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On Holonomic Systems for $\prod_{l=1}^{N}(f_l+\sqrt{-1}\,0)^{\lambda_l}$

By

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§ **0.** The purpose of this paper is to give a description of the characteristic variety of the holonomic $\mathcal{D}$-Module of which a hyperfunction of the form $\Phi(x)=\prod_{j=1}^{d}\delta(\varphi_j)\prod_{l=1}^{N}(f_l(x)+\sqrt{-1}\,0)^{\lambda_l}$ is a solution. The existence of such a holonomic system was shown in [7] (See also [3].) The description of the characteristic variety in terms of φ_j's and f_l's was announced in Lemma 1 of [8]. The result immediately gives an information on the singularity spectrum of $\Phi(x)$. (See Theorem 18 below.) Although a little more precise result was announced in [8] (Lemma 2), we have recently found a gap in our original proof of Lemma 2 of [8]. Even though we have not yet succeeded in filling the gap, we still believe that our original claim should be true and we feel it worth while presenting here as a conjecture. We also discuss some examples in order to show how subtle and delicate the conjecture is. In any case, we should correct our article [8], so that Lemma 2 still remains a conjecture and hence the last six lines of Theorem of [8] should be deleted at the moment.(***) Needless to say, Lemma 2 of [8] follows immediately from Lemma 1 of [8] if we replace $(\mathbb{R}^+)^N$ in line 16 of

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(***) As a consequence of this correction, all adjective "real" appearing in [9] p. 142 line 17 through line 20 should be replaced by "complex". See for details (and some improvements) the full paper which is being prepared by H. P. Stapp and the authors. (See reference [9].)

page 162 of [8] by $\boldsymbol{C}^N$.

§ **1.** The situation we shall discuss is the following:

Let M be a real analytic manifold and X a complexification of M. Let $\varphi=(\varphi_1, \cdots, \varphi_d)$ and $f=(f_1, \cdots, f_N)$ be sets of real-valued real analytic functions defined on M which extend to X. Let Y be a subvariety of X which satisfies the following conditions:

(1) Y is irreducible.

(2) $Y \subset \{x \in X; \varphi_1(x)=\cdots=\varphi_d(x)=0\}$

(3) There exists a proper complex analytic subset Y_{sing} of Y such that the following conditions are satisfied:

(3. a) $Y-Y_{\text{sing}}$ is a non-singular subvariety of codimension d.

(3. b) $d\varphi_1, \cdots, d\varphi_d$ are linearly independent at any point of $Y-Y_{\text{sing}}$.

(4) $f_l|_Y \not\equiv 0 \quad (l=1, \cdots, N)$

(5) $Y_{\text{sing}} \subset \{x \in X; \prod_{l=1}^{N} f_l(x)=0\}$

Under these assumptions we can easily show the following theorem by the desingularization theorem of Hironaka ([1], [5]).

Theorem 1. *For* $s=(s_1, \cdots, s_N) \in \boldsymbol{C}^N$ *with* $\operatorname{Re} s_l \gg 0$ *and a compactly supported* C^∞*-function* $g(x)$ *on* M, *the integral*

$$\int_{(Y-Y_{\text{sing}})\cap M} g(x) \prod_{j=1}^{d} \delta(\varphi_j(x)) \prod_{l=1}^{N} (f_l(x)+\sqrt{-1}\,0)^{s_l} dx$$

converges and it defines a distribution $\Phi(x;s)=\prod_{j=1}^{d} \delta(\varphi_j(x)) \times \times \prod_{l=1}^{N} (f_l(x)+\sqrt{-1}\,0)^{s_l}$ *with holomorphic parameters* s *when* $\operatorname{Re} s_l \gg 0$. *Furthermore this distribution can be extended as a meromorphic function in* $s=(s_1, \cdots, s_N) \in \boldsymbol{C}^N$. *More precisely, we can choose a* γ*-factor* $\gamma(s)$ *that makes* $\gamma(s)\Phi(x;s)$ *entire in* s *and has the form*

$$\gamma(s)=\prod_{k=1}^{M} \Gamma\Big(\sum_{l=1}^{N} \nu_{l,k} s_l + d_k\Big), \tag{6}$$

where $\nu_{l,k}$ *are non-negative integers such that* $(\nu_{1,k}, \cdots, \nu_{N,k}) \neq 0$ *for any* k *and* d_k *is an integer.*

Remark. If $Y_{\text{sing}}=\phi$, then we can choose d_k to be a strictly positive integer.

Since the proof is essentially the same as that given in [4] and [2], we omit the details. See also Section 5 and Section 6, especially the proof of Lemma 14.

§ **2.** In order to formulate our main results, we introduce the following notations:

(7) $W_{f,Y}$ is the closure of $\{(\sigma; x, \xi)\in \boldsymbol{C}^N\times T^*X;\ x\in Y-Y_{\text{sing}},\ f_l(x)\neq 0$ $(l=1, \cdots, N)$ and $\xi=\sum_{j=1}^{d} c_j d\varphi_j(x)+\sum_{l=1}^{N}\sigma_l d\log f_l(x)$ for some $c=$ $=(c_1, \cdots, c_d)\in \boldsymbol{C}^d\}$ in $\boldsymbol{C}^N\times T^*X$.

Remark. It is known that $W_{f,Y}$ is an irreducible analytic set. (Cf. Whitney [11]). Clearly it is involutory and of dimension $n+N$. Note that $W_{f,Y}$ depends only on $f|_Y$ and Y and not on φ.

(8) $W_0=W_{f,Y}\cap\{\sigma_l=0, l=1, \cdots, N\}$, which we consider as a subset of T^*X.

(9) We denote by $\boldsymbol{C}[s]$ the ring of polynomials in $s=(s_1, \cdots, s_N)$ and, for a set of N indeterminates $t=(t_1, \cdots, t_N)$ with commutation relations

$$\begin{cases}[t_j, t_k]=0 \\ [t_j, s_k]=\delta_{jk}t_j\end{cases} \quad \text{for } j, k=1, \cdots, N$$

we denote by $\boldsymbol{C}[s, t]$ the non-commutative algebra generated by s and t.

(10) $\mathscr{D}[s]=\mathscr{D}\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}[s]$ and $\mathscr{D}[s, t]=\mathscr{D}\underset{\boldsymbol{C}}{\otimes}\boldsymbol{C}[s, t]$. Here $\mathscr{D}=\mathscr{D}_X$ denotes the sheaf of linear differential operators of finite order defined on X. For an integer m and an Ideal $\mathscr{J}$ of $\mathscr{D}[s]$, we define $\mathscr{J}_m$ by

$$\{P(s)=\sum_{\alpha=(\alpha_1, \cdots, \alpha_N)} P_\alpha s^\alpha\in\mathscr{J};\ \operatorname{ord} P_\alpha\leqq m-|\alpha|\}. \tag{11}$$

For $P(s)$ in $\mathscr{J}_m$ we define the function $\tilde{\sigma}_m(P(s))$ of $(\sigma; x, \xi)\in\boldsymbol{C}^n\times T^*X$ by

$$\sum_{\alpha}\sigma_{m-|\alpha|}(P_\alpha)\sigma_1^{\alpha_1}\cdots\sigma_N^{\alpha_N}, \tag{12}$$

where $\sigma_j(Q)$ denotes the principal symbol of an operator Q of order at most j.

Let $\mathcal{J}$ be the Ideal of $P(s) \in \mathcal{D}[s]$ such that

$$P(s) \prod_{j=1}^{d} \delta(\varphi_j) \prod_{l=1}^{N} f_l^{s_l} = 0 \tag{13}$$

holds. This means that the equality (13) holds in $\mathcal{D} \prod_{j=1}^{d} \delta(\varphi_j)$ on $Y - Y_{\text{sing}} - (\bigcup_{l=1}^{N} f_l^{-1}(0))$ and for any complex numbers s. It is clear that $\gamma(s)\Phi(x; s)$ is annihilated by $P(s)$ in $\mathcal{J}$, where $\gamma(s)$ is the γ-factor introduced in Theorem 1.

We define $\mathcal{N}_{f,\varphi}$ by $\mathcal{D}[s]/\mathcal{J}$. We shall denote by u the differential operator 1 modulo $\mathcal{J}$. Note that the characteristic variety $\mathrm{SS}(\mathcal{N}_{f,\varphi})$ of $\mathcal{N}_{f,\varphi}$ contains $W_{f,Y}$, because $W_{f,Y}$ is contained in $\mathrm{SS}(\mathcal{N}_{f,\varphi})$ at generic points of $W_{f,Y}$.

If we define the multiplication of t_l by

$$t_l \colon P(s)u \mapsto P(s_1, \cdots, s_l + 1, \cdots, s_N) f_l u\,, \tag{14}$$

$\mathcal{N}_{f,\varphi}$ naturally acquires a structure of $\mathcal{D}[s, t]$-Module.

We also define

$$C_{f,Y} = \bigcap_m \{(\sigma; x, \xi) \in \boldsymbol{C}^N \times T^*X; \tilde{\sigma}_m(P(s))(\sigma; x, \xi) = 0 \quad \text{for } P(s) \in \mathcal{J}_m\}. \tag{15}$$

Then we have following

Theorem 2. *$C_{f,Y} = W_{f,Y}$ holds.*

In order to prove this theorem, we introduce auxiliary variables $w_l \in \boldsymbol{C}$ $(l = 1, \cdots, N)$ and define $\tilde{f} = (\tilde{f}_1, \cdots, \tilde{f}_N)$ by $(w_1 f_1(x), \cdots, w_N f_N(x))$. We denote $\boldsymbol{C}^N \times X$ (resp. $\boldsymbol{C}^N \times Y$) by $\tilde{X}$ (resp. $\tilde{Y}$). Then by defining the vector fields $\Theta_l (l = 1, \cdots, N)$ by $w_l \dfrac{\partial}{\partial w_l}$, we immediately find that

$$\Theta_l \tilde{f}_k = \delta_{lk} \tilde{f}_k \quad (l, k = 1, \cdots, N) \tag{16}$$

and

$$\Theta_l \varphi_j = 0 \quad (j = 1, \cdots, d, l = 1, \cdots, N) \tag{17}$$

hold. Then we have

$$s_l^m \prod_{j=1}^{d} \delta(\varphi_j) \prod_{l=1}^{N} \tilde{f}_l^{s_l} = \Theta_l^m \prod_{j=1}^{d} \delta(\varphi_j) \prod_{l=1}^{N} \tilde{f}_l^{s_l} \tag{18}$$

for any positive integer m. Hence $\mathcal{N}_{\tilde{f},\varphi}$ turns out to be a coherent $\mathcal{D}_{\tilde{X}}$-Module generated by the section $\tilde{u}=\prod_{j=1}^{d}\delta(\varphi_j)\prod_{l=1}^{N}\tilde{f}_l^{s_l}$. Since the function σ_l on $\boldsymbol{C}^N\times T^*\tilde{X}$ equals the principal symbol $\sigma(\Theta_l)$ of Θ_l on $W_{\tilde{f},\tilde{Y}}$, $W_{\tilde{f},\tilde{Y}}$ is imbedded into $T^*\tilde{X}$ by the projection from $\boldsymbol{C}^N\times T^*\tilde{X}$ onto $T^*\tilde{X}$. Therefore, we regard $W_{\tilde{f},\tilde{Y}}$ as a subset of $T^*\tilde{X}$. Then we can prove the following lemmas.

Lemma 3. $\mathrm{SS}(\mathcal{N}_{\tilde{f},\varphi})=\{(w,x;\tau,\xi)\in T^*(\boldsymbol{C}^N\times X);\ (w\tau;x,\xi)\in C_{f,Y}\}$.

Lemma 4. $\mathrm{SS}(\mathcal{N}_{\tilde{f},\varphi})=\{(w,x;\tau,\xi)\in T^*(\boldsymbol{C}^N\times X);$
$(w\tau;x,\xi)\in W_{f,Y}\}$.

It is clear that Theorem 2 follows from these two lemmas.

§ **3.** We first prove Lemma 3.

Proof of Lemma 3.

We first show that the left hand side of the formula in Lemma 3 is included in the right hand side. Let $\tilde{u}$ be the generator of $\mathcal{N}_{\tilde{f},\varphi}$. Let $P(s)$ be an element in $\mathcal{J}_m$. Then we have

$$P\left(w_1\frac{\partial}{\partial w_1},\cdots,w_N\frac{\partial}{\partial w_N}\right)u=0.$$

Furthermore

$$\sigma_m\left(P\left(w_1\frac{\partial}{\partial w_1},\cdots,w_N\frac{\partial}{\partial w_N}\right)\right)(w,x;\tau,\xi)$$
$$=\tilde{\sigma}_m(P(s))(w_1\tau_1,\cdots,w_N\tau_N,x,\xi)$$

holds. Hence the inclusion relation in question has been proved.

We now prove the opposite inclusion relaton. Let $P(w,x,D_w,D_x)$ be an annihilator of $\tilde{u}$. Let m be the order of P. We decompose P into the form

$$\sum_{\alpha\in\boldsymbol{Z}^N}P_\alpha, \tag{19}$$

where P_α is of degree α_l with respect to w_l (by counting $\partial/\partial w_l$ to be of degree -1). Then P_α has the form

$$(\prod_{\alpha_l>0} w_l^{\alpha_l}) P_\alpha^0 (\prod_{\alpha_l<0} (\partial/\partial w_l)^{-\alpha_l}), \tag{20}$$

where P_α^0 is of degree 0 in each w_l. It is clear that $P_\alpha u=0$ and hence we find $P_\alpha^0 u=0$. Since P_α^0 has the form $P_\alpha^0\left(w_1\frac{\partial}{\partial w_1}, \cdots, w_N\frac{\partial}{\partial w_N}, x, D_x\right)$, we can define $\tilde{P}_\alpha^0 \in \mathcal{D}[s]$ by

$$\tilde{P}_\alpha^0(s_1, \cdots, s_N, x, D_x). \tag{21}$$

Then we have

$$\sigma(P_\alpha^0)(w, x, \tau, \xi) = \tilde{\sigma}(\tilde{P}_\alpha^0)(w_1\tau_1, \cdots, w_N\tau_N, x, \xi). \tag{22}$$

On the other hand, it follows from the definition of P_α that

$$\sigma_m(P) = \sum_\alpha (\prod_{\alpha_l>0} w_l^{\alpha_l}) \sigma(P_\alpha^0) (\prod_{\alpha_l<0} \tau_l^{-\alpha_l}). \tag{23}$$

This proves the required inclusion relation. Q.E.D.

§ 4. Before giving the proof of Lemma 4, we prepare several auxiliary results. They have some interest in their own right.

Proposition 5. *Let $\mathcal{M}$ be a (non-zero) coherent $\mathcal{E}_X$-Module*[*]*, and let f_j and g_j $(j=1, \cdots, l)$ be endomorphisms of $\mathcal{M}$. Suppose the following commutation relations hold:*

$$\begin{cases} [f_j, f_k] = [g_j, g_k] = 0 \\ [f_j, g_k] = \delta_{jk} \qquad \text{for } j, k=1, \cdots, l. \end{cases} \tag{24}$$

Then we have

$$\text{codim Supp } \mathcal{M} + l \leqq n = \dim X.$$

Corollary 6. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-Module and let s_j and t_j $(j=1, \cdots, l)$ be endomorphisms of $\mathcal{M}$. Suppose the following conditions hold:*

$$t_j : \mathcal{M} \to \mathcal{M} \text{ is injective } (j=1, \cdots, l) \tag{25}$$

$$\begin{cases} [t_j, t_k] = [s_j, s_k] = 0 \\ [t_j, s_k] = \delta_{jk} t_j \qquad (j, k=1, \cdots, l) \end{cases} \tag{26}$$

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[*] $\mathcal{E}_X$ denotes the sheaf of micro-differential operators of finite order.

Then we have

$$\text{codim Supp } \mathcal{M}+l \leqq n=\dim X.$$

Proof of Corollary 6.

By the additive property of the multiplicity of coherent $\mathcal{D}$-Modules, the multiplicity of the cokernel of $t_j: \mathcal{M}\rightarrow\mathcal{M}$ along an irreducible component of the characteristic variety of $\mathcal{M}$ is the difference of the multiplicity of $\mathcal{M}$ and the same one. Hence the characteristic variety of the cokernel does not contain any irreducible components of the characteristic variety of $\mathcal{M}$. Therefore, we might assume from the first that t_j are isomorphisms. Then, $f_j=t_j$ and $g_j=t_j^{-1}s_j$ satisfy the commutation relations (24) and this result immediately follows from Proposition 5.

Proof of Proposition 5.

If $\mathcal{M}$ is holonomic, then $\mathrm{E}=\mathrm{End}(\mathcal{M})$ is a finite-dimensional vector space and hence $\mathrm{tr}\ 1=\mathrm{tr}[f_1, g_1]=0$, which is a contradiction if $l\geqq 1$. Thus the theorem holds in this case.

Let us assume that $\mathcal{M}$ is not holonomic. Let V be the support of $\mathcal{M}$. It is enough to prove the theorem at a generic point of V. Therefore, by a quantized contact transformation, we may assume that $V=\{(x, \xi); \xi_1=\cdots=\xi_d=0\}$ with $d=\mathrm{codim}\ V$. Set $Y=\{x\in X; x_1=\cdots=x_d=0\}$ and $\mathcal{N}=\mathcal{M}|_Y$. Then $\mathrm{Supp}\ \mathcal{N}=T^*Y$, and $\mathcal{N}$ has the endomorphisms $\tilde{f}_j$ and $\tilde{g}_j$ induced from f_j and g_j, respectively. The endomorphisms $\tilde{f}_j$ and $\tilde{g}_j$ satisfy also the same commutation relations (17). Hence, by replacing $\mathcal{M}$ with $\mathcal{N}$, we might assume from the first that $\mathrm{Supp}\ \mathcal{M}=T^*X$.

At a generic point of T^*X, $\mathcal{M}$ is a free $\mathcal{E}_X$-Module. Hence we can represent f_j and g_j by matrices of micro-differential operators. Thus the proposition immediately follows from the following

Proposition 7. *Let $P_1, \cdots, P_l, Q_1, \cdots, Q_l$ be $N\times N$ matrices of micro-differential operators on $\boldsymbol{C}^n$, where N is an integer. If the relations*

$$(27)\qquad \begin{cases}[P_j, P_k]=[Q_j, Q_k]=0 \\ [P_j, Q_k]=\delta_{jk} \qquad \text{for} \quad j, k=1, \cdots, l\end{cases}$$

hold, then $l \leqq n$.

In order to prove this proposition, we prepare the following lemma.

Lemma 8. *Let* $P=(P_{jk})_{j,k=1,\cdots,N}$ *be an* $N\times N$ *matrix of micro-differential operators of order* $\leqq m$. *Assume that the eigenvalues of the matrix* $\bar{P}=(\sigma_m(P_{jk}))_{j,k=1,\cdots,N}$ *are mutually different and do not vanish. Then there is an invertible matrix* A *of micro-differential operators defined at a generic point of* T^*X, *such that* APA^{-1} *is a diagonal matrix.*

Proof. Consider $\lambda D_t^m - P$ as a micro-differential operator defined on $\boldsymbol{C}_\lambda \times \boldsymbol{C}_t \times X$.

Then $\det(\sigma(\lambda D_t^m - P)) = \det(\lambda \tau^m - \bar{P})$. Hence, if one denotes by p_j the eigenvalues of $\bar{P}$, then $\lambda D_t^m - P$ is invertible provided $\lambda \neq p_j \tau^{-m}$.

We shall define A_j by

$$A_j = \frac{1}{2\pi\sqrt{-1}} \oint \frac{D_t^m}{\lambda D_t^m - P} d\lambda ,$$

where the integral is a contour integral along a path around $p_j \tau^{-m}$. It is easy to see that $A_j A_k = \delta_{jk} A_j$ and $1 = \sum_j A_j$ hold. Since A_j commutes with t and D_t, A_j is a matrix of micro-differential operators on X. Setting $\mathcal{L} = \mathcal{E}^N$, we regard P and A_j as endomorphisms of $\mathcal{L}$. Set $\mathcal{L}_j = A_j \mathcal{L}$. Since P is invertible, $\mathcal{L} = \bigoplus_{j=1}^{N} \mathcal{L}_j$. Since A_j is a matrix of micro-differential operators of order $\leqq m$ and $\sigma_m(A_j)$ is a projector onto the eigenspace for p_j, $\mathcal{L}_j$ is not zero. Hence $\mathcal{L}_j$ is with multiplicity 1. Thus $\mathcal{L}_j$ is isomorphic to $\mathcal{E}$ at a generic point with a base u_j. If we take an invertible matrix A corresponding to the base $u_1, \cdots, u_N$, then APA^{-1} is a diagonal matrix. Q.E.D.

Let us return to the proof of Proposition 7.

Let A be an invertible constant matrix whose eigenvalues are mutually different. For a sufficiently large integer N, we set

$$\tilde{P}_j = P_j + AD_t^N \quad \text{and} \quad \tilde{Q}_j = Q_j \quad \text{for } j=1, \cdots, l$$

$$\tilde{P}_{l+1} = AD_t^N$$

$$\tilde{Q}_{l+1}=\frac{1}{N}A^{-1}tD_t^{1-N}-\sum_{j=1}^{l}Q_j\,.$$

We shall regard $\tilde{P}_j$ and $\tilde{Q}_j$ as micro-differential operators on $\boldsymbol{C}\times X$. Then $\tilde{P}_1$ satisfies the condition in Lemma 8. Hence it is diagonalizable by an inner automorphism. Thus, replacing X with $\boldsymbol{C}\times X$ and P_j and Q_j with $\tilde{P}_j$ and $\tilde{Q}_j$, respectively, we might assume from the first that

(28) P_1 is a diagonal matrix with diagonal components $A_1, \cdots, A_N$.

(29) $A_1, \cdots, A_N$ are micro-differential operators of order m and $\sigma_m(A_j)$ are mutually different.

Here we note the following

Sublemma 9. *Let R be an $N\times N$ matrix of micro-differential operators such that $[P_1, R]$ is a diagonal matrix. Then R is also diagonal.*

Proof. Let $\{R_{jk}\}$ be components of R. Then $A_jR_{jk}=R_{jk}A_k$ for $j\neq k$. If $R_{jk}\neq 0$, then $\sigma(A_j)\sigma(R_{jk})=\sigma(R_{jk})\sigma(A_k)$. This is a contradiction. Q.E.D.

Now we resume proving Proposition 7. It follows from Sublemma 9 that all P_j and Q_j are diagonal matrices. Set $m_j=\operatorname{ord}P_j$ and $n_j=\operatorname{ord}Q_j$ $(j=1, \cdots, l)$. Then $m_j+n_j\geqq 1$. We shall prove the proposition by the induction on $\sum_{j=1}^{l}(m_j+n_j-1)$.

If $\sum_j(m_j+n_j-1)=0$, then $m_j+n_j=1$ for every j. Hence we obtain

$$\begin{cases}\{\sigma(P_j),\sigma(R_k)\}=\{\sigma(Q_j),\sigma(Q_k)\}=0\\ \{\sigma(P_j),\sigma(Q_k)\}=\delta_{jk} \qquad \text{for } j,k=1,\cdots,l.\end{cases}$$

Then, as is well-known in symplectic geometry, we have $l\leqq n$.

Next suppose that $\sum_j(m_j+n_j-1)>0$. Then either $m_j\geqq 1$ or $n_j\geqq 1$ holds for each j. If $n_j\geqq 1$, by replacing P_j and Q_j with $-Q_j$ and P_j, respectively, we might assume from the first that $m_j\geqq 1$. Set $\tilde{P}_j=P_j^{1/m_j}$ and $\tilde{Q}_j=m_jP_j^{1-1/m_j}Q_j$. Then it is easy to see that $\tilde{P}_j$ and $\tilde{Q}_j$ satisfy the same commutation relations as P_j and Q_j. Furthermore we have

$\text{ord}\,\tilde{P}_j=1$ and $\text{ord}\,\tilde{P}_j+\text{ord}\,\tilde{Q}_j=m_j+n_j$. Hence we may assume without loss of generality that $\text{ord}\,P_j=1$ $(j=1,\cdots,l)$ and $n_1\geqq n_2\geqq\cdots\geqq n_l\geqq 0$. Set $f_j=\sigma(P_j)$ $(j=1,\cdots,l)$. Suppose first that $df_1,\cdots,df_l$ are linearly independent. Since $\{f_j,f_k\}=0$ $(1\leqq j,k\leqq l)$, it is well-known in symplectic geometry that $l\leqq n$ holds in this case. Therefore we may assume that there exists r $(1\leqq r<l)$ such that $df_1,\cdots,df_r$ are linearly independent and $df_{r+1}\equiv 0 \mod df_1,\cdots,df_r$. This means that f_{r+1} is a function of $f_1,\cdots,f_r$, namely, there exists a homogeneous function $\varphi(t_1,\cdots,t_r)$ of degree 1 such that $f_{r+1}=\varphi(f_1,\cdots,f_r)$. Denote $\frac{\partial}{\partial t_j}\varphi(t_1,\cdots,t_r)$ by φ_j. Set $\tilde{P}_{r+1}=P_{r+1}-\varphi(P_1,\cdots,P_r)$, $\tilde{P}_j=P_j$ $(j\neq r+1)$, $\tilde{Q}_j=Q_j$ $(j>r)$ and $\tilde{Q}_j=Q_j+\varphi_j(P_1,\cdots,P_r)Q_{r+1}$ $(j\leqq r)$. Then it is easy to see that

$$\begin{cases}[\tilde{P}_j,\tilde{P}_k]=[\tilde{Q}_j,\tilde{Q}_k]=0\\ [\tilde{P}_j,\tilde{Q}_k]=\delta_{jk}.\end{cases}$$

Furthermore we have $\text{ord}\,\tilde{Q}_j\leqq n_j$, $\text{ord}\,\tilde{P}_j\leqq 1$ and $\text{ord}\,\tilde{P}_{r+1}<0$. Hence the induction proceeds and we conclude $l\leqq n$. This completes the proof of Proposition 7, hence also that of Proposition 5.

The second result we need for the proof of Lemma 4 is the following

Proposition 10. *Let* $f: X\to Y$ *be a proper map of complex manifolds. Let* V *be an involutory variety (possibly with singularities) of* T^*X *whose dimension is at most* $\dim X+l$ *at any point. Then the dimension of* $\varpi\rho^{-1}(V)$ *is at most* $\dim Y+l$ *at any point in* $\varpi\rho^{-1}(V)$. *Here* ϖ *(resp.* ρ*) is the canonical projection from* $X\underset{Y}{\times}T^*Y$ *to* T^*Y *(resp.* T^*X*).*

Proof. Set $W=\varpi\rho^{-1}(V)$. The question being local on W, we can choose a point p in $\rho^{-1}(V)$ such that W is non-singular at $\varpi(p)$, $\rho^{-1}(V)$ is non-singular at p and the projection $\rho^{-1}(V)\to W$ is smooth at p. Moreover it is enough to show the statement in a neighborhood of $\varpi(p)$. Let ω_X and ω_Y be the canonical 1-forms on T^*X and T^*Y. Then $\varpi^*(\omega_Y)=\rho^*(\omega_X)$.

Let $\varphi: V'\to V$ be a desingularization of V ([1], [5]). Then there exist a manifold U, $\psi: U\to\rho^{-1}(V)$ and $\phi: U\to V'$ such that $\varphi\phi=\rho\psi$ and that ψ is generically surjective.

$$
\text{(30)}\qquad
\begin{array}{ccc}
V' & \xrightarrow{\ \varphi\ } & V \\
{\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\rho} \\
U & \xrightarrow{\ \psi\ } & \rho^{-1}(V) \\
 & & {\scriptstyle\varpi}\downarrow \\
 & & W
\end{array}
$$

At a generic point of V, we have $(d\omega_X|_V)^{l+1}=0$, because V is involutory. Hence $(d(\phi^*\varphi^*\omega_X))^{l+1}=(\psi^*\rho^*d\omega_X)^{l+1}=(\psi^*\varpi^*d\omega_Y)^{l+1}$ vanishes. Since $\psi\varpi$ is generically surjective, $(d\omega_Y|_W)^{l+1}=0$. This implies immediately that the dimension of W is at most $\dim Y+l$ at any point in W. Q.E.D.

We shall further prove some related results in contact geometry, which will be used later.

Lemma 11. *Let Y be a submanifold of X (i.e., subvariety without singularities) and let V be an involutory variety in T^*X. Assume that V is non-characteristic with respect to Y, namely $\rho|_{Y\times_X V}: Y\times_X V\to T^*Y$ is a finite map. Then $W\equiv\rho(Y\times_X V)$ is an involutory subvariety of T^*Y.*

Proof. We denote by n (resp. l) $\dim X$ (resp. $\operatorname{codim}_{T^*X}V$). We may assume without loss of generality that $\operatorname{codim}_X Y=1$. Then $\operatorname{codim}_{T^*X}(Y\times_X V)=l+1$ and $\dim W=2(n-1)-(l-1)$. As in the proof of Proposition 10, we have

$$
(d\omega_Y|_W)^{n-l+1}=0\,, \tag{31}
$$

since $(d\omega_X|_V)^{n-l+1}=0$.

Let a be a generic point in W. Then by a suitable contact transformation we may assume that

$$
W=\{(x,\xi)\in T^*Y;\ x_1=\cdots=x_p=0,\ \xi_1=\cdots=\xi_q=0\}
$$

in a neighborhood of a. Here

(32) $$p+q=l-1 \ (=\operatorname{codim}_{T^*Y} W).$$

Then we have

(33) $$(d\omega_Y|_W)^{n-1-\max(p,q)} \neq 0 .$$

Comparing (31) and (33), we conclude

(34) $$n-l+1 > n-1-\max(p, q).$$

Combining (32) and (34), we have $p=0$ or $q=0$. This is equivalent to saying that W is involutory. Q.E.D.

Proposition 12. *Let V be an involutory variety in T^*X. Let $f(x, \xi)$ be a holomorphic function which is homogeneous in ξ. Assume that $df \neq 0$ on $\{f=0\}$. Assume furthermore that $V_0 = \{f=0\} \cap V$ is invariant under the Hamiltonian vector field*

$$H_f = \sum_j \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$$

associated with f. Then V_0 is an involutory variety.

Proof. By adding a dummy variable, we may assume without loss of generality that $df \wedge \omega_X \neq 0$ and f is homogeneous of degree 1 with respect to ξ. Hence by a suitable contact transformation we may assume that $f = \xi_1$. We may also assume that V is irreducible. If $V \subset f^{-1}(0)$, then there is nothing to prove. Therefore we may assume that V_0 is a hypersurface of V.

In order to prove the proposition, it is enough to discuss at a generic point of V_0. Hence, we may assume that V_0 and $V - V_0$ are non-singular and $(V - V_0, V_0)$ satisfies the condition of Whitney. Suppose that V_0 is not involutory. Then there is a point p in V_0 such that $T_p V_0$ is not involutory. Let us take a sequence $\{p_n\}$ in $V - V_0$ converging to p such that $T_{p_n} V$ converges to a linear subspace τ. The space τ is involutory and $T_p V_0$ is a hyperplane of τ. Next we show that $\tau \subset df^{-1}(0)$. If $\tau \not\subset df^{-1}(0)$, then $T_p V_0 = \tau \cap df^{-1}(0)$. Therefore $(T_p V_0)^\perp = (\tau \cap df^{-1}(0))^\perp = \tau^\perp + \boldsymbol{C} H_f$. Since $\tau^\perp \subset \tau$ and $H_f \in T_p V_0 \subset \tau$ we have $\tau^\perp + \boldsymbol{C} H_f \subset T_p V_0$, which contradicts the fact that $T_p V_0$ is not involutory. Thus we have seen τ is contained in $df^{-1}(0)$. Since we have assumed $f=\xi_1$, this implies that V is non-characteristic with respect to $Y=\{x_1=0\}$.

On the other hand, we may assume that V_0 has the form $\{(x, \xi); \xi_1=\xi_2=\cdots=\xi_\nu=0,\ x_2=\cdots=x_\mu=0\}$ at a generic point of V_0, because V_0 is invariant by H_{ξ_1}. Then we have

$$\rho(Y\underset{X}{\times}V)\supset\{(x_2,\cdots,x_n;\xi_2,\cdots,\xi_n)\in T^*Y;\ \xi_2=\cdots=\xi_\nu=0,\ x_2=\cdots=x_\mu=0\}. \tag{35}$$

Since $\operatorname{codim}_{T^*Y}\rho(Y\underset{X}{\times}V)=(\nu-1)+(\mu-1)$, the left hand side of (35) must coincide with the right hand side of (35).

On the other hand, the preceding lemma asserts that $\rho(Y\underset{X}{\times}V)$ is involutory. This means that $\nu=1$ or $\mu=1$.

Thus we have proved that V_0 is involutory. Q.E.D.

§ **5.** Now we embark on the proof of Lemma 4.

By the desingularization theorem of Hironaka ([1], [5]) we can find an analytic manifold X', a projective map F from X' to X and a proper subvariety Z of Y so that the following conditions (36) ~ (39) are satisfied.

$$Y_{\text{sing}}\subset Z\subset\{x\in Y;\ \prod_{l=1}^{N}f_l(x)=0\} \tag{36}$$

(37) $Z'=F^{-1}(Z)$ is a normally crossing hypersurface and $F|_{X'-Z'}$ is an isomorphism from $X'-Z'$ onto $X-Z$.

(38) The proper transform $Y'=\overline{F^{-1}(Y-Z)}$ is a non-singular subvariety of X'.

(39) At any point a in Y' we can find a local coordinate system $(v, y, z)\equiv(v_1,\cdots,v_d,y_1,\cdots,y_m,z_1,\cdots,z_k)$ $(d+m+k=n)$ so that the following conditions are satisfied:

(39. a) $v_1(a)=\cdots=v_d(a)=y_1(a)=\cdots=y_m(a)=z_1(a)=\cdots=z_k(a)=0$

(39. b) $Y'=\{(v, y, z);\ v_1=\cdots=v_d=0\}$

(39. c) $Z'\subset\{(v, y, z);\ \prod_{p=1}^{m}y_p\prod_{q=1}^{k}z_q=0\}$

(39. d) $\varphi'_j=\varphi_j\circ F=\psi_j(v, y, z)v_j\prod_{p=1}^{m} y_p^{\rho'_{j,p}}\prod_{q=1}^{k} z_q^{\rho_{j,q}}$

(39. e) $f'_l|_{Y'}=f_l\circ F|_{Y'}=\chi_l(y, z)\prod_{q=1}^{k} z_q^{r_{l,q}}$,

where $\rho_{j,q}$, $\rho'_{j,p}$ and $r_{l,q}$ are non-negative integers, $(r_{1,q}, \cdots, r_{N,q})\neq 0$ for each $q=1, \cdots, k$ and ψ_j and χ_l are non-vanishing holomorphic functions.

As a matter of fact, we have

(39. c′) $Y'\cap Z'\subset\{(v, y, z); v=0, \prod_{q=1}^{k} z_q=0\}$,

because $Z\subset f_1^{-1}(0)\cup\cdots\cup f_N^{-1}(0)$. We also find that $\rho'_{j,p}$ is actually zero, because $\{d\varphi'_1, \cdots, d\varphi'_d\}$ are linearly independent at any point in $Y'-Z'$.

We define $\tilde{X}'$ (resp. $\tilde{Y}'$) by $\boldsymbol{C}^N\times X'$ (resp. $\boldsymbol{C}^N\times Y'$) and extend F to $\tilde{F}:\tilde{X}'\to\tilde{X}$ $(=\boldsymbol{C}^N\times X)$. We denote $(w_1f_1', \cdots, w_Nf'_N)$ by $\tilde{f}'=(\tilde{f}_1', \cdots, \tilde{f}'_N)$. Then we have the following

Sublemma 13. *We have*

$$W_{\tilde{f}',\tilde{Y}'}=\Big\{(w, v, y, z; \tau, \sigma, \eta, \zeta)\in T^*\tilde{X}';\ v=0,\ \eta_p=\sum_{l=1}^{N} w_l\tau_l\frac{\partial}{\partial y_p}\log\chi_l,\ z_q\gamma_q=\sum_{l=1}^{N} r_{l,q}w_l\tau_l,\ \text{where}\ \gamma_q=\zeta_q-\sum_{l=1}^{N} w_l\tau_l\frac{\partial}{\partial z_q}\log\chi_l\Big\} \tag{40}$$

in a neighborhood $\tilde{a}=(0, a)\in\tilde{Y}'$.

Proof. Since the right hand side of (40) is a closed subset, it suffices to show that the following set $\mathring{W}_{\tilde{f}',\tilde{Y}'}$ is contained in the right hand side and that any point in the right hand side is reached by a sequence of points in $\mathring{W}_{\tilde{f}',\tilde{Y}'}$.

$$\mathring{W}_{\tilde{f}',\tilde{Y}'}=\{(w, v, y, z; \tau, \sigma, \eta, \zeta)\in T^*\tilde{X}'; v=0,\ \tilde{f}'_l\neq 0\ (l=1, \cdots, N),\ (\eta, \zeta)=\sum_{l=1}^{N} w_l\tau_l\,\mathrm{grad}_{(y,z)}\log\tilde{f}'_l\ (l=1, \cdots, N)\}. \tag{41}$$

Recall that $W_{\tilde{f}',\tilde{Y}'}$ is the closure of $\mathring{W}_{\tilde{f}',\tilde{Y}'}$ by the definition. Clearly $\mathring{W}_{\tilde{f}',\tilde{Y}'}$ coincides with the right hand side of (40) in a neighborhood of the points where $\prod_{q=1}^{k} z_q \neq 0$. It is also clear that $\mathring{W}_{\tilde{f}',\tilde{Y}'}$ is contained in the right hand side of (40). Hence it remains to be proved only that any point $b=(w, v, y, z; \tau, \sigma, \eta, \zeta)$ in the right hand side of (40) where $\prod_{q=1}^{k} z_q=0$ can be reached by a sequence of points in $\mathring{W}_{\tilde{f}',\tilde{Y}'}$. Assume that $z_1=\cdots=z_i=0$ and $z_{i+1}\neq 0, \cdots, z_k\neq 0$ hold at b. Here we may assume without loss of generality that $\gamma_q \neq 0$ $(q=1, \cdots, i)$ at b. Since $(r_{1,q}, \cdots, r_{N,q})\neq 0$ for any q, we can choose sequences $w_l^{(m)}\to w_l$ and $\tau_l^{(m)}\to\tau_l$ $(l=1, \cdots, N)$ so that

$$\sum_{l=1}^{N} r_{l,q} w_l^{(m)} \tau_l^{(m)} \neq 0 \quad (q=1, \cdots, i). \tag{42}$$

We also define sequences $\gamma_q^{(m)}$ and $z_q^{(m)}$ by the following:

$$\begin{cases} \gamma_q^{(m)}=\gamma_q \quad (q=1, \cdots, i) \\ \gamma_q^{(m)}=\dfrac{1}{z_q}\displaystyle\sum_{l=1}^{N} r_{l,q} w_l^{(m)} \tau_l^{(m)} \quad (q=i+1, \cdots, k) \\ z_q^{(m)}=\dfrac{1}{\gamma_q}\displaystyle\sum_{l=1}^{N} r_{l,q} w_l^{(m)} \tau_l^{(m)} \quad (q=1, \cdots, i) \\ z_q^{(m)}=z_q \quad (q=i+1, \cdots, k) \end{cases} \tag{43}$$

Then it is clear that $(w^{(m)}, v, y, z^{(m)}; \tau^{(m)}, \sigma, \eta, \gamma^{(m)}+\sum_{l=1}^{N} w_l^{(m)}\tau_l^{(m)} \frac{\partial}{\partial z_q}\log \chi_l)$ is a required limiting sequence which converges to b. Q.E.D.

By using this result we shall prove

$$W_{\tilde{f}',\tilde{Y}'} \underset{\tilde{X}}{\times} (\boldsymbol{C}^N \times Z) \text{ is an involutory variety.} \tag{44}$$

It follows from (39. c′) that $Y' \cap Z'$ is a union of hypersurfaces $\{(z, y, z);\ v=0,\ z_q=0\}$ of Y'. Hence it suffices to prove that $W_{\tilde{f}',\tilde{Y}'} \cap \{z_1=0\}$ is involutory. Since $W_{\tilde{f}',\tilde{Y}'}$ is involutory and $W_{\tilde{f}',\tilde{Y}'}\cap\{z_1=0\}$ is invariant by $H_{z_1}=-\partial/\partial\zeta_1$, this set is involutory by Proposition 12.

Now let us consider $\mathscr{D}_{\tilde{X}'}$-Module $\mathscr{N}_{\tilde{f}',\varphi'}$. Then by the aid of Sublemma 13 we find the following

Lemma 14. $SS(\mathscr{N}_{\tilde{f}',\varphi'})=W_{\tilde{f}',\tilde{Y}'}$ *holds.*

Proof of Lemma 14. Since $SS(\mathscr{N}_{\tilde{f}',\varphi'})$ clearly contains $W_{\tilde{f}',\tilde{Y}'}$, it suffices to prove the opposite inclusion relation. Let $\tilde{u}'$ be the generator

$$\prod_{j=1}^{d}\delta(\varphi_j')\prod_{l=1}^{N}\tilde{f}_l'^{s_l}=\prod_{l=1}^{N}(\chi_l(y,z)w_l\prod_{q=1}^{k}z_q^{r_{l,q}})^{s_l})\prod_{j=1}^{d}\delta(\psi_j(v,y,z)v_j\prod_{q=1}^{k}z_q^{\rho_{j,q}})=$$

$$=\prod_{j=1}^{d}\delta(v_j)(\prod_{l=1}^{N}\chi_l(y,z)^{s_l}w_l^{s_l})(\prod_{j=1}^{d}\psi_j(0,y,z)^{-1})(\prod_{q=1}^{k}z_q^{(\sum_{l=1}^{N}r_{l,q}s_l-\sum_j\rho_{j,q})})$$

Therefore, $\tilde{u}'$ satisfies the differential equations

$$(45)\quad\begin{cases} v_j\tilde{u}'=0 & (j=1,\cdots,d)\\ \left(\dfrac{\partial}{\partial y_p}-\displaystyle\sum_{l=1}^{N}\Big(\dfrac{\partial}{\partial y_p}\log\chi_l\Big)w_l\dfrac{\partial}{\partial w_l}\right)(\displaystyle\prod_{j=1}^{d}\psi_j(0,y,z)\Big)\tilde{u}'=0 & (p=1,\cdots,m)\\ \left[z_q\Big(\dfrac{\partial}{\partial z_q}-\displaystyle\sum_{l=1}^{N}\Big(\dfrac{\partial}{\partial z_q}\log\chi_l\Big)w_l\dfrac{\partial}{\partial w_l}\Big)-\displaystyle\sum_{l=1}^{N}r_{l,q}w_l\dfrac{\partial}{\partial w_l}+\sum_j\rho_{j,q}\right]\times & \\ \qquad\times\displaystyle\prod_{j=1}^{d}\psi_j(0,y,z)\tilde{u}'=0 & (q=1,\cdots,k). \end{cases}$$

Since the characteristic variety of $\mathcal{N}_{\tilde{f}',\varphi'}$ is contained in the common zeros of the principal symbols of the differential operators used in (45), it is contained in $W_{\tilde{f}',Y'}$. This completes the proof of the lemma.

Q.E.D.

Now we resume the proof of Lemma 4. Since $\tilde{F}$ is a projective map, $\mathcal{D}_{\tilde{X}}$-Module $\mathcal{N}'\equiv\int_{\tilde{F}^{-1}}\mathcal{N}_{\tilde{f}',\varphi'}$ is well-defined and coherent and its characteristic variety is contained in $\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'})$ (Theorem 4.2 of [6]). Let v be the section of $\mathcal{N}'$ corresponding to $1_{\tilde{X}\leftarrow\tilde{X}'}\otimes\prod_{l=1}^{N}\tilde{f}_l'^{s_l}\prod_{j=1}^{d}\delta(\varphi_j')$ and let $\mathcal{N}''$ be the sub $\mathcal{D}_{\tilde{X}}$-Module of $\mathcal{N}'$ generated by v. Then we can define a natural $\mathcal{D}_{\tilde{X}}$-linear surjective homomorphism from $\mathcal{N}''$ to $\mathcal{N}_{\tilde{f},\varphi}$ by assigning $\prod_{j=1}^{d}\delta(\varphi_j)\prod_{l=1}^{N}\tilde{f}_l^{s_l}$ to v. Since the characteristic variety of $\mathcal{N}''$ is contained in $\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'})$, we have

$$(46)\qquad SS(\mathcal{N}_{\tilde{f},\varphi})\subset\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'}).$$

On the other hand, we have

$$\begin{aligned}\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'})&=\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'}\underset{\tilde{X}'}{\times}(\tilde{X}'-\boldsymbol{C}^N\times Z'))\cup\\ &\cup\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'}\underset{\tilde{X}'}{\times}(\boldsymbol{C}^N\times Z'))\subset\\ &\subset W_{\tilde{f},\tilde{Y}}\cup\varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}}\underset{\tilde{X}'}{\times}(\boldsymbol{C}^N\times Z)),\end{aligned}$$

because $\tilde{F}$ is an isomorphism outside $\boldsymbol{C}^N\times Z$. Clearly

$$\dim W_{\tilde{f}',\tilde{Y}'}\underset{\tilde{X}}{\times}\boldsymbol{C}^N\times Z\leqq 2N+n-1 \tag{47}$$

Therefore Proposition 10 combined with (44) and (47) entails that

$$\dim \varpi\rho^{-1}(W_{\tilde{f}',\tilde{Y}'}\underset{\tilde{X}}{\times}(\boldsymbol{C}^N\times Z))\leqq 2N+n-1\,. \tag{48}$$

On the other hand, Corollary 6 claims that

$$\dim_p \mathrm{SS}(\mathcal{N}_{\tilde{f},\varphi})\geqq N+(n+N)=2N+n \tag{49}$$

holds at any point p of $\mathrm{SS}(\mathcal{N}_{\tilde{f},\varphi})$. Thus we finally conclude from (48) and (49) that

$$\mathrm{SS}(\mathcal{N}_{\tilde{f},\varphi})=W_{\tilde{f},Y}\,. \tag{50}$$

This completes the proof of Lemma 4, and, hence at the same time, that of Theorem 2.

§ **6.** In order to study the holonomic character of the distribution $\Phi(x;s)$ we further need the following geometric result.

Proposition 15. *The set W_0 is Lagrangian.*

Proof. We shall first show that W_0 is isotropic. Again by the desingularization theorem of Hironaka, we can find a complex manifold $\widetilde{W}$, a proper analytic subset and a proper surjective map G from $\widetilde{W}$ onto $W_{f,Y}$. Let F be the projection from $\boldsymbol{C}^N\times T^*X$ onto T^*X. Let $\omega_X=\sum\xi_j dx_j$ be the canonical 1-form on T^*X. Here we may assume the following:

$$\widetilde{W} \text{ is a closure of } G^{-1}(\{x;\prod_{l=1}^{N}f_l(x)\neq 0\}). \tag{51}$$

$$G^{-1}(W_0) \text{ is a hypersurface of } \widetilde{W}. \tag{52}$$

The question being local on W_0, it is enough to show that $\omega_X|_{W_0}=0$ at a generic point p of W_0. We may assume that W_0 is non-singular at p and that there exists a point p' in $G^{-1}(p)$ such that $G^{-1}(W_0)$ is non-singular at p' and $G^{-1}(W_0)\to W_0$ is smooth at p'. Let ψ be a defining function of $G^{-1}(W_0)$. Then, we have $f_l\circ G=\theta_l\psi^{\nu_l}$ and $\sigma_l\circ G=\chi_l\psi^{k_l}$, where θ_l and χ_l are holomorphic functions on W defined in a neighborhood of p such that their restrictions onto $G^{-1}(W_0)$ do not vanish identically. Here we have $k_l \geqq 1$. Then, we have

$$(FG)^*\omega_X = \sum_{l=1}^{N} (\sigma_l \circ G)\, d(\log f_l \circ G)$$

outside

$$\{x; \prod_{l=1}^{N} f_l(x) \neq 0\}.$$

Therefore we have

$$(FG)^*\omega_X = \sum_{l=1}^{N} \nu_l \chi_l \psi^{k_l - 1} d\psi + \sum_{l=1}^{N} \chi_l \psi^{k_l} d \log \theta_l$$

in a neighborhood of p'. Thus $(FG)^*\omega_X|_{W_0}$ vanishes at generic points and hence vanishes in a neighborhood of p'. Since $G^{-1}(W_0) \to W_0$ is smooth at p', $\omega_X|_{W_0}$ vanishes at p. Therefore W_0 is isotropic, and hence $\dim W_0 \leqq n$.

On the other hand, W_0 is defined as the common zeros of N functions on $W_{f,Y}$, and hence we have $\dim W_0 \geqq n$. Thus $\dim W_0 = n$, and this implies that W_0 is Lagrangian. Q.E.D.

Here we note the following interesting property of W_0 (Theorem 16), even though we do not need it in our subsequent discussions of this paper.

In stating the theorem we use the following notation:

For a set of complex numbers $a = (a_1, \cdots, a_N)$, $W(a)$ denotes the closure of $\{(\tau, x, \xi) \in \boldsymbol{C} \times T^*X; x \in Y - Y_{\text{sing}}, \prod_{l=1}^{N} f_l(x) \neq 0$ and $\xi = \sum_{j=1}^{d} c_j d\varphi_j(x) + {} + \tau \sum_{l=1}^{N} a_l \tau d \log f_l(x)$ for some complex numbers c_j and $\tau\}$ and $W_0(a)$ denotes the intersection of $W(a)$ and $\tau^{-1}(0)$. We identify $W_0(a)$ with the subset of T^*X.

Theorem 16. *The set $W_0(a)$ coincides with W_0, if $\sum_{l=1}^{N} a_l \nu_l \neq 0$ holds for $\nu_l \in \boldsymbol{Z}_+ = \{0, 1, 2, \cdots\}$ with $(\nu_1, \cdots, \nu_N) \neq (0, \cdots, 0)$.*

Proof. At least one of a_l does not vanish, say a_1. We can normalize a_1

to be 1. Let W' be the subvariety of $W_{f,\Gamma}$ defined by $\sigma_l=a_l\sigma_1$ ($l=1, \cdots, N$). Let E be the subvariety of $W_{f,\Gamma}$ defined by $\prod_{l=1}^{N} f_l(x)=0$. It is enough to show $\overline{W'-E}\supset W_0$.

We shall prove it by the reduction to absurdity.

If this were not true, there should be a point p in W_0 and a neighborhood U of p such that $U\cap E\supset U\cap W'$. Again by the desingularization theorem of Hironaka, we can find a complex manifold $\widetilde{W}$ and a surjective proper map $G: \widetilde{W}\to W_{f,\Gamma}$ which satisfy the following conditions.

(53) $\widetilde{W}$ is the closure of $G^{-1}(W_{f,\Gamma}-E)$.

(54) $G^{-1}(W')$ is a hypersurface of $\widetilde{W}$.

Let ψ be the defining function of $G^{-1}(W')$. Then, at a generic point of $G^{-1}(U\cap W')$, we have

(55) $$(\sigma_l-a_l\sigma_1)\circ G=\chi_l\psi^{k_l} \quad (l=2, \cdots, N)$$

(56) $$f_l\circ G=\theta_l\psi^{\nu_l} \quad (l=1, \cdots, N),$$

where χ_l and θ_l are non-vanishing and $k_l\geqq 1$ and $\nu_l\geqq 0$. Since we have supposed $\prod_{l=1}^{N} f_l(x)=0$ on W' in a neighborhood of p, we have also $\sum_{l=1}^{N}\nu_l\geqq 1$. Hence we have there

(57) $$\begin{aligned}\omega&=\sum_{l=1}^{N}(\sigma_l\circ G)\,d\log(f_l\circ G)\\&=(\sigma_1\circ G)\Big(\sum_{l=1}^{N}a_l d(\log f_l\circ G)\Big)+\sum_{l=2}^{N}((\sigma_l-a_l\sigma_1)\circ G)\,d\log(f_l\circ G)\\&=\Big(\sum_{l=1}^{N}a_l\nu_l\Big)\frac{(\sigma_1\circ G)}{\psi}d\psi+\text{(holomorphic form)}.\end{aligned}$$

Since ω is a holomorphic form, this is a contradiction if $\sum_{l=1}^{N}a_l\nu_l\neq 0$.

Q.E.D.

After proving these preparatory results, it is now easy to study the holonomic character of $\gamma(\lambda)\Phi(x;\lambda)$. Here $\gamma(\lambda)$ denotes the γ-factor introduced in Theorem 1. First we define the coherent $\mathcal{D}_X$-Module $\mathcal{N}_\lambda$ by

(58) $$\mathcal{N}_{f,\varphi}\Big/\Big(\sum_{l=1}^{N}(s_l-\lambda_l)\,\mathcal{N}_{f,\varphi}\Big)$$

for $\lambda=(\lambda_1,\cdots,\lambda_N)\in \boldsymbol{C}^N$. It immediately follows from the definition of $\mathcal{N}_\lambda$ that the distribution $\gamma(\lambda)\Phi(x;\lambda)$ satisfies the system of linear differential equations $\mathcal{N}_\lambda$.

The sheaf $\mathcal{D}_X[s]$ is contained in $\mathcal{D}_{X\times \boldsymbol{C}^N}$ if we regard s_j as the coordinate functions on $\boldsymbol{C}^N$. In the sequel we shall denote by $\tilde{\mathcal{N}}_{f,\varphi}$ the $\mathcal{D}_{X\times \boldsymbol{C}^N}$-Module

$$\mathcal{D}_{X\times \boldsymbol{C}^N}\underset{\mathcal{D}_X[s]}{\otimes}\mathcal{N}_{f,\varphi}=\mathcal{D}_{X\times \boldsymbol{C}^N}/\mathcal{D}_{X\times \boldsymbol{C}^N}\mathcal{J}\,.$$

Then we have the following

Theorem 17. *(a) The characteristic variety of $\mathcal{N}_\lambda$ is contained in W_0 (hence $\mathcal{N}_\lambda$ is a holonomic system).*

(b) The characteristic variety of $\tilde{\mathcal{N}}_{f,\varphi}$ is contained in

$$\{(s,x;\xi,\sigma)\in T^*(X\times \boldsymbol{C}^N)\,;\ (x,\xi)\in W_0\}.$$

Proof. Let $a(x,\xi)$ be a holomorphic function on T^*X vanishing on W_0 and homogeneous of degree r in ξ. Then, by Hilbert's Nullstellensatz, there exist an integer m and holomorphic functions $g_l(\sigma,x,\xi)$ which are homogeneous of degree $rm-1$ in (σ,ξ) such that

$$a(x,\xi)^m=\sum_{l=1}^N \sigma_l g_l(\sigma,x,\xi)\ \text{ on } W_{f,Y}\,.$$

By Theorem 2 we can find an integer m' and $P(s)\in\mathcal{J}_{mm'r}$ such that

$$\tilde{\sigma}_{mm'r}(P(s))=(a(x,\xi)^m-\sum_{l=1}^N \sigma_l g_l(\sigma,x,\xi))^{m'}.$$

Therefore, if we regard $P(s)$ as a differential operator on $\boldsymbol{C}^N\times X$ (or if we substitute λ into s), then the principal symbol of $P(s)$ is $a(x,\xi)^{mm'}$. This proves the desired results. Q.E.D.

§ **7.** Concerning the singularity spectrum of $\gamma(s)\Phi(x;s)$ (regarded as a distribution in (x,s) which depends holomorphically on s), Theorem 17(b) immediately gives the following Theorem 18. (See Theorem 2.1.1 of [10] Chap. III. § 2.1. Note also that a distribution on M with holomorphic parameters $s\in \boldsymbol{C}^N$ is a distribution on $M\times \boldsymbol{C}^N$ which satisfies the Cauchy-Riemann equations in s.)

Theorem 18. *The singularity spectrum of $\gamma(s)\Phi(x;s)$ is confined to*

$$\{(x,s;\sqrt{-1}(\langle\xi,dx\rangle+2\operatorname{Re}\langle\tau,ds\rangle)\infty)\in\sqrt{-1}S^*(M\times\boldsymbol{C}^N);\ \tau=0,\ (x,\xi)\in W_0\}.$$

However, the conclusion of this theorem is not the best possible one of the sort. One of the typical examples which manifest this fact is given by considering the case where

$$\sum_{l=1}^{N}\alpha_l\operatorname{grad}_x f_l(x)\quad(\alpha_l\geqq 0,\ \sum_{l=1}^{N}\alpha_l=1)$$

is contained in a proper convex cone. In order to take into account such phenomena, we introduced the following set $W_0'(+)$ in [8]. Unfortunately, our proof for the claim announced in Lemma 2 of [8] turns out to be incomplete.(*) Still we believe that the claim itself should be true. Hence we feel it worth while presenting here as a conjecture.

Conjecture. *The singularity spectrum of $\gamma(\lambda)\Phi(x;\lambda)$ (considered as a distribution in x) is contained in the following set $W_0'(+)$, if $Y_{\text{sing}}=\phi$.*

$W_0'(+)=\{(x,\sqrt{-1}\xi\infty)\in\sqrt{-1}S^*M$; *there exist a sequence $x_m\in X$ such that $\varphi_j(x_m)=0$ $(j=1,\cdots,d)$ and that converges to x with $\prod_{l=1}^{N}f_l(x)=0$ and sequences $\alpha_m=(\alpha_1^{(m)},\cdots,\alpha_N^{(m)})\in(\boldsymbol{R}^+)^N$ and $\beta_m=(\beta_1^{(m)},\cdots,\beta_d^{(m)})\in\boldsymbol{C}^d$ such that*

$$\alpha_l^{(m)}f_l(x_m)\to 0\quad(l=1,\cdots,N)\tag{59}$$

$$\sum_{l=1}^{N}\alpha_l^{(m)}\operatorname{grad}_x f_l(x_m)+\sum_{J=1}^{d}\beta_j^{(m)}\operatorname{grad}_{X\,j}\varphi(x_m)\to\eta.\}\tag{60}$$

The rest of this paper is devoted to the discussion on some examples which support this conjecture and indicate that it should be very difficult to improve it much. In the study of the examples given below, we concentrate our attention on delicate points and leave the complete argument to the reader.

(*) One rather trivial case covered by our original argument is the case where $N=1$.

Example 1. Let Φ be $(tx+\sqrt{-1}\,0)^{\lambda}(ty+\sqrt{-1}\,0)^{\mu}(\lambda, \mu\gg 0)$. Let $(t, x, y; \sqrt{-1}(\tau, \xi, \eta)\infty)$ denote a point in $\sqrt{-1}\,S^*\boldsymbol{R}^3$. At the origin, the prescription given by Theorem 18 allows the following set I as a possible singularity set over the origin for Φ.

$$I\equiv\{(0; \sqrt{-1}(\tau, \xi, \eta)\infty); (\tau, \xi, \eta)\neq 0\}. \tag{61}$$

On the other hand, the prescription given by the conjecture allows only the part I^+ of I where $\xi\eta\geqq 0$.

Actually, we find

$$\text{S.S.}\Phi\subset\{(t, x, y; \sqrt{-1}(\tau, \xi, \eta)\infty)\in\sqrt{-1}S^*\boldsymbol{R}^3; \xi\eta\geqq 0\}. \tag{62}$$

In order to see this, we first decompose Φ as follows:

$$\Phi=\Phi_+ +\Phi_-, \quad \text{where} \quad \Phi_+ = (tx+\sqrt{-1}\,0)^{\lambda}(ty+\sqrt{-1}\,0)^{\mu}Y(t) \tag{63}$$

$$\text{and} \quad \Phi_- = (tx+\sqrt{-1}\,0)^{\lambda}(ty+\sqrt{-1}\,0)^{\mu}Y(-t),$$

Here $Y(t)$ is the Heaviside function. Since Φ_+ (resp. Φ_-) is a hyperfunction which is holomorphic in $\{(x, y)\in\boldsymbol{C}^2; \operatorname{Im} x>0, \operatorname{Im} y>0\}$ (resp. $\{(x, y)\in\boldsymbol{C}^2; \operatorname{Im} x<0, \operatorname{Im} y<0\}$), we find by [10] Chap. I., § 3.2 (p. 308) that

$$\text{S.S.}\Phi_{\pm}\subset\{\xi\eta\geqq 0\}. \tag{64}$$

Hence we find (62). Thus in this case, the more delicate prescription based on the conjecture turns out to be a correct one.

Example 2. Let Φ be $(tx+z^2+\sqrt{-1}\,0)^{\lambda}(ty+z^2+\sqrt{-1}\,0)^{\mu}(\lambda, \mu\gg 1)$. Let $(t, x, y, z; \sqrt{-1}(\tau, \xi, \eta, \zeta)\infty)$ denote a point in $\sqrt{-1}\,S^*\boldsymbol{R}^4$. Again, at the origin the prescription based on the conjecture gives rise to an intriguing condition $\xi\eta\geqq 0$, which is not derived from the result stated in Theorem 18. In this case, again the prescription based on the conjecture is correct. To see this, we again decompose Φ into the sum $\Phi Y(t)+\Phi Y(-t)$. Then the same argument as in Example 1 succeeds.

From the viewpoint of applications to the problems in physics (e.g., [9], [12]), it would be more desirable if we could choose x_m to be real in the definition of $W_0'(+)$. The following Example 3 might give us a hope for such an improvement. Such a hope, however, is nullified by the subsequent Example 4. See also a very interesting paper [12] for related topics.

Example 3. Let Φ be $|ts|^{\lambda}(xt^2+ys^2)_+^{\mu}$ $(\lambda, \mu \gg 0)$.(*) Let $(x, y, t, s; \sqrt{-1}(\xi, \eta, \tau, \sigma)\infty)$ denote a point in $\sqrt{-1}\,S^*\boldsymbol{R}^4$. Then after some calculation we have

$$(65)\quad \int e^{i(x\xi+y\eta+t\tau+s\sigma)}\Phi\,dx\,dy\,dt\,ds$$

$$= \int e^{i(t\tau+s\sigma)}c(\mu)\,|ts|^{\lambda}\delta(\xi s^2-\eta t^2)\,(\xi+i0)^{-\mu-1}(t^2)^{-\mu-1}dt\,ds\,,$$

where $c(\mu)=2\pi i\exp(\mu\pi i/2)\Gamma(\mu+1)$. Then S.S.$\Phi$ is contained in the set $\{\xi\eta\geqq 0\}$ due to the factor $\delta(\xi s^2-\eta t^2)$. The prescription given by the conjecture does not give this result, while this constraint naturally appears if we assume x_m to be real in the definition of $W_0'(+)$.

Example 4. Let Φ be $(x^3-y^3)_+^{\lambda}$ $(\lambda\notin \boldsymbol{N}, \lambda\gg 0)$. Let $(x, y; \sqrt{-1}(\xi, \eta)\infty)$ denote a point in $\sqrt{-1}\,S^*\boldsymbol{R}^2$. If we use the recipe given by the conjecture, then there is no constraint on the cotangential component of the possible singularity spectrum of Φ at the origin. On the other hand, if we suppose that x_m should be chosen to be real in the description of $W_0'(+)$, then we have there an additional constraint $\xi\eta\leqq 0$. However, as we shall see below, a point $(0, 0; \sqrt{-1}(\xi, \eta)\infty)$ with $\xi\eta>0$ really appears in the singularity spectrum of Φ.(**) This implies that we cannot keep x_m to be real.

In order to see this, it suffices to show that the following Radon transform $R(\xi, \eta)$ of Φ does not vanish identically for $(\xi, \eta)\in\boldsymbol{R}^2$ with $\xi\eta\geqq 0$.(***)

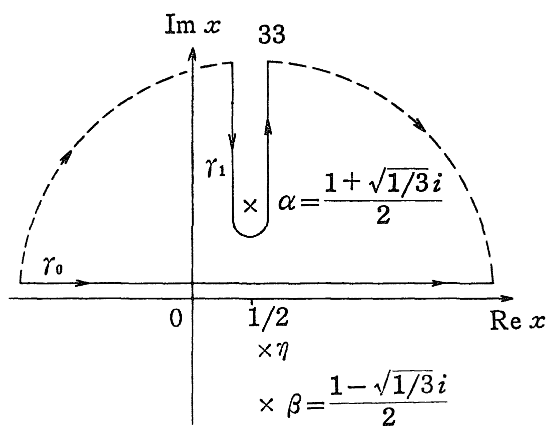
(*) Note that $|f(x)|^{\lambda}$ and $f(x)_+^{\lambda}$ are expressed as a linear combination of $(f(x)+\sqrt{-1}\,0)^{\lambda}$ and $(f(x)-\sqrt{-1}\,0)^{\lambda}$ for generic λ.

(**) As a matter of fact, the argument below implies that S. S. Φ contains any point $(0, 0; \sqrt{-1}(\xi, \eta)\infty)$ with $(\xi, \eta)\neq(0, 0)$.

(***) Since $(x^3-y^3)_+^{\lambda}$ is homogeneous in (x, y), we can use the Radon transform of Φ instead of its Fourier transform. Note also that there is no contribution to the set $\{\xi\eta\geqq 0\}$ from the points different from the origin. This is the reason why we do not need to use a cut-off function with respect to (x, y)-variables.

$$(66)\quad R(\xi,\eta)=\int_{S^1}(x^3-y^3)_+^{\lambda}(x\xi+y\eta+\sqrt{-1}\,0)^{-3\lambda-2}(ydx-xdy)$$
$$=\int_{x-y=1}(x^3-y^3)^{\lambda}(x\xi+y\eta+\sqrt{-1}\,0)^{-3\lambda-2}dx$$
$$=\int(3x^2-3x+1)^{\lambda}(x(\xi+\eta)-\eta+i0)^{-3\lambda-2}dx\,.$$

Here we may assume without loss of generality that $\xi+\eta=1$. Then what we should show is that the resulting integral $I(\eta)$ does not vanish identically. In order to see this we calculate $I(\eta)$ by shifting the path of integration $\gamma_0=(-\infty,\infty)$ to γ_1 described in the figure below.



Such a change of the path of integration is legitimate if $\lambda\gg 0$, as the integral along the dotted circle tends to zero as its radius tends to zero. In view of the $+\sqrt{-1}\,0$ in the integrand of $I(\eta)$, we may regard η to be a complex number running over the domain $\{\mathrm{Im}\ \eta<0\}$. If we move η so that it lies on the segment joining α and β, $I(\eta)$ acquires the form

$$-\sqrt{-1}(1-e^{\pi i\lambda})e^{-\pi i\lambda/2}\int_{1/2\sqrt{3}}^{\infty}\left(x-\frac{1}{2\sqrt{3}}\right)^{\lambda}\left(x+\frac{1}{2\sqrt{3}}\right)^{\lambda}\times$$
$$\times\left(x-\frac{1}{\sqrt{-1}}\left(\eta-\frac{1}{2}\right)\right)^{-3\lambda-2}dx\,.$$

Clearly the integral

$$\int_{1/2\sqrt{3}}^{\infty}\left(x-\frac{1}{2\sqrt{3}}\right)^{\lambda}\left(x+\frac{1}{2\sqrt{3}}\right)^{\lambda}\left(x-\frac{1}{\sqrt{-1}}\left(\eta-\frac{1}{2}\right)\right)^{-3\lambda-2}dx$$

converges for $\lambda\gg 0$. Furthermore, its integrand is non-negative if $\eta=\frac{1}{2}+$ $+\sqrt{-1}\,\zeta$ with $\zeta<\frac{1}{2\sqrt{3}}$. Therefore $I(\eta)$ is different from zero for $\eta=\frac{1}{2}+\sqrt{-1}\,\zeta$ with $\zeta<\frac{1}{2\sqrt{3}}$. This implies that $I(\eta)$ cannot be identically zero for real η. This completes the proof of the assertion that a point $(0,0;\sqrt{-1}(\xi,\eta)\infty)$ with $\xi\eta>0$ really appears in the singularity spectrum of Φ.

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Added in proof: The statement in page 9, line 4 is erroneous, because $\tilde{P}_j$'s and $\tilde{Q}_J$'s do not satisfy the commutation relation. Hence the proof of Proposition 7 is not correct. Although the proof of Theorem 2 depends on Proposition 7, if we replace $\mathcal{N}_{f,\varphi}$ with $\mathcal{N}_{f,\varphi}(\alpha,\beta)$, it does not depend on Proposition 7. Here $\alpha=(\alpha_1,\cdots,\alpha_l)$, $\beta=(\beta_1,\cdots,\beta_l)\in\mathbb{C}^l$ and $\mathcal{N}_{f,\varphi}(\alpha,\beta)$ is obtained from $\mathcal{N}_{f,\varphi}$ by letting s_j subject to the relation $s_j=\alpha_j s+\beta_j$ with one indeterminate s. Theorfore the proof of Theorem 18 is complete as it stands. The detailed corrections will be submitted to this journal. See also our paper "On the characteristic variety of a holonomic system with regular singularities," which will appear in Adv. in Math. It gives a complete proof for a generalization of Theorem 18.