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On a Sufficient Condition for Well-posedness of Weakly Hyperbolic Cauchy Problems

By

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Already, we have good results on this theme for 2-nd order equations ([1]). Recently, we have also many results for higher order equations with smooth characteristic roots ([2], [3]).

In this paper, we consider only weakly hyperbolic equations with double characteristic roots at most, which may be non-smooth.

Chapter I. Energy Inequalities

§ 1. Condition (A)

Let us consider the Cauchy problem:

$$\begin{cases} A(t, x'; D_t, D'_x)u = f(t, x') & \text{in } \{t>0, x'\in \mathbb{R}^n\}, \\ D_t^j u|_{t=0} = u_j(x') \quad (j=0, 1, \cdots, m-1) & \text{in } \{x'\in \mathbb{R}^n\}, \end{cases}$$

where $D_t = \dfrac{1}{i}\dfrac{\partial}{\partial t}$, $D'_x = \left(\dfrac{1}{i}\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{1}{i}\dfrac{\partial}{\partial x_n}\right)$ and

$$A(t, x'; \tau, \xi') = \sum_{i+|\nu|\leq m} a_{i\nu}(t, x')\tau^i \xi'^\nu .$$

Now we denote

$$A_0(t, x'; \tau, \xi') = \sum_{i+|\nu|=m} a_{i\nu}(t, x')\tau^i \xi'^\nu ,$$

$$A_1(t, x'; \tau, \xi') = \sum_{i+|\nu|=m-1} a_{i\nu}(t, x')\tau^i \xi'^\nu ,$$

then our consideration will be based on the assumption on A_0:

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Condition (A).

(A. 1) *$a_{i\nu}(t, x')$ are smooth and constant outside a ball in $\boldsymbol{R}^{n+1}$.*

(A. 2) *A_0 is hyperbolic with respect to τ, that is,*

$$\begin{cases} A_0(t, x'; 1, 0) \neq 0, \\ A_0(t, x'; \tau', \xi') \neq 0 \quad \text{for } \operatorname{Im} \tau \neq 0,\ \xi' \in \boldsymbol{R}^n. \end{cases}$$

(A. 3) *Roots of $A_0(t, x'; \tau, \xi') = 0$ with respect to τ are at most double for $(t, x', \xi') \in \boldsymbol{R}^{n+1} \times (\boldsymbol{R}^n - \{0\})$.*

For convenience, we denote

$$x = (t, x') = (x_0, x_1, \cdots, x_n), \quad \xi = (\tau, \xi') = (\xi_0, \xi_1, \cdots, \xi_n),$$

$$X' = (x, \xi') \in \boldsymbol{R}^{n+1} \times (\boldsymbol{R}^n - \{0\}), \quad X = (\tau, X'),$$

and we assume $A_0(x; 1, 0) = 1$. We denote

$$A_0(X) = \prod_{i=1}^{m} (\tau - \tau_i(X')).$$

In the following, let us consider the behaviour of characteristic roots $\tau_i(X')$ at $X' = X'_0$ and in its small neighbourhood U. We may assume

$$\tau_1(X'_0), \tau_2(X'_0), \cdots, \tau_{m-d}(X'_0)$$

are distinct each other, and

$$\tau_{m-d+1}(X'_0) = \tau_1(X'_0), \cdots, \tau_m(X'_0) = \tau_d(X'_0).$$

Then, there exists a neighbourhood U of $X' = X'_0$, where

(i) $$H_i(X) = (\tau - \tau_i(X'))(\tau - \tau_{m-d+i}(X'))$$
$$= (\tau - \alpha_i(X'))^2 - \beta_i(X') = h_i(X)^2 - \beta_i(X') \quad (i = 1, \cdots, d),$$

where $\alpha_i(X')$ and $\beta_i(X')$ are real-valued smooth function and $\beta_i(X') \geq 0$,

(ii) $\tau_i(X')$ $(i = d+1, \cdots, m-d)$ are real-valued smooth functions.

Now, we denote

$$h_i(X) = \tau - \tau_i(X') = \tau - \alpha_i(X') \quad (i = d+1, \cdots, m-d),$$

then

$$\{\alpha_1(X'), \cdots, \alpha_{m-d}(X')\}$$

are distinct, real-valued and smooth functions in U. Hence, we denote

$$\begin{cases} P_i(X) = h_i(X)\dfrac{A_0(X)}{H_i(X)} & (i=1, \cdots, d), \\ P_i'(X) = \dfrac{A_0(X)}{H_i(X)} = \dfrac{P_i(X)}{h_i(X)} & (i=1, \cdots, d), \\ P_i(X) = \dfrac{A_0(X)}{h_i(X)} & (i=d+1, \cdots, m-d), \end{cases}$$

then these m polynomials with respect to τ with smooth coefficients defined in U make a base of the linear space of polynomials of order less than m with respect to τ.

Now we proceed to the next step, assuming $U=\boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\})$, which will be justified in Section 5. Then we have smooth decompositions of $A_0(X)$:

$$\begin{cases} A_0(X) = H_i(X)P_i'(X) & (i=1, \cdots, d), \\ A_0(X) = h_i(X)P_i(X) & (i=d+1, \cdots, m-d) \end{cases}$$

for $X'\in\boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\})$, therefore we have decompositions of $A(D) = A(x, D)$:

Lemma 1.1.

(i) $A(D) = \tilde{H}_i(D)\tilde{P}_i'(D) + \{\{m-2\}\} \quad (i=1, \cdots, d)$,

where

$\tilde{H}_i(D) = (D_t - \tilde{\alpha}_i(D'))^2 - \tilde{\beta}_i(D') = \tilde{h}_i(D)^2 - \tilde{\beta}_i(D')$,

$\tilde{\alpha}_i(D') = \alpha_i(D') + \alpha_i^1(D')$ ($\alpha_i^1(X')$: *homogeneous of degree* 0),

$\tilde{\beta}_i(D') = \beta_i(D') + \beta_i^1(D')$ ($\beta_i^1(X')$: *homogeneous of degree* 1),

$\tilde{P}_i'(D) = P_i'(D) + P_i'^1(D)$ ($P_1'^1(X)$: *homogeneous of degree* $m-3$).

(ii) $A(D) = \tilde{h}_i(D)\tilde{P}_i(D) + \{\{m-2\}\} \quad (i=d+1, \cdots, m-d)$,

where

$\tilde{h}_i(D) = D_t - \tilde{\alpha}_i(D')$,

$\tilde{\alpha}_i(D') = \alpha_i(D') + \alpha_i^1(D')$ ($\alpha_i^1(X')$: *homogeneous of degree* 0),

$\tilde{P}_i(D) = P_i(D) + P_i^1(D)$ ($P_i^1(X)$: *homogeneous of degree* $m-2$).

Notation 1. $\{\{m-2\}\}$ means an operator of order not greater than $m-2$.

Notation 2. Homogeneous of degree k means homogeneous of degree k with respect to $\xi=(\tau, \xi')$, which is a polynomial of degree not greater than k with respect to τ.

Proof of (i). Let $a(D)\circ b(D)$ be a singular integral operator with simbol $a(X)b(X)$, then we have

$$\begin{aligned} A_0(D) &= H_i(D)\circ P_i'(D) \\ &= H_i(D)P_i'(D)-(D_\xi H_i)(D)\circ(\partial_x P_i')(D)+\{\{m-2\}\}, \end{aligned}$$

where

$$(D_\xi H_i)(X)(\partial_x P_i')(X)=\sum_{j=0}^{n}(D_{\xi_j}H_i)(X)(\partial_{x_j}P_i')(X),$$

$$\partial_t=\frac{\partial}{\partial t}, \quad \partial_{x_j}=\frac{\partial}{\partial x_j}.$$

Therefore we have

$$A(D)=H_i(D)P_i'(D)+R_i(D)+\{\{m-2\}\},$$

where

$$R_i(X)=A_1(X)-(D_\xi H_i)(X)(\partial_x P_i')(X).$$

Using the representation of $R_i(X)$ by $\{P_j'(X)\ (j=1, \cdots, d),\ P_j(X)\ (j=1, \cdots, m-d)\}$, we have

$$R_i(X)=c_i(X')P_i(X)+d_i(X')P_i'(X)+S_i(X)H_i(X),$$

where $c_i(X')$, $d_i(X')$, $S_i(X)$ are smooth and homogeneous. Hence we have

$$\begin{aligned} A(D) &= \{H_i(D)+c_i(D')\circ h_i(D)+d_i(D')\} \\ &\qquad \times\{P_i'(D)+S_i(D)\}+\{\{m-2\}\} \\ &= \{h_i(D)\circ h_i(D)-\beta_i(D')+c_i(D')\circ h_i(D)+d_i(D')\} \\ &\qquad \times\{P_i'(D)+S_i(D)\}+\{\{m-2\}\} \end{aligned}$$

$$=\left\{\left(h_i(D)+\frac{1}{2}c_i(D')\right)^2-(\beta_i(D')-d_i(D')\right.$$

$$\left.+(D_\xi h_i)(D')\circ(\partial_x h_i)(D'))\right\}\{P_i'(D)+S_i(D)\}+\{\{m-2\}\}$$

$$=\tilde{H}_i(D)\tilde{P}_i'(D)+\{\{m-2\}\},$$

where

$$\tilde{H}_i(D)=\tilde{h}_i(D)^2-\tilde{\beta}_i(D'),\quad h_i(D)=D_t-\tilde{\alpha}_i(D'),$$

$$\tilde{\alpha}_i(D')=\alpha_i(D')-\frac{1}{2}c_i(D'),$$

$$\tilde{\beta}_i(D')=\beta_i(D')-d_i(D')+(D_\xi h_i)(D')\circ(\partial_x h_i)(D'),$$

$$\tilde{P}_i'(D)=P_i'(D)+S_i(D).$$

Proof of (ii). Since

$$A_0(D)=h_i(D)\circ P_i(D)$$

$$=h_i(D)P_i(D)-(D_\xi h_i)(D')\circ(\partial_x P_i)(D)+\{\{m-2\}\},$$

we have

$$A(D)=h_i(D)P_i(D)+R_i(D)+\{\{m-2\}\},$$

where

$$R_i(X)=A_1(X)-(D_\xi h_i)(X)(\partial_x P_i)(X).$$

Using the representation of $R_i(X)$ by $\{P_j'(j=1,\cdots,d),\ P_j(j=1,\cdots,m-d)\}$, we have

$$R_i(X)=c_i(X')P_i(X)+S_i(X)h_i(X),$$

where $c_i(X'), S_i(X)$ are smooth and homogeneous. Hence we have

$$A(D)=\{h_i(D)+c_i(D')\}\{P_i(D)+S_i(D)\}+\{\{m-2\}\}$$

$$=\tilde{h}_i(D)\tilde{P}_i(D)+\{\{m-2\}\},$$

where

$$\tilde{h}_i(D)=D_t-\tilde{\alpha}_i(D'),\quad \tilde{\alpha}_i(D')=\alpha_i(D')-c_i(D'),$$

$$\tilde{P}_i(D)=P_i(D)+S_i(D).$$

Q.E.D.

§ 2. $\widetilde{H}(D)$ and Green's Formulas

Let

$$\widetilde{H}(D)=(D_t-\tilde{\alpha}(D'))^2-\tilde{\beta}(D')=\tilde{h}(D)^2-\tilde{\beta}(D')$$

be one of $\{\widetilde{H}_i(D)\}_{i=1,\dots,d}$ in § 1. At first, we consider

$$\tilde{\beta}(X')=\beta(X')+\beta^1(X'),$$

where $\beta(X')\geq 0$ is homogeneous of degree 2 and $\beta^1(X')$ is homogeneous of degree 1. In the following, we present a sufficient condition for $\tilde{\beta}(D')$ to be a positive operator:

$$\operatorname{Re}(\tilde{\beta}(D')u, u)_{L^2(\boldsymbol{R}^n)}\geq -C\|u\|^2_{L^2(\boldsymbol{R}^n)}.$$

Lemma 2. 1. *Let us assume*

(i) $\beta(X')=\sum_{j=1}^{N} b_j(X')^2$

where $b_j(X')$ are smooth real-valued functions,

(ii) $\operatorname{Re}\beta^1(X')\equiv 0 \mod \boldsymbol{b}(X')$,

that is,

$$\operatorname{Re}\beta^1(X')=\sum_{k=1}^{N} c_k(X')b_k(X'),$$

where $c_k(X')$ are smooth.

Then there exist $c>0$ and $C>0$ such that

$$\operatorname{Re}(\tilde{\beta}(D')u, u)_{L^2(\boldsymbol{R}^n)}\geq c\sum_{j=1}^{N}\|b_j(D')u\|^2_{L^2(\boldsymbol{R}^n)}-C\|u\|^2_{L^2(\boldsymbol{R}^n)}.$$

Remark. If

$$\{\partial'_X b_1(X'_0), \cdots, \partial'_X b_N(X'_0)\}$$

are linearly independent, then (ii) is equivalent to one of the following:

(ii)′ $\operatorname{Re}\beta^1(X')=0$ on $\boldsymbol{b}(X')=0$,

(ii)″ $|\operatorname{Re}\beta^1(X')|\leq C|\boldsymbol{b}(X')|$.

In fact, there exist smooth functions

$$\boldsymbol{b}'(X') = (b_{N+1}(X'), \cdots, b_{2n+1}(X')),$$

where $\boldsymbol{b}'(X'_0) = 0$ and

$$\{\partial'_X b_1(X'_0), \cdots, \partial'_X b_{2n+1}(X'_0)\}$$

are linearly independent. Let $f(X')$ be a smooth function, then

$$\begin{aligned} f(X') &= f(X'(\boldsymbol{b}, \boldsymbol{b}')) = f(X'(0, \boldsymbol{b}')) + \sum_{j=1}^{N} c_j(\boldsymbol{b}, \boldsymbol{b}') b_j \\ &= f(X')|_{\boldsymbol{b}(X')=0} + \sum_{j=1}^{N} c'_j(X') b_j(X'). \end{aligned}$$

Therefore

$$|f(X')| \leq C \sum_{j=1}^{N} |b_j(X')|$$

is equivalent to

$$f(X') = 0 \quad \text{on} \quad \boldsymbol{b}(X') = 0,$$

that is,

$$f(X') = \sum_{j=1}^{N} c'_j(X') b_j(X').$$

Proof. Since

$$b_j^*(D') = b_j(D') + (D_\xi \partial_x b_j)(D') + \{\{-1\}\},$$

we have

$$\begin{aligned} \beta(D') &= \sum_{j=1}^{N} b_j(D') \circ b_j(D') \\ &= \sum b_j(D')^2 - \sum (D_\xi b_j)(D') \circ (\partial_x b_j)(D') + \{\{0\}\} \\ &= \sum b_j^*(D') b_j(D') - \sum (D_\xi \partial_x b_j)(D') \circ b_j(D') \\ &\qquad - \sum (D_\xi b_j)(D') \circ (\partial_x b_j)(D') + \{\{0\}\} \\ &= \sum b_j^*(D') b_j(D') - \frac{1}{2}(D_\xi \partial_x \beta)(D') + \{\{0\}\}. \end{aligned}$$

Then we have from the assumption (ii)

$$\tilde{\beta}(D') = \beta(D') + \beta^1(D')$$

$$=\sum b_j^*(D')\,b_j(D')+\left\{\beta^1(D')-\frac{1}{2}(D_\xi\partial_x\beta)(D')\right\}+\{\{0\}\}$$

$$=\sum b_j^*(D')\,b_j(D')+\sum c_j(D')\,b_j(D')+id(D')+\{\{0\}\}.$$

$$(d(x');\ \text{real})$$

Hence we have

$$\operatorname{Re}(\tilde\beta(D')u,u)\geq\sum\|b_j(D')u\|^2-C\{\sum\|b_j(D')u\|\cdot\|u\|+\|u\|^2\}$$

$$\geq\frac{1}{2}\sum\|b_j(D')u\|^2-C'\|u\|^2.$$ Q.E.D.

Next, we consider $\tilde H(D)$.

Lemma 2.2 (Green's formula). *Let us assume*

(i) $\beta(X')=\sum_{j=1}^N b_j(X')^2$,

(ii) $\beta^1(X')-\frac{1}{2}D_\xi\partial_x\beta(X')\equiv 0 \mod \boldsymbol{b}(X')$,

(iii) $\{h(X),b_j(X')\}\equiv 0 \mod \boldsymbol{b}(X')$,

where

$$\{h(X),b_j(X')\}=(D_\xi h)(X')\,\partial_x b_j(X')-D_\xi b_j(X')\,(\partial_x h)(X').$$

Then we have

$$(\tilde H(D)u,\tilde h(D)u)-(\tilde h(D)u,\tilde H(D)u)$$

$$=D_t\{\|\tilde h(D)u\|^2+\sum_{j=1}^N\|b_j(D')u\|^2\}+\cdots,$$

where

$$|\cdots|\leq C\{\|h(D)u\|^2+\sum_{j=1}^N\|b_j(D')u\|^2+\|u\|^2\}.$$

Proof. From the assumptions (i) and (ii), we have

$$(\tilde Hu,\tilde hu)-(\tilde hu,\tilde Hu)$$

$$=\{((D_t-\tilde\alpha)\tilde hu,\tilde hu)-(\tilde hu,(D_t-\tilde\alpha)\tilde hu)\}$$

$$-\{(\tilde\beta u,\tilde hu)-(\tilde hu,\tilde\beta u)\}$$

$$=D_t\|\tilde{h}u\|^2-\sum_{j=1}^{N}\{(b_ju,\, b_j\tilde{h}u)-(b_j\tilde{h}u,\, b_ju)\}+\cdots.$$

From the assumption (iii), we have

$$\|(b_j\tilde{h}-\tilde{h}b_j)u\|\leq C\{\sum_{j=1}^{N}\|b_ju\|+\|u\|\},$$

therefore

$$\sum_{j=1}^{N}\{(b_ju,\, b_j\tilde{h}u)-(b_j\tilde{h}u,\, b_ju)\}$$

$$=\sum_{j=1}^{N}\{(b_ju,\, (D_t-\alpha)b_ju)-((D_t-\alpha)b_ju,\, b_ju)\}+\cdots$$

$$=-D_t\sum_{j=1}^{N}\|b_ju\|^2+\cdots. \qquad \text{Q.E.D.}$$

Corollary. *There exists* $\gamma_0>0$ *such that for* $\gamma>\gamma_0$

$$e^{-2\gamma t}\{\|\tilde{h}u(t)\|^2+\sum_{j=1}^{N}\|b_ju(t)\|^2\}$$

$$+\gamma\int_0^t e^{-2\gamma t}\{\|\tilde{h}u(t)\|^2+\sum_{j=1}^{N}\|b_ju(t)\|^2\}\,dt$$

$$\leq\{\|\tilde{h}u(0)\|^2+\sum_{j=1}^{N}\|b_ju(0)\|^2\}$$

$$+C\left\{\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|\tilde{H}u(t)\|^2dt+\int_0^t e^{-2\gamma t}\|u(t)\|^2dt\right\}.$$

Proof. In general, we have

$$\int_0^t e^{-2\gamma t}D_tf(t)\,dt=\int_0^t(D_t-2i\gamma)\,e^{-2\gamma t}f(t)\,dt$$

$$=-i\left\{e^{-2\gamma t}f(t)-f(0)+2\gamma\int_0^t e^{-2\gamma t}f(t)\,dt\right\}.$$

Let us multiply the both sides of green's formula in Lemma 2.2 by $ie^{-2\gamma t}$ and integrate them, then we have

$$-2\int_0^t e^{-2\gamma t}\,\mathrm{Im}\,(\tilde{H}u(t),\,\tilde{h}u(t))\,dt$$

$$=e^{-2\gamma t}\{\|\tilde{h}u(t)\|^2+\sum_{j=1}^{N}\|b_ju(t)\|^2\}-\{\|\tilde{h}u(0)\|^2+\sum_{j=1}^{N}\|b_ju(0)\|^2\}$$

$$+2\gamma\int_0^t e^{-2\gamma t}\{\|\tilde{h}u(t)\|^2+\sum_{j=1}^N\|b_ju(t)\|^2\}dt+\cdots,$$

where

$$|\cdots|\le C\int_0^t e^{-2\gamma t}\{\|\tilde{h}u(t)\|^2+\sum_{j=1}^N\|b_ju(t)\|^2+\|u(t)\|^2\}dt. \text{ (Q.E.D.)}$$

Finally, we consider of

$$\tilde{h}(D)=D_t-\tilde{\alpha}(D'),\quad \tilde{\alpha}(D')=\alpha(D')+\alpha^1(D'),$$

where $\alpha(X')$ is real-valued and homogeneous of degree 1 and $\alpha^1(X')$ is homogeneous of degree 0. Then we have

$$\begin{aligned}(\tilde{h}u,u)-(u,\tilde{h}u)&=((D_t-\tilde{\alpha})u,u)-(u,(D_t-\tilde{\alpha})u)\\&=D_t\|u\|^2+\cdots,\end{aligned}$$

where

$$|\cdots|\le C\|u\|^2.$$

Hence we have

Lemma 2.3. *There exists* $\gamma_0>0$ *such that for* $\gamma>\gamma_0$

$$\begin{aligned}&e^{-2\gamma t}\|u(t)\|^2+\gamma\int_0^t e^{-2\gamma t}\|u(t)\|^2dt\\&\qquad\le\|u(0)\|^2+\frac{C}{\gamma}\int_0^t e^{-2\gamma t}\|\tilde{h}u(t)\|^2dt.\end{aligned}$$

Here we have from the corollary of Lemma 2.2 and Lemma 2.3

Proposition 2.4. *Let us assume the conditions* (i) $\sim$ (iii) *stated in Lemma* 2.2, *then there exists* $\gamma_0>0$ *such that for* $\gamma>\gamma_0$

$$\begin{aligned}&e^{-2\gamma t}\{\|hu(t)\|^2+\sum_{j=1}^N\|b_ju(t)\|^2+\gamma^2\|u(t)\|^2\}\\&\qquad+\gamma\int_0^t e^{-2\gamma t}\{\|hu(t)\|^2+\sum_{j=1}^N\|b_ju(t)\|^2+\gamma^2\|u(t)\|^2\}dt\\&\quad\le C\Big\{\|hu(0)\|^2+\sum_{j=1}^N\|b_ju(0)\|^2+\gamma^2\|u(0)\|^2\end{aligned}$$

$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|\tilde{H}u(t)\|^2 dt\Big\}.$$

§ 3. Condition (B)

Let the decomposition of $A_0(X)$ at $X'=X'_0$ be

$$A_0(X)=C\prod_{i=1}^{d}H_i(X)\prod_{i=d+1}^{m-d}h_i(X),$$

where

$$H_i(X)=h_i(X)^2-\beta_i(X')=(\tau-\alpha_i(X'))^2-\beta_i(X') \quad (i=1,\cdots,d),$$

$$h_i(X)=\tau-\alpha_i(X') \quad (i=d+1,\cdots,m-d).$$

Condition (B). *It holds in a neighborhood of X'_0 for $i=1,\cdots,d$*

(B. 1) $$\beta_i(X')=\sum_{j=1}^{N_i}b_j^{(i)}(X')^2,$$

(B. 2) $$L_A(\alpha_i(X'),X')\equiv 0 \mod \boldsymbol{b}^{(i)}(X'),$$

where

$$L_A(X)=A_1(X)-\frac{1}{2}(D_\xi\partial_x A_0)(X)$$

(B. 3) $$\{h_i(X),b_j^{(i)}(X')\}\equiv 0 \mod \boldsymbol{b}^{(i)}(X') \quad (j=1,\cdots,N_i),$$

(B. 4) $$\{b_j^{(i)}(X'),b_k^{(i)}(X')\}\equiv 0 \mod \boldsymbol{b}^{(i)}(X') \quad (j,k=1,\cdots,N_i).$$

Lemma 3.1. (B. 2) *is equivalent to*

$$L_{\beta_i}(X')=\beta_i^1(X')-\frac{1}{2}D_\xi\partial_x\beta_i(X')\equiv 0 \mod \boldsymbol{b}^{(i)}(X').$$

Proof. Since

$$A(D)=A_0(D)+A_1(D)+\{\{m-2\}\}=\tilde{H}_i(D)\tilde{P}'_i(D)+\{\{m-2\}\}$$
$$=H_i(D)\circ P'_i(D)+\{H_i^1(D)\circ P'_i(D)+H_i(D)\circ P'^{\,1}_i(D)$$
$$+D_\xi H_i(D)\circ\partial_x P'_i(D)\}+\{\{m-2\}\},$$

where

$$\tilde{H}_i(D) = H_i(D) + H_i^1(D) + \{\{0\}\}$$
$$= H_i(D) + \{-2h_i(D) \circ \alpha_i^1(D') + (D_\xi h_i)(D')(\partial_x h_i)(D')$$
$$-\beta_i^1(D')\} + \{\{0\}\},$$
$$\tilde{P}_i'(D) = P_i'(D) + P_i'^1(D) + \{\{m-4\}\},$$

we have

$$\begin{cases} A_0(X) = H_i(X) P_i'(X), \\ A_1(X) = H_i^1(X) P_i'(X) + H_i(X) P_i'^1(X) + D_\xi H_i(X) \partial_x P_i'(X). \end{cases}$$

Hence we have

$$L_A(X) = A_1(X) - \frac{1}{2} D_\xi \partial_x A_0(X)$$
$$= H_i^1(X) P_i'(X) + H_i(X) P_i'^1(X) + D_\xi H_i(x) \partial_x P_i'(X)$$
$$- \frac{1}{2}\{(D_\xi \partial_x H_i)(X) P_i'(X) + (D_\xi H_i)(X)(\partial_x P_i')(X)$$
$$+ (\partial_x H_i)(X)(D_\xi P_1')(X) + H_i(X)(D_\xi \partial_x P_i')(X)\},$$

therefore

$$L_A(\alpha_i(X'), X') \equiv \Big\{H_i^1(\alpha_i(X'), X')$$
$$- \frac{1}{2} D_\xi \partial_x H_i(\alpha_i(x'), X')\Big\} P_1'(\alpha_i(X'), X')$$
$$= -\Big\{\beta_i^1(X') - \frac{1}{2} D_\xi \partial_x \beta_i(X')\Big\} P_i'(\alpha_i(X'), X')$$
$$\mod \boldsymbol{b}^{(i)}(X'). \qquad \text{Q.E.D.}$$

Now, for examples, let us consider A with constant coefficients. Of course, (B. 3) and (B. 4) are trivially satisfied. When (B. 1) is satisfied, (B. 2) is stated as

$$A_1(\alpha_i(\xi'), \xi') \equiv 0 \mod \boldsymbol{b}^{(i)}(\xi'),$$

that is,

$$\beta_i^1(\xi') \equiv 0 \mod \boldsymbol{b}^{(i)}(\xi').$$

We shall see that $A(\tau,\xi')$ is hyperbolic with respect to τ if and only if

$$|\beta_i^1(\xi')|\leq C\sqrt{\beta_i(\xi')}\quad (i=1,\cdots,d).$$

In fact, we know that $A(\tau,\xi')$ is hyperbolic with respect to τ, if and only if ([4])

$$(*)\qquad |A_1(\tau,\xi')|\leq\frac{C}{|\mathrm{Im}\,\tau|}|A_0(\tau,\xi')|\quad \text{for}\ \ \mathrm{Im}\,\tau\neq 0\,,\ \xi'\in\boldsymbol{R}^n\,.$$

Let us denote

$$A_1(\tau,\xi')=\sum_{j=1}^{m-d}c_j(\xi')P_j(\tau,\xi')+\sum_{j=1}^{d}d_j(\xi')P_j'(\tau,\xi')\quad (d_j=-\beta_j^1).$$

Since

$$H_i(\tau,\xi')=\{(\mathrm{Re}\,\tau-\alpha_i(\xi'))^2-\beta_i(\xi')-(\mathrm{Im}\,\tau)^2\}$$
$$+2i\,\mathrm{Im}\,\tau(\mathrm{Re}\,\tau-\alpha_i(\xi')),$$

we have

$$|H_i(\tau,\xi')|\geq c|\mathrm{Im}\,\tau|\{\sqrt{\beta_i(\xi')}+|\mathrm{Im}\,\tau|\}\quad (c>0).$$

Therefore $(*)$ is equivalent to

$$|d_i(\xi')|\leq C(\sqrt{\beta_i(\xi')}+|\mathrm{Im}\,\tau|),$$

that is,

$$|d_i(\xi')|\leq C\sqrt{\beta_i(\xi')}.$$

Hence we may say that (B. 2) is an almost necessary condition for A to be hyperbolic in constant coefficient case. In fact, if

$$\{\partial_\xi' b_1^{(i)}(\xi_0'),\cdots,\partial_\xi' b_{N_i}^{(i)}(\xi_0')\}$$

are linearly independent, (B. 2) is really necessary for A to be hyperbolic.

Lemma 3. 2. *Let*

$$\beta_i(X')=\sum_{j=1}^{N_i}b_j^{(i)}(X')^2,$$

$$b_j^{(i)}(X')=\sum_{k=1}^{N_i}c_{jk}^{(i)}(X')B_k^{(i)}(X'),\ \det(c_{jk}^{(i)}(X'))_{j,k=1,\cdots,N_i}\neq 0\,,$$

then (B. 2), (B. 3), (B. 4) *are equivalent to*

(B. 2)′ $L_A(\alpha_i(X'), X') \equiv 0 \mod \boldsymbol{B}^{(i)}(X')$,

(B. 3)′ $\{h_i(X), B_j^{(i)}(X')\} \equiv 0 \mod \boldsymbol{B}^{(i)}(X') \quad (j=1, \cdots, N_i)$,

(B. 4)′ $\{B_j^{(i)}(X'), B_k^{(i)}(X')\} \equiv 0 \mod \boldsymbol{B}^{(i)}(X') \quad (j, k=1, \cdots, N_i)$,

respectively.

Proof. (i) Let

$$f(X') = \sum_{j=1}^{N_i} \varphi_j(X') b_j^{(i)}(X'),$$

then

$$f(X') = \sum_{k=1}^{N_i} \left(\sum_{j=1}^{N_i} \varphi_j(X') c_{jk}^{(i)}(X')\right) B_k^{(i)}(X').$$

Hence,

$$f(X') \equiv 0 \mod \boldsymbol{b}^{(i)}(X')$$

is equivalent to

$$f(X') \equiv 0 \mod \boldsymbol{B}^{(i)}(X').$$

(ii)
$$\{h_i(X), b_j^{(i)}(X')\} = \{h_i(X), \sum_{k=1}^{N_i} c_{jk}^{(i)}(X') B_k^{(i)}(X')\}$$
$$\equiv \sum_{k=1}^{N_i} c_{jk}^{(i)}(X') \{h_i(X), B_k^{(i)}(X')\} \mod \boldsymbol{B}^{(i)}(X').$$

(iii)
$$\{b_j^{(i)}(X'), b_k^{(i)}(X')\} \equiv \sum_{p,q=1}^{N_i} c_{jp}^{(i)}(X') c_{kq}^{(i)}(X') \{B_p^{(i)}(X'), B_q^{(i)}(X')\} \mod \boldsymbol{B}^{(i)}(X').$$

Q.E.D.

Lemma 3. 3. *Let us consider a special case when*

$$\beta_i(X') = |\boldsymbol{c}(x')|^2 |\boldsymbol{B}(\xi')|^2 = \sum_{j=1}^{M} c_j(x')^2 \sum_{k=1}^{M} B_k(\xi')^2$$

and

$$\alpha_i(X') \equiv 0 \mod \boldsymbol{b}(X') = (c_j(x') B_k(\xi'))_{j,k=1,\cdots,M}.$$

Then, (B. 3) *and* (B. 4) *are satisfied. Moreover* (B. 2)$_i$ *is equivalent* *to*

$$A_1(0, X') \equiv 0 \quad \mathrm{mod}\ \boldsymbol{b}(X').$$

Proof. Since

$$\alpha_i = \sum_{j,k=1}^{M} \varphi_{jk}(X') c_j(x') B_k(\xi'),$$

we have

$$\partial'_x \alpha_i = \sum_{j,k} \partial'_x \{\varphi_{jk}(X') c_j(x')\} B_k(\xi') \equiv 0 \quad \mathrm{mod}\ \boldsymbol{B}(\xi'),$$

$$\partial'_\xi \alpha_i = \sum_{j,k} \partial'_\xi \{\varphi_{jk}(X') B_k(\xi')\} c_j(x') \equiv 0 \quad \mathrm{mod}\ \boldsymbol{c}(x'),$$

$$\partial_t \alpha_i = \sum_{j,k} \partial_t \varphi_{jk}(X') c_j(x') B_k(\xi') \equiv 0 \quad \mathrm{mod}\ \boldsymbol{b}(x', \xi').$$

Hence

$$\begin{aligned}\{h_i, c_j B_k\} &= -\{\alpha_i, c_j B_k\} \\ &= -(D'_\xi \alpha_i)(\partial'_x c_j) B_k + (\partial'_x \alpha_i) c_j (D'_\xi B_k) \equiv 0 \quad \mathrm{mod}\ \boldsymbol{b},\end{aligned}$$

$$\{c_j B_k, c_p B_q\} = c_j (D'_\xi B_k)(\partial'_x c_p) B_q - (\partial'_x c_j) B_k c_p (D'_\xi B_q) \equiv 0 \ \mathrm{mod}\ \boldsymbol{b},$$

and

$$\begin{aligned}L_A(\alpha_i(X'), X') &\equiv L_A(0, X') \\ &= A_1(0, X') - \frac{1}{2}(D_\xi \partial_x H_i)(0, X') P'_i(0, X'),\end{aligned}$$

where

$$\frac{1}{2}(D_\xi \partial_x H_i)(0, X') \equiv (D_\xi h_i)(\partial_x h_i) = -\frac{1}{i}\partial_t \alpha_i + (D'_\xi \alpha_i)(\partial'_x \alpha_i) \equiv 0 \quad \mathrm{mod}\ \boldsymbol{b}.$$

Q.E.D.

Example 1. Let

$$A_0 = \tau^2 - 2c(x')(\xi_1 + \xi_2)\tau + 2c(x')^2 \xi_1 \xi_2,$$

that is,

$$A_0 = (\tau - \alpha)^2 - \beta, \quad \alpha = c(x')(\xi_1 + \xi_2), \quad \beta = c(x')^2(\xi_1^2 + \xi_2^2).$$

Then, (B. 3) and (B. 4) are satisfied, owing to Lemma 3. 3. Let

$$A_1=a_0(t,x')\tau+a_1(t,x',\xi'),$$

then (B. 2) is equivalent to

$$a_1(t,x',\xi')=\varphi(t,x')c(x')\xi_1+\psi(t,x')c(x')\xi_2.$$

Example 2. Let

$$\begin{aligned}A_0&=\tau^4-2|\xi'|^2\tau^2+c(x')^2(\xi_1^4+\xi_2^4)\qquad(0\leq c(x')<1)\\&=\{\tau^2-(|\xi'|^2+[|\xi'|^4-c(x')^2(\xi_1^4+\xi_2^4)]^{1/2})\}\\&\quad\times\left\{\tau^2-\frac{c(x')^2(\xi_1^4+\xi_2^4)}{|\xi'|^2+[|\xi'|^4-c(x')^2(\xi_1^4+\xi_2^4)]^{1/2}}\right\}.\end{aligned}$$

Then, $A_0=0$ has a unique double root $\tau=0$ at $c(x')^2(\xi_1^4+\xi_2^4)=0$. (B. 3) and (B. 4) are satisfied, owing to Lemma 3. 3. Let

$$A_1=a_0(t,x')\tau^3+a_1(t,x',\xi')\tau^2+a_2(t,x',\xi')\tau+a_3(t,x',\xi'),$$

then (B. 2) is equivalent to

$$a_3(t,x',\xi')=\sum_{j=1}^n\varphi_j(t,x')\xi_jc(x')\xi_1^2+\sum_{j=1}^n\psi_j(t,x')\xi_jc(x')\xi_2^2.$$

Example 3. Let the characteristic roots of A_0 be smooth and $d=1$ at most. Let

$$A_0(X)=(\tau-\lambda(X'))(\tau-\mu(X'))P'(X)=H(X)P'(X),$$

where $\lambda(X_0')=\mu(X_0')$, then

$$H(X)=(\tau-\alpha(X'))^2-\beta(X'),$$

where

$$\alpha(X')=\frac{\lambda(X')+\mu(X')}{2},\quad\beta(X')=\left(\frac{\lambda(X')-\mu(X')}{2}\right)^2.$$

Hence, (B. 1) is satisfied by setting

$$b(X')=\lambda(X')-\mu(X').$$

Then, (B. 2) means

$$A_1(\lambda(X'),X')-\frac{1}{2}\{D_\xi(\tau-\lambda(X'))\partial_x(\tau-\mu(X'))$$

$$+D_\xi(\tau-\mu(X'))\partial_x(\tau-\lambda(X'))\}$$

$$\times\frac{1}{2}\partial_\tau^2 A_0(\lambda(X'),X')\equiv 0 \mod \lambda(X')-\mu(X'),$$

that is,

$$A_1(\lambda(X'),X')-\frac{1}{2i}\{-\partial_t\lambda(X')-\partial_t\mu(X')$$

$$+\partial'_\xi\lambda(X')\partial'_x\mu(X')+\partial'_\xi\mu(X')\partial'_x\lambda(X')\}$$

$$\times\frac{1}{2}\partial_\tau^2 A_0(\lambda(X'),X')\equiv 0 \mod \lambda(X')-\mu(X').$$

(B. 3) means

$$\left\{\tau-\frac{\lambda(X')+\mu(X')}{2},\lambda(X')-\mu(X')\right\}$$

$$=\{\tau-\lambda(X'),\tau-\mu(X')\}\equiv 0 \mod \lambda(X')-\mu(X'),$$

that is,

$$D_\xi(\tau-\lambda(X'))\partial_x(\tau-\mu(X'))-D_\xi(\tau-\mu(X'))\partial_x(\tau-\lambda(X'))\equiv 0$$
$$\mod \lambda(X')-\mu(X').$$

Moreover, (B. 2) and (B. 3) are equivalent to

$$\begin{cases} A_1(\lambda(X'),X')-D_\xi(\tau-\lambda(X'))\partial_x(\tau-\mu(X')) \\ \qquad\times\frac{1}{2}\partial_\tau^2 A_0(\lambda(X'),X')\equiv 0 \mod \lambda(X')-\mu(X'), \\ A_1(\lambda(X'),X')-D_\xi(\tau-\mu(X'))\partial_x(\tau-\lambda(X')) \\ \qquad\times\frac{1}{2}\partial_\tau^2 A_0(\lambda(X'),X')\equiv 0 \mod \lambda(X')-\mu(X'). \end{cases}$$

§ 4. Energy Inequalities

Already, we have had in Section 1

$$A(D)=\begin{cases} \tilde{H}_i(D)\tilde{P}'_i(D)+\{\{m-2\}\} & (i=1,\cdots,d), \\ \tilde{h}_i(D)\tilde{P}_i(D)+\{\{m-2\}\} & (i=d+1,\cdots,m-d), \end{cases}$$

by the simplification of $U=\boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\})$. Now, we assume conditions (B. 1) $\sim$ (B. 3), then we can apply results in Section 2.

Lemma 4. 1. *There exist* $\gamma_0>0$ *and* $C>0$ *such that for* $\gamma>\gamma_0$

$$e^{-2\gamma t}\|\boldsymbol{P}u(t)\|^2+\gamma\int_0^t e^{-2\gamma t}\|\boldsymbol{P}u(t)\|^2dt$$

$$\leq C\Big\{\|\boldsymbol{P}u(0)\|^2+|||u(0)|||_{m-2}^2+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|Au(t)\|^2dt$$

$$+e^{-2\gamma t}|||u(t)|||_{m-2}^2+\gamma\int_0^t e^{-2\gamma t}|||u(t)|||_{m-2}^2dt\Big\},$$

where

$$|||u(t)|||_k^2=\sum_{j=0}^{k}\|D_t^ju(t)\|_{H^{k-j}(R_x^n)}^2,$$

$$\|\boldsymbol{P}u(t)\|^2=\sum_{i=1}^{m-d}\|P_iu(t)\|^2+\sum_{i=1}^{d}\sum_{j=1}^{N_i}\|b_j^{(i)}P_i'u(t)\|^2.$$

Remark. Let $Q(X)$ be homogeneous of degree $m-1$ and

$$Q(\alpha_i(X'),X')\equiv 0 \mod \boldsymbol{b}^{(i)}(X') \quad (i=1,\cdots,d),$$

then we have

$$\|Q(D)u\|\leq C(\|\boldsymbol{P}u\|+|||u|||_{m-2}).$$

Proof. We apply the corollary of Lemma 2. 2 to

$$A(D)u=\tilde{H}_i(D)\tilde{P}_i'(D)u+\{\{m-2\}\}u \quad (i=1,\cdots,d),$$

then we have

$$e^{-2\gamma t}\{\|\tilde{h}_i\tilde{P}_i'u(t)\|^2+\sum_{j=1}^{N_i}\|b_j^{(i)}\tilde{P}_i'u(t)\|^2\}$$

$$+\gamma\int_0^t e^{-2\gamma t}\{\|\tilde{h}_i\tilde{P}_i'u(t)\|^2+\sum_{j=1}^{N_i}\|b_j^{(i)}\tilde{P}_i'u(t)\|^2\}dt$$

$$\leq\{\|\tilde{h}_i\tilde{P}_i'u(0)\|^2+\sum_{j=1}^{N_i}\|b_j^{(i)}\tilde{P}_i'u(0)\|^2\}$$

$$+C\Big\{\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|Au(t)\|^2dt+\int_0^t e^{-2\gamma t}|||u(t)|||_{m-2}^2dt\Big\}.$$

On the other hand, we apply Lemma 2. 3 to

$$A(D)u=\tilde{h}_i(D)\tilde{P}_i(D)u+\{\{m-2\}\}u \quad (i=d+1, \cdots, m-d),$$

then we have

$$e^{-2\gamma t}\|\tilde{P}_i u(t)\|^2+\gamma\int_0^t e^{-2\gamma t}\|\tilde{P}_i u(t)\|^2 dt$$
$$\leq\|\tilde{P}_i u(0)\|^2+\frac{C}{\gamma}\int_0^t e^{-2\gamma t}(\|Au(t)\|^2+|||u(t)|||_{m-2}^2)\,dt\,.$$

We have only to remark

$$\begin{cases} \tilde{h}_i(D)\tilde{P}_i'(D)=P_i(D)+\{\{m-2\}\} \quad (i=1, \cdots, d), \\ b_j^{(i)}(D')\tilde{P}_i'(D)=b_j^{(i)}(D')P_i'(D)+\{\{m-2\}\} \quad (i=1, \cdots, d), \\ \tilde{P}_i(D)=P_i(D)+\{\{m-2\}\} \quad (i=d+1, \cdots, m-d). \end{cases}$$

Q.E.D.

Since $\partial_t A_0(X)$ is strictly hyperbolic, we have

Lemma 4. 2.

$$e^{-2\gamma t}\sum_{k=0}^{m-2}\gamma^{2(m-2-k)}|||u(t)|||_k^2+\gamma\int_0^t e^{-2\gamma t}\sum_{k=0}^{m-2}\gamma^{2(m-2-k)}|||u(t)|||_k^2 dt$$
$$\leq C\left\{\sum_{k=0}^{m-2}\gamma^{2(m-2-k)}|||u(0)|||_k^2+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|\partial_\tau A_0(D)u(t)\|^2 dt\right\}.$$

Here we remark that

$$\partial_\tau A_0(X)=2\sum_{i=1}^{d}P_i(X)+\sum_{i=d+1}^{m-d}P_i(X),$$

then we have

Proposition 4. 3. Let us assume condition (B. 1) $\sim$ (B. 3), then there exist $\gamma_0>0$ and $C>0$ such that for $\gamma>\gamma_0$

$$e^{-2\gamma t}(\|\boldsymbol{P}u(t)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(t)|||_k^2)$$
$$+\gamma\int_0^t e^{-2\gamma t}(\|\boldsymbol{P}u(t)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(t)|||_k^2)\,dt$$
$$\leq C\left\{(\|\boldsymbol{P}u(0)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(0)|||_k^2)+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|Au(t)\|^2 dt\right\}.$$

Let us denote

$$\Lambda(\xi')=|\xi'|,$$

then we have

Corollary.

$$\gamma^2 e^{-2\gamma t}|||(\Lambda+\gamma)^{-1}u(t)|||_{m-1}^2+\gamma^3\int_0^t e^{-2\gamma t}|||(\Lambda+\gamma)^{-1}u(t)|||_{m-1}^2 dt$$
$$\leq C\left\{\|\boldsymbol{P}u(0)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(0)|||_k^2+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|Au(t)\|^2 dt\right\}.$$

Proof. Since

$$P_1(D)=D_t^{m-1}+a_1(D')D_t^{m-2}+\cdots+a_{m-1}(D'),$$

where $a_i(X')$ is homogeneous of degree i, we have

$$\gamma^2\|(\Lambda+\gamma)^{-1}D_t^{m-1}u\|^2\leq C\gamma^2\{\|(\Lambda+\gamma)^{-1}P_1(D)u\|^2+|||u|||_{m-2}^2\}$$
$$\leq C(\|\boldsymbol{P}u\|^2+\gamma^2|||u|||_{m-2}^2).$$

Moreover, since

$$\gamma^2\|(\Lambda+\gamma)^{-1}\Lambda^{m-1-j}D_t^j u\|^2\leq C\sum_{k=j}^{m-2}\gamma^{2(m-1-k)}|||u|||_k^2 \quad (j=0, 1, \cdots, m-2),$$

we have

$$\gamma^2|||(\Lambda+\gamma)^{-1}u|||_{m-1}^2\leq C(\|\boldsymbol{P}u\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u|||_k^2).$$

Q.E.D.

Chapter II. Energy Inequalities (continued)

§ 5. Localizations

Let $X_0'\in \boldsymbol{R}^{n+1}\times S^{n-1}$, then there is an r such that in the neighbourhood

$$V(X_0')=\{X'\in\boldsymbol{R}^{n+1}\times S^{n-1};\ |X'-X_0'|<r\},$$

the decomposition of $A_0(X)$ holds and condition (B) is satisfied. Of course, r depends on X_0'. Here we denote

$$W(X_0')=\left\{X'\in\boldsymbol{R}^{n+1}\times S^{n-1};\ |X'-X_0'|<\frac{r}{2}\right\}.$$

Owing to the assumption (A), $A_0(X)$ is independent of (t, x') in $\{t^2+|x'|^2>R^2\}$, therefore there exist $\{X'_p\}_{p=1,\cdots,M}$ such that

$$\{t^2+|x'|^2\leq R^2\}\times S^{n-1}\subset\bigcup_{p=1}^{M}W(X'_p).$$

Especially, we may think

$$(t_0, x'_0)\times S^{n-1}\subset\bigcup_{p=1}^{M_0}W(X'_p)\quad(t_0{}^2+|x'_0|^2=R^2, M_0<M).$$

Now we denote

$$W_0(X'_p)=W(X'_p)\cup\{(t, x', \xi')\in R^{n+1}\times S^{n-1}; t^2+|x'|^2>R^2, \\ (t_0, x'_0, \xi')\in W(X'_p)\},$$

and

$$\begin{cases}\widetilde{W}(X'_p)=\left\{(t, x', \xi')\in \boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\}); \left(t, x', \dfrac{\xi'}{|\xi'|}\right)\in W_0(X'_p)\right\} \\ \qquad\qquad (p=1, \cdots, M_0), \\ \widetilde{W}(X'_p)=\left\{(t, x', \xi')\in \boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\}); \left(t, x', \dfrac{\xi'}{|\xi'|}\right)\in W(X'_p)\right\} \\ \qquad\qquad (p=M_0+1, \cdots, M),\end{cases}$$

then we have

$$\boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\})\subset\bigcup_{p=1}^{M}\widetilde{W}(X'_p).$$

Let $W(X'_p)\cap W(X'_q)\neq\emptyset$, then we have $X'_p\in V(X'_q)$ or $X'_q\in V(X'_p)$. Let $X'_q\in V(X'_p)$, then a simple root of $A_0(X)=0$ at $X'=X'_p$ is always simple along the straight line from X'_p to X'_q and double roots at $X'=X'_p$ may be simple or may be double there with same pair.

Here we introduce a decomposition of unity depending on $\{\widetilde{W}(X'_p)\}$, that is, there exist $\{\varphi_p(X')\}_{p=1,\cdots,M}$ such that

$$\sum_{p=1}^{M}\varphi_p(X')=1\quad\text{in } R^{n+1}\times(R^n-\{0\}),\quad \operatorname{supp}[\varphi_p(X')]\subset\widetilde{W}(X'_p),$$

where $\varphi_p(X')$ are smooth and homogeneous of degree 0 with respect to ξ.

Let

$$A_0(X)=\prod_{i=1}^{d_p} H_{pi}(X)\prod_{i=d_p+1}^{m-d_p} h_{pi}(X)$$

be the decomposition of $A_0(X)$ at $X'=X'_p$ in $\widetilde{W}(X'_p)$, where

$$H_{pi}(X)=h_{pi}(X)^2-\beta_{pi}(X')=(\tau-\alpha_{pi}(X'))^2-\beta_{pi}(X'),$$

$$h_{pi}(X)=\tau-\alpha_{pi}(X'),$$

and

$$\beta_{pi}(X')=\sum_{k=1}^{N_{pi}} b_k^{(pi)}(X')^2.$$

Suitable extensions of α_{pi}, $b_k^{(pi)}$ outside $\widetilde{W}(X'_p)$ will be denoted by the same notations. Hence we have

$$A_0^{(p)}(X)=\prod_{i=1}^{d_p} H_{pi}(X)\prod_{i=d_p+1}^{m-d_p} h_{pi}(X),\quad X'\in \boldsymbol{R}^{n+1}\times(\boldsymbol{R}^n-\{0\}),$$

where

$$A_0^{(p)}(X)=A_0(X),\quad X'\in\widetilde{W}(X'_p).$$

Now, we denote

$$\begin{cases} P_{pi}(X)=h_{pi}(X)\dfrac{A_0^{(p)}(X)}{H_{pi}(X)},\quad P'_{pi}(X)=\dfrac{A_0^{(p)}(X)}{H_{pi}(X)} \quad (i=1,\cdots,d_p), \\ P_{pi}(X)=\dfrac{A_0^{(p)}(X)}{h_{pi}(X)} \quad (i=d_p+1,\cdots,m-d_p), \end{cases}$$

and

$$\|\boldsymbol{P}^{(p)}(D)u\|^2=\sum_{i=1}^{m-d_p}\|P_{pi}(D)u\|^2+\sum_{i=1}^{d_p}\sum_{k=1}^{N_{pi}}\|b_k^{(pi)}(D')P'_{pi}(D)u\|^2.$$

Lemma 5. 1. *Let $\varphi(X')$ be smooth and homogeneous of degree 0, and let its support be in $\widetilde{W}(X'_p)\cap\widetilde{W}(X'_q)$ then*

$$\|\boldsymbol{P}^{(q)}(D)\varphi(D')u\|\le C\{\|\boldsymbol{P}^{(p)}(D)\varphi(D')u\|+|||u|||_{m-2}\}.$$

Proof. Let

$$A_0(X)=\prod_{i=1}^{d_p} H_{pi}(X)\prod_{i=d_p+1}^{m-d_p} h_{pi}(X)=\prod_{j=1}^{d_q} H_{qj}(X)\prod_{j=d_q+1}^{m-d_q} h_{qj}(X),$$

$$X'\in\widetilde{W}(X'_p)\cap\widetilde{W}(X'_q),$$

then the following cases may happen:

(i) for $j\leq d_q$, there exists $i\leq d_p$ such that $H_{qj}(X)=H_{pi}(X)$,

(ii) for $j\leq d_q$, there exist $i>d_p$ and $i'>d_p$ such that $H_{qj}(X)=h_{pi}(X)h_{pi'}(X)$,

(iii) for $j>d_q$, there exist $j'>d_q$ and $i\leq d_p$ such that $h_{qj}(X)h_{qj'}(X)=H_{pi}(X)$,

(iv) for $j>d_q$, there exists $i>d_p$ such that $h_{qj}(X)=h_{pi}(X)$.

Case (i) It is sufficient to show that

$$\sum_{k=1}^{N_{qj}}\|b_k^{(qj)}(D')P'_{qj}(D)\varphi(D')u\|$$
$$\leq C\{\sum_{k=1}^{N_{pi}}\|b_k^{(pi)}(D')P'_{pi}(D)\varphi(D')u\|+|||u|||_{m-2}\}$$

and

$$\|P_{qj}(D)\varphi(D')u\|\leq C(\|P_{pi}(D)\varphi(D')u\|+|||u|||_{m-2}).$$

The latter is trivial, because

$$\|P_{qj}\varphi u\|\leq\|P_{qj}\circ\varphi u\|+C|||u|||_{m-2}$$
$$=\|P_{pi}\circ\varphi u\|+C|||u|||_{m-2}\leq\|P_{pi}\varphi u\|+C'|||u|||_{m-2}.$$

Now, let us consider

$$I_{pi}=\sum_{k=1}^{N_{pi}}\|b_k^{(pi)}P'_{pi}\varphi u\|^2$$
$$=(\beta_{pi}P'_{pi}\varphi u,\ P'_{pi}\varphi u)+((\sum_{k=1}^{N_{pi}}b_k^{(pi)*}b_k^{(pi)}-\beta_{pi})P'_{pi}\varphi u,\ P'_{pi}\varphi u)$$
$$=K_{pi}+L_{pi}.$$

Since

$$\beta_{pi}P'_{pi}\varphi-\beta_{pi}\circ P'_{pi}\circ\varphi=D_\xi\beta_{pi}\circ\partial_x P'_{pi}\circ\varphi+D_\xi(\beta_{pi}\circ P'_{pi})\circ\partial_x\varphi+\{\{m-2\}\},$$

we have

$$K_{pi}=((\beta_{pi}\circ P'_{pi}\circ\varphi)u,\ P'_{pi}\circ\varphi u)+((\beta_{pi}\circ P'_{pi}\circ\varphi)u,\ D_\xi P'_{pi}\circ\partial_x\varphi u)$$
$$+(\{D_\xi\beta_{pi}\circ\partial_x P'_{pi}\circ\varphi+D_\xi(\beta_{pi}\circ P'_{pi})\circ\partial_x\varphi\}u,\ P'_{pi}\circ\varphi u)+\cdots,$$

where

$$|\cdots|\leq C|||u|||_{m-2}^2,$$

therefore we have

$$|K_{pi}-((\beta_{pi}\circ P'_{pi}\circ\varphi)u,\ (P'_{pi}\circ\varphi)u)|\leq C(\sqrt{I_{pi}}+|||u|||_{m-2})|||u|||_{m-2}\,.$$

On the other hand, since

$$\sum_{k=1}^{N_{pi}} b_k^{(pi)*}b_k^{(pi)}-\beta_{pi}=\frac{1}{2}D_\xi\partial_x\beta_{pi}+\{\{0\}\}\,.$$

we have

$$L_{pi}=\frac{1}{2}(D_\xi\partial_x\beta_{pi}\circ P'_{pi}\varphi u,\ P'_{pi}\circ\varphi u)+\cdots,$$

where

$$|\cdots|\leq C|||u|||^2_{m-2},$$

therefore we have

$$\left|L_{pi}-\frac{1}{2}(D_\xi\partial_x\beta_{pi}\circ P'_{pi}\circ\varphi u,\ P'_{pi}\circ\varphi u)\right|\leq C(\sqrt{I_{pi}}+|||u|||_{m-2})|||u|||_{m-2}\,.$$

Since

$$\begin{cases} ((\beta_{pi}\circ P'_{pi}\circ\varphi)u,\ (P'_{pi}\circ\varphi)u)=((\beta_{qj}\circ P'_{qj}\circ\varphi)u,\ (P'_{qj}\circ\varphi)u), \\ ((D_\xi\partial_x\beta_{pi}\circ P'_{pi}\circ\varphi)u,\ (P'_{pi}\circ\varphi)u)=((D_\xi\partial_x\beta_{qj}\circ P'_{qj}\circ\varphi)u,\ (P'_{qj}\circ\varphi)u), \end{cases}$$

we have

$$|I_{pi}-I_{qj}|\leq C(\sqrt{I_{pi}}+\sqrt{I_{qi}}+|||u|||_{m-2})|||u|||_{m-2}\,.$$

Here we have

$$I_{qj}\leq C'(I_{pi}+|||u|||^2_{m-2})\,.$$

Case (ii). It is sufficient to show that

$$\sum_{k=1}^{N_{qj}}\|b_k^{(qj)}(D')P'_{qj}(D)\varphi(D')u\|+\|P_{qj}(D)\varphi(D')u\|$$
$$\leq C(\|P_{pi}(D)\varphi(D')u\|+\|P_{pi'}(D)\varphi(D')u\|+|||u|||_{m-2})\,.$$

Since for $X'\in\widetilde{W}(X'_p)\cap\widetilde{W}(X'_q)$

$$P'_{qj}(X)=\frac{A_0(X)}{H_{qj}(X)}=\frac{A_0(X)}{h_{pi}(X)h_{pi'}(X)}$$
$$=\frac{A_0(X)}{(\tau-\alpha_{pi}(X'))(\tau-\alpha_{pi'}(X'))}$$

$$=\frac{1}{\alpha_{pi}(X')-\alpha_{pi'}(X')}(P_{pi}(X)-P_{pi'}(X)),$$

$$P_{qj}(X)=(\tau-\alpha_{qj}(X'))\frac{A_0(X)}{H_{qj}(X)}=\frac{(\tau-\alpha_{qj}(X'))A_0(X)}{(\tau-\alpha_{pi}(X'))(\tau-\alpha_{pi'}(X'))}$$

$$=\frac{\alpha_{pi}(X')-\alpha_{qj}(X')}{\alpha_{pi}(X')-\alpha_{pi'}(X')}P_{pi}(X)+\frac{\alpha_{pi'}(X')-\alpha_{pj}(X')}{\alpha_{pi'}(X')-\alpha_{pi}(X')}P_{pi'}(X),$$

we have

$$\|b_k^{(qj)}P'_{qj}\varphi u\|+\|P_{qj}\varphi u\|\leq\|b_k^{(qj)}\circ P'_{qj}\circ\varphi u\|+\|P_{qj}\circ\varphi u\|+C|||u|||_{m-2}$$

$$\leq C'\{\|P_{qj}\circ\varphi u\|+\|P_{pi'}\circ\varphi u\|+|||u|||_{m-2}\}$$

$$\leq C''\{\|P_{pi}\varphi u\|+\|P_{pi'}\varphi u\|+|||u|||_{m-2}\}.$$

Case (iii). It is sufficient to show that

$$\|P_{qj}\varphi u\|+\|P_{qj'}\varphi u\|\leq C(\|P_{pi}\varphi u\|+\sum_{k=1}^{N_{pi}}\|b_k^{(pi)}P'_{pi}\varphi u\|+|||u|||_{m-2}).$$

Since for $X'\in\widetilde{W}(X'_p)\cap\widetilde{W}(X'_q)$

$$P_{qj}(X)=\frac{A_0(X)}{h_{qj}(X)}=\frac{h_{qj'}(X)A_0(X)}{h_{qj'}(X)h_{qj}(X)}=(\tau-\alpha_{qj'}(X'))P'_{pi}(X)$$

$$=P_{pi}(X)+(\alpha_{pi}(X')-\alpha_{qj'}(X'))P'_{pi}(X),$$

$$P_{qj'}(X)=(\tau-\alpha_{qj}(X'))P'_{pi}(X)=P_{pi}(X)+(\alpha_{pi}(X')-\alpha_{qj}(X'))P'_{pi}(X),$$

we have

$$\|P_{qj}\varphi u\|+\|P_{qj'}\varphi u\|\leq\|P_{qj}\circ\varphi u\|+\|P_{qj'}\circ\varphi u\|+C|||u|||_{m-2}$$

$$\leq C'(\|P_{pi}\circ\varphi u\|+\|P'_{pi}\circ\varphi u\|_1+|||u|||_{m-2}).$$

On the other hand, since

$$(\tau-\alpha_{qj})(\tau-\alpha_{qj'})=(\tau-\alpha_{pi})^2-\beta_{pi},$$

we have $\beta_{pi}(X')\neq 0$ for $X'\in\widetilde{W}(X'_p)\cap\widetilde{W}(X'_q)$, hence we have

$$\|P'_{pi}\circ\varphi u\|_1\leq C(\sum_{k=1}^{N_{pi}}\|b_k^{(pi)}\circ P'_{pi}\circ\varphi u\|+|||u|||_{m-2}).$$

Case (iv). Since

$$P_{qj}(X)=\frac{A_0(X)}{h_{qj}(X)}=\frac{A_0(X)}{h_{pi}(X)}=P_{pi}(X),$$

we have

$$\|P_{qj}\varphi u\| \leq \|P_{qj}\circ\varphi u\| + C|||u|||_{m-1} = \|P_{pi}\circ\varphi u\| + C|||u|||_{m-1}$$
$$\leq \|P_{pi}\varphi u\| + C'|||u|||_{m-1}. \qquad \text{Q.E.D.}$$

Now, we return to the problem of the localization of $A(D)$. Using the decomposition of unity $\{\varphi_p(X')\}_{p=1,\dots,M}$, we have

$$\varphi_p(D')A(D) - A^{(p)}(D)\varphi_p(D')$$
$$= D_\xi\varphi_p(D')\circ\partial_x A_0(D) - D_\xi A_0(D)\circ\partial_x\varphi_p(D') + L_p,$$

where

$$\|L_p u\| \leq C|||(\Lambda+1)^{-1}u|||_{m-1}.$$

Let $\psi_p(X')$ be smooth and homogeneous of degree 0, whose support is in $\widetilde{W}(X'_p)$, and $\psi_p(X')=1$ on the support of $\varphi_p(X')$. Then we have

$$\|(D_\xi\varphi_p\circ\partial_x A_0 - D_\xi A_0\circ\partial_x\varphi_p)u\| \leq C\|\boldsymbol{P}^{(p)}\circ\psi_p u\|$$
$$\leq C'\{\|\boldsymbol{P}^{(p)}\psi_p u\| + |||u|||_{m-2}\}.$$

Moreover, we have from Lemma 5.1

$$\|\boldsymbol{P}^{(p)}\psi_p u\| = \|\boldsymbol{P}^{(p)}\psi_p\sum_{q=1}^{M}\varphi_q u\| \leq \sum_{q=1}^{M}\|\boldsymbol{P}^{(p)}\psi_p\circ\varphi_q u\| + C|||u|||_{m-2}$$
$$\leq C'\left(\sum_{q=1}^{M}\|\boldsymbol{P}^{(q)}\psi_p\circ\varphi_q u\| + |||u|||_{m-2}\right)$$
$$\leq C''\left(\sum_{q=1}^{M}\|\boldsymbol{P}^{(q)}\varphi_q u\| + |||u|||_{m-2}\right).$$

Hence we have

Lemma 5.2 (Localization of A).

$$\|A^{(p)}\varphi_p u\| \leq C\{\|\varphi_p A u\| + \sum_{q=1}^{M}\|\boldsymbol{P}^{(q)}\varphi_q u\| + |||(\Lambda+1)^{-1}u|||_{m-1}\}.$$

Now we apply Proposition 4.3 and its corollary to localized operators $A^{(p)}$, then we have for $\gamma>\gamma_0$

$$e^{-2\gamma t}\{\|\boldsymbol{P}^{(p)}\varphi_p u(t)\|^2 + \sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||\varphi_p u(t)|||_k^2 + \gamma^2|||(\Lambda+\gamma)^{-1}\varphi_p u(t)|||_{m-1}^2\}$$

$$+\gamma\int_0^t e^{-2\gamma t}\{\|\boldsymbol{P}^{(p)}\varphi_p u(t)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||\varphi_p u(t)|||_k^2$$

$$+\gamma^2|||(\Lambda+\gamma)^{-1}\varphi_p u(t)|||_{m-1}^2\Big\}dt$$

$$\leq C\Big\{\|\boldsymbol{P}^{(p)}\varphi_p u(0)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||\varphi_p u(0)|||_k^2$$

$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|A^{(p)}\varphi_p u(t)\|^2 dt\Big\}$$

$$\leq C'\Big\{\|\boldsymbol{P}^{(p)}\varphi_p u(0)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||\varphi_p u(0)|||_k^2$$

$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}(\|\varphi_p A u(t)\|^2+\sum_{q=1}^{M}\|\boldsymbol{P}^{(q)}\varphi_q u(t)\|^2$$

$$+|||(\Lambda+1)^{-1}u(t)|||_{m-1}^2)dt\Big\}.$$

Summing them up with respect to p, we have

Theorem I. *Let us assume Conditions (A) and (B), then there exist $\gamma_0>0$ and $C>0$ such that for $\gamma>\gamma_0$*

$$e^{-2\gamma t}\Big\{\|\boldsymbol{P}u(t)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(t)|||_k^2+\gamma^2|||(\Lambda+\gamma)^{-1}u(t)|||_{m-1}^2\Big\}$$

$$+\gamma\int_0^t e^{-2\gamma t}\{\|\boldsymbol{P}u(t)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(t)|||_k^2$$

$$+\gamma^2|||(\Lambda+\gamma)^{-1}u(t)|||_{m-1}^2\}dt$$

$$\leq C\Big\{\|\boldsymbol{P}u(0)\|^2+\sum_{k=0}^{m-2}\gamma^{2(m-1-k)}|||u(0)|||_k^2+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|Au(t)\|^2 dt\Big\},$$

where

$$\|\boldsymbol{P}u\|^2=\sum_{p=1}^{M}\|\boldsymbol{P}^{(p)}\varphi_p u\|^2.$$

Remark Let us denote

$$|||u|||_{k,\gamma}^2=\sum_{l=0}^{k}\gamma^{2(k-l)}|||u|||_l^2,$$

then

$$c_1|||(\Lambda+\gamma)^{-1}u|||^2_{m-1,\gamma}\leq|||u|||^2_{m-2,\gamma}+|||(\Lambda+\gamma)^{-1}u|||^2_{m-1}\leq c_2|||(\Lambda+\gamma)^{-1}u|||^2_{m-1,\gamma}.$$

Corollary. *Let*

$$\int_{-\infty}^{\infty}e^{-2\gamma t}|||u(t)|||^2_m dt<+\infty,$$

then

$$\int_{-\infty}^{\infty}e^{-2\gamma t}|||(\Lambda+\gamma)^{-1}u(t)|||^2_{m-1,\gamma}\leq\frac{C}{\gamma^4}\int_{-\infty}^{\infty}e^{-2\gamma t}\|Au(t)\|^2 dt.$$

Finally, we consider energy inequalities of higher order or lower order.

Lemma 5. 3.

$$\|(D_\xi A_0)(D)u\|+\|(\Lambda+1)^{-1}(D_x A_0)(D)u\|$$
$$\leq C\{\|\boldsymbol{P}u\|+\|(\Lambda+1)^{-1}u\|_{m-1}\}.$$

Proof. We have

$$\|(D_\xi A_0)(D)u\|\leq\sum_{p=1}^{M}\|(D_\xi A_0)(D)\varphi_p(D')u\|$$
$$\leq\sum_{p=1}^{M}\|(D_\xi A_0^{(p)})(D)\varphi_p(D')u\|+C|||(\Lambda+1)^{-1}u|||_{m-1},$$

and

$$\|(\Lambda+1)^{-1}(D_x A_0)(D)u\|\leq\sum_{p=1}^{M}\|(\Lambda+1)^{-1}(D_x A_0)(D)\varphi_p(D')u\|$$
$$\leq\sum_{p=1}^{M}\|(\Lambda+1)^{-1}(D_x A_0^{(p)})(D)\varphi_p(D')u\|$$
$$+C|||(\Lambda+1)^{-1}u|||_{m-1}.$$

On the other hands, we have

$$(D_\xi A_0^{(p)})(\alpha_{pi}(X'),X')\equiv 0 \mod \boldsymbol{b}^{(pi)}(X'),$$
$$(D_x A_0^{(p)})(\alpha_{pi}(X'),X')\equiv 0 \mod \boldsymbol{b}^{(pi)}(X'),$$

then we have

$$\|D_\xi A_0^{(p)}(D)v\| + \|(\Lambda+1)^{-1}(D_x A_0^{(p)})(D)v\|$$
$$\leq C(\|\boldsymbol{P}^{(p)}v\| + |||v|||_{m-2}). \qquad \text{Q.E.D.}$$

Now, we remark

$$\|(D_x A(D) - A(D)D_x)u\| = \|(D_x A)(D)u\| \leq C\{\|\boldsymbol{P}\Lambda u\| + |||u|||_{m-1}\},$$

and we apply Theorem I, replacing $D_x u$ instead of u, then we have

$$e^{-2\gamma t}(\|\boldsymbol{P}D_x u(t)\|^2 + \gamma^2 |||D_x u(t)|||^2_{m-2,\gamma})$$
$$+\gamma \int_0^t e^{-2\gamma t}(\|\boldsymbol{P}D_x u(t)\|^2 + \gamma^2 |||D_x u(t)|||^2_{m-2,\gamma})\,dt$$
$$\leq C\Big\{\|(\boldsymbol{P}D_x u(0)\|^2 + \gamma^2 |||D_x u(0)|||^2_{m-2,\gamma}$$
$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}(\|D_x(Au)(t)\|^2 + \|\boldsymbol{P}\Lambda u(t)\|^2 + |||u(t)|||^2_{m-1})\,dt\Big\}.$$

Therefore, together with Theorem I, we have for large γ

$$e^{-2\gamma t}\{|||\boldsymbol{P}u(t)|||_1^2 + \gamma^2 |||u(t)|||^2_{m-1,\gamma}\}$$
$$+\gamma \int_0^t e^{-2\gamma t}\{|||\boldsymbol{P}u(t)|||_1^2 + \gamma^2 |||u(t)|||^2_{m-1,\gamma}\}\,dt$$
$$\leq C\Big\{|||\boldsymbol{P}u(0)|||_1^2 + \gamma^2 |||u(0)|||^2_{m-1,\gamma} + \frac{1}{\gamma}\int_0^t e^{-2\gamma t}|||Au(t)|||_1^2\,dt\Big\}.$$

Then we have step by step

Theorem I′. *Let us assume Conditions* (*A*) *and* (*B*), *then there exist* $\gamma_k > 0$ *and* $C_k > 0$ *such that for* $\gamma > \gamma_k$

$$e^{-2\gamma t}\{|||\boldsymbol{P}u(t)|||_k^2 + \gamma^2 |||u(t)|||^2_{m-2+k,\gamma}\}$$
$$+\gamma \int_0^t e^{-2\gamma t}\{|||\boldsymbol{P}u(t)|||_k^2 + \gamma^2 |||u(t)|||^2_{m-2+k,\gamma}\}\,dt$$
$$\leq C_k\Big\{|||\boldsymbol{P}u(0)|||_k^2 + \gamma^2 |||u(0)|||^2_{m-2+k,\gamma} + \frac{1}{\gamma}\int_0^t e^{-2\gamma t}|||Au(t)|||_k^2\,dt\Big\}$$
$$(k = 0, 1, 2, \cdots).$$

Lemma 5.4.

$$\|\{(\Lambda+\gamma)^{-1}A-A(\Lambda+\gamma)^{-1}\}u\|$$
$$\leq C\{\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u\|+|||(\Lambda+\gamma)^{-2}u|||_{m-1}\}.$$

Proof. We have

$$(\Lambda+\gamma)^{-1}A-A(\Lambda+\gamma)^{-1}=(\Lambda+\gamma)^{-1}(A\Lambda-\Lambda A)(\Lambda+\gamma)^{-1}$$
$$=(\Lambda+\gamma)^{-1}(-D_\xi\Lambda\cdot\partial_x A_0+\{\{m-1\}\})(\Lambda+\gamma)^{-1}.$$

From Lemma 5. 3, we have

$$\|(\Lambda+\gamma)^{-1}(D_\xi\Lambda)(\partial_x A_0)(\Lambda+\gamma)^{-1}u\|$$
$$\leq C\|(\Lambda+\gamma)^{-1}(\partial_x A_0)(\Lambda+\gamma)^{-1}u\|$$
$$\leq C'(\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u\|+|||(\Lambda+\gamma)^{-2}u|||_{m-1}). \qquad \text{Q.E.D.}$$

Now we apply Theorem I, replacing $(\Lambda+\gamma)^{-1}u$ instead of u, then we have

$$e^{-2\gamma t}\{\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(t)\|^2+\gamma^2|||(\Lambda+\gamma)^{-2}u(t)|||^2_{m-1,\gamma}\}$$
$$+\gamma\int_0^t e^{-2\gamma t}\{\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(t)\|^2+\gamma^2|||(\Lambda+\gamma)^{-2}u(t)|||^2_{m-1,\gamma}\}dt$$
$$\leq C\{\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(0)\|^2+\gamma^2|||(\Lambda+\gamma)^{-2}u(0)|||^2_{m-1,\gamma}\}$$
$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\{\|(\Lambda+\gamma)^{-1}Au(t)\|^2+\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(t)\|^2$$
$$+|||(\Lambda+\gamma)^{-2}u(t)|||^2_{m-1}\}dt\,.$$

Hence we have for large γ

$$e^{-2\gamma t}\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(t)\|^2+\gamma^2|||(\Lambda+\gamma)^{-2}u(t)|||^2_{m-1,\gamma}$$
$$+\gamma\int_0^t e^{-2\gamma t}\{\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(t)\|^2+\gamma^2|||(\Lambda+\gamma)^{-2}u(t)|||^2_{m-1,\gamma}\}dt$$
$$\leq C\Big\{\|\boldsymbol{P}(\Lambda+\gamma)^{-1}u(0)\|^2+|||(\Lambda+\gamma)^{-2}u(0)|||^2_{m-1,\gamma}$$
$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|(\Lambda+\gamma)^{-1}Au(t)\|^2dt\Big\}.$$

In this way, we have step by step

Theorem I″. *Let us assume Conditions* (*A*) *and* (*B*), *then there exist* $\gamma_k>0$ *and* $C_k>0$ *such that for* $\gamma>\gamma_k$

$$e^{-2\gamma t}\{\|\boldsymbol{P}(\Lambda+\gamma)^{-k}u(t)\|^2+\gamma^2|||(\Lambda+\gamma)^{-1-k}u(t)|||^2_{m-1,\gamma}\}$$

$$+\gamma\int_0^t e^{-2\gamma t}\{\|\boldsymbol{P}(\Lambda+\gamma)^{-k}u(t)\|^2+\gamma^2|||(\Lambda+\gamma)^{-1-k}u(t)|||^2_{m-1,\gamma}\}\,dt$$

$$\leq C_k\Big\{\|\boldsymbol{P}(\Lambda+\gamma)^{-k}u(0)\|^2+\gamma^2|||(\Lambda+\gamma)^{-1-k}u(0)|||^2_{m-1,\gamma}$$

$$+\frac{1}{\gamma}\int_0^t e^{-2\gamma t}\|(\Lambda+\gamma)^{-k}Au(t)\|^2dt\Big\}\qquad(k=0,1,2,\cdots).$$

§ 6. Existence Theorems

First, we consider a slight variation of energy inequalities. Let $\alpha(t)$ be a smooth function of t such that

$$\begin{cases}\alpha(t)=1 & \text{for}\quad t\geq 1,\\ \alpha(t)=0 & \text{for}\quad t\leq -1,\end{cases}$$

and

$$0<\alpha(t)<1\qquad\text{for}\quad -1<t<1\,.$$

Let us denote a weight function

$$e_\delta(t)=\alpha(t)+\alpha(-t)e^{\delta t}$$

for $|\delta|<1$, then we have

$$\left|\left(\frac{d}{dt}\right)^j e_\delta(t)\right|\leq C_j e_\delta(t).$$

Hence, let us denote

$$\frac{e'_\delta(t)}{e_\delta(t)}=\psi_\delta(t),$$

then we have

$$e_\delta(t)D_t u=(D_t+i\psi_\delta(t))e_\delta(t)u\,,$$

therefore we have

$$e_\delta(t)A(D_t,D'_x)u=A(D_t+i\psi_\delta(t),D'_x)e_\delta(t)u=\tilde{A}(D_t,D'_x)e_\delta(t)u\,.$$

Since

$$\tilde{A}_0(X) = A_0(X), \quad \tilde{A}_1(X) = A_1(X) + \partial_\tau A_0(X) i\psi_\delta(t),$$

we have

Lemma 6.1. *Let Conditions (A) and (B) be satisfied by $A(D)$, then they are satisfied also by $\tilde{A}(D)$.*

Now let us apply Corollary of Theorem I on $\tilde{A}$, then we have

$$\int_{-\infty}^{\infty} e^{-2\gamma t} |||(\Lambda+\gamma)^{-1}(e_\delta u)(t)|||^2_{m-1,\gamma} dt \leq \frac{C}{\gamma^4} \int_{-\infty}^{\infty} e^{-2\gamma t} \tilde{A} \|(e_\delta u)(t)\|^2 dt .$$

Since

$$\begin{aligned} |||(\Lambda+\gamma)^{-1} e_\delta u|||^2_{m-1,\gamma} &\geq c \{\|(\Lambda+\gamma)^{-1} D_t^{m-1}(e_\delta u)\|^2 \\ &\quad + \sum_{k=0}^{m-2} \gamma^{2(m-2-k)} \|D_t^k(e_\delta u)\|^2\} \\ &= c\, e_\delta^2 \{\|(\Lambda+\gamma)^{-1}(D_t - i\psi_\delta)^{m-1} u\|^2 + \sum_{k=0}^{m-2} \gamma^{2(m-2-k)} \|(D_t - i\psi_\delta)^k u\|^2\} \\ &\geq c\, e_\delta^2 \Big\{ \|(\Lambda+\gamma)^{-1} D_t^{m-1} u\|^2 + \sum_{k=0}^{m-2} \gamma^{2(m-2-k)} \|D_t^k u\|^2 \\ &\quad - C \frac{1}{\gamma^2} \sum_{k=0}^{m-2} \gamma^{2(m-2-k)} \|D_t^k u\|^2 \Big\}, \end{aligned}$$

we have for large γ

$$\int_{-\infty}^{\infty} e^{-2\gamma t} e_\delta(t)^2 |||(\Lambda+\gamma)^{-1} u(t)|||^2_{m-1,\gamma} dt \leq \frac{C}{\gamma^4} \int_{-\infty}^{\infty} e^{-2\gamma t} e_\delta(t)^2 \|Au(t)\|^2 dt .$$

Let us denote

$$e_{\gamma,\gamma'}(t) = \alpha(t) e^{\gamma t} + \alpha(-t) e^{\gamma' t}$$

for $|\gamma - \gamma'| \leq 1$, then

$$e_{\gamma,\gamma'}(t) = e^{\gamma t} e_\delta(t) \quad (\delta = \gamma' - \gamma).$$

Hence we have

Proposition 6.2. *Let $\gamma, \gamma' \geq \gamma_0$ and $|\gamma - \gamma'| \leq 1$, then*

$$\int_{-\infty}^{\infty} e_{-\gamma,-\gamma'}(t)^2 |||u(t)|||^2_{m-2,\gamma} dt \leq \frac{C}{\gamma^4} \int_{-\infty}^{\infty} e_{-\gamma,-\gamma'}(t)^2 \|Au(t)\|^2 dt .$$

Corollary. *Let* $\gamma, \gamma' \geq \gamma_0$, $|\gamma - \gamma'| \leq 1$, *and*

$$\int_{-\infty}^{\infty} e^{-2\gamma t} |||u(t)|||_m^2 dt < +\infty, \quad \int_{-\infty}^{\infty} e^{-2\gamma' t} |||v|||_m^2 dt < +\infty .$$

Then $Au = Av$ *implies* $u = v$.

Proof. Let $w = u - v$, then we have $Aw = 0$ and

$$\int_{-\infty}^{\infty} e_{-\gamma, -\gamma'}(t)^2 |||w(t)|||_m^2 dt < +\infty .$$

Then we have from Proposition 6.2

$$\int_{-\infty}^{\infty} e_{-\gamma,\gamma'}(t)^2 |||w(t)|||_{m-2,\gamma}^2 dt \leq \frac{C}{\gamma^4} \int_{-\infty}^{\infty} e_{-\gamma,-\gamma'}(t)^2 \| Aw(t) \|^2 dt = 0 .$$

Q.E.D.

Next, let us consider the formal adjoint A^* of A. Since

$$A^*(D) = A_0(D) + \{\bar{A}_1(D) + (D_\xi \partial_x A_0)(D)\} + \{\{m-2\}\},$$

we have

$$(A^*)_0(X) = A_0(X),$$

$$L_{A^*}(X) = \overline{A_1(X)} + D_\xi \partial_x A_0(X) - \frac{1}{2} D_\xi \partial_x A_0(X)$$

$$= \overline{A_1(X) - \frac{1}{2} D_\xi \partial_x A_0(X)} = \overline{L_A(X)}.$$

Here we have

Lemma 6.3. *Let Conditions* (*A*) *and* (*B*) *be satisfied by* $A(D)$, *then they are satisfied also by* $A^*(D)$.

Hence, we have energy inequalities for $A^*(D)$, corresponding to Theorems I, I′, I″ for $A(D)$. Therefore, in usual technique, ([5])

Theorem II. *Let us assume Conditions* (*A*) *and* (*B*) *for* $A(D)$. *Then Cauchy problem*:

$$\begin{cases} A(D)u = f(t, x') & \text{in } \{t > 0, x' \in R^n\}, \\ D_t^j u|_{t=0} = u_j(x') \quad (j = 0, 1, \cdots, m-1) & \text{in } \{x' \in R^n\} \end{cases}$$

is well posed in sense of Sobolev's space.

Chapter III. Finiteness of Propagation Speed

§ 7. Change of Variables

Let us consider basic properties about the change of differential operators caused by a change of variables. Let us denote

$$(t, x') = x = (x_0, x_1, \cdots, x_n), \quad (\tau, \xi') = \xi = (\xi_0, \xi_1, \cdots, \xi_n),$$

and let $x = \varphi(y)$ be a change of variables x into y, and $y = \psi(x)$ be its inverse. Then we have

$$\partial_{y_i} = \sum_{j=0}^{n} \varphi_{j(i)}(y) \partial_{x_j} \quad (\partial_y = {}^t\Phi(y) \partial_x),$$

$$\partial_{x_i} = \sum_{j=0}^{n} \psi_{j(i)}(x) \partial_{y_j} \quad (\partial_x = {}^t\Psi(x) \partial_y),$$

where

$$\Phi(y) = (\varphi_{i(j)}(y))_{i,j=0,\cdots,n} = (\partial_{y_j} \varphi_i(y))_{i,j=0,\cdots,n},$$

$$\Psi(x) = (\psi_{i(j)}(x))_{i,j=0,\cdots,n} = (\partial_{x_j} \psi_i(x))_{i,j=0,\cdots,n},$$

and

$$\Phi(\psi(x)) \Psi(x) = I.$$

Hence we have another change of variables $X = (x, \xi)$ into $Y = (y, \eta)$ and its inverse:

$$X(Y) = (\varphi(y), {}^t\Psi(\varphi(y))\eta), \quad Y(X) = (\psi(x), {}^t\Phi(\psi(x))\xi).$$

Then we have

$$\begin{cases} \partial_{x_i} = \sum_{j=0}^{n} \psi_{j(i)}(x) \partial_{y_j} + \sum_{j,k,l=0}^{n} \xi_k \varphi_{k(l,j)}(y) \psi_{l(i)}(x) \partial_{\eta_j} \\ \qquad = \sum_{j=0}^{n} \psi_{j(i)}(x) \partial_{y_j} - \sum_{j,k,l=0}^{n} \eta_k \psi_{k(i,l)}(x) \varphi_{l(j)}(y) \partial_{\eta_j}, \\ \partial_{\xi_i} = \sum_{j=0}^{n} \varphi_{i(j)}(y) \partial_{\eta_j}, \end{cases}$$

that is,

$$\begin{pmatrix} \partial_x \\ \partial_\xi \end{pmatrix} = \begin{pmatrix} {}^t\Psi(\varphi(y)) & -\sum_{k=0}^{n} \eta_k H_{\psi_k}(\varphi(y))\Phi(y) \\ 0 & \Phi(y) \end{pmatrix} \begin{pmatrix} \partial_y \\ \partial_\eta \end{pmatrix},$$

where

$$H_{\varphi_k}(y) = (\varphi_{k(i,j)}(y))_{i,j=0,\dots,n} = (\partial_{y_i}\partial_{y_j}\varphi_k(y))_{i,j=0,\dots,n},$$

$$H_{\psi_k}(x) = (\psi_{k(i,j)}(x))_{i,j=0,\dots,n} = (\partial_{x_i}\partial_{x_j}\psi_k(x))_{i,j=0\dots,n},$$

and

$${}^t\Psi H_{\varphi_k}\Psi = -\sum_{l=0}^{n} \varphi_{k(l)} H_{\psi_l}.$$

Hence we have

$$\begin{aligned} \partial_\xi a(X)\cdot\partial_x b(X) &= (\Phi\partial_\eta a)\cdot({}^t\Psi\partial_y b - \sum_{k=0}^{n}\eta_k H_{\psi_k}\Phi\partial_\eta b) \\ &= \partial_\eta a\cdot\partial_y b - (\Phi\partial_\eta a)\cdot(\sum_{k=0}^{n}\eta_k H_{\psi_k}\Phi\partial_\eta b) \\ &= \partial_\eta a\cdot\partial_y b - \partial_\xi a\cdot(\sum_{k=0}^{n}\eta_k H_{\psi_k}\partial_\xi b), \end{aligned}$$

therefore we have

Lemma 7.1. *Let $a=a(X)$ and $b=b(X)$, then*

$$\begin{aligned} &\partial_\xi a(X)\cdot\partial_x b(X) - \partial_\eta a(X(Y))\cdot\partial_y b(X(Y)) \\ &\qquad = -\partial_\xi a(X)\cdot(\sum_{k=0}^{n}\eta_k H_{\psi_k}(x)\partial_\xi b(X)), \end{aligned}$$

and moreover

$$\{a(X), b(X)\}_X = \{a(X(Y)), b(X(Y))\}_Y.$$

Now, let us consider

$$\mathcal{A}(y, D_y) = A(\varphi(y), {}^t\Psi(\varphi(y))D_y).$$

We denote

$$\mathcal{A}(Y) = \mathcal{A}_0(Y) + \mathcal{A}_1(Y) + \cdots, \quad Y = (y, \eta),$$

where $\mathcal{A}_j(Y)$ is homogeneous of degree $m-j$ with respect to η, and we denote

$$L_{\mathcal{A}}(Y) = \mathcal{A}_1(Y) - \frac{1}{2} D_\eta \partial_y \mathcal{A}_0(Y),$$

then we have

Lemma 7. 2.

$$\mathcal{A}_0(Y) = A_0(X(Y)),$$

and

$$\mathcal{A}_1(Y) = A_1(X(Y)) + \frac{1}{2} \left\{D_\eta \partial_y A_0(X(Y)) - (D_\xi \partial_x A_0)(X(Y))\right\}$$

$$+ \sum_{i,k=0}^{n} \partial_{x_i} \{\varphi_{i(k)}(\psi(x))\} D_{\eta_k} A_0(X(Y)),$$

that is,

$$L_{\mathcal{A}}(Y) - L_A(X(Y)) = \frac{1}{2} \sum_{i,k=0}^{n} \partial_{x_i} \{\varphi_{i(k)}(\psi(x))\} D_{\eta_k} A_0(X(Y)).$$

Proof Let us denote

$$A_0(X) = \sum_{i_1=0}^{n} \cdots \sum_{i_m=0}^{n} a_{i_1 \cdots i_m}(x) \xi_{i_1} \cdots \xi_{i_m},$$

where $a_{i_1 \cdots i_m}$ are invariant with respect to permutations of suffixes. Now we consider

$$A_0(\varphi(y), {}^t\Psi(\varphi(y)) D_y) = \sum_{i_1, \cdots, i_m} a_{i_1 \cdots i_m}(\varphi(y)) \Xi_{i_1}(y, D_y) \cdots \Xi_{i_m}(y, D_y),$$

where

$$\Xi(y, D_y) = {}^t\Psi(\varphi(y)) D_y,$$

then we have

$$A_0(\varphi(y), {}^t\Psi(\varphi(y)) D_y) = \sum a_{i_1 \cdots i_m}(\varphi(y)) \Xi_{i_1}(y, D_y) \circ \cdots \circ \Xi_{i_m}(y, D_y)$$

$$+ \frac{m(m-1)}{2} \sum a_{i_1 \cdots i_m}(\varphi(y)) (D_\eta \Xi_{i_1})(y, D_y) \circ (\partial_y \Xi_{i_2})(y, D_y)$$

$$\circ \Xi_{i_3}(y, D_y) \circ \cdots \circ \Xi_{i_m}(y, D_y) + \{\{m-2\}\}.$$

Since we have from Lemma 7. 1

$$D_\eta \Xi_i(Y) \cdot \partial_y \Xi_j(Y)$$

$$=D_\xi\xi_i\cdot\partial_x\xi_j+\frac{1}{i}\partial_\xi\xi_i\cdot(\sum_{k=0}^{n}\eta_k H_{\phi_k}\partial_\xi\xi_j)=\frac{1}{i}\sum_{k=0}^{n}\eta_k\psi_{k(i,j)}(x),$$

we have

$$A_0(\varphi(y),{}^t\Psi(\varphi(y)D_y)=\sum a_{i_1\cdots i_m}(\varphi(y))\Xi_{i_1}(y,D_y)\circ\cdots\circ\Xi_{i_m}(y,D_y)$$

$$+\frac{1}{i}\frac{m(m-1)}{2}\sum a_{i_1\cdots i_m}(\varphi(y))\sum_{k=0}^{n}\psi_{k(i_1,i_2)}(\varphi(y))D_{y_k}$$

$$\circ\Xi_{i_3}(y,D_y)\circ\cdots\circ\Xi_{i_m}(y,D_y)+\{\{m-2\}\}.$$

Here we have

$$\mathcal{A}_0(Y)=\sum a_{i_1\cdots i_m}(\varphi(y))\Xi_{i_1}(Y)\cdots\Xi_{i_m}(Y)=A_0(X(Y)),$$

and

$$\mathcal{A}_1(Y)-A_1(X(Y))$$

$$=\frac{1}{i}\frac{m(m-1)}{2}\sum_{i_1,\cdots,i_m}a_{i_1\cdots i_m}(\varphi(y))\sum_{k=0}^{n}\eta_k\psi_{k(i_1,i_2)}(\varphi(y))\Xi_{i_3}(Y)\cdots\Xi_{i_m}(Y)$$

$$=\frac{1}{2i}\sum_{i,j,k}\eta_k\psi_{k(i,j)}(\varphi(y))(\partial_{\xi_i}\partial_{\xi_j}A_0)(X(Y))$$

$$=\frac{1}{2i}\sum_i\{\sum_j\psi_{j(i)}(x)\partial_{y_j}-\partial_{x_i}\}\partial_{\xi_i}A_0$$

$$=\frac{1}{2i}\{\sum_{i,j}\psi_{j(i)}(x)\partial_{y_j}(\sum_k\varphi_{i(k)}(y)\partial_{\eta_k}A_0)-\sum_i\partial_{x_i}\partial_{\xi_i}A_0\}$$

$$=\frac{1}{2i}\{\sum_j\partial_{y_j}\partial_{\eta_j}A_0+\sum_{i,j,k}\psi_{j(i)}(x)\varphi_{i(j,k)}(y)\partial_{\eta_k}A_0-\sum_i\partial_{x_i}\partial_{\xi_i}A_0\},$$

where

$$\sum_{i,j,k}\psi_{j(i)}(x)\varphi_{i(j,k)}(y)\partial_{\eta_k}A_0=\sum\partial_{x_i}\{\varphi_{i(k)}(\psi(x))\}\partial_{\eta_k}A_0$$

$$=-\sum\varphi_{i(j)}(y)\psi_{j(i,k)}(x)\partial_{\xi_k}A_0=-\sum\partial_{y_j}\{\psi_{j(k)}(\varphi(y))\}\partial_{\xi_k}A_0.$$

Q.E.D.

§ 8. Space-like Change of Variables

Let Γ_x be the open connected component of $\{\xi;A_0(x,\xi)\neq0\}$ containing $\xi=(1,0,\cdots,0)$. Let us say that $y=\psi(x)=(\psi_0(x),\cdots,\psi_n(x))$ is a space-like change of variables if $\partial_x\psi_0(x)\in\Gamma_x$.

Lemma 8. 1. *Let* $y=\psi(x)$ *be a space-like change of variables. If the Condition* (*A*) *is satisfied by* $A(x, D_x)$, *then it is satisfied also by* $\mathcal{A}(y, D_y)=A(\varphi(y), {}^t\Psi(\varphi(y))D_y)$.

Proof. We have from Lemma 7. 2

$$\mathcal{A}_0(y, \eta)=A_0(\varphi(y), {}^t\Psi(\varphi(y))\eta)=A_0(\varphi(y), \sum_{j=0}^{n}\partial_x\psi_j(\varphi(y))\eta_j),$$

therefore we have

$$\mathcal{A}_0(y, (1, 0, \cdots, 0))=A_0(\varphi(y), (\partial_x\psi_0)(\varphi(y)))\neq 0 .$$

Moreover, since

$$A_0(x, \xi)\neq 0 \qquad \text{for} \quad \pm\operatorname{Im}\xi\in\Gamma_x,$$

we have

$$\mathcal{A}_0(y, \eta)\neq 0 \qquad \text{for} \quad \eta=(\omega, \eta'), \quad \operatorname{Im}\omega\neq 0, \quad \eta'\in R^n.$$

Q.E.D.

Now, let us consider the decomposition of $A_0(x, \xi)$ at $X'=X_0'=(x_0, \xi_0')$:

$$A_0(x, \xi)=\prod_{i=1}^{d}H_i(x, \xi)\prod_{i=d+1}^{m-d}h_i(x, \xi),$$

then we have

Lemma 8. 2. *Let* $\zeta\in\Gamma_{x_0}$, *then there exist* $\varepsilon_0>0$ *and* $c>0$ *such that*

(i) $H_i(x_0, \xi_0+\varepsilon\zeta)\geq c\,\varepsilon^2 \quad (i=1, \cdots, d)$,

(ii) $h_i(x_0, \xi_0+\varepsilon\zeta)\geq c\,\varepsilon \quad (i=d+1, \cdots, m-d)$

for $0<\varepsilon<\varepsilon_0$, *where* $\xi_0=(\alpha_i(X_0'), \xi_0')$.

Proof. (i) Let

$$H_i(x_0, \xi)=(\tau-\alpha_i(\xi'))^2-\beta_i(\xi')=(\tau-\tau_+(\xi'))(\tau-\tau_-(\xi')),$$

where $\tau_+(\xi') \geq \tau_-(\xi')$, and let

$$\Gamma_{x_0}^{(1)} = \{\zeta \in \Gamma_{x_0};\ |\zeta| < 1\}.$$

Then there exists $\varepsilon_0 > 0$ such that

$$\xi_0 + \varepsilon_0 \Gamma_{x_0}^{(1)} \subset \{\tau > \tau_+(\xi')\},$$

that is,

$$\tau_0 + \varepsilon\theta \geq \tau_+(\xi_0' + \varepsilon\zeta') \quad \text{for } 0 < \varepsilon < \varepsilon_0,\ \zeta = (\theta, \zeta') \in \Gamma_{x_0}^{(1)},$$

because $A_0(x_0, \xi)$ is hyperbolic with respect to each $\zeta \in \Gamma_{x_0}$. Let $\zeta = (\theta, \zeta') \in \Gamma_{x_0}^{(1)}$ be fixed, then there exists $c > 0$ such that

$$\tau_0 + \varepsilon\theta - \tau_+(\xi_0' + \varepsilon\zeta') \geq c\varepsilon \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Because, choosing $\theta' < \theta$ such that $(\theta', \zeta) \in \Gamma_{x_0}^{(1)}$, we have

$$\tau_0 + \varepsilon\theta' \geq \tau_+(\xi_0' + \varepsilon\zeta'),$$

that is

$$\tau_0 + \varepsilon\theta - \tau_+(\xi_0' + \varepsilon\zeta') = \{\tau_0 + \varepsilon\theta' - \tau_+(\xi_0' + \varepsilon\zeta')\} + \varepsilon(\theta - \theta') \geq \varepsilon(\theta - \theta').$$

Here we have

$$\begin{aligned} H_i(x_0, \xi_0 + \varepsilon\zeta) &= (\tau_0 + \varepsilon\theta - \tau_+(\xi_0' + \varepsilon\zeta'))(\tau_0 + \varepsilon\theta - \tau_-(\xi_0' + \varepsilon\zeta')) \\ &\geq (\tau_0 + \varepsilon\theta - \tau_+(\xi_0' + \varepsilon\zeta'))^2 \geq \varepsilon^2 c^2. \end{aligned}$$

(ii) Regarding $\alpha_i(\xi')$ as $\tau_+(\xi')$, we have in the same way

$$h_i(x_0, \xi_0 + \varepsilon\zeta) = \tau_0 + \varepsilon\theta - \alpha_i(\xi_0' + \varepsilon\zeta') \geq \varepsilon c \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Q.E.D.

Corollary. *Let* $\zeta = (\theta, \zeta') \in \Gamma_{x_0}$, *then*

(i) $\{\theta - \partial_\xi' \alpha_i(X_0') \cdot \zeta'\}^2 - \dfrac{1}{2} \sum_{j,k=1}^{n} \partial_{\xi_j} \partial_{\xi_k} \beta_i(X_0') \zeta_j \zeta_k > 0 \quad (i = 1, \cdots, d),$

(ii) $\theta - \partial_\xi' \alpha_i(X_0') \cdot \zeta' > 0 \quad (i = d+1, \cdots, m-d).$

Remark. $\theta - \partial_\xi' \alpha_i(X_0') \cdot \zeta' > 0$ $(i = 1, \cdots, d)$ follow from (i), because $\beta_i(X') \geq 0, \beta_i(X_0') = 0$.

Proof. (i) Let $X_0 = (x_0, \alpha_i(x_0, \xi_0'), \xi_0')$, then

$$H_i(X_0)=\partial_\tau H_i(X_0)=\partial'_\xi H_i(X_0)=0\,,$$

and

$$\begin{cases}\partial_\tau^2 H_i(X_0)=2\,, \quad \partial_\tau\partial_{\xi_j}H_i(X_0)=-2\partial_{\xi_j}\alpha_i(X'_0)\,,\\ \partial_{\xi_j}\partial_{\xi_k}H_i(X_0)=2\partial_{\xi_j}\alpha_i(X'_0)\,\partial_{\xi_k}\alpha_i(X'_0)-\partial_{\xi_j}\partial_{\xi_k}\beta_i(X'_0)\,.\end{cases}$$

Then we have by Taylor's expansion

$$\begin{aligned}H_i(x_0, \xi_0+\varepsilon\zeta) &= \frac{\varepsilon^2}{2}\{\partial_\tau^2 H_i(X_0)\,\theta^2+2\sum_{j=1}^{n}\partial_\tau\partial_{\xi_j}H_i(X_0)\,\theta\zeta_j\\ &\qquad +\sum_{j,k=1}^{n}\partial_{\xi_j}\partial_{\xi_k}H_i(X_0)\,\zeta_j\zeta_k\}+0(\varepsilon^3)\\ &=\varepsilon^2\Big\{(\theta-\sum_{j=1}^{n}\partial_{\xi_j}\alpha_i(X'_0)\,\zeta_j)^2-\frac{1}{2}\sum_{j,k=1}^{n}\partial_{\xi_j}\partial_{\xi_k}\beta_i(X'_0)\,\zeta_j\zeta_k\}+0(\varepsilon^3)\\ &\geq c\varepsilon^2 \quad (c>0)\,,\end{aligned}$$

which means

$$(\theta-\sum_{j=1}^{n}\partial_{\xi_j}\alpha_i(X'_0)\,\zeta_j)^2-\frac{1}{2}\sum_{j,k=1}^{n}\partial_{\xi_j}\partial_{\xi_k}\beta_i(X'_0)\,\zeta_j\zeta_k>0\,.$$

(ii) Since

$$h_i(X_0)=0,\ \partial_\tau h_i(X_0)=1\,, \quad \partial_{\xi_j}h_i(X_0)=-\partial_{\xi_j}\alpha_i(X'_0)\,,$$

we have

$$h_i(x_0, \xi_0+\varepsilon\zeta)=\varepsilon\{\theta-\sum_{j=1}^{n}\partial_{\xi_j}\alpha_i(X'_0)\,\zeta_j\}+0(\varepsilon^2)\geq c\varepsilon \quad (c>0)\,,$$

which means

$$\theta-\sum_{j=1}^{n}\partial_{\xi_j}\alpha_i(X'_0)\,\zeta_j>0\,.$$ Q.E.D.

Now denoting

$$Y=(y, \eta)=(y, \omega, \eta')\,, \quad Y'=(y, \eta')\,,$$

we consider the decomposition of $\mathcal{A}_0(Y)$ at $Y'=Y'_0=(y_0, \eta'_0)$. Let ω_0 be a root of $\mathcal{A}_0(\omega, Y')=0$, and let $Y_0=(\omega_0, Y'_0)$, then we have $A_0(X_0)=0$, where

$$X_0=(x_0, \xi_0)=(\varphi(y_0),\ {}^t\Psi(\varphi(y_0))\,\eta_0)\,.$$

Here, we consider the decomposition of $A_0(X)$ at $X'=X'_0=(x_0, \xi'_0)$:

$$A_0(X)=\prod_{i=1}^{d} H_i(X) \prod_{i=d+1}^{m-d} h_i(X),$$

then there exists unique number $i(\leq d)$ such that $H_i(X_0)=0$ or $i(>d)$ such that $h_i(X_0)=0$. First, we consider the case of $i\leq d$, then we have

$$\partial_\omega \{H_i(x_0, \partial_x\psi_0(x_0)\omega+\sum_{j=1}^{n}\partial_x\psi_j(x_0)\eta_j^0)\}|_{\omega=\omega_0}$$

$$=(\partial_\tau H_i)(X_0)(\partial_t\psi_0)(x_0)+\sum_{j=1}^{n}(\partial_{\xi_j}H_i)(X_0)(\partial_{x_j}\psi_0)(x_0)=0\,,$$

$$\partial_\omega^2 \{H_i(x_0, \partial_x\psi_0(x_0)\omega+\sum_{j=1}^{n}\partial_x\psi_j(x_0)\eta_j^0)\}|_{\omega=\omega_0}$$

$$=(\partial_\tau^2 H_i)(X_0)\{(\partial_t\psi_0)(x_0)\}^2+2\sum_{j=1}^{n}(\partial_\tau\partial_{\xi_j}H_i)(X_0)(\partial_t\psi_0)(x_0)$$

$$\times(\partial_{x_j}\psi_0)(x_0)+\sum_{j,k=1}^{n}(\partial_{\xi_j}\partial_{\xi_k}H_i)(X_0)(\partial x_{x_j}\psi_0)(x_0)(\partial_{x_k}\psi_0)(x_0)$$

$$=2\Big[\{(\partial_t\psi_0)(x_0)-\sum_{j=1}^{n}(\partial_{\xi_j}\alpha_i)(X'_0)(\partial_{x_j}\psi_0)(x_0)\}^2$$

$$-\frac{1}{2}\sum_{j,k=1}^{n}(\partial_{\xi_j}\partial_{\xi_k}\beta_i)(X'_0)(\partial_{x_j}\psi_0)(x_0)(\partial_{x_k}\psi_0)(x_0)\Big]>0\,.$$

Then, using Weierstrass' preparation theorem, we have

$$H_i(\varphi(y), {}^t\Psi(\varphi(y))\eta)=H_i(x, \partial_x\psi_0(x)\omega+\sum_{j=1}^{n}\partial_x\psi_j(x)\eta_j)$$

$$=\{(\omega-\tilde{\alpha}(y,\eta'))^2-\tilde{\beta}(y,\eta')\}f(y,\omega,\eta')$$

$$=\{(\omega-\tilde{\alpha}(Y'))^2-\tilde{\beta}(Y')\}f(Y),$$

where

$$\tilde{\alpha}(Y'_0)=\omega_0\,,\quad \tilde{\beta}(Y'_0)=0\,,\quad f(Y_0)>0\,.$$

In case of $i>d$, we have by the similar reasoning

$$h_i(\varphi(y), {}^t\Psi(\varphi(y))\eta)=(\omega-\tilde{\alpha}(Y'))f(Y),$$

where

$$\tilde{\alpha}(Y'_0)=\omega_0,\quad f(Y_0)>0\,.$$

Hence roots of $\mathcal{A}_0(\omega, Y'_0)=0$ are double at most, and therefore we have its smooth decomposition at $Y'=Y'_0$:

$$\mathcal{A}_0(Y)=c(y)\prod_{j=1}^{\delta}\tilde{H}_j(Y)\prod_{j=\delta+1}^{m-\delta}\tilde{h}_j(Y),$$

where

$$\begin{cases}\tilde{H}_j(Y)=(\omega-\tilde{\alpha}_j(Y'))^2-\tilde{\beta}_j(Y') & (j=1,\cdots,\delta),\\ \tilde{h}_j(Y)=\omega-\tilde{\alpha}_j(Y') & (j=\delta+1,\cdots,m-\delta).\end{cases}$$

Let $Y_0=(\tilde{\alpha}_j(Y'_0),Y'_0)$, then it corresponds

$$X_0=X(Y_0)=(\varphi(y_0),{}^t\Psi(\varphi(y_0))\eta_0),$$

and let

$$A_0(X)=\prod_{i=1}^{d}H_i(X)\prod_{i=d+1}^{m-d}h_i(X)$$

be the decomposition of $A_0(X)$ at $X'=X'_0$, then we have

Lemma 8. 3. *If $j\leq\delta$, then there exists $i(\leq d)$ such that*

$$H_i(X(Y))=\tilde{H}_j(Y)f_j(Y),\quad f_j(Y_0)>0,$$
$$h_i(X(Y))=(\omega-\tilde{\tilde{\alpha}}_j(Y'))g_j(Y),\ g_j(Y_0)\neq0,$$

and if $j>\delta$, then there exists $i(>d)$ such that

$$h_i(X(Y))=\tilde{h}_j(Y)f_j(Y),\quad f_j(Y_0)>0,$$

where $f_j(Y)$ and $g_j(Y)$ are smooth in a neighbourhood of $Y=Y_0$.

§ 9. Space-like Change of Variables and Condition (B)

Now we consider whether condition (B) is satisfied or not for $\mathcal{A}$, given by space-like change of variables of A. Let

$$\mathcal{A}_0(Y)=c\prod_{j=1}^{\delta}\tilde{H}_j(Y)\prod_{j=\delta+1}^{m-\delta}\tilde{h}_j(Y)$$
$$=c\prod_{j=1}^{\delta}\{(\omega-\tilde{\alpha}_j(Y'))^2-\tilde{\beta}_j(Y')\}\prod_{j=\delta+1}^{m-\delta}(\omega-\tilde{\alpha}_j(Y')).$$

be the decomposition of $\mathcal{A}_0(Y)$ at $Y'=Y'_0$. Let $j\leq\delta$, then there exists the decomposition of $A_0(X)$ at $X'=X'_0=X'(Y_0)=X'(\alpha_j(Y'_0),Y'_0)$:

$$A_0(X)=c\prod_{i=1}^{d}H_i(X)\prod_{i=d+1}^{m-d}h_i(X)$$

$$=c\prod_{i=1}^{d}\{(\tau-\alpha_i(X'))^2-\beta_i(X')\}\prod_{i=d+1}^{m-d}(\tau-\alpha_i(X')),$$

where there exists $i(\leq d)$ such that

$$H_i(X)=\tilde{H}_j(Y)f_j(Y),\quad f_j(Y)>0,$$

that is,

$$(\tau-\alpha_i(X'))^2-\beta_i(X')=\{(\omega-\tilde{\alpha}_j(Y'))^2-\tilde{\beta}_j(Y')\}f_j(Y).$$

Moreover, we have

$$\tau-\alpha_i(X')=(\omega-\tilde{\tilde{\alpha}}_j(Y'))g_j(Y),\quad g_j(Y)\neq 0,$$

then we have

$$(*)\qquad (\omega-\tilde{\tilde{\alpha}}_j(Y'))^2 g_j(Y)^2-\beta_i(X'(Y))$$
$$=\{(\omega-\tilde{\alpha}_j(Y'))^2-\tilde{\beta}_j(Y')\}f_j(Y).$$

Let $\omega=\tilde{\tilde{\alpha}}_j(Y')$, then we have

$$\tilde{\beta}_j(Y')=f_j(\tilde{\tilde{\alpha}}_j(Y'),Y')^{-1}\beta_i(X'(\tilde{\tilde{\alpha}}_j(Y'),Y')$$
$$+(\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y'))^2.$$

Let

$$\beta_i(X')=\sum_{k=1}^{N_i}b_k^{(i)}(X')^2,$$

then we have

$$\tilde{\beta}_j(Y')=\sum_{k=1}^{\tilde{N}_j}b_k^{(j)}(Y')^2\quad(\tilde{N}_j=N_i+1),$$

where

$$\tilde{b}_k^{(j)}(Y')=f_j(\tilde{\tilde{\alpha}}_j(Y'),Y')^{-1/2}b_k^{(i)}(X'(\tilde{\tilde{\alpha}}_j(Y'),Y'))\quad(k=1,\cdots,N_i),$$

$$b_{N_i+1}^{(j)}(Y')=\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y').$$

Hence

Lemma 9.1. *Let* (*B.* 1) *be satisfied for* A, *then it is also satisfied for* $\mathcal{A}$.

Next, differentiate $(*)$ with respect to ω and let $\omega=\tilde{\tilde{\alpha}}_j(Y')$, then we have

$$-(\partial_\omega\beta_i)(X'(\tilde{\tilde{\alpha}}_j(Y'),Y'))=2(\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y'))f_j(\tilde{\tilde{\alpha}}_j(Y'),Y')$$
$$+\{(\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y'))^2-\tilde{\beta}_j(Y')\}(\partial_\omega f_j)(\tilde{\tilde{\alpha}}_j(Y'),Y')$$
$$=2(\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y'))f_j(\tilde{\tilde{\alpha}}_j(Y'),Y')$$
$$-\beta_i(X'(\tilde{\tilde{\alpha}}_j(Y'),Y'))f_j(\tilde{\tilde{\alpha}}_j(Y'),Y')^{-1}(\partial_\omega f_j)(\tilde{\tilde{\alpha}}_j(Y'),Y'),$$

that is,

$$\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y')=\frac{1}{2}f_j(\tilde{\tilde{\alpha}}_j(Y'),Y')^{-1}\{-(\partial_\omega\beta_i)(X'(\tilde{\tilde{\alpha}}_j(Y'),Y'))$$
$$+\beta_i(X'(\tilde{\tilde{\alpha}}_j(Y'),Y'))f_j(\tilde{\tilde{\alpha}}_j(Y'),Y'))^{-1}(\partial_\omega f_j)(\tilde{\tilde{\alpha}}_j(Y),Y)).$$

Hence

Lemma 9.2. *Let (B.1) be satisfied for A, then*

$$\tilde{\tilde{\alpha}}_j(Y')-\tilde{\alpha}_j(Y')\equiv 0 \mod (\tilde{b}_1^{(j)}(Y'),\cdots,\tilde{b}_{N_j-1}^{(j)}(Y')).$$

Corollary. *Let (B.1) and (B.2) be satisfied for A, then (B.2) is satisfied also for $\mathcal{A}$.*

Proof. We have from Lemma 7.2 and Lemma 9.2

$$L_{\mathcal{A}}(\tilde{\alpha}_j(Y'),Y')\equiv L_{\mathcal{A}}(\tilde{\tilde{\alpha}}_j(Y'),Y')\equiv L(\alpha_i(X'),X') \mod \tilde{\boldsymbol{b}}^{(j)}(Y').$$

Q.E.D.

Finally, we consider of Conditions (B.3) and (B.4). Condition (B.3) for A is

$$\{h_i(X),b_k^{(i)}(X')\}\equiv 0 \mod \boldsymbol{b}^{(i)}(X'),$$

which is equivalent to

$$\{(\omega-\tilde{\tilde{\alpha}}_j(Y'))g_j(Y),b_k^{(i)}(X'(Y))\}\equiv 0 \mod \boldsymbol{b}^{(i)}(X'(Y))$$

from the corollary of Lemma 7.1. Let $\omega=\tilde{\tilde{\alpha}}_j(Y')$ in the latter equality, then we have

$$\{\omega-\tilde{\tilde{\alpha}}_j(Y'),\tilde{b}_k^{(j)}(Y')\}\equiv 0 \mod \tilde{\boldsymbol{b}}^{(j)}(Y').$$

In the same way, Condition (B. 4) is

$$\{b_k^{(i)}(X'), b_l^{(i)}(X')\} \equiv 0 \mod \boldsymbol{b}^{(i)}(X'),$$

which is equivalent to

$$\{b_k^{(i)}(X'(Y)), b_l^{(i)}(X'(Y))\} \equiv 0 \mod \boldsymbol{b}^{(i)}(X'(Y)).$$

Let $\omega = \tilde{\tilde{\alpha}}_j(Y')$, then we have

$$\{\tilde{b}_k^{(j)}(Y') + \partial_\omega b_k^{(i)}(X'(\tilde{\tilde{\alpha}}_j(Y'), Y'))(\omega - \tilde{\tilde{\alpha}}_j(Y')),$$
$$\tilde{b}_l^{(j)}(Y') + \partial_\omega b_l^{(i)}(X'(\tilde{\tilde{\alpha}}_j(Y'), Y'))(\omega - \tilde{\tilde{\alpha}}_j(Y'))\}_{\omega = \tilde{\tilde{\alpha}}_j(Y')} \equiv 0$$
$$\mod \tilde{\boldsymbol{b}}^{(j)}(Y').$$

Since

$$\{\omega - \tilde{\tilde{\alpha}}_j(Y'), \tilde{b}_k^{(j)}(Y')\} \equiv 0 \mod \tilde{\boldsymbol{b}}^{(j)}(Y'),$$

we have

$$\{\tilde{b}_k^{(j)}(Y'), \tilde{b}_l^{(j)}(Y')\} \equiv 0 \mod \tilde{\boldsymbol{b}}^{(j)}(Y').$$

On the other hands, we have from Lemma 9. 2

$$\tilde{\tilde{\alpha}}_j(Y') - \tilde{\alpha}_j(Y') = \sum_l c_l(Y') \tilde{b}_l^{(j)}(Y'),$$

therefore

$$\{\tilde{\tilde{\alpha}}_j(Y') - \tilde{\alpha}_j(Y'), \tilde{b}_k^{(j)}(Y')\} \equiv \sum_l c_l(Y') \{\tilde{b}_l^{(j)}(Y'), \tilde{b}_k^{(j)}(Y')\} \equiv 0$$
$$\mod \tilde{\boldsymbol{b}}^{(j)}(Y').$$

Moreover, we have

$$\{\omega - \tilde{\alpha}_j(Y'), \tilde{b}_k^{(j)}(Y')\} = \{\omega - \tilde{\tilde{\alpha}}_j(Y'), \tilde{b}_k^{(j)}(Y')\}$$
$$+ \{\tilde{\tilde{\alpha}}_j(Y') - \tilde{\alpha}_j(Y'), \tilde{b}_k^{(j)}(Y')\} \equiv 0 \mod \tilde{\boldsymbol{b}}^{(j)}(Y').$$

Here we have

Lemma 9. 3. *Let us assume conditions* (*B.* 1), (*B.* 3), (*B.* 4) *for A, then we have* (*B.* 3), (*B.* 4) *for* $\mathcal{A}$,

Proposition 9. 4. *Let us assume that Conditions* (*A*) *and* (*B*) *are satisfied by A, then they are satisfied also by* $\mathcal{A}$ *which is given by a space-like change of variables of A.*

§ 10. Finiteness of Propagation Speed

Let us denote Γ_x be the connected component of

$$\{\xi\in \boldsymbol{R}^{n+1};\ A_0(x,\xi)\neq 0\},$$

containing $\xi=(1,0,\cdots,0)$. Γ_x is an open convex cone. Let

$$A_0(x,\xi)=\prod_{j=1}^{m}(\tau-\tau_j(x,\xi')),\quad \xi=(\tau,\xi'),$$

$$\tau(x,\xi')=\max_j \tau_j(x,\xi'),$$

then

$$\Gamma_x=\{\xi\in \boldsymbol{R}^{n+1};\ \tau>\tau(x,\xi')\}.$$

Moreover, we denote

$$\Gamma=\text{interior of } \bigcap_x \Gamma_x,$$

then Γ is also an open convex cone and

$$\Gamma=\{\xi=(\tau,\xi')\in R^{n+1};\ \tau>\tau(\xi')\},\quad \tau(\xi')=\sup_x \tau(x,\xi'),$$

where $\tau(\xi')$ is continuous in $\boldsymbol{R}^n$-$\{0\}$, because of the convexity of Γ.

Next, we consider the dual cone Γ' of Γ:

$$\Gamma'=\{x\in \boldsymbol{R}^{n+1};\ x\cdot\xi\geq 0,\ \forall\xi\in\Gamma\}.$$

Let us fix $a=(1,a')\in\mathring{\Gamma}'$, then we have a change of variables

$$(\eta_0,\eta')=(a\cdot\xi,\xi')\quad \text{i.e.}\quad (\xi_0,\xi')=(\eta_0-a'\cdot\eta',\eta').$$

If $\tau(\xi')$ is smooth, then $S=\partial\Gamma\cap\{a\cdot\xi=1\}$ is represented by

$$S=\{\xi=(\eta_0-a'\cdot\eta',\eta');\ \eta_0=1,\ 1-a'\cdot\eta'=\tau(\eta')\},$$

and $S'=\partial\Gamma'\cap\{x_0=1\}$ is represented by

$$S'=\{x=(x_0,x');\ x_0=1,\ x'=-\partial'_\xi\tau(\xi'),\ \xi'\in\boldsymbol{R}^n-\{0\}\}.$$

Moreover if

$$\text{rank}\ (\partial_{\xi_i}\partial_{\xi_j}\tau(\xi'))_{i,j=1,\cdots,n}=n-1$$

for $\xi'\in R^n-\{0\}$, then S' becomes a smooth manifold of dimension $n-1$. Then there exists a smooth function $f(x')$ such that

$$\begin{cases} S' = \{x' ; f(x') = 0\}, \\ \text{interior domain of } S' = \{x' ; f(x') > 0\}, \end{cases}$$

that is,

$$\Gamma' = \left\{x = (x_0, x') ; x_0 > 0, f\left(\frac{x'}{x_0}\right) > 0\right\}.$$

In general, we shall see that Γ can be approximated from the inside by such cones stated above, that is,

Lemma 10. 1. *There exist smooth open convex cones* $\{\Gamma_k\}_{k=1,2,\dots}$ *such that*

$$\bar{\Gamma}_k \subset \Gamma_{k+1}, \quad \bigcup_{k=1}^{\infty} \Gamma_k = \Gamma,$$

and

$$\text{rank } (\partial_{\xi_i} \partial_{\xi_j} \tau_k(\xi'))_{i,j=1,\dots,n} = n-1,$$

where

$$\Gamma_k = \{\xi = (\tau, \xi') ; \tau > \tau_k(\xi')\}.$$

Proof. Since $\Omega = \Gamma \cap \{a \cdot \xi = 1\}$ is convex, there exist smooth and strictly convex domains $\{\Omega_k\}_{k=1,2,\dots}$ such that

$$\bar{\Omega}_k \subset \Omega_{k+1}, \quad \bigcup_{k=1}^{\infty} \Omega_k = \Omega.$$

Let

$$\Gamma_k = \text{convex hull of } \Omega_k \text{ and } \{0\},$$

then we have the desired properties. Q.E.D.

Corollary. *Γ'_k and Γ' are represented by*

$$\Gamma'_k = \left\{x = (x_0, x') ; x_0 > 0, f_k\left(\frac{x'}{x_0}\right) > 0\right\},$$

$$\Gamma' = \bigcap_{k=1}^{\infty} \Gamma'_k = \left\{x = (x_0, x') ; x_0 > 0, f\left(\frac{x'}{x_0}\right) > 0\right\},$$

where

$$f(x')=\inf_k f_k(x'),\quad \partial_x\left\{f_k\left(\frac{x'}{x_0}\right)\right\}\in\overline{\Gamma}_k\subset\Gamma .$$

Lemma 10. 2. *Let $T>0$, then there exist smooth approximating functions $\{\varphi_j(x)\}_{j=1,2\dots}$ such that $\partial_x\varphi_j(x)\in\Gamma$ and*

$$D_1\subset D_2\subset\cdots,\quad \bigcup_{j=1}^{\infty}D_j=\mathring{\Gamma}'\cap\{0<x_0<T\},$$

where

$$D_j=\{x=(x_0,x');\ \phi_j(x)>0,\ 0<x_0<T\}.$$

Proof In the corollary of Lemma 10. 1, we have

$$\Gamma'_k=\left\{x=(x_0,x');x_0>0,\ f_k\left(\frac{x'}{x_0}\right)>0\right\},$$

where

$$a\cdot\partial_x f_k\left(\frac{x'}{x_0}\right)=(\partial_{x_0}f_k+a'\cdot\partial'_x f_k)\left(\frac{x'}{x_0}\right)>0 .$$

Let us consider the change of variable: $(x_0,x')=(y_0,y'+a'\cdot y_0)$ i.e. $(y_0,y')=(x_0,x'-a'\cdot x_0)$. Then we have $\partial_{y_0}=\partial_{x_0}+a'\cdot\partial'_x$, and $\partial'_y=\partial'_x$. Hence, $f_k\left(\frac{x'}{x_0}\right)=0$ can be solved with respect to y_0:

$$y_0=\varphi_k(y'),$$

that is,

$$x_0=\varphi_k(x'-a'\cdot x_0).$$

Since $\partial_x\left\{f_k\left(\frac{x'}{x_0}\right)\right\}\in\overline{\Gamma}_k$, we have

$$\begin{aligned}((\partial_{y_0}-a'\cdot\partial'_y)&(y_0-\varphi_k(y')),\ \partial'_y(y_0-\varphi_k(y')))\\ &=(1+a'\cdot\partial'_y\varphi_k(y'),\ -\partial'_y\varphi_k(y'))\in\overline{\Gamma}_k\subset\Gamma .\end{aligned}$$

Moreover

$$\varphi(y')=\sup_k\varphi_k(y'),\quad 0<\rho_k=\sup_{\varphi(y')<T}\{\varphi(y')-\varphi_k(y')\}\xrightarrow[k\to+\infty]{}0,$$

where we may assume

$$\sum_{k=1}^{\infty}\rho_k<+\infty .$$

Let

$$\psi_j(y') = \varphi_j(y') + \sum_{k=j}^{\infty} \rho_k ,$$

then we have

$$\psi_1(y') > \psi_2(y') > \cdots \to \varphi(y') \quad \text{in } \varphi(y') < T .$$

Now, we operate Friedrichs' mollifier $\rho_\varepsilon *$ on each $\psi_j(y')$, then there exist $\varepsilon_j > 0$ such that

$$\tilde{\psi}_1(y') > \tilde{\psi}_2(y') > \cdots \to \varphi(y') \quad \text{in } \varphi(y') < T,$$

where

$$\tilde{\psi}_j(y') = \rho_{\varepsilon_j}(y') * \psi_j(y').$$

Let

$$\phi_j(x) = x_0 - \tilde{\psi}_j(x' - a'x_0), \quad D_j = \{\phi_j(x) > 0\},$$

then

$$\begin{aligned} \partial_x \phi_j(x) &= (1 + a' \cdot (\partial'_y \tilde{\psi}_j)(x' - a'x_0), -\partial'_y \tilde{\psi}_j(x' - a'x_0)) \\ &= \rho_{\varepsilon_j} * (1 + a' \cdot \partial'_y \psi_j, -\partial'_y \psi_j) \in \Gamma. \end{aligned}$$

Q.E.D.

Using the sweeping out method, we have

Theorem III. *Let us assume Conditions (A) and (B). Let $a = (a_0, a') \in \boldsymbol{R}^{n+1} (a_0 > 0)$ and u satisfy*

$$\begin{cases} Au = 0 & in \ \{a - \Gamma'\} \cap \{x_0 > 0\}, \\ D_t^j u = 0 \, (j = 0, \cdots, m-1) & on \ \{a - \Gamma'\} \cap \{x_0 = 0\}, \end{cases}$$

then

$$u = 0 \quad in \ \{a - \Gamma'\} \cap \{x_0 > 0\}.$$

Corollary. *Theorems II and III mean that Conditions (A) and (B) are sufficient for the Cauchy problem of A to be $\mathcal{E}$-well posed.*

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