_n}^{\mathrm{strict}}$ in the notation of [3]) is self-dual, and we have the $\boldsymbol{R}$-linear subspace $W_{F,a_1,\cdots,a_n}^{\mathrm{strict},\boldsymbol{R}}$ consisting of "real" elements $w=w^*\in W_{F,a_1,\cdots,a_n}^{\mathrm{strict}}$. Clearly $W_{F,a_1,\cdots,a_n}^{\mathrm{strict}}=W_{F,a_1,\cdots,a_n}^{\mathrm{strict},\boldsymbol{R}}\oplus\sqrt{-1}W_{F,a_1,\cdots,a_n}^{\mathrm{strict},\boldsymbol{R}}$, $\dim_{\boldsymbol{R}}W_{F,a_1,\cdots,a_n}^{\mathrm{strict},\boldsymbol{R}}=n$. An element $w\in W_{F,a_1,\cdots,a_n}^{\mathrm{strict}}$ (or correspondingly $v\in W_{B,a_1,\cdots,a_n}^{1/2,\cdots,1/2}$) is regarded as a function defined on the 2-fold ramified covering manifold of X^{Euc}: $\mathcal{R}_{a_1,\cdots,a_n}=\{(z,\bar{z},\zeta,\bar{\zeta})\,|\,\zeta^2=\prod_{\nu=1}^{n}(z-a_\nu),\ \bar{\zeta}^2=\prod_{\nu=1}^{n}(\bar{z}-\bar{a}_\nu)\}=(\tilde{X}'/\mathrm{Ker}\ \rho_{-1/2,\cdots,-1/2})\cup\{(a_\nu,\bar{a}_\nu)\}_{\nu=1,\cdots,n}$, and satisfies there the Euclidean Dirac (resp. Klein-Gordon) equation with point sources at the branch points $(a_\nu,\bar{a}_\nu)$; namely, in the local parameter $\zeta_\nu=\sqrt{z-a_\nu}=\overset{+1}{\xi}_\nu+i\overset{+2}{\xi}_\nu$, the local expansion $(3.2.16)'_\nu$ (resp. $(3.2.4)'_\nu$) implies

$$(3.2.25)\qquad \begin{pmatrix} 2m\zeta_\nu & -\dfrac{\partial}{\partial\zeta_\nu} \\ -\dfrac{\partial}{\partial\bar{\zeta}_\nu} & 2m\bar{\zeta}_\nu \end{pmatrix} w = -\sqrt{\frac{\pi}{m}}\cdot\begin{pmatrix} c_0^{*(\nu)}(w) \\ c_0^{(\nu)}(w) \end{pmatrix}\delta(\xi_\nu^1)\,\delta(\xi_\nu^2),$$

$$\left(4m^2\zeta_\nu\bar{\zeta}_\nu - \frac{\partial^2}{\partial\zeta_\nu\partial\bar{\zeta}_\nu}\right)v = -\sqrt{\frac{\pi}{m}}\,c_{-1/2}^{(\nu)}(v)\cdot\frac{\partial}{\partial\zeta_\nu}\delta(\xi_\nu^1)\,\delta(\xi_\nu^2).$$

Here we have used the formula $\dfrac{\partial}{\partial\zeta_\nu}\left(\dfrac{1}{\bar{\zeta}_\nu}\right)=\pi\delta(\xi_\nu^1)\,\delta(\xi_\nu^2)$.

The results in this paragraph are generalized to the case of an n-dimensional monodromy representation of special type, as shown below.

Let $\Lambda=(\lambda_{\mu\nu})_{\mu,\nu=1,\cdots,n}$ be a positive definite real symmetric matrix such that $\lambda_{\nu\nu}=1$ $(\nu=1,\cdots,n)$. For $l_1,\cdots,l_n\in\boldsymbol{R}$, we introduce the monodromy representation

$$(3.2.26)\qquad \rho_{l_1,\cdots,l_n,\Lambda}:\pi_1(X';x_0)\to GL(n,\boldsymbol{C})$$

$$\rho_{l_1,\cdots,l_n,\Lambda}(\gamma_\nu)=M_\nu=1+(e^{-2\pi i l_\nu}-1)E_\nu\Lambda,$$

$$E_\nu=\begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \overset{\nu}{\smile} & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}\quad(\nu=1,\cdots,n).$$

Since M_ν satisfies $M_\nu\Lambda^{-1}\,{}^t\overline{M}_\nu=\Lambda^{-1}$, $\rho_{l_1,\cdots,l_n,\Lambda}$ is a unitary representation in the sense $\rho_{l_1,\cdots,l_n,\Lambda}(\pi_1(X';x_0))\subset U(n,\Lambda)=\{g\in GL(n,\boldsymbol{C})\mid {}^t\bar{g}\Lambda g=\Lambda\}$. In place of (3.2.3), let us now consider the following monodromy property for an n-tuple $v=(v^{(1)},\cdots,v^{(n)})$ of functions on $\tilde{X}'$:

$$(3.2.27)\qquad \gamma\cdot(v^{(1)},\cdots,v^{(n)})=(v^{(1)},\cdots,v^{(n)})\cdot\rho_{l_1,\cdots,l_n,\Lambda}(\gamma)$$

$$(\gamma\in\pi_1(X';x_0)).$$

The condition (3.2.27) will be motivated later in Chapter IV. (3.2.26) and (3.2.27) imply in particular

$$(3.2.28)\qquad \gamma_\nu v^{(\nu)}=e^{-2\pi i l_\nu}v^{(\nu)},\ \gamma_\nu(v^{(\mu)}-\lambda_{\mu\nu}v^{(\nu)})=v^{(\mu)}-\lambda_{\mu\nu}v^{(\nu)}$$

$$(\mu,\nu=1,\cdots,n).$$

Note also that if v satisfies (3.2.27), then $|v|_\Lambda=(v\Lambda^{-1}\,{}^t\bar{v})^{1/2}\geqq 0$ is single-valued on X'.

Definition 3.2.10. *For* $l_1, \cdots, l_n \notin \boldsymbol{Z}$, $W^{l_1, \cdots, l_n}_{B, a_1, \cdots, a_n}(\Lambda)$ *will denote the space of* n*-tuples* $v = (v^{(1)}, \cdots, v^{(n)})$ *of real analytic functions on* $\tilde{X}'$ *satisfying* (3.2.2), (3.2.27) *and*

$$(3.2.29)_\nu \qquad v^{(\mu)} = \lambda_{\mu\nu}\Big(\sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(v)\cdot v_{-l_\nu+j}[a_\nu] + \sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu^*+j}(v)\cdot v^*_{l_\nu^*+j}[a_\nu]\Big)$$

$$+ \textit{single-valued regular function at } (a_\nu, \bar{a}_\nu)$$

$$(l_\nu^* = l_\nu - 2[l_\nu];\ \mu, \nu = 1, \cdots, n)$$

$$(3.2.29)_\infty \qquad |v|_\Lambda = O(e^{-2m|z|}) \quad \textit{as} \quad |z| \to \infty.$$

Expansion $(3.2.29)_\nu$ is consistent with (3.2.28) (in particular the regular function term for $v^{(\nu)}$ is missing at $(a_\nu, \bar{a}_\nu)$).

Definition 3.2.11. *We define* $W^{l_1, \cdots, l_n}_{F, a_1, \cdots, a_n}(\Lambda)$ *for* $l_\nu \not\equiv \frac{1}{2} \bmod \boldsymbol{Z}$ $(\nu = 1, \cdots, n)$ *to be the space of* $2n$*-tuples* $w = (w^{(1)}, \cdots, w^{(n)})$, $w^{(\nu)} = \begin{pmatrix} w^{(\nu)}_+ \\ w^{(\nu)}_- \end{pmatrix}$ $(\nu = 1, \cdots, n)$, *of real analytic functions on* $\tilde{X}'$, *such that each* $w^{(\nu)}$ *satisfies* (3.2.14) *and*

$$(3.2.30) \qquad \gamma \cdot (w^{(1)}_\varepsilon, \cdots, w^{(n)}_\varepsilon) = (w^{(1)}_\varepsilon, \cdots, w^{(n)}_\varepsilon)\,\rho_{l_1 - 1/2, \cdots, l_n - 1/2, \Lambda}(\gamma)$$

$$(\varepsilon = + \text{ or } -;\ \gamma \in \pi_1(X'; x_0))$$

$$(3.2.31)_\nu \qquad w^{(\mu)} = \lambda_{\mu\nu}\Big(\sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(w)\cdot w_{-l_\nu+j}[a_\nu] + \sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu^*-j}(w)\cdot w^*_{l_\nu^*+j}[a_\nu]\Big)$$

$$+ \textit{single-valued regular function at } (a_\nu, \bar{a}_\nu)$$

$$\Big(l_l^* = l_\nu - 2\Big[l_\nu + \frac{1}{2}\Big],\ \mu, \nu = 1, \cdots, n\Big)$$

$$(3.2.31)_\infty \qquad |w_\pm|_\Lambda = O(e^{-2m|z|}) \quad \text{as} \quad |z| \to \infty$$

$$(w_\pm = (w_{1\pm}, \cdots, w_{n\pm})).$$

The space $W^{l_1, \cdots, l_n, \text{strict}}_{*, a_1, \cdots, a_n}(\Lambda)$ $(* = B \text{ or } F)$ are defined analogously. Namely we replace $(3.2.29)_\nu$ and $(3.2.31)_\nu$ respectively by

$$(3.2.29)^{\text{strict}}_\nu \qquad v^{(\mu)} = \lambda_{\mu\nu}\Big(\sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(v)\cdot v_{-l_\nu+j}[a_\nu]$$

$$+\sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu+j}(v)\cdot v^*_{l_\nu+j}[a_\nu]) + \textit{regular function}$$

$$(3.2.31)^{\text{strict}}_{\nu} \qquad w^{(\mu)} = \lambda_{\mu\nu}\Big(\sum_{j=0}^{\infty} c^{(\nu)}_{-l_\nu+j}(w)\cdot w_{-l_\nu+j}[a_\nu]$$

$$+\sum_{j=0}^{\infty} c^{*(\nu)}_{l_\nu+j}(w)\cdot w^*_{l_\nu+j}[a_\nu]) + \textit{regular function}$$

for $\mu, \nu = 1, \cdots, n$. (We note that the definition of $W^{l_1,\cdots,l_n}_{a_1,\cdots,a_n}(\Lambda)$ given in VII [5] coincides with $W^{l_1,\cdots,l_n,\text{strict}}_{F,a_1,\cdots,a_n}(\Lambda)$ here). We also define their dual spaces by $W^{*l_1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_n}(\Lambda) = \{{}^t\bar{v} \mid v \in W^{l_1,\cdots,l_n,\text{strict}}_{B,a_1,\cdots,a_n}(\Lambda)\}$ and $W^{*l_1,\cdots,l_n,\text{strict}}_{F,a_1,\cdots,a_n}(\Lambda) = \{{}^tw^* \mid w \in W^{l_1,\cdots,l_n,\text{strict}}_{F,a_1,\cdots,a_n}(\Lambda)\}$. In this case as well we have

$$(3.2.32) \qquad W^{l_1+1/2,\cdots,l_n+1/2}_{B,a_1,\cdots,a_n}(\Lambda) \xrightarrow{\sim} W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}(\Lambda), \quad v \mapsto \begin{pmatrix} v \\ m^{-1}\bar{\partial}v \end{pmatrix}.$$

Remark. In the trivial case $\Lambda = 1$, $W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}(\Lambda)$ $(* = B \text{ or } F)$ splits into the direct sum $W^{l_1}_{*,a_1}\oplus\cdots\oplus W^{l_n}_{*,a_n}$ (the case "with no interaction"). On the other hand, the space $W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}$ discussed above is understood as the degenerating limit $\lambda_{\mu\nu}\to 1$ for all $\mu, \nu = 1, \cdots, n$. See Proposition 3.2.14 below.

A hermitian inner product is introduced by setting

$$(3.2.33) \qquad I_{B,\Lambda}(v, v') = \frac{1}{2}\int\!\!\int_{X^{\text{Euc}}} idz\wedge d\bar{z}\,(\bar{\partial}v\cdot\Lambda^{-1}\cdot{}^t(\partial\bar{v}')$$

$$+ m^2 v\cdot\Lambda^{-1}\cdot{}^t\bar{v}')$$

for $v, v' \in W^{l_1,\cdots,l_n}_{B,a_1,\cdots,a_n}(\Lambda)$ $(0 < l_1, \cdots, l_n < 1)$, and

$$(3.2.34) \qquad I_{F,\Lambda}(w, w') = \frac{m^2}{2}\int\!\!\int_{X^{\text{Euc}}} idz\wedge d\bar{z}\,(w_+\cdot\Lambda^{-1}\cdot{}^t\bar{w}'_+ + w_-\cdot\Lambda^{-1}\cdot{}^t\bar{w}'_-)$$

for $w, w' \in W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}(\Lambda)$ $\left(-\frac{1}{2} < l_1, \cdots, l_n < \frac{1}{2}\right)$. To see the convergence of the integral at $|z|\to\infty$, we note the following.

Lemma. Let $v = (v^{(1)}, \cdots, v^{(n)})$ be a solution of (3.2.2) on $\tilde{X}' \cap \{2|z| \geqq R\}$ such that $v(e^{2\pi i}z, e^{-2\pi i}\bar{z}) = v(z, \bar{z})\cdot M$ for some $M \in U(n, \Lambda)$. If $\sup_{2|z|\geqq R} |v|_\Lambda < \infty$, we have

$$|(pv)(z,\bar{z})|_1 = O\left(\frac{e^{-2m|z|}}{\sqrt{|z|}}\right) \quad (|z|\to\infty)$$

for any $p\in\boldsymbol{C}[\partial,\bar{\partial},M_B]$.

Proof. Since M is unitary, there exists a $P\in GL(n,\boldsymbol{C})$ which diagonalizes M, i.e. $P^{-1}MP=\begin{pmatrix}\varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n\end{pmatrix}$, $|\varepsilon_\nu|=1$ $(\nu=1,\cdots,n)$. Setting $vP=(v^{(1)\prime},\cdots,v^{(n)\prime})$, we see that $v^{(\nu)\prime}$ satisfies the assumption of Proposition 3.1.5. Hence our assertion follows.

By a similar calculation we obtain

$$I_{B,\Lambda}(v,v') = -\sum_{\nu=1}^{n} c_{-l_\nu}^{(\nu)}(v)\cdot\overline{c_{l_\nu}^{*(\nu)}(v')}\cdot\sin\pi l_\nu \tag{3.2.35}$$

$$I_{F,\Lambda}(w,w') = -\sum_{\nu=1}^{n} c_{-l_\nu}^{(\nu)}(w)\cdot\overline{c_{l_\nu}^{*(\nu)}(w')}\cdot\cos\pi l_\nu \tag{3.2.36}$$

where $\{c_{-l_\nu}^{(\nu)}(v), c_{l_\nu}^{*(\nu)}(v)\}$ and $\{c_{-l_\nu}^{(\nu)}(w), c_{l_\nu}^{*(\nu)}(w)\}$ denote those appearing in $(3.2.29)_\nu$ and $(3.2.31)_\nu$, respectively.

Theorem 3.2.4, Proposition 3.2.5, Theorem 3.2.6 (and the corresponding results for the Dirac case) are also generalized to the case of $W_{*,a_1,\cdots,a_n}^{l_1,\cdots,l_n}(\Lambda)$ $(*=B \text{ or } F)$. We omit their proofs, which are almost the same as in the original ones.

Theorem 3.2.12. *For $0<l_1,\cdots,l_n<1$, there exists a unique canonical basis $v_\mu=v_\mu(L;\Lambda)=(v_\mu^{(1)}(L;\Lambda),\cdots,v_\mu^{(n)}(L;\Lambda))$ $(\mu=1,\cdots,n)$ of $W_{B,a_1,\cdots,a_n}^{l_1,\cdots,l_n}(\Lambda)$ such that*

$$\begin{aligned} v_\mu^{(\sigma)}(L;\Lambda) &= \lambda_{\sigma\nu}\{\delta_{\mu\nu}v_{-l_\nu}[a_\nu]+\alpha_{\mu\nu}(L;\Lambda)v_{-l_\nu+1}[a_\nu]+\cdots \\ &\quad +\sum_{\kappa=1}^{n}\beta_{\mu\kappa}(L;\Lambda)\cdot(\delta_{\kappa\nu}v_{l_\nu}^{*}[a_\nu]+\alpha_{\kappa\nu}(-\bar{L}+1;\bar{\Lambda})\cdot v_{l_\nu+1}^{*}[a_\nu]+\cdots)\} \\ &\quad +\textit{single-valued regular function at } (a_\nu,\bar{a}_\nu) \end{aligned} \tag{3.2.37}$$

for $\mu,\nu,\sigma=1,\cdots,n$.

Likewise for $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$, there exists a unique canonical basis $w_\mu=w_\mu(L;\Lambda)=(w_\mu^{(1)}(L;\Lambda),\cdots,w_\mu^{(n)}(L;\Lambda))$ $(\mu=1,\cdots,n)$ of $W_{F,a_1,\cdots,a_n}^{l_1,\cdots,l_n}(\Lambda)$ such that

$$(3.2.38)\qquad w_\mu^{(\sigma)}(L;\Lambda)=\lambda_{\sigma\nu}\Big\{\delta_{\mu\nu}\cdot w_{-l_\nu}[a_\nu]+\alpha_{\mu\nu}\Big(L-\frac{1}{2};\Lambda\Big)\times w_{-l_\nu+1}[a_\nu]+\cdots$$
$$+\sum_{k=1}^{n}\beta_{\mu k}\Big(L-\frac{1}{2};\Lambda\Big)\cdot\Big(\delta_{k\nu}\cdot w_{l_\nu}^{*}[a_\nu]+\overline{\alpha_{k\nu}\Big(-L+\frac{1}{2};\Lambda\Big)}\times w_{l_\nu+1}^{*}[a_\nu]+\cdots\Big)\Big\}$$
$$+\textit{single-valued regular function at } (a_\nu,\bar a_\nu).$$
$$(\mu,\nu,\sigma=1,\cdots,n).$$

The coefficients $\alpha_{\mu\nu}(L;\Lambda)$ *and* $\beta_{\mu\nu}(L;\Lambda)$ *satisfy the same relations* (*i*), (*ii*) *of Proposition* 3.2.5.

Theorem 3.2.13. *Assume* $0<l_1,\cdots,l_n<1$ *for* $*=B$ *and* $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$ *for* $*=F$. *Then* $\bigcup_{j=0}^{\infty}W^{l_1+j,\cdots,l_n+j}_{*,a_1,\cdots,a_n}(\Lambda)$ *is a left* $\mathbb{C}[\partial,\bar\partial,M_*]$-*module. We have the isomorphism*

$$(3.2.39)\qquad (\mathbb{C}[\partial,\bar\partial]/(m^2-\partial\bar\partial))\otimes_{\mathbb{C}}W^{l_1,\cdots,l_n}_{*,a_1,\cdots,a_n}(\Lambda)\xrightarrow{\sim}W^{l_1+j,\cdots,l_n+j}_{*,a_1,\cdots,a_n}(\Lambda)$$

given by $p(\partial,\bar\partial)\otimes w\mapsto p(\partial,\bar\partial)w$.

By the same argument as in p. 16-17, combined with the existence theorem for the basis to be proved in Chapter IV, we may show that $\dim_{\mathbb{C}}W^{l_1,\cdots,l_n,\mathrm{strict}}_{*,a_1,\cdots,a_n}(\Lambda)=n$ for $*=B$ or F.

Next choose $\lambda=(\lambda_1,\cdots,\lambda_n)\in\mathbb{C}^n$ and $(z^*,\bar z^*)\in X^{\mathrm{Euc}}-\{(a_\nu,\bar a_\nu)\}_{\nu=1,\cdots,n}$ arbitrarily. We let $W^{0,l_1,\cdots,l_n,\mathrm{strict}}_{B,z^*,a_1,\cdots,a_n}(\Lambda,\lambda)$ $(l_1,\cdots,l_n\notin\boldsymbol{Z})$ be the space of $v=(v^{(1)},\cdots,v^{(n)})$'s satisfying (3.2.2), (3.2.27), $(3.2.29)_\nu^{\mathrm{strict}}$, $(3.2.29)_\infty$ and having at $(z^*,\bar z^*)$ an additional singularity of the form

$$(3.2.40)\qquad v^{(\mu)}=\lambda_\mu\cdot\tilde c_0^{(0)}(v)\cdot\tilde v_0[z^*]+\text{regular function},$$
$$(\tilde c_0^{(0)}(v)\in\mathbb{C};\ \mu=1,\cdots,n).$$

By the same method as in the case of $v_0(z^*,z;L)$, we can show, at least for $0<l_1,\cdots,l_n<1$, the existence of a unique $v_0(L,\Lambda,\lambda)=v_0(z^*,z;L,\Lambda,\lambda)$ such that $\tilde c_0^{(0)}(v_0)=1$ and $c_{-l_\nu}^{(\nu)}(v_0)=0$ $(\nu=1,\cdots,n)$. Clearly $v_0(L,\Lambda,\lambda)$ is linear in $\lambda\in\mathbb{C}^n$. Denote by $\lambda_{(\mu)}$ the μ-th row vector of

Λ, and set $v_0(z^*, z; L, \Lambda)_{\mu\nu} = v_0^{(\nu)}(z^*, z; L, \Lambda, \lambda_{(\mu)})$. We have then for $\mu, \nu = 1, \cdots, n$

$$(3.2.41)\quad v_0(z^*, z; L, \Lambda)_{\mu\nu} = -\frac{\pi}{2\sin \pi l_\nu}(v_\nu^{(\mu)}(z^*; 1-L, \Lambda)\, v_{-l_\nu+1}[a_\nu] + \cdots + \overline{v_\nu^{(\mu)}(z^*; L, \Lambda)} \cdot v_{l_\nu}^*[a_\nu] + \cdots)$$

$$(3.2.42)\quad v_0(z^*, z; L, \Lambda)_{\mu\nu} = \overline{v_0(z, z^*; L, \Lambda)}_{\nu\mu}.$$

Similarly, for $-1/2 < l_1, \cdots, l_n < 1/2$ there exist unique $w_0^{(\pm)}(L, \Lambda, \lambda) = w_0^{(\pm)}(z^*, z; L, \Lambda, \lambda) = (w_0^{(\pm)(1)}, \cdots, w_0^{(\pm)(n)})$ that satisfy (3.2.14), (3.2.30), (3.2.31)$_\nu$ with $c_{-l_\nu}^{(\nu)}(w_0^{(\pm)}) = 0$ $(\nu = 1, \cdots, n)$, (3.2.31)$_\infty$ and at $(z^*, \bar{z}^*)$

$$(3.2.43)\quad w_0^{(\pm)(\mu)} = \lambda_\mu \begin{cases} -\tilde{w}_{1/2}^*[z^*] \\ \tilde{w}_{1/2}[z^*] \end{cases} + \text{regular function.}$$

Setting $w_0^{(\pm)}(L, \Lambda)_{\mu\nu} = w_0^{(\pm)(\nu)}(L, \Lambda, \lambda_{(\mu)})$ we find

$$(3.2.44)\quad (w_0^{(+)}(z^*, z; L, \Lambda)_{\mu\nu}, w_0^{(-)}(z^*, z; L, \Lambda)_{\mu\nu}) = \frac{\pi}{2\cos \pi l_\nu}(w_{-l_\nu+1}[a_\nu] \cdot {}^t w_\nu^{(\mu)}(z^*; 1-L, \Lambda) + \cdots + w_{l_\nu}^*[a_\nu] \cdot {}^t w_\nu^{*(\mu)}(z^*; L, \Lambda) + \cdots)$$

$$(3.2.45)\quad w_+^{(+)}(z^*, z; L, \Lambda)_{\mu\nu} = \overline{w_-^{(-)}(z, z^*; L, \Lambda)}_{\nu\mu}$$

$$w_\mp^{(\pm)}(z^*, z; L, \Lambda)_{\mu\nu} = \overline{w_\mp^{(\pm)}(z, z^*; L, \Lambda)}_{\nu\mu}$$

for $\mu, \nu = 1, \cdots, n$.

For later convenience we shall refer to as $W_{F, z^*, a_1, \cdots, a_n}^{(\pm)0, l_1, \cdots, l_n, \text{strict}}(\Lambda, \lambda)$ the space spanned by $w_0^{(\pm)}(L, \Lambda, \lambda)$ and $w_\nu(L, \Lambda)$ $(\nu = 1, \cdots, n)$. We set also $W_{B, a_1, \cdots, a_n, z}^{*l_1, \cdots, l_n, 0, \text{strict}}(\Lambda, {}^t\lambda) = \{{}^t\overline{v} \mid v \in W_{B, z, a_1 \cdots, a_n,}^{0, l_1, \cdots, l_n, \text{strict}}(\Lambda, \lambda)\}$, $W_{F, a_1, \cdots, a_n, z}^{*(\pm)l_1, \cdots, l_n, 0, \text{strict}}(\Lambda, {}^t\lambda) = \{{}^t w^* \mid w \in W_{F, z, a_1, \cdots, a_n}^{(\mp)0, l_1, \cdots, l_n, \text{strict}}(\Lambda, \lambda)\}$.

For notational convenience we introduce the matrices $v = v(L; \Lambda) = (v_\mu^{(\nu)}(L, \Lambda))_{\mu, \nu = 1, \cdots, n}$ and $w = w(L; \Lambda) = (w_\mu^{(\nu)}(L; \Lambda))_{\mu, \nu = 1, \cdots, n}$. Their local expansions then read as

$$(3.2.46)\quad v(L; \Lambda) = 1_n \cdot v_{-l_\nu}[a_\nu] + \alpha(L; \Lambda) \cdot v_{-l_\nu+1}[a_\nu] + \cdots + \beta(L; \Lambda) \cdot v_{l_\nu}^*[a_\nu] + \beta(L; \Lambda)\overline{\alpha(-L+1; \Lambda)} \times v_{l_\nu+1}^*[a_\nu] + \cdots) \cdot E_\nu \Lambda$$

$$+\text{regular function}$$

$$w(L;\Lambda)=\Big(1_n\cdot w_{-l_\nu}[a_\nu]+\alpha\Big(L-\frac{1}{2};\Lambda\Big)\cdot w_{-l_\nu+1}[a_\nu]+\cdots$$

$$+\beta\Big(L-\frac{1}{2};\Lambda\Big)\cdot w^*_{l_\nu}[a_\nu]+\beta\Big(L-\frac{1}{2};\Lambda\Big)\cdot\alpha\Big(-L+\frac{1}{2};\Lambda\Big)$$

$$\times w^*_{l_\nu+1}[a_\nu]+\cdots\Big)\cdot E_\nu\Lambda$$

$$+\text{regular function.}$$

Since Λ is non-singular, it is clear that the n column vectors $v^{(\nu)}(L;\Lambda)$ (resp. $w^{(\nu)}(L;\Lambda)$), $\nu=1,\cdots,n$, of $v(L;\Lambda)$ (resp. $w(L;\Lambda)$) are linearly independent.

This linear independence fails in the limit where Λ tends to a degenerate matrix $\Lambda^0={}^t\Lambda^0=\bar{\Lambda}^0$ with $\lambda^0_{\nu\nu}=1$ $(1\leqq\nu\leqq n)$. For the sake of definiteness we consider $w(L;\Lambda)$. Set $\operatorname{rank}\Lambda^0=n-r$, and let $p_1,\cdots,p_r$ be a basis of $\operatorname{Ker}\Lambda^0$. Without loss of generality we may assume that the first $r\times r$ block of $P=(p_1,\cdots,p_r)$ to be non-degenerate: $P={}^t({}^tP_1,{}^tP_3)$, $\det P_1\neq 0$. Setting $w(L;\Lambda^0)=\lim_{\Lambda\to\Lambda^0}w(L;\Lambda)$ we have

$$\gamma_\nu w(L;\Lambda^0)P=w(L;\Lambda^0)\cdot(1-(e^{-2\pi i l_\nu}+1)E_\nu\Lambda)P$$

$$=w(L;\Lambda^0)P\,.$$

Hence by (3.2.46) $w(L;\Lambda^0)P$ is single-valued and regular at each $(a_\nu,\bar{a}_\nu)$. Regarding $(3.2.31)_\infty$ we then conclude $w(L;\Lambda^0)P\equiv 0$. That is, the first r-columns of $w(L;\Lambda^0)$ are linearly dependent on $w^{(r+1)}(L;\Lambda^0),\cdots,w^{(n)}(L;\Lambda^0)$ (linear independence of the latter follows from (3.2.46)).

In order to recover the rest we proceed as follows. Choose $\Lambda'={}^t\Lambda'=\bar{\Lambda}'$ so that $\lambda'_{\nu\nu}=0$ $(\nu=1,\cdots,n)$ and $\Lambda^0+\varepsilon\Lambda'$ is positive definite for $1\gg\varepsilon>0$. (In particular $\det{}^tP\Lambda'P\neq 0$). We set

$$w_\varepsilon(L;\Lambda^0,\Lambda')=w(L;\Lambda^0+\varepsilon\Lambda')\cdot\begin{pmatrix}\varepsilon^{-1}P_1 & 0\\ \varepsilon^{-1}P_3 & 1\end{pmatrix} \tag{3.2.47}$$

$$w(L;\Lambda^0,\Lambda')=\lim_{\varepsilon\downarrow 0}w_\varepsilon(L;\Lambda^0,\Lambda')\,.$$

Proposition 3.2.14. *Assume* $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$. *Then the*

column vectors of $\boldsymbol{w}(L; \Lambda^0, \Lambda')$ *are linearly independent. They satisfy the monodromy property*

$$(3.2.48)\qquad \gamma_\nu \boldsymbol{w}(L; \Lambda^0, \Lambda') = \boldsymbol{w}(L; \Lambda^0, \Lambda') \cdot M_\nu(L; \Lambda^0, \Lambda')$$

$$M_\nu(L; \Lambda^0, \Lambda') = 1 - (e^{-2\pi i l_\nu} + 1) \times \begin{pmatrix} 0 & 0 \\ -P_3 P_1^{-1} & 1 \end{pmatrix} E_\nu \begin{pmatrix} \Lambda_1' P_1 + \Lambda_2' P_3 & \Lambda_2^0 \\ \Lambda_3' P_1 + \Lambda_4' P_3 & \Lambda_4^0 \end{pmatrix}.$$

Proof. From (3.2.31) we have

$$(3.2.49)\qquad \gamma_\nu \boldsymbol{w}_\varepsilon(L; \Lambda^0, \Lambda')$$

$$= \boldsymbol{w}(L; \Lambda^0 + \varepsilon\Lambda') \cdot (1 - (e^{-2\pi i l_\nu} + 1) E_\nu(\Lambda^0 + \varepsilon\Lambda')) \begin{pmatrix} \varepsilon^{-1} P & 0 \\ & 1 \end{pmatrix}$$

$$= \boldsymbol{w}_\varepsilon(L; \Lambda^0, \Lambda') - (e^{-2\pi i l_\nu} + 1) \times \boldsymbol{w}(L; \Lambda^0 + \varepsilon\Lambda') \cdot E_\nu \cdot \left(\Lambda' P,\ (\Lambda^0 + \varepsilon\Lambda') \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

On the other hand, $\boldsymbol{w}(L; \Lambda^0) P = 0$ implies $\boldsymbol{w}(L; \Lambda^0) = \boldsymbol{w}(L; \Lambda^0) \begin{pmatrix} 0 & 0 \\ -P_3 P_1^{-1} & 1 \end{pmatrix} = \boldsymbol{w}(L; \Lambda^0, \Lambda') \cdot \begin{pmatrix} 0 & 0 \\ -P_3 P_1^{-1} & 1 \end{pmatrix}$. Substituting into (3.2.49) we obtain in the limit $\varepsilon \to 0$

$$\gamma_\nu \boldsymbol{w}(L; \Lambda^0, \Lambda') = \boldsymbol{w}(L; \Lambda^0, \Lambda') \cdot (1 - (e^{-2\pi i l_\nu} + 1) L_\nu),$$

$$L_\nu = \begin{pmatrix} 0 & 0 \\ -P_3 P_1^{-1} & 1 \end{pmatrix} E_\nu \left(\Lambda' P,\ \Lambda^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

This shows (3.2.48).

To prove the linear independence of the columns of $\boldsymbol{w}(L; \Lambda^0, \Lambda')$, set $N = \{c \in \mathbb{C}^n \mid \boldsymbol{w}(L; \Lambda^0, \Lambda') \cdot c = 0\}$. Since $\boldsymbol{w}^{(\nu)}(L; \Lambda^0, \Lambda') = \boldsymbol{w}^{(\nu)}(L; \Lambda^0)$ are linearly independent for $\nu = r+1, \cdots, n$, $N \cap \left\{ \begin{pmatrix} 0 \\ c'' \end{pmatrix} \in \mathbb{C}^n \right\} = \{0\}$. So we are only to show that, if N were not $\{0\}$, it would contain a non-zero vector of the form $\begin{pmatrix} 0 \\ c'' \end{pmatrix}$.

For $c \in N$ we have $0 = \gamma_\nu \boldsymbol{w}(L; \Lambda^0, \Lambda') c = \boldsymbol{w}(L; \Lambda^0, \Lambda') M_\nu(L; \Lambda^0, \Lambda') c$ so that $M_\nu(L; \Lambda^0, \Lambda') N \subset N$, i.e. $L_\nu N \subset N$ since $e^{-2\pi i l_\nu} + 1 \neq 0$. From the form of L_ν, if $L_\nu c \neq 0$ for some $c \in N$ and ν, then we are done. Assume $L_\nu c = 0$, $0 \neq c \in N$ for all ν. Since $\Lambda^0 P = 0$ is equivalently written as $\Lambda^0 = \Lambda^0 \begin{pmatrix} 0 & 0 \\ -P_3 P_1^{-1} & 1 \end{pmatrix}$, we have

$$\sum_{\nu=1}^{n} E_\nu A^0 L_\nu = \sum_{\nu=1}^{n} E_\nu A^0 \begin{pmatrix} 0 & 0 \\ -P_3 P_1^{-1} & 1 \end{pmatrix} E_\nu \left(A'P, A^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= \left(\sum_{\nu=1}^{n} E_\nu A^0 E_\nu \right) \cdot \left(A'P, A^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= \left(A'P, A^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

where we have used $\lambda_{\nu\nu}^0 = 1$ $(\nu = 1, \cdots, n)$. Hence $L_\nu c = 0$ for all ν implies

$$0 = {}^tP \sum_{\nu=1}^{n} E_\nu A^0 L_\nu c$$

$$= {}^tP \left(A'P, A^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \cdot c$$

$$= ({}^tP A'P, 0)\, c\,,$$

because ${}^tP A^0 = {}^t(A^0 P) = 0$. Since ${}^tP A'P$ is non-degenerate, we obtain $c = \begin{pmatrix} 0 \\ c'' \end{pmatrix} \neq 0$. This completes the proof of Proposition 3. 2. 14.

Remark. In the above proof we have tacitly assumed the existence of $\lim_{\varepsilon \downarrow 0} w_\varepsilon(L; A^0, A')$. This point is justified if we show that the family $w(L; A)$ depends holomorphically on A. This will be done in Chapter IV.

As an example of Proposition 3. 2. 14, consider the case

$$A^0 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad A' = A^0 - 1_n, \quad P = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix}.$$

In this case we have, setting $\varepsilon_\nu = e^{-2\pi i l_\nu}$ $(\nu = 1, \cdots, n)$,

$$M_\nu = \begin{cases} \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \hline 0 \cdots \varepsilon_\nu + 1 \cdots 0 & & & -\varepsilon_\nu \end{array} \right) & (\nu = 1, \cdots, n-1) \\ \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \hline -\varepsilon_n - 1, \cdots, & & -\varepsilon_n - 1 & -\varepsilon_n \end{array} \right) & (\nu = n) \end{cases}$$

and the n-th column $\boldsymbol{w}^{(n)}(L;\Lambda^0,\Lambda')=\boldsymbol{w}^{(n)}(L;\Lambda^0)$ coincides with the vector ${}^t({}^tw_1(L),\cdots,{}^tw_n(L))$, where $w_\mu(L)$ denotes the canonical basis (3.2.19) of $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$.

§ 3.3. Holonomic System and the Deformation Equations

One of the most important consequences of Theorems 3.2.6, 3.2.9, and 3.2.13 is the existence of a holonomic(*) system of linear differential equations satisfied by a basis of the space $W^{l_1,\cdots,l_n,\mathrm{strict}}_{*,a_1,\cdots,a_n}$ or $W^{l_1,\cdots,l_n,\mathrm{strict}}_{*,a_1,\cdots,a_n}(\Lambda)$ ($*=B$ or F). We shall now proceed to discuss this topic. For the sake of definiteness we shall mainly be concerned with the case $*=F$ in the sequel. (In view of the isomorphism (3.2.17) or (3.2.32), this is no restriction.)

Proposition 3.3.1. *Let $w_1,\cdots,w_n$ be an arbitrary basis of $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$ (resp. $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}(\Lambda)$) with $-\frac{1}{2}<l_1,\cdots,l_n<\frac{1}{2}$. Let $\boldsymbol{w}={}^t({}^tw_1,\cdots,{}^tw_n)$ denote the $2n\times 1$ (resp. $2n\times n$) matrix. Then there exist unique $n\times n$ constant matrices B, B^* and E, depending on $L=(\delta_{\mu\nu}l_\nu)$, $A=(\delta_{\mu\nu}a_\nu)$ (resp. L, A and Λ) and also on the choice of $\boldsymbol{w}$, such that*

$$M_F\boldsymbol{w}=(B\partial-B^*\bar{\partial}+E)\boldsymbol{w} \tag{3.3.1}$$

holds.(**)

Proof. From Theorem 3.2.9 (resp. Theorem 3.2.13) we have $M_Fw_\mu\in W^{l_1+1,\cdots,l_n+1}_{F,a_1,\cdots,a_n}$ (resp. $W^{l_1+1,\cdots,l_n+1}_{F,a_1,\cdots,a_n}(\Lambda)$) for each $\mu=1,\cdots,n$. By virtue of the isomorphism (3.2.20) (resp. (3.2.39)), there exist unique constants $b_{\mu\nu}$, $b^*_{\mu\nu}$ and $e_{\mu\nu}\in\boldsymbol{C}$ such that

$$M_Fw_\mu=\sum_{\nu=1}^{n}(b_{\mu\nu}\partial-b^*_{\mu\nu}\bar{\partial}+e_{\mu\nu})w_\nu,\quad \mu=1,\cdots,n \tag{3.3.2}$$

holds. This proves Proposition 3.3.1.

(*) Concerning holonomic (=maximally overdetermined) systems we refer the reader to [9], [10].

(**) To be precise B, B^*, E are understood as $B\otimes I_2$, etc. Namely (3.3.1) stands for (3.3.2).

Let $C_j=(c^{(\nu)}_{-l_\nu+j}(w_\mu))_{\mu,\nu=1,\cdots,n}$, $C^*_j=(c^{*(\nu)}_{l_\nu+j}(w_\mu))_{\mu,\nu=1,\cdots,n}$ be the matrices of the coefficients of the local expansion (3. 2. 16)$'_\nu$ (resp. (3. 2. 31)$_\nu$). Comparing the local expansions of both hand sides of (3. 3. 1), we obtain the following recursion relations for all $j\in\boldsymbol{Z}$:

$$(3.3.3)_j \qquad C_{j+1}mA-mBC_{j+1}+C_j(-L+j)$$
$$-EC_j-C_{j-1}m\bar{A}+mB^*C_{j-1}=0$$

$$(3.3.3)^*_j \qquad C^*_{j+1}m\bar{A}-mB^*C^*_{j+1}+C^*_j(L+j)$$
$$+EC^*_j-C^*_{j-1}mA+mBC^*_{j-1}=0\,.$$

Notice that $C_j=0$, $C^*_j=0$ for $j<0$. In particular $(3.3.3)_j$ and $(3.3.3)^*_j$ read for $j=-1,0,1$

$$(3.3.4) \qquad B=C_0AC_0^{-1},\ B^*=C_0^*\bar{A}C_0^{*-1}$$

$$(3.3.5) \qquad E=[C_1C_0^{-1},C_0mAC_0^{-1}]-C_0LC_0^{-1}$$
$$=-[C_1^*C_0^{*-1},C_0^*m\bar{A}C_0^{*-1}]-C_0^*LC_0^{*-1}$$

$$(3.3.6) \qquad [C_2C_0^{-1},C_0mAC_0^{-1}]+C_1(-L+1)C_0^{-1}$$
$$-EC_1C_0^{-1}-C_0m\bar{A}C_0^{-1}+C_0^*m\bar{A}C_0^{*-1}=0$$
$$[C_2^*C_0^{*-1},C_0^*m\bar{A}C_0^{*-1}]+C_1^*(L+1)C_0^{*-1}$$
$$+EC_1^*C_0^{*-1}-C_0^*mAC_0^{*-1}+C_0mAC_0^{-1}=0\,.$$

For the canonical basis $\boldsymbol{w}=\boldsymbol{w}(L)$ (resp. $\boldsymbol{w}(L;A)$) satisfying (3. 2. 19) (resp. (3. 2. 38)), we have

$$(3.3.7) \qquad C_0=1,\ C_0^*=\beta,\ C_1=\alpha,\ C_1^*=\beta\bar{\alpha}'$$
$$B=A,\ B^*=\beta\bar{A}\beta^{-1}$$
$$E=[\alpha,mA]-L=-\beta([\bar{\alpha}',m\bar{A}]+L)\beta^{-1}$$

where $\alpha=\alpha\left(L+\frac{1}{2}\right)$, $\alpha'=\alpha\left(-L+\frac{1}{2}\right)$ and $\beta=\beta\left(L+\frac{1}{2}\right)$ $\left(\text{resp. }\alpha=\alpha\left(L+\frac{1}{2};A\right),\alpha'=\alpha\left(-L+\frac{1}{2};A\right)\text{ and }\beta=\beta\left(L+\frac{1}{2};A\right)\right)$. In either case Proposition 3. 2. 5 implies $\beta=-e^{2H}(\cos\pi L)^{-1}$ with $H=H\left(L+\frac{1}{2}\right)$ $\left(\text{resp. }H\left(L+\frac{1}{2};A\right)\right)$ and $[\alpha',mA]=(\cos\pi L)\cdot[{}^t\alpha,mA]\cdot(\cos\pi L)^{-1}$. This shows the following:

Theorem 3.3.2. *The canonical basis* $w(L)$ *(resp.* $w(L, \Lambda)$*) satisfies the following holonomic system of first order linear differential equations*:

$$(3.3.8) \qquad (m-\Gamma)w=0$$

$$M_F w=(A\cdot\partial-G^{-1}\bar{A}G\cdot\bar{\partial}+F)w$$

where we have set

$$(3.3.9) \qquad F=[\alpha, mA]-L, \quad G=e^{-2H}$$

$$\alpha=\alpha\left(L+\frac{1}{2}\right), \quad H=H\left(L+\frac{1}{2}\right) \quad \textit{for} \quad w=w(L)$$

$$\alpha=\alpha\left(L+\frac{1}{2};\Lambda\right), \quad H=H\left(L+\frac{1}{2};\Lambda\right) \quad \textit{for} \quad w=w(L;\Lambda).$$

These matrices are subject to the algebraic conditions

$$(3.3.10) \qquad {}^t\bar{F}=GFG^{-1}, \ \textit{diagonal of} \ F=-L$$

$$G={}^t\bar{G} \ \textit{is positive definite.}$$

Remark. Comparing the diagonal part of (3.3.6) in the case (3.3.7), we see that the diagonal of α is expressible in terms of β and the off-diagonal part of α as follows.

$$(3.3.11) \qquad \alpha_{\mu\mu}=-m\sum_{\nu(\neq\mu)}(a_\mu-a_\nu)\alpha_{\mu\nu}\alpha_{\nu\mu}+m(\bar{a}_\mu-(\beta\bar{A}\beta^{-1})_{\mu\mu}),$$

$$(\mu=1, \cdots, n).$$

Noting $\alpha'_{\mu\nu}\alpha'_{\nu\mu}=\alpha_{\mu\nu}\alpha_{\nu\mu}(\mu\neq\nu)$ and $\beta\left(-L+\frac{1}{2}\right)\cdot\bar{A}\cdot\beta\left(-L+\frac{1}{2}\right)^{-1}=(\cos\pi L)\cdot{}^t\left(\beta\left(L+\frac{1}{2}\right)\cdot\bar{A}\cdot\beta\left(L+\frac{1}{2}\right)^{-1}\right)\cdot(\cos\pi L)^{-1}$, we have then ${}^t\left(\alpha\left(L+\frac{1}{2}\right)\cdot\cos\pi L\right)=\alpha\left(-L+\frac{1}{2}\right)\cos\pi L$.

It is sometimes useful to rewrite (3.3.8) into the form of a Pfaffian system with regular singularities:

$$(3.3.12) \qquad d_{z,\bar{z}}w=\Omega' w, \quad \Omega'=Pdz+P^*d\bar{z}$$

where

$$(3.3.13)\qquad P=(z-A)^{-1}\begin{pmatrix} F-\frac{1}{2} & G^{-1}m(\bar{z}-\bar{A})G \end{pmatrix}+\begin{pmatrix} 0 \\ m \end{pmatrix}$$

$$P^*=G^{-1}(\bar{z}-\bar{A})^{-1}G\begin{pmatrix} m(z-A) & -F-\frac{1}{2} \end{pmatrix}+\begin{pmatrix} m \\ 0 \end{pmatrix}.$$

Here we arrange the blocks according to the partition ${}^t\boldsymbol{w}=({}^t\boldsymbol{w}_+, {}^t\boldsymbol{w}_-)$, ${}^t\boldsymbol{w}_\pm=(w_{1,\pm}, \cdots, w_{n,\pm})$.

Up to present we have fixed the position of branch points $(a_\nu, \bar{a}_\nu)$. We now consider how a basis of $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$ or $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}(\Lambda)$ depends on $(a_\nu, \bar{a}_\nu)$. Denote by d', d'' and $d=d'+d''$ the exterior differentiation with respect to the variables $(z, \bar{z})$, $(A, \bar{A})=(a_1, \bar{a}_1, \cdots, a_n, \bar{a}_n)$ and $(z, \bar{z}, A, \bar{A})$, respectively.

Proposition 3.3.3. *Notations being the same as in Proposition* 3.3.1, *assume that* $\boldsymbol{w}$ *depends differentiably on* $(A, \bar{A})$. *Then there exist unique matrices* Φ, Φ^* *and* Ψ, *whose entries are* $(z, \bar{z})$*-independent* 1*-forms in* $(A, \bar{A})$(*), *such that*

$$(3.3.14)\qquad d''\boldsymbol{w}=(\Phi\cdot\partial+\Phi^*\cdot\bar{\partial}+\Psi)\boldsymbol{w}$$

holds.

Proof. Clearly $d''\boldsymbol{w}$ (that is, each of its coefficient of da_ν or $d\bar{a}_\nu$) satisfies the Euclidean Dirac equation. Differentiation of the expansion $(3.2.16)'_\nu$ or $(3.2.31)_\nu$ shows that $d''\boldsymbol{w}$ also has the local expansion of the same type except for the growth order. Regarding Proposition 3.1.5-(iii) we see that the exponentially decreasing property is preserved. Hence we have $d''\boldsymbol{w}\in W^{l_1+1,\cdots,l_n+1}_{F,a_1,\cdots,a_n}$ or $\in W^{l_1+1,\cdots,l_n+1}_{F,a_1,\cdots,a_n}(\Lambda)$. (3.3.14) follows by the same argument as in Proposition 3.3.1.

The coefficients Φ, Φ^* and Ψ are related to C_j, C_j^* (cf. $(3.3.3)_j$, $(3.3.3)^*_j$) through the formulas

$$(3.3.15)_j\qquad C_{j+1}mdA+m\Phi C_{j+1}-dC_j+\Psi C_j$$

(*) In the sequel capital roman and Greek letters are used to indicate matrices of 0-forms and 1-(sometimes 2-) forms, respectively.

$$+C_{j-1}md\bar{A}+m\Phi^*C_{j-1}=0$$

(3. 3. 15)$_j^*$ $$C_{j+1}^*md\bar{A}+m\Phi^*C_{j+1}^*-dC_j^*+\Psi C_j^*$$
$$+C_{j-1}^*mdA+m\Phi C_{j-1}^*=0$$

for all $j\in\boldsymbol{Z}$. For $j=-1, 0, 1$ they read

(3. 3. 16) $$\Phi=-C_0dA\cdot C_0^{-1},\quad \Phi^*=-C_0^*d\bar{A}\cdot C_0^{*-1}$$

(3. 3. 17) $$\Psi=dC_0\cdot C_0^{-1}+[C_0mdA\cdot C_0^{-1}, C_1C_0^{-1}]$$
$$=dC_0^*\cdot C_0^{*-1}+[C_0^*md\bar{A}\cdot C_0^{*-1}, C_1^*C_0^{*-1}]$$

(3. 3. 18) $$[C_2C_0^{-1}, C_0mdA\cdot C_0^{-1}]-dC_1\cdot C_0^{-1}+\Psi C_1C_0^{-1}$$
$$+C_0md\bar{A}\cdot C_0^{-1}-C_0^*md\bar{A}\cdot C_0^{*-1}=0,$$
$$[C_2^*C_0^{*-1}, C_0^*md\bar{A}\cdot C_0^{*-1}]-dC_1^*\cdot C_0^{*-1}+\Psi C_1^*C_0^{*-1}$$
$$+C_0^*mdA\cdot C_0^{*-1}-C_0mdA\cdot C_0^{-1}=0.$$

Now we apply Proposition 3. 3. 3 to the case $w=w(L)$ or $w(L;\Lambda)$. Although their differentiability in $(A, \bar{A})$ is not a priori clear, it will eventually turn out to be true (Corollary 3. 3. 11). We assume this for the moment, but it is logically independent of the arguments given below. For $w=w(L)$ or $w(L;\Lambda)$ we have, as in (3. 3. 7),

(3. 3. 19) $$\Phi=-dA,\quad \Phi^*=-G^{-1}d\bar{A}\cdot G,\quad \Psi=-[\alpha, mdA].$$

To sum up we have the following extended system of linear differential equations

(3. 3. 20) $$\begin{cases}(m-\Gamma)w=0,\\ M_Fw=(A\partial-G^{-1}\bar{A}G\bar{\partial}+F)w\\ d''w=(-dA\cdot\partial-G^{-1}d\bar{A}\cdot G\partial+\Theta)w\end{cases}$$

where we have set

(3. 3. 21) $$\Theta=-[\alpha, mdA].$$

Remark. The system (3. 3. 20) contains the equations

(3. 3. 22) $$(\partial+\sum_{\nu=1}^{n}\partial_{a_\nu})w=0,\quad (\bar{\partial}+\sum_{\nu=1}^{n}\bar{\partial}_{a_\nu})w=0,$$

$$(M_F+\sum_{\nu=1}^{n} M_{B,a_\nu}+L)\,\boldsymbol{w}=0 \quad (M_{B,a_\nu}=a_\nu\partial_{a_\nu}-\bar{a}_\nu\bar{\partial}_{a_\nu})$$

expressing the Euclidean covariance of $\boldsymbol{w}$.

The system (3. 3. 20) is equivalently rewritten as

$$d\boldsymbol{w}=\Omega\boldsymbol{w}, \quad \boldsymbol{w}=\begin{pmatrix}\boldsymbol{w}_+\\ \boldsymbol{w}_-\end{pmatrix} \tag{3. 3. 23}$$

$$\Omega=d\log(z-A)\cdot\begin{pmatrix}F-\frac{1}{2} & G^{-1}m(\bar{z}-\bar{A})G\end{pmatrix}$$

$$+G^{-1}d\log(\bar{z}-\bar{A})\cdot G\begin{pmatrix}m(z-A) & -F-\frac{1}{2}\end{pmatrix}$$

$$+\begin{pmatrix}\Theta & G^{-1}md(\bar{z}-\bar{A})G\\ md(z-A) & \Theta\end{pmatrix}.$$

Thus we have

Theorem 3. 3. 4. *The canonical basis* $\boldsymbol{w}(L)$ *or* $\boldsymbol{w}(L;A)$ *satisfies the extended holonomic system* (3. 3. 20) (*or its equivalent* (3. 3. 23)).

Theorem 3. 3. 5. *The coefficient matrices* F, G *appearing in* (3. 3. 20) *for* $\boldsymbol{w}(L)$ *or* $\boldsymbol{w}(L;A)$ *satisfy the following total differential equations* ("*the deformation equations*")

$$dF=[\Theta,F]+m^2([dA,G^{-1}\bar{A}G]+[A,G^{-1}d\bar{A}\cdot G]) \tag{3. 3. 24}$$

$$dG=-G\Theta-\Theta^*G\,.$$

Here Θ, Θ^* *denote matrices of* 1*-forms characterized by*

$$[\Theta,A]+[F,dA]=0\,, \quad \textit{diagonal of } \Theta=0\,. \tag{3. 3. 25}$$

$$[\Theta^*,\bar{A}]+[GFG^{-1},d\bar{A}]=0\,, \quad \textit{diagonal of } \Theta^*=0\,.$$

Proof. We set $\Theta=-[\alpha,mdA]$ and $\Theta^*={}^t\bar{\Theta}$. The relations (3. 3. 25) are immediately verified by using (3. 3. 9), (3. 3. 10). Notice that in terms of matrix elements $\Theta=(\theta_{\mu\nu})$, $\Theta^*=(\theta^*_{\mu\nu})$ (3. 3. 25) is equivalent to

$$\text{(3. 3. 26)}\qquad \theta_{\mu\nu}=\begin{cases} -f_{\mu\nu}\dfrac{d(a_\mu-a_\nu)}{a_\mu-a_\nu} & (\mu\neq\nu) \\ 0 & (\mu=\nu), \end{cases}$$

$$\theta^*_{\mu\nu}=\begin{cases} -(GFG^{-1})_{\mu\nu}\dfrac{d(\bar{a}_\mu-\bar{a}_\nu)}{\bar{a}_\mu-\bar{a}_\nu} & (\mu\neq\nu) \\ 0 & (\mu=\nu). \end{cases}$$

Now (3. 3. 6) and (3. 3. 18) gives in the case of (3. 3. 7)

$$\text{(3. 3. 27)}\qquad [C_2, mA]+\alpha\cdot(-L+1)-F\alpha-m\bar{A}+G^{-1}m\bar{A}G=0$$

$$[C_2, mdA]-d\alpha+\Theta\alpha+md\bar{A}-G^{-1}md\bar{A}\cdot G=0\,.$$

Since $dF=d([\alpha, mA]-L)=[d\alpha, mA]-\Theta$, we have

$$\begin{aligned} dF&=[[C_2, mdA]+\Theta\alpha+md\bar{A}-G^{-1}md\bar{A}\cdot G, mA]-\Theta \\ &=-[[mA, C_2], mdA]+[\Theta, mA]\alpha+\Theta[\alpha, mA] \\ &\quad -m^2[G^{-1}d\bar{A}\cdot G, A]-\Theta \\ &=-[\alpha\cdot(-L+1)-F\alpha-m\bar{A}+G^{-1}m\bar{A}G, mdA] \\ &\quad +m[\Theta, A]\alpha+\Theta(F+L)+m^2[A, G^{-1}d\bar{A}\cdot G]-\Theta \\ &=[\Theta, F]+m^2([dA, G^{-1}\bar{A}G]+[A, G^{-1}d\bar{A}\cdot G]). \end{aligned}$$

Similarly (3. 3. 17) and (3. 3. 7) yield

$$\begin{aligned} \Theta&=dG^{-1}\cdot G+[G^{-1}md\bar{A}\cdot G, G^{-1t}\bar{\alpha}G] \\ &=-G^{-1}dG-G^{-1t}\bar{\Theta}G. \end{aligned}$$

This proves (3. 3. 24).

Let us now consider in general Pfaffian systems of the form (3. 3. 12)-(3. 3. 13), (3. 3. 23) and (3. 3. 24) without assuming the algebraic conditions (3. 3. 10).

First we treat the linear system (3. 3. 12)-(3. 3. 13) in the complex domain. The integrability condition

$$\text{(3. 3. 28)}\qquad -\bar{\partial}P+\partial P^*-[P, P^*]=0$$

is easily verified. We denote the diagonal of F and GFG^{-1} by $-L=-(\delta_{\mu\nu}l_\nu)$ and $-L^*=-(\delta_{\mu\nu}l^*_\nu)$, respectively. In the system (3. 3. 8) for $w(L)$ or $w(L; A)$ we have $L=L^*$.

Proposition 3. 3. 6. *Assume* $l_\nu \not\equiv \frac{1}{2}$, $l_\nu^* \not\equiv \frac{1}{2} \bmod \boldsymbol{Z}$ $(\nu=1, \cdots, n)$.

(i) *At* $z=a_\mu$, $\bar{z}=\bar{z}_0 \neq \bar{a}_\nu$ $(\nu=1, \cdots, n)$, *there exist* $2n-1$ *independent holomorphic solutions and one that has the form*

$$(z-a_\mu)^{-l_\mu-1/2} \times (\textit{holomorphic function}). \tag{3. 3. 29}$$

Likewise at $z=z_0 \neq a_\mu$ $(\mu=1, \cdots, n)$ *and* $\bar{z}=\bar{a}_\nu$, *there exist* $2n-1$ *independent holomorphic solutions and one that has the form*

$$(\bar{z}-\bar{a}_\nu)^{l_\nu^*-1/2} \times (\textit{holomorphic function}). \tag{3. 3. 30}$$

(ii) *At* $z=a_\mu$ *and* $\bar{z}=\bar{a}_\nu$, *there exist* $2n-2$ *independent holomorphic solutions and two that behaves like* (3. 3. 29) *and* (3. 3. 30), *respectively.*

Proof. Set $P_\mu = P_\mu(\bar{z}) = \operatorname{Res}_{z=a_\mu} P$, $P_\nu^* = P_\nu^*(z) = \operatorname{Res}_{\bar{z}=\bar{a}_\nu} P^*$. A simple calculation shows $P_\mu\left(P_\mu + l_\mu + \frac{1}{2}\right)=0$, $P_\nu^*\left(P_\nu^* - l_\nu^* + \frac{1}{2}\right)=0$. It is also clear that P_μ, P_ν^* are of rank 1. Hence P_μ and P_ν^* are semi-simple matrices with eigenvalues $-l_\mu-\frac{1}{2}, 0, \cdots, 0$ and $l_\nu^*-\frac{1}{2}, 0, \cdots, 0$, respectively. Moreover the integrability condition (3. 3. 28) implies $[P_\mu(\bar{a}_\nu), P_\nu^*(a_\mu)]=0$, so that $P_\mu(\bar{a}_\nu)$ and $P_\nu^*(a_\mu)$ are simultaneously diagonalizable. If $c=\begin{pmatrix} c' \\ 0 \end{pmatrix}$ is an eigenvector of $P_\mu(\bar{a}_\nu)$ corresponding to $-l_\mu-\frac{1}{2}$ (and hence an eigenvector of $P_\nu^*(a_\mu)$), then $P_\nu^*(a_\mu)c$ has the form $\begin{pmatrix} 0 \\ c'' \end{pmatrix}$, so that $P_\nu^*(a_\mu)c=0$. Hence at $(z, \bar{z}) = (a_\mu, \bar{a}_\nu)$, P_μ and P_ν^* are simultaneously diagonalized into the form $\begin{pmatrix} -l_\mu-\frac{1}{2} & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & & \\ & l_\nu^*-\frac{1}{2} & \\ & & \ddots \\ & & & 0 \end{pmatrix}$, respectively.

On the other hand, from the assumption $l_\nu \not\equiv \frac{1}{2}$, $l_\nu^* \not\equiv \frac{1}{2} \bmod \boldsymbol{Z}$ $(\nu=1, \cdots, n)$ we conclude as follows.(*) In the case (i) the system (3. 3. 12) admits a fundamental matrix solution of the form $Y(z, \bar{z}) = U(z, \bar{z}) \cdot (z-a_\mu)^{P_\mu(\bar{z}_0)}$ or $U(z, \bar{z}) \cdot (\bar{z}-\bar{a}_\nu)^{P_\nu^*(z_0)}$, and in the case (ii) of the form $Y(z, \bar{z}) = U(z, \bar{z}) \cdot (z-a_\mu)^{P_\mu(\bar{a}_\nu)} (\bar{z}-\bar{a}_\nu)^{P_\nu^*(a_\mu)}$. Here $U(z, \bar{z})$ denotes an invertible holomorphic matrix at $(a_\mu, \bar{z}_0)$, $(z_0, \bar{a}_\nu)$ or $(a_\mu, \bar{a}_\nu)$,

(*) For the local theory of a linear Pfaffian system with regular singularities, see [7] where complete results are obtained.

respectively. The assertions of Proposition 3. 3. 6 follows from the above observations.

Next consider (3. 3. 23) and the associated non-linear system (3. 3. 24).

Proposition 3. 3. 7. *The non-linear system* (3. 3. 24) *is completely integrable.*

Proof. Define matrices of 1- and 2-forms $\Omega_1, \Omega_2, \Omega_3$ and Ω_3^* as follows.

$$\Omega_1 = dF - [\Theta, F] - m^2([dA, G^{-1}\bar{A}G] + [A, G^{-1}d\bar{A}\cdot G])$$
$$\Omega_2 = dG + G\Theta + \Theta^* G$$
$$\Omega_3 = d\Theta - \Theta \wedge \Theta - m^2[dA, G^{-1}d\bar{A}\cdot G]_+ \text{(*)}$$
$$\Omega_3^* = d\Theta^* + \Theta^* \wedge \Theta^* + m^2[d\bar{A}, GdA\cdot G^{-1}]_+ .$$

(3. 3. 31)

We have then

$$[\Omega_3, A] + [\Omega_1, dA]_+ = 0, \quad \text{diagonal of } \Omega_3 = 0$$
$$[\Omega_3^*, \bar{A}] + [G\Omega_1 G^{-1} + [\Omega_2 G^{-1}, GFG^{-1}], dA]_+ = 0,$$
$$\text{diagonal of } \Omega_3^* = 0 .$$

(3. 3. 32)

These equations are obtained by differentiating the defining equations (3. 3. 26) for Θ, Θ^* and using (3. 3. 27) (in particular $[\Theta, dA]_+ = 0$ and $[\Theta^*, d\bar{A}]_+ = 0$). Noting the Jacobi identity $[[X, \Phi], \Psi]_+ + [[\Phi, \Psi]_+, X] - [[\Psi, X], \Phi]_+ = 0$ (X: 0-form, Φ, Ψ: 1-form), we calculate $d\Omega_1$ and $d\Omega_2$. After a little computation we get

$$d\Omega_1 = [\Omega_1, \Theta]_+ + [F, \Omega_3] + m^2([dA, [G^{-1}\bar{A}G, G^{-1}\Omega_2]]_+ + [A, [G^{-1}d\bar{A}\cdot G, G^{-1}\Omega_2]_+]),$$
$$d\Omega_2 = \Omega_2 \wedge \Theta - \Theta^* \wedge \Omega_2 + G\Omega_3 + \Omega_3^* G.$$

(3. 3. 33)

On the other hand, (3. 3. 32) shows that the matrix elements of Ω_3 and Ω_3^* are linear combinations of (1-form) $\wedge$ (matrix elements of Ω_1 or Ω_2),

(*) $[X, Y]_+ = XY + YX$.

hence so are the right hand sides of (3. 3. 33). Therefore by Frobenius' theorem the Pfaffian system $\Omega_1=0$, $\Omega_2=0$ is completely integrable.

Proposition 3. 3. 8. *The system* (3. 3. 24) *has the following properties.*

(i) *For any solution* F *and* G, $F'={}^t\bar{G}^{-1t}\bar{F}{}^t\bar{G}$ *and* $G'={}^t\bar{G}$ *give another solution. In particular we have* ${}^t\bar{F}=GFG^{-1}$ *and* ${}^t\bar{G}=G$ *if they hold at some* $(A^0, \bar{A}^0)$.

(ii) $L=-$*diagonal of* F, $L^*=-$*diagonal of* GFG^{-1} *and* $\det G$ *are first integrals of* (3. 3. 24).

(iii) *For any solution* F, G, *we have*

$$F(e^{i\theta}A+b, e^{-i\theta}\bar{A}+\bar{b})=e^{-iL\theta}F(A, \bar{A})e^{iL\theta} \tag{3. 3. 34}$$
$$G(e^{i\theta}A+b, e^{-i\theta}\bar{A}+\bar{b})=e^{-iL^*\theta}G(A, \bar{A})e^{iL\theta}.$$

Proof. Set $\Theta'={}^t\bar{\Theta}^*$ and $\Theta^{*\prime}={}^t\bar{\Theta}$. We have then in the notation of (3. 3. 31)

$$\begin{aligned}
&G'^{-1t}\bar{\Omega}_1G'+[F', G'^{-1t}\bar{\Omega}_2] \qquad (3. 3. 35)\\
&\quad=dF'-[\Theta', F']-m^2([dA, G'^{-1}\bar{A}G']\\
&\qquad+[A, G'^{-1}d\bar{A}\cdot G']),\\
&{}^t\bar{\Omega}_2=dG'+G'\Theta'+\Theta^{*\prime}G',\\
&[\Theta', A]+[F', dA]=0, \qquad \text{diagonal of } \Theta'=0\\
&[\Theta^{*\prime}, \bar{A}]+[G'F'G'^{-1}, d\bar{A}]=0, \quad \text{diagonal of } \Theta^{*\prime}=0.
\end{aligned}$$

Assertion (i) follows from (3. 3. 35). To see (ii) we note that diagonal of $[\Theta, F]=0$ by (3. 3. 26). Hence (3. 3. 24) implies $dL=0$. From (i) the diagonal of $GFG^{-1}={}^t\bar{F}'$ is also constant. On the other hand, we have by (3. 3. 24)

$$\begin{aligned}
d\log\det G&=d\,\mathrm{trace}\,\log G\\
&=\mathrm{trace}\,G^{-1}dG\\
&=\mathrm{trace}\,(-\Theta-G^{-1}\Theta^*G)\\
&=0.
\end{aligned}$$

This proves (ii). Finally let $\sigma: E(2) \times (X^{\mathrm{Euc}})^n \to (X^{\mathrm{Euc}})^n$, $(b, \bar{b}, \theta; A, \bar{A}) \mapsto (e^{i\theta}A + b, e^{-i\theta}\bar{A} + \bar{b})$ denote the action of the Euclidean motion group $E(2)$ on $(X^{\mathrm{Euc}})^n$. Set $\tilde{F} = \sigma^*(F)$, $\tilde{G} = \sigma^*(G)$, $\tilde{\Theta} = \sigma^*(\Theta) + (\tilde{F} + L)\, id\theta$, $\tilde{\Theta}^* = \sigma^*(\Theta^*) - (\tilde{G}\tilde{F}\tilde{G}^{-1} + L^*)\, id\theta$. Then the pullbacks of (3. 3. 24), (3. 3. 26) to $E(2) \times (X^{\mathrm{Euc}})^n$ read as follows.

$$
\begin{aligned}
(3.3.36) \qquad & d\tilde{F} = [\tilde{\Theta}, \tilde{F}] - i[L, \tilde{F}]\, d\theta \\
& \qquad + m^2([dA, \tilde{G}^{-1}\bar{A}\tilde{G}] + [A, \tilde{G}^{-1}d\bar{A}\cdot\tilde{G}]) \\
& d\tilde{G} = -\tilde{G}\tilde{\Theta} - \tilde{\Theta}^*\tilde{G} + i(\tilde{G}L - L^*\tilde{G})\, d\theta \\
& [\tilde{\Theta}, A] + [\tilde{F}, dA] = 0, \quad \text{diagonal of } \tilde{\Theta} = 0 \\
& [\tilde{\Theta}^*, \bar{A}] + [\tilde{G}\tilde{F}\tilde{G}^{-1}, d\bar{A}] = 0, \quad \text{diagonal of } \tilde{\Theta}^* = 0.
\end{aligned}
$$

In particular $\tilde{\Theta}$, $\tilde{\Theta}^*$ do not contain the terms of db, $d\bar{b}$ and $d\theta$. Hence (3. 3. 36) implies

$$
\frac{\partial \tilde{F}}{\partial b} = 0, \quad \frac{\partial \tilde{F}}{\partial \bar{b}} = 0, \quad \frac{\partial \tilde{F}}{\partial \theta} = -i[L, \tilde{F}]
$$

$$
\frac{\partial \tilde{G}}{\partial b} = 0, \quad \frac{\partial \tilde{G}}{\partial \bar{b}} = 0, \quad \frac{\partial \tilde{G}}{\partial \theta} = i(\tilde{G}L - L^*\tilde{G}).
$$

This shows (3. 3. 34), and the proof of Proposition 3. 3. 8 completes. We remark that $\sigma^*(\Theta) = e^{-iL\theta}(\Theta - (F + L)\, id\theta)\, e^{iL\theta}$, $\sigma^*(\Theta^*) = e^{-iL^*\theta}(\Theta^* + (GFG^{-1} + L^*)\, id\theta)\, e^{iL^*\theta}$ hold.

Remark. In Chapter IV we shall construct a family of solutions $w_\nu(z; L)$ of (3. 3. 8) which depend holomorphically on L and coincide with those in § 3. 2 for real L. The corresponding matrices $F = [\alpha, mA] - L$, $G = -(\cos \pi L)^{-1}\beta^{-1}$ are then contained in those leafs of (3. 3. 24) satisfying diagonal of $F =$ diagonal of GFG^{-1}, because both members are holomorphic in L and coincide if L is real.

Proposition 3. 3. 9. *The linear system* (3. 3. 20) (*or its equivalent* (3. 3. 23)) *is completely integrable if F and G satisfy the non-linear system* (3. 3. 24).

Proof. Set $\Omega = \Omega' + \Omega''$, where Ω' is given in (3. 3. 12) and Ω''

$=-dA\cdot P-G^{-1}d\bar{A}\cdot GP^*+\Theta$. Noting (3. 3. 28), we see that the integrability condition reads $d''\Omega'+d'\Omega''+d''\Omega''=[\Omega', \Omega'']_+ +\Omega''\wedge\Omega''$. Singling out the coefficients of dz and $d\bar{z}$, we obtain

$$
\begin{aligned}
&d''P+dA\cdot\partial P+G^{-1}dA\cdot G\partial P^* \\
&\quad =[P, dA]P+[P, G^{-1}d\bar{A}\cdot GP^*]-[P, \Theta] \\
&d''P^*+dA\cdot\bar{\partial}P+G^{-1}d\bar{A}\cdot G\bar{\partial}P^* \\
&\quad =[P^*, dA\cdot P]+[P^*, G^{-1}d\bar{A}\cdot G]P^*-[P^*, \Theta] \\
&dA\wedge d''P+G^{-1}d\bar{A}\cdot G\wedge d''P^*+[G^{-1}d\bar{A}\cdot G, G^{-1}dG]_+P^*+d\Theta \\
&\quad =dA\cdot P\wedge dA\cdot P+G^{-1}d\bar{A}\cdot GP^*\wedge G^{-1}d\bar{A}\cdot GP^* \\
&\qquad +\Theta\wedge\Theta+[d\bar{A}\cdot P, G^{-1}d\bar{A}\cdot GP^*]_+ \\
&\qquad -[dA\cdot P, \Theta]_+-[G^{-1}dA\cdot GP^*, \Theta]_+ .
\end{aligned}
$$

After some calculation these equations are equivalently rewritten as

$$
\begin{aligned}
&dF=[\Theta, F]+m^2([dA, G^{-1}\bar{A}G]+[A, G^{-1}d\bar{A}\cdot G]) \\
&[\bar{A}, dG\cdot G^{-1}+G\Theta G^{-1}+\Theta^*]=0 \\
&d\Theta=\Theta\wedge\Theta+m^2[dA, G^{-1}d\bar{A}\cdot G]_+ ,
\end{aligned}
\tag{3. 3. 37}
$$

which are direct consequences of (3. 3. 25). This proves Proposition 3. 3. 9.

Remark. Conditions (3. 3. 37) are derived more directly as follows. Set $Q=M_F-(A\partial-G^{-1}\bar{A}G\bar{\partial}+F)$, $\hat{\Omega}''=-dA\cdot\partial-G^{-1}d\bar{A}\cdot G\bar{\partial}+\Theta$. (3. 3. 20) is then written as

$$
(m-\Gamma)\boldsymbol{w}=0, \quad Q\boldsymbol{w}=0, \quad d''\boldsymbol{w}=\hat{\Omega}''\boldsymbol{w}. \tag{3. 3. 20)'}
$$

Noting that Q, $d''-\hat{\Omega}''$ commutes with $m-\Gamma$, we obtain as the consistency condition for (3. 3. 20)' $0=(d''Q-[\hat{\Omega}'', Q])\boldsymbol{w}$ and $0=(d''\hat{\Omega}''-\hat{\Omega}''\wedge\hat{\Omega}'')\boldsymbol{w}$. These conditions hold if

$$
d''Q-[\hat{\Omega}'', Q]\equiv 0, \quad d''\hat{\Omega}''-\hat{\Omega}''\wedge\hat{\Omega}''\equiv 0 \mod m^2-\partial\bar{\partial}. \tag{3. 3. 37)'}
$$

It is easy to verify that (3. 3. 37)' is equivalent to (3. 3. 37). Note that in the case of $\boldsymbol{w}=\boldsymbol{w}(L)$ or $\boldsymbol{w}(L;\Lambda)$, (3. 3. 37)' is also necessary, since $d''Q-[\hat{\Omega}'', Q]$ and $d\hat{\Omega}''-\hat{\Omega}''\wedge\hat{\Omega}''$ both belong to $\boldsymbol{C}[\partial, \bar{\partial}]\otimes(n\times n$ matrices

of differential forms in $(A, \bar{A})$).

Now let $(A^0, \bar{A}^0) = \{(a_\nu^0, \bar{a}_\nu^0)\}_{\nu=1,\cdots,n}$ be distinct n-points of X^{Euc}. Choose matrices F^0, G^0 arbitrarily. Since the right hand side of the system (3. 3. 24) is analytic in F, G and $(A, \bar{A})$ provided $(a_\mu, \bar{a}_\mu) \neq (a_\nu, \bar{a}_\nu)$ $(\mu \neq \nu)$, the complete integrability ensures that there exists a unique solution F, G of (3. 3. 24) in a sufficiently small (simply connected) neighborhood U_A of $(A^0, \bar{A}^0)$ such that $F = F^0, G = G^0$ at $(A^0, \bar{A}^0)$. Next let $W^0(z, \bar{z})$ be a $2n \times 2n$ fundamental matrix solution of (3. 3. 12) corresponding to F^0, G^0 and $(A^0, \bar{A}^0)$. Then Proposition 3. 3. 9 guarantees the existence of a unique solution $W(z, \bar{z}; A, \bar{A})$ of the extended system (3. 3. 23), such that it assumes the initial value $W^0(z, \bar{z})$ at $(A, \bar{A}) = (A^0, \bar{A}^0)$. Clearly W is analytically prolongable to the universal covering manifold of $\{(z, \bar{z}, A, \bar{A}) \in (X^{\mathrm{Euc}})^{n+1} | (A, \bar{A}) \in U_A, (z, \bar{z}) \neq (a_\nu, \bar{a}_\nu)$ $(\nu = 1, \cdots, n)\}$. For each fixed $(A, \bar{A}) \in U_A$, let

$$\rho_{A,\bar{A}} : \pi_1(X'_{a_1,\cdots,a_n}; x_0) \to GL(n, \boldsymbol{C}) \tag{3. 3. 38}$$

$$\gamma \cdot W = W \cdot \rho_{A,\bar{A}}(\gamma), \quad \gamma \in \pi_1(X'_{a_1,\cdots,a_n}; x_0)$$

be the associated monodromy representation. Since $\gamma d'' W = d''(\gamma W)$ $= d''(W \cdot \rho_{A,\bar{A}}(\gamma))$, we have $\Omega'' W \rho_{A,\bar{A}}(\gamma) = \Omega'' W \rho_{A,\bar{A}}(\gamma) + W d'' \rho_{A,\bar{A}}(\gamma)$, hence $d'' \rho_{A,\bar{A}}(\gamma) = 0$. Observe also that, as $|z| \to \infty$, $|W(z, \bar{z}; A, \bar{A})|$ $\leqq$const.$|W^0(z, \bar{z})|$ hold since the matrix elements of Ω'' are bounded in $(z, \bar{z})$ there. Summing up we have

Proposition 3. 3. 10. *Notations being as above, the monodromy representation* (3. 3. 38) *stays constant along each integral manifold of* (3. 3. 24). *Moreover the exponentially decreasing property for a column of W is also preserved.*

Corollary 3. 3. 11. *The canonical basis $\boldsymbol{w}(L)$ (resp. $\boldsymbol{w}(L; A)$) of $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}$ (resp. $W^{l_1,\cdots,l_n}_{F,a_1,\cdots,a_n}(A)$) depends analytically on $(A, \bar{A})$ provided that these n-points are distinct.*

Proof. We prove the case $\boldsymbol{w}(L)$ for definiteness. Let $\boldsymbol{w}^0(L)$ and F^0, G^0 denote the canonical basis and the cprresponding matrices at

$(A^0, \bar{A}^0)$, respectively. Let $\boldsymbol{w}(z, \bar{z}; A, \bar{A})$ denote the unique solution of (3. 3. 23) obtained from $\boldsymbol{w}^0(L)$, F^0 and G^0 by the procedure described above. We shall prove that $\boldsymbol{w}(z, \bar{z}; A, \bar{A})$ coincides with the canonical basis of $W_{F, a_1, \cdots, a_n}^{l_1, \cdots, l_n}$ for each fixed $(A, \bar{A})$. Since the analyticity of $\boldsymbol{w}(z, \bar{z}; A, \bar{A})$ is obvious, our assertion then follows.

Clearly (3. 3. 23) implies the Euclidean Dirac equation (3. 2. 14). The monodromy property (3. 2. 15) and the exponential fall-off condition $(3.2.16)_\infty$ follows from Proposition 3. 3. 10. The local behavior $(3.2.16)_\nu$ is a consequence of (3. 3. 23) and Proposition 3. 3. 6. It remains to prove that the 0-th coefficient matrix C_0 is identically 1. From the extended system (3. 3. 23) we have (3. 3. 4) with $B=A$, so that C_0 must be diagonal. Comparing the diagonal of (3. 3. 17) with $\Psi=\Theta$ we see that $dC_0=0$, hence $C_0=\text{constant}=1$. This proves Corollary 3. 3. 11.

It is instructive to rewrite the system (3. 3. 8), (3. 3. 20) by introducing the formal Laplace transformation

$$\boldsymbol{w}(z, \bar{z}) = \int \frac{du}{2\pi u} \begin{pmatrix} \sqrt{u} \\ \sqrt{u}^{-1} \end{pmatrix} e^{m(zu+\bar{z}u^{-1})} \hat{\boldsymbol{w}}(u). \tag{3. 3. 39}$$

Then (3. 3. 8) is transformed into a system of ordinary linear differential equations

$$\left(u\frac{d}{du} + mAu - G^{-1}m\bar{A}Gu^{-1} + F\right)\hat{\boldsymbol{w}}(u) = 0 \tag{3. 3. 40}$$

having irregular singular points of rank 1 at $u=0$ and at $u=\infty$. The deformation equations (3. 3. 24) are then regarded as an irregular-singular version of the Schlesinger's equations (cf. Chapter II [2], (2. 3. 38)$\sim$(2. 3. 43)). Setting $F=[\alpha, mA]-L$ (L: diagonal), the extended system (3. 3. 20) reads

$$\begin{aligned} &(d+\hat{\Omega})\hat{\boldsymbol{w}}=0 \\ &\hat{\Omega} = d(umA) + G^{-1}d(u^{-1}m\bar{A})\cdot G \\ &\qquad\qquad + [\alpha, u^{-1}d(umA)] - Lu^{-1}du. \end{aligned} \tag{3. 3. 41}$$

Example. Let us write down the system (3. 3. 24) more explicitly in the case $n=2$. We assume the algebraic conditions (3. 3. 10). Let

$\begin{pmatrix} l_1 & \\ & l_2 \end{pmatrix}$ = −diagonal of $F(l_1, l_2 \in \boldsymbol{R})$ and let $A = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}$ with $a_1 - a_2 = te^{i\theta}/2m$, $t > 0$. Regarding the Euclidean covariance (3. 3. 34) and the hermiticity of G, we see that F, G are of the form

$$(3.3.42)\qquad F = \begin{pmatrix} -l_1 & e^{-il\theta} f_+ \\ e^{il\theta} \bar{f}_- & -l_2 \end{pmatrix},\quad G = c \begin{pmatrix} \kappa \cosh\psi & e^{-il\theta}\varepsilon \sinh\psi \\ e^{il\theta}\bar{\varepsilon} \sinh\psi & \kappa^{-1}\cosh\psi \end{pmatrix}$$

$$l = l_1 - l_2,\quad c = (\det G)^{1/2} = \text{constant} > 0$$

$$\kappa = \bar{\kappa},\quad \psi = \bar{\psi},\quad \varepsilon\bar{\varepsilon} = 1$$

where $f_\pm$, κ, ψ and ε are functions of $t > 0$ independent of θ. The condition ${}^t\bar{F}G = GF$ implies further

$$(3.3.43)\qquad (\bar{\varepsilon} f_- - \varepsilon \bar{f}_-)\sinh\psi = 0,\quad (\bar{\varepsilon} f_+ - \varepsilon \bar{f}_+)\sinh\psi = 0$$

$$(\kappa f_+ - \kappa^{-1} f_-)\cosh\psi + l\varepsilon \sinh\psi = 0\,.$$

If $\psi \equiv 0$ the system (3. 3. 24) decouples into linear ones which are immediately integrated. We omit this case. From (3. 3. 43) we have then

$$(3.3.44)\qquad f_+ = \varepsilon\kappa^{-1}(f - l\tanh\psi)/2,\quad f_- = \varepsilon\kappa(f + l\tanh\psi)/2$$

$$f = \bar{f}\,.$$

Substituting (3. 3. 42) and (3. 3. 44) into (3. 3. 24) we obtain after some calculations

$$(3.3.45)\qquad f = t\frac{d\psi}{dt},\quad t\frac{d}{dt}\log\kappa = l\tanh^2\psi,\quad \frac{d\varepsilon}{dt} = 0$$

$$(3.3.46)\qquad \frac{d}{dt}\left(t\frac{d\psi}{dt}\right) = \frac{l^2}{t}\tanh\psi(1-\tanh^2\psi) + \frac{t}{2}\sinh 2\psi\,.$$

For $l = l_1 - l_2 = 0$, equation (3. 3. 46) coincides with an equivalent of the Painlevé equation of the third kind of restricted type ($\nu = 0$) studied by McCoy-Tracy-Wu [8]. In general (3. 3. 46) is converted into the following Painlevé equation of the fifth kind by the substitution $s = t^2$, $\sigma = \tanh^2\psi$:

$$(3.3.47)^{(*)}\qquad \frac{d^2\sigma}{ds^2} = \left(\frac{d\sigma}{ds}\right)^2\left(\frac{1}{2\sigma} + \frac{1}{\sigma - 1}\right) - \frac{1}{s}\frac{d\sigma}{ds} + \frac{(\sigma-1)^2}{s^2}\cdot\frac{l^2}{2}\sigma + \frac{1}{2s}\sigma\,.$$

---

* The general form of the Painlevé equation of the fifth kind reads $y'' = y'^2\left(\frac{1}{2y} + \frac{1}{y-1}\right) - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}$, where $'$ means $\frac{d}{dx}$.

Arguments for the case of half integral l_ν go quite parallel. Let $w_0^{(\pm)}(L)$ be the solutions (3. 2. 19) which satisfy (3. 2. 13), (3. 2. 14) with $-\frac{1}{2}<l_1, \cdots, l_n<\frac{1}{2}$, $c_{l_\nu}^{(\nu)}(w_0^{(\pm)})=0$ $(\nu=1, \cdots, n)$, and have an additional singularity at $(z^*, \bar{z}^*)$. We set ${}^t\tilde{\boldsymbol{w}}^{(\pm)}(L)=({}^tw_0^{(\pm)}(L), {}^t\boldsymbol{w}(L))$. By the same argument we have the following extended holonomic system for $\tilde{\boldsymbol{w}}^{(\pm)}=\tilde{\boldsymbol{w}}^{(\pm)}(L)$.

$$\text{(3. 3. 48)}\qquad \begin{cases} (m-\Gamma)\tilde{\boldsymbol{w}}^{(\pm)}=0 \\ M_F\tilde{\boldsymbol{w}}^{(\pm)}=(\tilde{A}\partial-\tilde{G}^{(\pm)-1}\bar{\tilde{A}}\tilde{G}^{(\pm)}\bar{\partial}+\tilde{F}^{(\pm)})\tilde{\boldsymbol{w}}^{(\pm)} \\ d''\tilde{\boldsymbol{w}}^{(\pm)}=(-d\tilde{A}\cdot\partial-\tilde{G}^{(\pm)-1}\,d\bar{\tilde{A}}\cdot\tilde{G}^{(\pm)}\bar{\partial}+\tilde{\Theta}^{(\pm)})\tilde{\boldsymbol{w}}^{(\pm)}. \end{cases}$$

Here the coefficients are given as follows.

$$\text{(3. 3. 49)}\qquad \tilde{A}=\begin{pmatrix} z^* & \\ & A \end{pmatrix},\quad A=\begin{pmatrix} a_1 & \\ & \ddots \\ & & a_n \end{pmatrix}$$

$$\tilde{F}^{(\pm)}=\left(\begin{array}{c|c} \mp\frac{1}{2} & -{}^t\alpha_0^{(\pm)}m(z^*-A) \\ \hline & F \end{array}\right),$$

$$\tilde{G}^{(\pm)}=\left(\begin{array}{c|c} 1 & {}^t\beta_0^{(\pm)}(\cos\pi L)G \\ \hline 0 & G \end{array}\right),$$

$$\tilde{\Theta}^{(\pm)}=\left(\begin{array}{c|c} 0 & {}^t\alpha_0^{(\pm)}md(z^*-A) \\ \hline 0 & \Theta \end{array}\right).$$

In (3. 3. 49) F, G, Θ are those corresponding to $\boldsymbol{w}(L)$, and ${}^t\alpha_0^{(\pm)}=(\alpha_{01}^{(\pm)}, \cdots, \alpha_{0n}^{(\pm)})$, ${}^t\beta_0^{(\pm)}=(\beta_{01}^{(\pm)}, \cdots, \beta_{0n}^{(\pm)})$ are given in (3. 2. 21)$_\nu^{(\pm)}$. Moreover $\tilde{F}^{(\pm)}$ and $\tilde{G}^{(\pm)}$ satisfy the deformation equations of the same type as those for F and G, namely

$$\text{(3. 3. 50)}\qquad d\tilde{F}^{(\pm)}=[\tilde{\Theta}^{(\pm)}, \tilde{F}^{(\pm)}]+m^2([d\tilde{A}, \tilde{G}^{(\pm)-1}\bar{\tilde{A}}\tilde{G}^{(\pm)}] + [\tilde{A}, \tilde{G}^{(\pm)-1}d\bar{\tilde{A}}\cdot\tilde{G}^{(\pm)}])$$

$$d\tilde{G}^{(\pm)}=-\tilde{G}^{(\pm)}\tilde{\Theta}^{(\pm)}-\tilde{\Theta}^{*(\pm)}\tilde{G}^{(\pm)}$$

$$[\tilde{\Theta}^{(\pm)}, \tilde{A}]+[\tilde{F}^{(\pm)}, d\tilde{A}]=0, \text{ diagonal of } \tilde{\Theta}^{(\pm)}=0.$$

$$[\tilde{\Theta}^{*(\pm)}, \bar{\tilde{A}}]+[\tilde{G}^{(\pm)}\tilde{F}^{(\pm)}\tilde{G}^{(\pm)-1}, d\bar{\tilde{A}}]=0, \text{ diagonal of } \tilde{\Theta}^{*(\pm)}=0.$$

Writing down the equations (3. 3. 48) involving $w_0=(w_0^{(+)}(L), w_0^{(-)}(L))$ and using (3. 2. 22),

$$(m-\Gamma_z)w_0=0 \tag{3. 3. 51}$$

$$\begin{cases} (\partial_{z^*}+\sum_{\nu=1}^{n}\partial_{a_\nu}+\partial_z)w_0=0 \\ (\bar{\partial}_{z^*}+\sum_{\nu=1}^{n}\bar{\partial}_{a_\nu}+\bar{\partial}_z)w_0=0 \\ (M^*_{F,z^*}+\sum_{\nu=1}^{n}M_{B,a_\nu}+M_{F,z})w_0=0 \end{cases} \tag{3. 3. 52}$$

$$\begin{cases} m^{-1}\partial_{a_\nu}w_0=-\dfrac{\pi}{2\cos\pi l_\nu}w_\nu(z;L)\cdot{}^t w_\nu(z^*;1-L) \\ m^{-1}\bar{\partial}_{a_\nu}w_0=-\dfrac{\pi}{2\cos\pi l_\nu}w^*_\nu(z;1-L)\cdot{}^t w^*_\nu(z^*;L) \end{cases} \tag{3. 3. 53}$$

$$(\nu=1,\cdots,n).$$

Here we have set $M^*_{F,z^*}w_0=(z^*\partial_{z^*}-\bar{z}^*\bar{\partial}_{z^*})w_0+w_0\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}$, $M_{B,a_\nu}=a_\nu\partial_{a_\nu}-\bar{a}_\nu\bar{\partial}_{a_\nu}$. Equations (3. 3. 52) are consequences of the Euclidean invariance of w_0. It is also possible to derive (3. 3. 53) directly by comparing the local behavior of both hand sides (notice that the singularities at $z=z^*$ are absent in $m^{-1}\partial_{a_\nu}w_0$, $m^{-1}\bar{\partial}_{a_\nu}w_0$). The integrability condition (3. 3. 50) for (3. 3. 48) splits into the deformation equations (3. 3. 24) for F, G, and a linear system for $\alpha_0^{(\pm)}$ and $\beta_0^{(\pm)}$. The latter is nothing but the linear total differential equations (3. 3. 23) for $w_\nu^*(z^*;L)$.

Equations for $w^{(\pm)}(z^*,z;L,\Lambda,\lambda)$ are obtained in the same fashion. Set $w_0(z^*,z;L,\Lambda)_{\mu\nu}=(w_0^{(+)(\nu)}(z^*,z;L,\Lambda,\lambda_{(\mu)}),\ w_0^{(-)(\nu)}(z^*,z;L,\Lambda,\lambda_{(\mu)}))$ where $\lambda_{(\mu)}=$ the μ-th row of Λ. Then $w_0=w_0(z^*,z;L,\Lambda)_{\mu\nu}$ satisfies (3. 3. 51), (3. 3. 52) and

$$\begin{cases} m^{-1}\partial_{a_\sigma}w_0(z^*,z;L,\Lambda)_{\mu\nu} \\ \qquad=-\dfrac{\pi}{2\cos\pi l_\sigma}w_\sigma^{(\nu)}(z;L,\Lambda)\cdot{}^t w_\sigma^{(\mu)}(z^*;1-L,\Lambda) \\ m^{-1}\bar{\partial}_{a_\sigma}w_0(z^*,z;L,\Lambda)_{\mu\nu} \\ \qquad=-\dfrac{\pi}{2\cos\pi l_\sigma}w_\sigma^{*(\nu)}(z;1-L,\Lambda)\cdot{}^t w_\sigma^{*(\mu)}(z^*;L,\Lambda) \end{cases} \tag{3. 3. 54}$$

$$(\sigma=1,\cdots,n).$$

Remark. Likewise the "Green's functions" $v_0=v_0(z^*, z; L)$ or $v_0(z^*, z; L, \Lambda)_{\mu\nu}=v_0^{(\nu)}(z^*, z; L, \Lambda, \lambda_{(\mu)})$ satisfies the following (besides the Euclidean Klein-Gordon equation (3. 1. 2)):

$$\text{(3. 3. 55)}\qquad \begin{cases} (\partial_{z^*}+\sum_{\nu=1}^{n}\partial_{a_\nu}+\partial_z)\,v_0=0 \\ (\bar\partial_{z^*}+\sum_{\nu=1}^{n}\bar\partial_{a_\nu}+\bar\partial_z)\,v_0=0 \\ (M_{B,z^*}+\sum_{\nu=1}^{n}M_{B,a_\nu}+M_{B,z})\,v_0=0 \end{cases}$$

$$\text{(3. 3. 56)}\qquad \begin{cases} m^{-1}\partial_{a_\nu}v_0(z^*, z; L)=-\dfrac{\pi}{2\sin\pi l_\nu}v_\nu(z^*; 1-L)\cdot v_\nu(z; L) \\ m^{-1}\bar\partial_{a_\nu}v_0(z^*, z; L)=-\dfrac{\pi}{2\sin\pi l_\nu}\overline{v_\nu(z^*; L)}\cdot\overline{v_\nu(z; 1-L)} \end{cases}$$

$$\begin{cases} m^{-1}\partial_{a_\sigma}v_0(z^*, z; L, \Lambda)_{\mu\nu} \\ \qquad =-\dfrac{\pi}{2\sin\pi l_\sigma}v_\sigma^{(\mu)}(z^*; 1-L, \Lambda)\cdot v_\sigma^{(\nu)}(z; L, \Lambda) \\ m^{-1}\bar\partial_{a_\sigma}v_0(z^*, z; L, \Lambda)_{\mu\nu} \\ \qquad =-\dfrac{\pi}{2\sin\pi l_\sigma}\overline{v_\sigma^{(\mu)}(z^*; L, \Lambda)}\cdot\overline{v_\sigma^{(\nu)}(z; 1-L, \Lambda)}\,. \end{cases}$$

We shall now introduce a closed 1-form ω associated with a solution of the non-linear system (3. 3. 24). It will be shown to coincide with the logarithmic derivative of the τ-function in Chapter IV.

Proposition 3. 3. 12. *For a solution F, G of* (3. 3. 24), *set*

$$\text{(3. 3. 57)}\qquad \omega=-\frac{1}{2}\operatorname{trace}(F\Theta+\Theta^*GFG^{-1})$$
$$+m^2\operatorname{trace}(d(A\bar A)-G^{-1}\bar AGdA-GAG^{-1}d\bar A).$$

Then ω is a closed 1*-form.*

Proof. Making use of (3. 3. 24) and

$$\text{(3. 3. 58)}\qquad d\Theta=\Theta\wedge\Theta+m^2[dA, G^{-1}d\bar A\cdot G]_+$$
$$d\Theta^*=-\Theta^*\wedge\Theta^*-m^2[d\bar A, GdA\cdot G^{-1}]_+$$

$$d(GFG^{-1}) = -[\Theta^*, GFG^{-1}] - m^2([\bar{A}, GdA\cdot G^{-1}] + [d\bar{A}, GAG^{-1}]),$$

we calculate $d\omega$. We have

$$d\omega = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\text{(3. 3. 59)}\quad \mathrm{I} = -\frac{1}{2}\,\mathrm{trace}\{([\Theta, F] + m^2([dA, G^{-1}\bar{A}G] + [A, G^{-1}d\bar{A}\cdot G]))\Theta + F(\Theta\wedge\Theta + m^2[dA, G^{-1}d\bar{A}\cdot G]_+)\}$$

$$= -\frac{m^2}{2}\,\mathrm{trace}\{F[dA, G^{-1}d\bar{A}\cdot G]_+ + ([dA, G^{-1}\bar{A}G] + [A, G^{-1}d\bar{A}\cdot G])\Theta\}$$

$$\text{(3. 3. 60)}\quad \mathrm{II} = -\frac{1}{2}\,\mathrm{trace}\{-(\Theta^*\wedge\Theta^* + m^2[d\bar{A}, GdA\cdot G^{-1}]_+)GFG^{-1} - \Theta^*(-[\Theta^*, GFG^{-1}] - m^2([\bar{A}, GdA\cdot G^{-1}] + [d\bar{A}, GAG^{-1}]))\}$$

$$= -\frac{m^2}{2}\,\mathrm{trace}\{-[G^{-1}d\bar{A}\cdot G, dA]_+F + \Theta^*([\bar{A}, GdA\cdot G^{-1}] + [d\bar{A}, GAG^{-1}])\}$$

$$\text{(3. 3. 61)}\quad \mathrm{III} = -m^2\,\mathrm{trace}([G^{-1}\bar{A}G, G^{-1}dG]dA + [dG\cdot G^{-1}, GAG^{-1}]d\bar{A})$$

$$= m^2\,\mathrm{trace}([G^{-1}\bar{A}G, \Theta]dA + G[\Theta, A]G^{-1}d\bar{A} + [\Theta^*, GAG^{-1}]d\bar{A} + G^{-1}[\bar{A}, \Theta^*]GdA).$$

Here in deriving (3. 3. 59) we have used

$$\text{(3. 3. 62)}\quad \mathrm{trace}\,\Theta F\Theta = \sum_{\substack{\lambda\mu\nu\\ \text{distinct}}} \Theta_{\lambda\mu}f_{\mu\nu}\wedge\Theta_{\nu\lambda} - \sum_{\mu} l_{\mu}\sum_{\lambda(\neq\mu)}\Theta_{\lambda\mu}\wedge\Theta_{\mu\lambda}$$

$$= \frac{1}{3}\sum_{\substack{\lambda\mu\nu\\ \text{distinct}}} f_{\lambda\mu}f_{\mu\nu}f_{\nu\lambda}\sum_{\text{cyclic}} d\log(a_\lambda - a_\mu)\wedge d\log(a_\nu - a_\lambda)$$

$$= 0,$$

and similarly for (3. 3. 60). Hence we have

$$d\omega=\frac{m^2}{2}\operatorname{trace}(-[dA, G^{-1}\bar{A}G]\Theta-\Theta^*[d\bar{A}, GAG^{-1}]$$
$$+2[G^{-1}\bar{A}G, \Theta]dA+2[\Theta^*, GAG^{-1}]d\bar{A})$$
$$+\frac{m^2}{2}\operatorname{trace}(-[A, G^{-1}d\bar{A}\cdot G]\Theta-\Theta^*[\bar{A}, GdA\cdot G^{-1}]$$
$$+2G[\Theta, A]G^{-1}d\bar{A}+2G^{-1}[\bar{A}, \Theta^*]GdA).$$

The first term vanishes by virtue of $[\Theta, dA]_+=0$, $[\Theta^*, d\bar{A}]_+=0$. The second term reads

$$\frac{m^2}{2}\operatorname{trace}(-G[F, dA]G^{-1}d\bar{A}-G^{-1}[d\bar{A}, GAG^{-1}]GdA)$$
$$=-\frac{m^2}{2}\operatorname{trace}(FdA\cdot G^{-1}d\bar{A}\cdot G-dA\cdot F\cdot G^{-1}d\bar{A}\cdot G$$
$$+G^{-1}d\bar{A}\cdot G\cdot FdA-FG^{-1}d\bar{A}\cdot G\cdot dA)$$
$$=0.$$

This proves Proposition 3. 3. 12.

Remark. In the case of the system corresponding to $\boldsymbol{w}=\boldsymbol{w}(L)$ or $\boldsymbol{w}(L; A)$, the 1-form ω is given by $\omega=\operatorname{trace}(\alpha m dA+\bar{\alpha}m d\bar{A})$, $\alpha=\alpha\left(L+\frac{1}{2}\right)$ or $\alpha\left(L+\frac{1}{2}; A\right)$ (cf. (3. 3. 11)).

The transformation property of ω under the Euclidean motion group is deduced from Proposition 3. 3. 8. We set

$$\sigma: E(2)\times(X^{\mathrm{Euc}})^n\to(X^{\mathrm{Euc}})^n, \tag{3. 3. 63}$$
$$(b, \bar{b}, \theta; A, \bar{A})\mapsto(e^{i\theta}A+b, e^{-i\theta}\bar{A}+\bar{b}).$$

Proposition 3. 3. 13. *We have*

$$\sigma^*\omega=\omega-\frac{1}{2}\operatorname{trace}(L^2-L^{*2})id\theta. \tag{3. 3. 64}$$

Proof. Using (3. 3. 34) and the remark at the end of the proof of Proposition 3. 3. 8, we have

$$(3.3.65)\quad \sigma^* \operatorname{trace}(F\Theta + \Theta^* GFG^{-1}) = \operatorname{trace}(F(\Theta - (F+L)id\theta) + (\Theta^* + (GFG^{-1}+L^*)id\theta)GFG^{-1})$$
$$= \operatorname{trace}(F\Theta + \Theta^* GFG^{-1}) - \operatorname{trace}(FL - L^*GFG^{-1})id\theta,$$

$$(3.3.66)\quad \sigma^* \operatorname{trace}(d(A\bar{A}) - G^{-1}\bar{A}GdA - GAG^{-1}d\bar{A})$$
$$= \operatorname{trace}(d(e^{i\theta}A+b)(e^{-i\theta}\bar{A}+\bar{b}) - G^{-1}(e^{-i\theta}\bar{A}+\bar{b})Gd(e^{i\theta}A+b) - G(e^{i\theta}A+b)G^{-1}d(e^{-i\theta}\bar{A}+\bar{b})).$$

A little calculation shows that (3.3.66) reduces to trace $(d(A\bar{A}) - G^{-1}\bar{A}GdA - GAG^{-1}d\bar{A})$. Noting $L = -$diagonal of F and $L^* = -$diagonal of GFG^{-1}, we obtain (3.3.64).

Remark. As mentioned in p. 43, we shall be concerned only with the case $L^* = L$ in Chapter IV. In this case the 1-form ω is invariant under $E(2)$.

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Note added in proof: After the preparation of the manuscript the authors have come to know the existence of the short letter (*Phys. Rev. Lett.*, **31** (1973) 1409-1411) by E. Barouch, B. M. McCoy and T. T. Wu, in which there is already announced the Painlevé representation of the 2-point function for the Ising model. They wish to thank Prof. E. Barouch for drawing their attention to this article, which should be included in the references of our previous papers [1], [2].