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Propagation of Singularities of Fundamental Solutions of Hyperbolic Mixed Problems

By

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§ 1. Introduction

In this paper we shall deal with hyperbolic mixed problems with constant coefficients in a quarter-space and study the wave front sets of the fundamental solutions under the only assumption that the hyperbolic mixed problems are $\mathcal{E}$-well posed. Recently Garnir has studied the wave front sets of fundamental solutions for hyperbolic systems [2]. The author was stimulated by his work. For the detailed literatures we refer the reader to [7], [8].

Now let us state our problems, assumptions and main results. Let $\boldsymbol{R}^n$ denote the n-dimensional euclidean space and write $x'=(x_1, \cdots, x_{n-1})$ for the coordinate $x=(x_1, \cdots, x_n)$ in $\boldsymbol{R}^n$ and $\xi'=(\xi_1, \cdots, \xi_{n-1})$, $\tilde{\xi}=(\xi, \xi_{n+1})$ for the dual coordinate $\xi=(\xi_1, \cdots, \xi_n)$. We shall also denote by $\boldsymbol{R}^n_+$ the half-space $\{x=(x', x_n)\in\boldsymbol{R}^n; x_n>0\}$. For differentiation we will use the symbol $D=i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. Let $P=P(\xi)$ be a hyperbolic polynomial of order m of n variables ξ with respect to $\vartheta=(1,0, \cdots, 0)\in\boldsymbol{R}^n$ in the sense of Gårding, i.e.

$$P^0(-i\vartheta)\neq 0 \text{ and } P(\xi-s\vartheta)\neq 0 \text{ when } \xi \text{ is real and } \operatorname{Im} s<\gamma_0,$$

where P^0 denotes the principal part of P, i.e.

$$P(t\xi)=t^m(P^0(\xi)+o(1)) \text{ as } t\to\infty,\ P^0(\xi)\not\equiv 0.$$

Let $\Gamma=\Gamma(P,\vartheta)$ $(\subset\boldsymbol{R}^n)$ be the component of the set $\{\xi\in\boldsymbol{R}^n; P^0(-i\xi)\neq 0\}$ which contains ϑ. We also write $\Gamma(P)=\Gamma(P,\vartheta)$. Put

$$\Gamma_0=\{\xi'\in\boldsymbol{R}^{n-1};\ (\xi', 0)\in\Gamma\},$$
$$\dot{\Gamma}=\{\xi'\in\boldsymbol{R}^{n-1};\ (\xi', \xi_n)\in\Gamma \text{ for some } \xi_n\in\boldsymbol{R}\}.$$

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The localization $P_{\xi^0}(\eta)$ of $P(\xi)$ at ξ^0 and the multiplicity m_{ξ^0} of ξ^0 relative to P are defined by

$$\nu^m P(\nu^{-1}\xi^0+\eta)=\nu^{m_{\xi^0}}(P_{\xi^0}(\eta)+o(1)) \text{ as } \nu\downarrow 0,\ P_{\xi^0}(\eta)\not\equiv 0$$

(see [1]). We note that

$$\Gamma\subset\Gamma_{\xi^0}\equiv\Gamma(P_{\xi^0}).$$

Now write

$$P(\xi)=\sum_{j=0}^{m'}P_j(\xi')\xi_n^j,\qquad P_{m'}(\xi')\not\equiv 0.$$

Then we see that

$$P_{m'}(\xi')\neq 0 \text{ for } \xi'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{(0,1)}.$$

In fact, $P_{m'}(\xi')=P_{(0,1)}(\xi)$ and $\Gamma_{(0,1)}=\dot{\Gamma}_{(0,1)}\times\boldsymbol{R}$. It easily follows that $\Gamma_0\subset\dot{\Gamma}\subset\dot{\Gamma}_{(0,1)}$. When $\xi'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\Gamma_0$, we can denote the roots of $P(\xi',\lambda)=0$ with respect to λ by $\lambda_1^+(\xi'),\cdots,\lambda_l^+(\xi'),\lambda_1^-(\xi'),\cdots,\lambda_{m'-l}^-(\xi')$, which are enumerated so that $\operatorname{Im}\lambda_k^{\pm}(\xi')\gtrless 0$. We consider the mixed initial-boundary value problem for the hyperbolic operator $P(D)$ in a quarter-space

$$\begin{aligned}&P(D)u(x)=f(x),\quad x\in\boldsymbol{R}_+^n,\qquad x_1>0,\\&D_1^k u(x)|_{x_1=0}=0,\quad 0\leq k\leq m-1,\quad x_n>0,\\&B_j(D)u(x)|_{x_n=0}=0,\quad 1\leq j\leq l,\quad x_1>0.\end{aligned}$$

Here the $B_j(D)$ are boundary operators with constant coefficients. Put

$$P_+(\xi',\lambda)=\prod_{j=1}^{l}(\lambda-\lambda_j^+(\xi')),\quad \xi'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\Gamma_0.$$

Then Lopatinski's determinant for the system $\{P,B_j\}$ is defined by

$$R(\xi')=\det L(\xi')\qquad\text{for } \xi'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\Gamma_0,$$

where

$$L(\xi')=\left(\frac{1}{2\pi i}\oint B_j(\xi',\lambda)\lambda^{k-1}P_+(\xi',\lambda)^{-1}d\lambda\right)_{j,k=1,\cdots,l}.$$

We impose the following assumption on $\{P,B_j\}$:

(A) The system $\{P,B_j\}$ is $\mathcal{E}$-well posed, i.e.

$$R^0(-i\vartheta')\neq 0,\ R(\xi'+s\vartheta')\neq 0 \text{ when } \xi' \text{ is real and } \operatorname{Im}s<-\gamma_1,$$

where $R^0(\xi')$ denotes the principal part of $R(\xi')$ and $\gamma_1>\gamma_0$ (see [3]).

Now we can construct the fundamental solution $G(x, y)$ for $\{P, B_j\}$ which describes the propagation of waves produced by unit impulse given at position $y=(0, y_2, \cdots, y_n)$ in $\boldsymbol{R}^n_+$. Write

$$G(x, y)=E(x-y)-F(x, y),$$
$$x\in\boldsymbol{R}^n_+,\quad x_1>0,\quad y=(0, y_2, \cdots, y_n)\in\boldsymbol{R}^n_+,$$

where $E(x)$ is the fundamental solution of the Cauchy problem represented by

$$E(x)=(2\pi)^{-n}\int_{\boldsymbol{R}^n-i\eta}\exp[ix\cdot\xi]P(\xi)^{-1}d\xi,\qquad \eta\in\gamma_0\vartheta+\Gamma.$$

Then $F(x, y)$ is written in the form

$$F(x, y)=(2\pi)^{-(n+1)}\int_{\boldsymbol{R}^{n+1}-i\gamma\tilde{\vartheta}} i^{-1}\sum_{j,k=1}^{l}\exp[i\{(x'-y')\cdot\xi'$$
$$+x_n\xi_n-y_n\xi_{n+1}\}]R_{jk}(\xi')B_k(\xi', \xi_{n+1})$$
$$\times\xi_n^{j-1}(R(\xi')P_+(\xi)P(\xi', \xi_{n+1}))^{-1}d\tilde{\xi},$$

where $\gamma>\gamma_1$, $\tilde{\vartheta}=(\vartheta, 0)\in\boldsymbol{R}^{n+1}$ and $R_{jk}(\xi')=(k, j)$-cofactor of $L(\xi')$ (see [3], [4], [6]). $F(x, y)$ has to be interpreted in the sense of distribution with respect to (x, y) in $\boldsymbol{R}^n_+\times\boldsymbol{R}^n_+$. We put

$$\tilde{F}(\tilde{z})=F(z', z_n, 0, -z_{n+1}),\quad \tilde{z}=(z, z_{n+1})\in X=\boldsymbol{R}^{n-1}\times\boldsymbol{R}^1_+\times\boldsymbol{R}^1_-,$$

where $\boldsymbol{R}^1_-=\{\lambda\in\boldsymbol{R}; \lambda<0\}$, and regard $\tilde{F}(\tilde{z})$ as a distribution on X. We note that $\tilde{F}(\tilde{z})$ can be regarded as a distribution on $\boldsymbol{R}^{n-1}$ and that supp $\tilde{F}$ $\subset\{\tilde{z}\in\boldsymbol{R}^{n+1}; z_n\geq 0\}$. In order to investigate the wave front set $WF(G)$ of $G(x, y)$ it suffices to study $WF(\tilde{F})$. Our main result is stated as follows:

Theorem 1.1. *Assume that the condition* (A) *is satisfied and that* $\tilde{\xi}^0\in\boldsymbol{R}^{n+1}$. *Then we have*

$$t^{N/L}\{t^{p_0}\exp[-it\tilde{z}\cdot\tilde{\xi}^0]\tilde{F}(\tilde{z})-\sum_{j=0}^{N}\tilde{F}_{\tilde{\xi}^0,j}(\tilde{z})t^{-j/L}\}\to 0$$
$$\text{as } t\to\infty,\ \text{in } \mathscr{D}'(X),\ N=0, 1, 2, \cdots,$$

where p_0 *is a rational number and* L *is a positive integer. Moreover we have*

$$\bigcup_{\tilde{\xi}\in \boldsymbol{R}^{n+1}\setminus\{0\}} \bigcup_{j=0}^{\infty} \operatorname{supp} \tilde{F}_{\tilde{\xi},j}(\tilde{z}) \times \{\tilde{\xi}\} \subset WF(\tilde{F}(\tilde{z}))$$

$$\subset WF_A(\tilde{F}(\tilde{z})) \subset \bigcup_{\tilde{\xi}\in \boldsymbol{R}^{n+1}\setminus\{0\}} K^0_{\tilde{\xi}} \times \{\tilde{\xi}\},$$

$$\overline{\text{ch}}^{\dagger}[\bigcup_{j=0}^{\infty} \operatorname{supp} \tilde{F}_{\xi^0,j}(\tilde{z})] \subset K_{\xi^0} \tag{1.1}$$

$$K_{\xi^0} \subset K^0_{\xi^0}, \tag{1.2}$$

where

$$K_{\xi^0} = \{\tilde{z}\in X; \tilde{z}\cdot\tilde{\eta} \geq 0 \text{ for all } \tilde{\eta}\in \Gamma_{\xi^0}\},$$

$$K^0_{\xi^0} = \{\tilde{z}\in X; \tilde{z}\cdot\tilde{\eta} \geq 0 \text{ for all } \tilde{\eta}\in \Gamma^0_{\xi^0}\}$$

and Γ_{ξ^0} and $\Gamma^0_{\xi^0}$ are defined by (3. 3) and (3. 4), respectively.

Remark. The inclusion of (1. 1) can be replaced by the equality except in certain exceptional cases (see Example 5. 1 in [8]).

The remainder of this paper is organized as follows. In Section 2 we shall study some properties of symmetric functions of $\lambda_1^+(\xi'), \cdots, \lambda_l^+(\xi')$. In Section 3 Theorem 1. 1 will be proved. In Section 4 we shall give some remarks and examples.

§ 2. Algebraic Considerations

In this section we assume without loss of generality that $P(\xi)$ is irreducible. Let $\xi^{0\prime}$ be fixed in $\boldsymbol{R}^{n-1}$ and $m'_{\xi^{0\prime}}$ the multiplicity of $\xi^{0\prime}$ relative to $P_{m'}(\xi')$. Let $\xi^0_n \in \boldsymbol{R}$ and write

$$\nu^m P(\nu^{-1}\xi^0+\eta) = \sum_{j=m_{\xi^0}}^{m} \nu^j Q_{\xi^0,j}(\eta), \ Q_{\xi^0,m_{\xi^0}}(\eta) \not\equiv 0 .$$

It is easy to see that $Q_{\xi^0,m_{\xi^0}}(\eta) = P_{\xi^0}(\eta)$,

$$Q_{\xi^0,j}(\eta) = \sum_{|\alpha|+k=j} \frac{1}{\alpha!}\partial^{|\alpha|}/\partial\xi^{\alpha} P^k(\xi^0)\cdot\eta^{\alpha},$$

where $P(\xi) = P^0(\xi) + P^1(\xi) + \cdots + P^m(\xi)$ and $P^k(\xi)$ is a homogeneous

† $\overline{\text{ch}}[M]$ denotes the closed convex hull of M in X.

polynomial of degree $m-k$. We can write

$$Q_{\xi^0,j}(\eta)=\sum_{k=0}^{r_j} q_{\xi^0,jk}(\eta')\eta_n^k, \qquad q_{\xi^0,jr_j}(\eta')\not\equiv 0 \quad \text{if } Q_{\xi^0,j}(\eta)\not\equiv 0,$$

where $r_j \equiv r_j(\xi^0)$ depends on ξ^0. It follows that $r_{m'+m'_{\xi^{0\prime}}}=m'$ and $r_j<m'$ if $j<m'+m'_{\xi^{0\prime}}$. We put

$$j_1 \equiv j_1(\xi^0)=m'+m'_{\xi^{0\prime}},$$

$$l_k \equiv l_k(\xi^0)=\min\{(r_{j_k}-r_j)/(j_k-j);\, m_{\xi^0}\leq j<j_k\},$$

$$j_{k+1}\equiv j_{k+1}(\xi^0)=\min\{j;\, m_{\xi^0}\leq j<j_k \text{ and } (r_{j_k}-r_j)/(j_k-j)=l_k\},$$

and obtain the sequence $\{j_k, l_k\}_{k=0,\dots,s+1}$ so that

$$j_0=m>j_1=m'+m'_{\xi^{0\prime}}>j_2>\cdots>j_s>j_{s+1}=m_{\xi^0},$$

$$l_0=0<l_1<l_2<\cdots<l_s<l_{s+1}=\infty,$$

where $s\equiv s(\xi^0)$ depends on ξ^0. For $\rho>0$ we define the modified localization $P_{\rho,\xi^0}(\eta;\lambda)$ of P at ξ^0 by

$$\nu^m P(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n)=\nu^{m_{\xi^0}(\rho)}(P_{\rho,\xi^0}(\eta;\lambda)+o(1))$$

$$\text{as } \nu\downarrow 0,\ P_{\rho,\xi^0}(\eta;\lambda)\not\equiv 0 \text{ in } (\eta,\lambda).$$

Then we have

$$P_{\rho,\xi^0}(\eta;\lambda)=q_{\xi^0,j_k r_{j_k}}(\eta')\lambda^{r_{j_k}}, \tag{2.1}$$

$$m_{\xi^0}(\rho)=j_k-r_{j_k}/\rho,$$

if $l_k>\rho>l_{k-1}$, $1\leq k\leq s+1$, and we have

$$P_{\rho,\xi^0}(\eta;\lambda)=[q_{\xi^0,j_k r_{j_k}}(\eta')\lambda^{r_{j_k}-r_{j_{k+1}}} \tag{2.2}$$

$$+\cdots+q_{\xi^0,j_{k+1}r_{j_{k+1}}}(\eta')]\lambda^{r_{j_{k+1}}},$$

$$m_{\xi^0}(\rho)=j_k-r_{j_k}/\rho=j_{k+1}-r_{j_{k+1}}/\rho,$$

if $\rho=l_k$, $1\leq k\leq s$. Moreover we have

$$P_{\rho,\xi^0}(\eta;\lambda)=P_{\xi^0}(\eta',\lambda+\eta_n), \qquad m_{\xi^0}(\rho)=m_{\xi^0},$$

if $\rho=l_{s+1}=\infty$. We note that $j_k(\xi^0)$ and $l_{k-1}(\xi^0)$ are independent of ξ_n^0 if $l_{k-1}<1$. In fact, we have

$$P_{\rho,\xi^0}(\eta;\lambda)=P_{\rho,(\xi^{0\prime},0)}(\eta;\lambda) \quad \text{if} \quad l_{k-1}<\rho<\min(1,l_k).$$

Now we define the modified principal part $p_\rho^0(\eta;\lambda)$ and modified degree $\deg_\rho p=\sigma$ for a polynomial $p(\eta;\lambda)$ by

$$p(t\eta; t^{(\rho-1)/\rho}\lambda)=t^\sigma(p_\rho^0(\eta;\lambda)+o(1)) \quad \text{as} \quad t\to\infty,$$

$$p_\rho^0(\eta;\lambda)\not\equiv 0 \quad \text{in} \quad (\eta,\lambda).$$

Lemma 2.1. *Let* $\rho>0$ *and put* $P_{\rho,\xi^0}^0(\eta;\lambda)=(P_{\rho,\xi^0})_\rho^0(\eta;\lambda)$. *Then we have*

$$P_{\rho,\xi^0}^0(\eta;\lambda)=(P^0)_{\rho,\xi^0}(\eta;\lambda), \qquad \deg_\rho P_{\rho,\xi^0}=m_{\xi^0}(\rho).$$

Proof.

$$\nu^m P^0(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n)$$
$$=\nu^{\sigma_0}((P^0)_{\rho,\xi^0}(\eta;\lambda)+Q(\eta,\lambda;\nu)),$$

where $Q(\eta,\lambda;\nu)$ is a polynomial in (η,λ), continuous in (η,λ,ν) and $Q(\eta,\lambda;0)=0$. Therefore we have

$$\nu^m\partial^{|\alpha|}/\partial\eta^\alpha P^0(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n)$$
$$=\nu^{\sigma_0}(\partial^{|\alpha|}/\partial\eta^\alpha(P^0)_{\rho,\xi^0}(\eta;\lambda)+\partial^{|\alpha|}/\partial\eta^\alpha Q(\eta,\lambda;\nu)).$$

From this it follows that

$$\nu^m\tilde{P}^0(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n)$$
$$=\nu^{\sigma_0}(((P^0)_{\rho,\xi^0})^\sim(\eta;\lambda)^2+o(1))^{1/2} \quad \text{as} \quad \nu\downarrow 0,$$

where $\tilde{p}(\eta;\lambda)^2=\sum|\partial^{|\alpha|}/\partial\eta^\alpha p(\eta;\lambda)|^2$. Hyperbolicity of P implies that

$$|P(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n)|$$
$$\le \text{const.}\times\tilde{P}^0(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n), \quad \lambda\in\boldsymbol{R}, \ \eta\in\boldsymbol{R}^n$$

(see [5]). Since there exists $(\eta^0,\lambda_0)\in\boldsymbol{R}^{n+1}$ such that $P_{\rho,\xi^0}(\eta^0;\lambda_0)\neq 0$, it follows that $\sigma_0\le m_{\xi^0}(\rho)$. Put

$$\nu^m P^k(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda+\eta_n)$$
$$=\nu^{\sigma_k}((P^k)_{\rho,\xi^0}(\eta;\lambda)+o(1)) \quad \text{as} \quad \nu\downarrow 0.$$

Then we have $\deg_\rho(P^k)_{\rho,\xi^0}=\sigma_k-k$ and $(P^k)_{\rho,\xi^0}^0=(P^k)_{\rho,\xi^0}$. Therefore it follows that $\sigma_0=m_{\xi^0}(\rho)$. This proves the lemma. Q.E.D.

Lemma 2. 2. *Let* $\rho>0, \rho\neq 1$ *and* $\lambda_0\in \mathbf{R}\setminus\{0\}$. *Then* $P_{\rho,\xi^0}(\eta;\lambda_0)$ *is a hyperbolic polynomial with respect to* ϑ. *Moreover we have*

$$(2.3)\quad P_{\rho,\xi^0}(\eta;\lambda_0)\neq 0 \quad \textit{for } \eta\in\begin{cases}\mathbf{R}^n-i\gamma_0\vartheta-i\Gamma((P_{(0,1)})_{\xi^0}) & \textit{if } 1>\rho>0,\\ \mathbf{R}^n-i\gamma_0\vartheta-i\Gamma((P_{\xi^0})_{(0,1)}) & \textit{if } \infty>\rho>1,\\ \mathbf{R}^n-i\gamma_0\vartheta-i\Gamma_{\xi^0} & \textit{if } \rho=l_{s+1}=\infty .\end{cases}$$

In particular,

$$(2.4)\quad \Gamma(P_{\rho,\xi^0}(\eta;\lambda_0))\supset\begin{cases}\Gamma((P_{(0,1)})_{\xi^0}) & \textit{if } 1>\rho>0,\\ \Gamma((P_{\xi^0})_{(0,1)}) & \textit{if } \infty>\rho>1,\\ \Gamma_{\xi^0} & \textit{if } \rho=l_{s+1}=\infty ,\end{cases}$$

and

$$(2.5)\quad (P_{\rho,\xi^0})^0(\eta;\lambda_0)=\begin{cases}(P^0_{(0,1)})_{\xi^0}(\eta)\lambda_0^{m'} & \textit{if } l_1\geq\rho>0,\\ (P^0_{\xi^0})_{(0,1)}(\eta)\lambda_0^{r_{j_{s+1}}} & \textit{if } \infty>\rho\geq l_s\\ P^0_{\xi^0}(\eta) & \textit{if } \rho=l_{s+1}=\infty ,\end{cases}$$

where $(P_{\rho,\xi^0})^0(\eta;\lambda_0)$ *denotes the principal part of a polynomial* $P_{\rho,\xi^0}(\eta;\lambda_0)$ *in* η.

Remark. We note that $\Gamma_{\xi^0}\subset\Gamma((P_{\xi^0})_{(0,1)})$ and that $(P_{(0,1)})_{\xi^0}(\eta)$ is independent of ξ^0_n.

Proof. Since $\rho\neq 1$, it follows that $P_{\rho,\xi^0}(\eta;\lambda_0)\not\equiv 0$ in η. In fact, from Lemma 2. 1 we have

$$\deg q_{\xi^0,j_kr_{j_k}}(\eta')=j_k-r_{j_k}.$$

Thus

$$(2.6)\quad (P_{\rho,\xi^0})^0(\eta;\lambda_0)=\begin{cases}(q_{\xi^0,j_kr_{j_k}})^0(\eta')\lambda_0^{r_{j_k}} & \text{if } l_k\geq\rho>l_{k-1} \text{ and } 1>\rho>0,\\ (q_{\xi^0,j_kr_{j_k}})^0(\eta')\lambda_0^{r_{j_k}} & \text{if } l_k>\rho\geq l_{k-1} \text{ and } \rho>1,\\ P^0_{\xi^0}(\eta) & \text{if } \rho=l_{s+1}=\infty .\end{cases}$$

Now let us assume that there exists $\eta^0\in\mathbf{R}^n-i\gamma_0\vartheta-i\Gamma$ such that $P_{\rho,\xi^0}(\eta^0;\lambda_0)=0$. Then there exist positive numbers ε, δ and $\zeta^0\in\mathbf{C}^n$ such that

$$|P_{\rho,\xi^0}(\eta^0+\mu\zeta^0;\lambda_0)|>\varepsilon>0 \quad \text{for } |\mu|=\delta>0,$$

$$\eta^0+\mu\zeta^0\in\mathbf{R}^n-i\gamma_0\vartheta-i\Gamma \quad \text{for } |\mu|\leq\delta.$$

Therefore from Rouché's theorem it follows that there exists a positive number ν_0 such that $P(\nu^{-1}\xi^{0\prime}+\eta^{0\prime}+\mu\zeta^{0\prime}, \nu^{-1}\xi_n^0+\nu^{-1/\rho}\lambda_0+\eta_n^0+\mu\zeta_n^0)$ has zeros within $|\mu|<\delta$ if $0<\nu\leq\nu_0$, which is a contradiction to $P(\xi)\neq 0$ for $\xi\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma$. So we have

$$P_{\rho,\xi^0}(\eta;\lambda_0)\neq 0 \quad \text{for } \eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma.$$

This implies that $P_{\rho,\xi^0}(\eta;\lambda_0)$ is a hyperbolic polynomial with respect to ϑ and that $\Gamma(P_{\rho,\xi^0}(\eta;\lambda_0))\supset\Gamma$. Next let us prove (2.4). We note that (2.3) follows from (2.4) (see [1], [3]). One can easily verify (2.5). Therefore (2.4) holds when $\infty\geq\rho\geq l_s$ or $l_1\geq\rho>0$. Let us prove (2.4) when $1>\rho>0$. For we can prove (2.4) in the same manner when $\rho>1$. Now assume that $\Gamma(P_{\rho,\xi^0}(\eta;\lambda_0))\supset\Gamma((P_{(0,1)})_{\xi^0})$ when $1>l_k>\rho>0$. Then by (2.1) we have

$$\Gamma(q_{\xi^0,j_k r_{j_k}}(\eta'))\supset\Gamma((P_{(0,1)})_{\xi^0}).$$

Thus from (2.6) it follows that

$$P_{l_k,\xi^0}(\eta;\lambda_0)\neq 0 \quad \text{for } \eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{(0,1)})_{\xi^0}). \tag{2.7}$$

Assume that

$$q_{\xi^0,j_{k+1}r_{j_{k+1}}}(\eta^{0\prime})=0 \quad \text{for some } \eta^0\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{(0,1)})_{\xi^0}).$$

From (2.2) we have

$$\lambda^{-r_{j_{k+1}}}P_{l_k,\xi^0}(\eta;\lambda)\to q_{\xi^0,j_{k+1}r_{j_{k+1}}}(\eta') \text{ as } \lambda\downarrow 0$$

(locally uniform), which leads us to a contradiction, using Rouché's theorem. Therefore,

$$P_{\rho,\xi^0}(\eta;\lambda_0)=q_{\xi^{0\prime},j_{k+1}r_{j_{k+1}}}(\eta')\lambda_0^{r_{j_{k+1}}}\neq 0 \tag{2.8}$$

when $l_{k+1}>\rho>l_k$ and $\eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{(0,1)})_{\xi^0})$. From (2.7) and (2.8) it follows that

$$\Gamma(P_{\rho,\xi^0}(\eta;\lambda_0))\supset\Gamma((P_{(0,1)})_{\xi^0}) \quad \text{when} \quad l_{k+1}>\rho>0.$$

Q.E.D.

We define $q\equiv q(\xi^{0\prime})$ by

$$\partial^k/\partial\xi_1^k P^0(\xi^{0\prime},\lambda)\equiv 0, \quad 0\leq k\leq q-1, \quad \partial^q/\partial\xi_1^q P^0(\xi^{0\prime},\lambda)\not\equiv 0 \text{ in } \lambda. \tag{2.9}$$

Put

$$p \equiv p(\xi^{0\prime}) = \deg \partial^q/\partial\xi_1^q P^0(\xi^{0\prime}, \lambda),$$

and define $r \equiv r(\xi^0)$ by

$$\partial^{q+k}/\partial\xi_1^q\partial\xi_n^k P^0(\xi^0) = 0, \quad 0 \leq k \leq r-1, \tag{2.10}$$

$$\partial^{q+r}/\partial\xi_1^q\partial\xi_n^r P^0(\xi^0) \neq 0. \tag{2.11}$$

Then we have the following

Lemma 2.3.

$$q \leq m'_{\xi^{0\prime}} \leq m - m', \qquad p \leq \min(m', m-q), \tag{2.12}$$

$$q \leq m_{\xi^0} \leq q + r \leq q + p \leq m' + m'_{\xi^{0\prime}} \leq m,$$

$$p = m', \quad m'_{\xi^{0\prime}} = m - m' \quad \textit{if} \ q = m - m'.$$

Moreover

$$r_j \leq j - q \quad \textit{for} \ m_{\xi^0} \leq j \leq m, \tag{2.13}$$

$$r_j < j - q \quad \textit{if} \ m_{\xi^0} \leq j < q + r \quad \textit{or} \quad q + p < j \leq m, \tag{2.14}$$

$$r_j < m' \quad \textit{if} \ m_{\xi^0} \leq j < m' + m'_{\xi^{0\prime}}, \tag{2.15}$$

$$r_{q+r} = r, \quad r_{q+p} = p \quad \textit{and} \quad r_{m' + m'_{\xi^{0\prime}}} = m'. \tag{2.16}$$

Remark. This lemma yields us the following Newton polygon (Fig. 1).

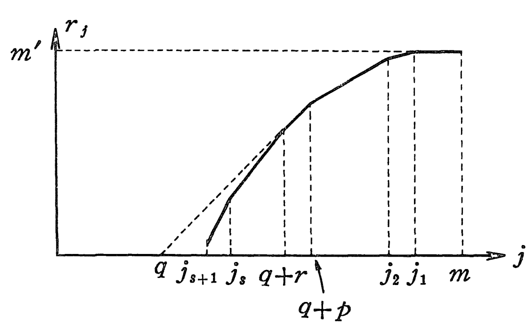

Fig. 1.

Proof. If $|\alpha| + k < q$,

$$\partial^{|\alpha|}/\partial\xi^\alpha P^k(\xi^{0\prime}, \lambda) \equiv 0 \quad \text{in } \lambda. \tag{2.17}$$

In fact, for each $\lambda_0 \in \boldsymbol{R}$

$$\nu^m P(\nu^{-1}\xi^{0\prime}+\eta', \nu^{-1}\lambda_0+\eta_n)$$
$$=\sum_{j=0}^{m}\nu^j\sum_{|\alpha|+k=j}\frac{1}{\alpha!}\partial^{|\alpha|}/\partial\xi^\alpha P^k(\xi^{0\prime}, \lambda_0)\eta^\alpha.$$

If $\partial^{|\alpha|}/\partial\xi^\alpha P^k(\xi^{0\prime}, \lambda_0)\neq 0$ for some α and k with $|\alpha|+k<q$, hyperbolicity of P implies that there exists a non-negative integer h such that $h\leq|\alpha|+k<q$ and $\partial^h/\partial\xi_1^h P^0(\xi^{0\prime}, \lambda_0)=h!P^0_{(\xi^{0\prime},\lambda_0)}(\vartheta)\neq 0$, which is a contradiction to (2.9). (2.13) easily follows from (2.17). (2.12), (2.15) and (2.16) are obvious. Now assume that

$$p'\equiv\max\{\deg\partial^{|\alpha'|}/\partial\xi'^{\alpha'}P^k(\xi^{0\prime}, \lambda); |\alpha'|+k=q\}>p.$$

Then we have

$$P^0_{\rho,(\xi^{0\prime},0)}(\vartheta;\lambda)=0 \quad \text{for } 1>\rho>(m'-p)/(m'+1-p), \tag{2.18}$$

which is a contradiction to hyperbolicity of $P_{\rho,(\xi^{0\prime},0)}(\eta;\lambda_0)$, $\lambda_0\in\boldsymbol{R}\setminus\{0\}$. In fact, we have $r_{q+p'}=p'$ and $r_j<j-q$ for $q+p'<j\leq m$. Therefore, $j-r_j/\rho>q+p'-p'/\rho$ when $1>\rho>(m'-p')/(m'+1-p')$ and $j\neq q+p'$. For it is obvious that $j-r_j/\rho\geq j(1-1/\rho)+q/\rho>q+p'-p'/\rho$ if $j<q+p'$. If $j>q+p'$, then

$$j-r_j/\rho=j-r_j+(1-1/\rho)r_j\geq q+1+(1-1/\rho)m'>q+p'-p'/\rho.$$

Thus we have $P_{\rho,(\xi^{0\prime},0)}(\eta;\lambda)=q_{(\xi^{0\prime},0),q+p'p'}(\eta')\lambda^{p'}$. Since $q^0_{(\xi^{0\prime},0),q+p'p'}(\vartheta')=(q!p'!)^{-1}\partial^{q+p'}/\partial\xi_1^q\partial\xi_n^{p'}P^0(\xi^{0\prime},0)$ we obtain (2.18). Therefore we have

$$p=\max\{\deg\partial^{|\alpha'|}/\partial\xi'^{\alpha'}P^k(\xi^{0\prime},\lambda); |\alpha'|+k=q\}.$$

This implies that $r_j<j-q$ if $q+p<j\leq m$. Next let us prove that

$$\partial^{|\alpha'|+h}/\partial\xi'^{\alpha'}\partial\xi_n^h P^k(\xi^0)=0 \quad \text{for } |\alpha'|+k=q \text{ and } 0\leq h\leq r-1. \tag{2.19}$$

Assume that

$$r'=\min\{h; \partial^{|\alpha'|+h}/\partial\xi'^{\alpha'}\partial\xi_n^h P^k(\xi^0)\neq 0 \quad \text{for some } \alpha' \text{ and } k \text{ with } |\alpha'|+k=q\}<r.$$

Then similarly we have

$$P^0_{\rho,\xi^0}(\vartheta;\lambda)=0 \quad \text{for } (q+r'-m_{\xi^0}+1)/(q+r'-m_{\xi^0})>\rho>1,$$

which is a contradiction to hyperbolicity of $P_{\rho,\xi^0}(\eta;\lambda_0)$, $\lambda_0\in\boldsymbol{R}\setminus\{0\}$. From (2.19) it follows that $r_j<j-q$ if $m_{\xi^0}\leq j<q+r$. Q.E.D.

From Lemma 2.3 it follows that there exist positive integers $t \equiv t(\xi^{0\prime})$ and $t' \equiv t'(\xi^0)$ such that $1 \leq t \leq t' \leq s+1$, $j_t = q+p$ and $j_{t'} = q+r$. If $r < p$, then $t' = t+1$ and $l_t = 1$. If $r = p$, then $t = t'$, $l_t > 1$ and $l_{t-1} < 1$. Thus $j_k(\xi^0)$ and $l_{k-1}(\xi^0)$, $0 \leq k \leq t(\xi^{0\prime})$, are independent of ξ_n^0. Put

$$P_{l_k, \xi^0}(\eta; \lambda) = P_{k, \xi^0}(\eta'; \lambda) \lambda^{r_{j_{k+1}}},$$

$$P_{k, \xi^0}(\eta'; \lambda) = q_{\xi^0, j_k r_{j_k}}(\eta') \lambda^{r_{j_k} - r_{j_{k+1}}} + \cdots + q_{\xi^0, j_{k+1} r_{j_{k+1}}}(\eta').$$

By Lemma 2.2 we obtain the following

Lemma 2.4. *For $1 \leq k < t$ $P_{k, \xi^0}(\eta'; \lambda)$ has no real zeros when $\eta' \in \boldsymbol{R}^{n-1} - i\gamma_0 \vartheta' - i\dot{\Gamma}((P_{(0,1)})_{\xi^0})$. For $t' \leq k \leq s$ $P_{k, \xi^0}(\eta'; \lambda)$ has no real zeros when $\eta' \in \boldsymbol{R}^{n-1} - i\gamma_0 \vartheta' - i\dot{\Gamma}((P_{\xi^0})_{(0,1)})$.*

Denote the roots of $P_{\xi^0}(\eta', \lambda) = 0$ by $\lambda^+_{\xi^0, 1}(\eta'), \cdots, \lambda^+_{\xi^0, l'}(\eta'), \lambda^-_{\xi^0, 1}(\eta'), \cdots, \lambda^-_{\xi^0, r_{m_{\xi^0}} - l'}(\eta')$ so that the $\lambda^{\pm}_{\xi^0, j}(\eta')$ are continuous and that

$$\operatorname{Im} \lambda^{\pm}_{\xi^0, j}(\eta' - i\gamma\vartheta') \gtreqless 0 \qquad \text{for } \gamma > \gamma_0 \text{ and } \eta' \in \boldsymbol{R}^{n-1},$$

when $r_{m_{\xi^0}} \neq 0$. Then we easily obtain the following

Lemma 2.5. *Assume that $r_{m_{\xi^0}} \neq 0$. Then*

$$\{\lambda^+_{\xi^0, j}(\eta')\}_{1 \leq j \leq l'} \cap \{\lambda^-_{\xi^0, j}(\eta')\}_{1 \leq j \leq r_{m_{\xi^0}} - l'} = \emptyset$$

if $\eta' \in \boldsymbol{R}^{n-1} - i\gamma_0 \vartheta' - i\mathring{\Gamma}_{\xi^0}$.

Put

$$P_{\xi^{0\prime}, q}(\eta; \lambda) = \sum_{|\alpha| + k = q} \frac{1}{\alpha!} \partial^{|\alpha|} / \partial \xi^\alpha P^k(\xi^{0\prime}, \lambda) \eta^\alpha.$$

We note that $P_{\xi^{0\prime}, q}(\eta; \lambda)$ is independent of η_n. From the proof of Lemma 2.3 it follows that $\deg P_{\xi^{0\prime}, q}(\eta; \lambda) \leq p$ in λ for fixed η. The coefficient of λ^p in $P_{\xi^{0\prime}, q}(\eta; \lambda)$ is equal to $q_{\xi^0, q+pp}(\eta')$, where $\xi_n^0 \in \boldsymbol{R}$. Since $q+p = j_t$, $p = r_{j_t}$ and $l_{t-1} < 1$, it follows from (2.1) and Lemma 2.2 that

$$q_{\xi^0, q+pp}(\eta') \neq 0 \text{ for } \eta \in \boldsymbol{R}^n - i\gamma_0 - i\Gamma((P_{(0,1)})_{\xi^0}).$$

Therefore we have

$\deg P_{\xi^{0\prime},q}(\eta;\lambda)=p$ in λ for fixed $\eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{(0,1)})_{(\xi^{0\prime},0)})$.

Lemma 2.6. *Let* $\xi_n^0\in\boldsymbol{R}$ *and* $\eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{\xi^0})_{(0,1)})$. *Then* $\lambda=\xi_n^0$ *is a root of* $\partial^q/\partial\xi_1^q P^0(\xi^{0\prime},\lambda)=0$ *with multiplicity* r† *if and only if* $\lambda=\xi_n^0$ *is a root of* $P_{\xi^{0\prime},q}(\eta;\lambda)=0$ *with multiplicity* r.

Proof. Now assume that $P_{\xi^{0\prime},q}(\eta;\xi_n^0)=0$ for some $\eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{\xi^0})_{(0,1)})$. Then we have $\partial^q/\partial\xi_1^q P^0(\xi^0)=0$. In fact, if $\partial^q/\partial\xi_1^q P^0(\xi^0)\neq0$, we have $(P_{\xi^0})_{(0,1)}(\xi)=P_{\xi^{0\prime},q}(\xi;\xi_n^0)\neq0$ for $\xi\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{\xi^0})_{(0,1)})$, which is a contradiction to $P_{\xi^{0\prime},q}(\eta;\xi_n^0)=0$. Next assume that there exists a non-negative integer k such that $k\leq r-1$ and $\partial^k/\partial\lambda^k P_{\xi^{0\prime},q}(\eta;\xi_n^0)\not\equiv0$ in η. Then we have $r_{q+k}=k$, which is a contradiction to (2.14). For $l_{t'}>\rho>1$ we have

$$P_{\rho,\xi^0}(\eta;\lambda)=\frac{1}{r!}\lambda^r\partial^r/\partial\lambda^r P_{\xi^{0\prime},q}(\eta;\xi_n^0).$$

From Lemma 2.2 and this it follows that

$$\partial^r/\partial\lambda^r P_{\xi^{0\prime},q}(\eta;\xi_n^0)\neq0 \quad \text{for } \eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{\xi^0})_{(0,1)}).$$

This proves the lemma. Q.E.D.

Lemma 2.6 yields the following

Lemma 2.7. *Let* $\eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\{\bigcap_{\xi_n^0\in\boldsymbol{R}}\Gamma((P_{\xi^0})_{(0,1)})\cap\Gamma((P_{(0,1)})_{(\xi^{0\prime},0)})\}$. *The real zeros of* $\partial^q/\partial\xi_1^q P^0(\xi^{0\prime},\lambda)$ *agree with those of* $P_{\xi^{0\prime},q}(\eta;\lambda)$ *(including multiplicities). Moreover the number of the roots with positive imaginary part of* $P_{\xi^{0\prime},q}(\eta;\lambda)=0$ *is equal to that of the roots with positive imaginary part of* $\partial^q/\partial\xi_1^q P^0(\xi^{0\prime},\lambda)=0$.

Remark. The non-real zeros of $\partial^q/\partial\xi_1^q P^0(\xi^{0\prime},\lambda)$ do not always agree with those of $P_{\xi^{0\prime},q}(\eta;\lambda)$. In fact, for $P(\xi)=P^0(\xi)=\xi_1^4-2\xi_1^2(\xi_2^2+\xi_3^2+\xi_4^2)+(\xi_2^2+\xi_3^2+\xi_4^2/2)\xi_2^2$ we have

† $r=r(\xi^0)$ is defined by (2.10) and (2.11). Lemma 2.6 implies that $\lambda=\xi_n^0$ is a root of $P_{\xi^{0\prime},q}(\eta;\lambda)=0$ with multiplicity r and that $\partial^q/\partial\xi_1^q P^0(\xi^0)=0$ if $P_{\xi^{0\prime},q}(\eta;\xi_n^0)=0$ for some $\eta\in\boldsymbol{R}^n-i\gamma_0\vartheta-i\Gamma((P_{\xi^0})_{(0,1)})$.

$$\partial^2/\partial\xi_1^2 P^0(0,0,1,\lambda) = -4(1+\lambda^2),$$

$$P_{(0,0,1),2}(\eta;\lambda) = -2(1+\lambda^2)\eta_1^2 + (1+\lambda^2/2)\eta_2^2.$$

Moreover if $\partial^q/\partial\xi_1^q P^0(\xi^0)\neq 0$, then $P_{\xi^0}(\eta) = (P_{\xi^0})_{(0,1)}(\eta)$ and $P^0_{\xi^{0\prime},q}(-i\eta^0;\xi_n^0) = P^0_{\xi^0}(-i\eta^0) = 0$ for $\eta^0\in\partial\Gamma(P_{\xi^0})$†.

Put

$$\sigma^k(\xi') = \sum_{j=1}^{l}\lambda_j^+(\xi')^k, \quad 1\leq k\leq l^{\dagger\dagger},$$

$$\dot{\Gamma}_{\xi^{0\prime}} = \bigcap_{\xi_n^0\in\boldsymbol{R}} \dot{\Gamma}_{\xi^0}\cap\dot{\Gamma}((P_{(0,1)})_{(\xi^{0\prime},0)}).$$

Lemma 2.8. *Let* $1\leq k\leq l$. *For any compact set* K *in* $\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$ *there exists* ν_K (>0) *such that* $\sigma^k(\nu^{-1}\xi^{0\prime}+\eta')$ *is well-defined for* $\eta'\in K$ *and* $0<\nu\leq\nu_K$ *and*

$$\nu^{s_k}\sigma^k(\nu^{-1}\xi^{0\prime}+\eta') = \sum_{j=0}^{\infty}\sigma^k_{\xi^{0\prime},j}(\eta')\nu^{j/L}, \quad \sigma^k_{\xi^{0\prime},0}(\eta')\not\equiv 0,$$

whose convergence is uniform in $K\times\{\nu;0\leq\nu\leq\nu_K\}$, *where* s_k *is a rational number and* L *is a positive integer. Moreover the* $\sigma^k_{\xi^{0\prime},j}$ *are holomorphic in* $\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$.

Proof. We can assume without loss of generality that K is small so that

$$\{\lambda^+_{\xi^0,j}(\eta');1\leq j\leq l' \text{ and } \eta'\in K\}\cap\{\lambda^-_{\xi^0,j}(\eta');1\leq j\leq r_{m_{\xi^0}}-l' \text{ and } \eta'\in K\}=\emptyset \quad \text{if } \xi_n^0\in\boldsymbol{R} \text{ and } r_{m_{\xi^0}}\neq 0$$

(see Lemma 2.5). Let $\xi_n^0\in\boldsymbol{R}$ and $\mathscr{C}^+_{\xi^0,j}$ $(1\leq j<t(\xi^{0\prime}), t'(\xi^0)\leq j\leq s(\xi^0))$ be simple closed curves enclosing only the roots with positive imaginary part of $P_{j,\xi^0}(\eta';\lambda)=0$ for $\eta'\in K$ (see Lemma 2.4). Let $\mathscr{C}^+_{\xi^0,0}$ be a simple closed curve enclosing only the roots $\lambda^+_{\xi^0,j}(\eta')$, $1\leq j\leq l'$, of $P_{\xi^0}(\eta',\lambda)=0$ for $\eta'\in K$ if $r_{m_{\xi^0}}\neq 0$ and $\mathscr{C}^+_{\xi^{0\prime}}$ a simple closed curve enclosing only the roots with positive imaginary part of $P_{\xi^{0\prime},q}(\eta;\lambda)=0$ for $\eta'\in K$ (see Lemma 2.7). From the relations between the roots of $P(\nu^{-1}\xi^{0\prime}+\eta',\lambda)=0$ and

† ∂M denotes the boundary of M.

†† The $\lambda_j^+(\xi')$ are continuous and Im $\lambda_j^+(\xi'-i\gamma\vartheta')>0$ for $\xi'\in\boldsymbol{R}^{n-1}$ and $\gamma>\gamma_0$.

the roots of $P_{j,\xi^0}(\eta';\lambda)=0$, $P_{\xi^0}(\eta',\lambda)=0$ and $P_{\xi^{0\prime},q}(\eta;\lambda)=0$ there exists $\nu'_K(>0)$ such that $\{\lambda_j^+(\nu^{-1}\xi^{0\prime}+\eta')\}_{1\le j\le l}\cap\{\lambda_j^-(\nu^{-1}\xi^{0\prime}+\eta')\}_{1\le j\le m'-l}=\emptyset$ for $\eta'\in K$, $0<\nu\le\nu'_K$. So we can take $\mathscr{C}_\nu^+$ to be a simple closed curve enclosing only the roots $\lambda_j^+(\nu^{-1}\xi^{0\prime}+\eta')$, $1\le j\le l$, of $P(\nu^{-1}\xi^{0\prime}+\eta',\lambda)=0$ for $\eta'\in K$, $0<\nu\le\nu'_K$. For $\eta'\in K$ and $0<\nu\le\nu'_K$ we have

$$
\begin{aligned}
(2.20)\quad \sigma^k(\nu^{-1}\xi^{0\prime}+\eta') &= (2\pi i)^{-1}\int_{\mathscr{C}_\nu^+}\lambda^k\partial/\partial\xi_n P(\nu^{-1}\xi^{0\prime}+\eta',\lambda)\\
&\qquad\times P(\nu^{-1}\xi^{0\prime}+\eta',\lambda)^{-1}d\lambda\\
&=\sum_{j=1}^{t-1}(2\pi i)^{-1}\int_{\mathscr{C}^+_{(\xi^{0\prime},0),j}}\lambda^k\partial/\partial\xi_n P(\nu^{-1}\xi^{0\prime}+\eta',\nu^{-1/l_j}\lambda)\\
&\quad\times(P_{l_j,(\xi^{0\prime},0)}(\eta',0;\lambda)+o(1))^{-1}\nu^{m-(k+1)/l_j-m_{(\xi^{0\prime},0)}(l_j)}d\lambda\\
&\quad+(2\pi i)^{-1}\int_{\mathscr{C}^+_{\xi^{0\prime}}}\lambda^k\partial/\partial\xi_n P(\nu^{-1}\xi^{0\prime}+\eta',\nu^{-1}\lambda)\\
&\quad\times(P_{\xi^{0\prime},q}(\eta',0;\lambda)+o(1))^{-1}\nu^{m-q-k-1}d\lambda\\
&\quad+\sum_{\xi_n^0\in\mathbf{R},\,\partial^q/\partial\xi_n^q P^0(\xi^{0\prime},\xi_n^0)=0}\Big[\sum_{j=t'(\xi^0)}^{s(\xi^0)}(2\pi i)^{-1}\\
&\quad\times\int_{\mathscr{C}^+_{\xi^0,j}}(\nu^{-1}\xi_n^0+\nu^{-1/l_j}\lambda)^k\partial/\partial\xi_n P(\nu^{-1}\xi^{0\prime}+\eta',\nu^{-1}\xi_n^0+\nu^{-1/l_j}\lambda)\\
&\quad\times(P_{l_j,\xi^0}(\eta',0;\lambda)+o(1))^{-1}\nu^{m-1/l_j-m_{\xi^0}(l_j)}d\lambda\\
&\quad+(1-\delta_{0r_{m_{\xi^0}}(\xi^0)})\times(2\pi i)^{-1}\int_{\mathscr{C}^+_{\xi^0,0}}(\nu^{-1}\xi_n^0+\lambda)^k\\
&\quad\times\partial/\partial\xi_n P(\nu^{-1}\xi^{0\prime}+\eta',\nu^{-1}\xi_n^0+\lambda)\\
&\quad\times(P_{\xi^0}(\eta',\lambda)+o(1))^{-1}\nu^{m-m_{\xi^0}}d\lambda\Big],
\end{aligned}
$$

where each $o(1)$ is a polynomial of η', λ and $\nu^{1/L}$ and vanishes for $\nu=0$ and L is a positive integer. So there exists ν_K (>0) such that each integrand in (2.20) can be expanded in a power series of $\nu^{1/L}$, which converges uniformly in $\eta'\in K$ and $0\le\nu\le\nu_K$. From this the lemma easily follows. Q.E.D.

Lemma 2.9. *Let $1\le k\le l$. For any compact set K in $\mathbf{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}}$, there exist ν_K and r_K (>0) such that $\sigma^k(\nu^{-1}r\xi^{0\prime}+r\eta')$ is well-defined* *when* $r_K\eta'\in\mathbf{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$, $\alpha\eta'\in K$ *for some* $\alpha\in\mathbf{C}$ $(|\alpha|=1)$, $0<\nu\leq\nu_K$ *and* $r\geq r_K$. *We have*

$$(\nu r^{-1})^{s_k}\sigma^k(\nu^{-1}r\xi^{0\prime}+r\eta')=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}r^{q_{kj}}\sigma^{ki}_{\xi^{0\prime},j}(\eta')\nu^{j/L}r^{-i},$$

$$\sigma^{k0}_{\xi^{0\prime},j}(\eta')\neq 0 \quad \textit{if } \sigma^{k}_{\xi^{0\prime},j}(\eta')\neq 0,$$

whose convergence is uniform in $\{(\eta',\nu,r);r_K\eta'\in\mathbf{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}},$ $\alpha\eta'\in K$ *for some* $\alpha\in\mathbf{C}$ $(|\alpha|=1)$, $0\leq\nu\leq\nu_K$ *and* $r\geq r_K\}$, *where the* q_{kj} *are rational numbers. Moreover the* $\sigma^{ki}_{\xi^{0\prime},j}(\eta')$ *are holomorphic in* $\mathbf{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}}$ *and homogeneous and*

$$\sigma^{k}_{\xi^{0\prime},j}(r\eta')=r^{q_{kj}+j/L}\sum_{i=0}^{\infty}\sigma^{ki}_{\xi^{0\prime},j}(\eta')r^{-i},$$

whose convergence is uniform in $\{(\eta',r);r_K\eta'\in\mathbf{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}},$ $\alpha\eta'\in K$ *for some* $\alpha\in\mathbf{C}$ $(|\alpha|=1)$ *and* $r\geq r_K\}$.

Proof. Modifying the curves $\mathscr{C}^+_\nu$, $\mathscr{C}^+_{(\xi^{0\prime},0),j}$, $\mathscr{C}^+_{\xi^{0\prime}}$ and $\mathscr{C}^+_{\xi^0,j}$ in the proof of Lemma 2. 8, we have

$$\text{(2. 21)}\quad \sigma^k(\nu^{-1}r\xi^{0\prime}+r\eta')=(2\pi_i)^{-1}\int_{\mathscr{C}^+_\nu}\lambda^k\partial/\partial\xi_nP(\nu^{-1}r\xi^{0\prime}+r\eta',\lambda)$$

$$\times P(\nu^{-1}r\xi^{0\prime}+r\eta',\lambda)^{-1}d\lambda$$

$$=[\sum_{j=1}^{t-1}(2\pi i)^{-1}\int_{\mathscr{C}^+_{(\xi^{0\prime},0),j}}\lambda^k\partial/\partial\xi_nP(\nu^{-1}r\xi^{0\prime}+r\eta',\nu^{-1/l_j}r\lambda)$$

$$\times(P^0_{l_j,(\xi^{0\prime},0)}(\eta',0;\lambda)+o(1))^{-1}\nu^{m-(k+1)/l_j-m_{(\xi^{0\prime},0)}(l_j)}d\lambda$$

$$+(2\pi i)^{-1}\int_{\mathscr{C}^+_{\xi^{0\prime}}}\lambda^k\partial/\partial\xi_nP(\nu^{-1}r\xi^{0\prime}+r\eta',\nu^{-1}r\lambda)$$

$$\times(P^0_{\xi^{0\prime},q}(\eta',0;\lambda)+o(1))^{-1}\nu^{m-k-1-q}d\lambda$$

$$+\sum_{\xi^0_n\in\mathbf{R},\,\partial^q/\partial\xi_1^qP^0(\xi^{0\prime},\xi^0_n)=0}\{\sum_{j=t'(\xi^0)}^{s(\xi^0)}(2\pi i)^{-1}$$

$$\times\int_{\mathscr{C}^+_{\xi^0,j}}(\nu^{-1}\xi^0_n+\nu^{-1/l_j}\lambda)^k$$

$$\times\partial/\partial\xi_nP(\nu^{-1}r\xi^{0\prime}+r\eta',\nu^{-1}r\xi^0_n+\nu^{-1/l_j}r\lambda)$$

$$\times(P^0_{l_j,\xi^0}(\eta',0;\lambda)+o(1))^{-1}\nu^{m-1/l_j-m_{\xi^0}(l_j)}d\lambda$$

$$+(1-\delta_{0r_{m_{\xi^0}(\xi^0)}})\times(2\pi i)^{-1}\int_{\mathscr{C}^+_{\xi^0,0}}(\nu^{-1}\xi_n^0+\lambda)^k$$

$$\times\partial/\partial\xi_n P(\nu^{-1}r\xi^{0\prime},\nu^{-1}r\xi_n^0+r\lambda)(P^0_{\xi^0}(\eta',\lambda)+o(1))^{-1}$$

$$\times\nu^{m-m_{\xi^0}}d\lambda\}]\times r^{-m+k+1},$$

where each $o(1)$ is a polynomial of η', λ, $\nu^{1/L}$ and r^{-1} and vanishes for $\nu=0$, $r^{-1}=0$. In fact, for example, we have

$$(\nu r^{-1})^m P(\nu^{-1}r\xi^{0\prime}+r\eta',\nu^{-1}r\xi_n^0+\nu^{-1/l_j}r\lambda)$$

$$=(\nu r^{-1})^{m_{\xi^0}(l_j)}P_{l_j,\xi^0}(r\eta',0;r^{(l_j-1)/l_j}\lambda)$$

$$+\sum_{n_h>m_{\xi^0}(l_j)}(\nu r^{-1})^{n_h}P_{l_j,\xi^0,h}(r\eta',0;r^{(l_j-1)/l_j}\lambda),$$

$$\deg_{l_j}P_{l_j,\xi^0,h}(\eta',0;\lambda)\le n_h\,.$$

Therefore we have

$$(\nu r^{-1})^m P(\nu^{-1}r\xi^{0\prime}+r\eta',\nu^{-1}r\xi_n^0+\nu^{-1/l_j}r\lambda)$$

$$=\nu^{m_{\xi^0}(l_j)}(P^0_{l_j,\xi^0}(\eta',0;\lambda)+o(1))\quad\text{as }\nu,\ r^{-1}\to 0\,.$$

So there exist ν_K and r_K (>0) such that each integrand in (2. 21) can be expanded in a power series of $\nu^{1/L}$ and r^{-1}, which converges uniformly in $\{(\eta',\nu,r);\eta'\in K,\ r\in\boldsymbol{C},\ \nu\in\boldsymbol{C},\ |r|\ge r_K\text{ and }0\le|\nu|\le\nu_K\}$. We note that

$$\sigma^{ki}_{\xi^{0\prime},j}(\alpha\eta')=\alpha^{q_{kj}+j/L-i}\sigma^{ki}_{\xi^{0\prime},j}(\eta')$$

when $\alpha\eta',\eta'\in\boldsymbol{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}}$, where $1^{q_{kj}+j/L-i}=1$. This completes the proof.

Q.E.D.

Let us consider $\dot{\Gamma}_{\xi^{0\prime}}$. Although $\Gamma(P_{(0,1)})=\Gamma((P_{(0,1)})_{(\xi^{0\prime},0)})$ does not always hold, we can prove the inner semi-continuity of $\dot{\Gamma}_{\xi'}$.

Lemma 2. 10. *Let $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$ and assume that $0<\rho<l_1(\xi^{0\prime},0)$. Then for any compact set $\widetilde{M}$ in $\Gamma((P_{(0,1)})_{(\xi^{0\prime},0)})$ there exist a neighborhood U of $\xi^{0\prime}$ and positive numbers r_0, t_0 such that*

$$P(r\xi'-irt\eta'-i\gamma_0\vartheta',r^{1/\rho}\lambda-irt\eta_n)\neq 0$$

when $\eta\in\widetilde{M}$, $\xi'\in U$, $\lambda\in\boldsymbol{R}$, $|\lambda|\ge 1$, $r\ge r_0$ and $0<t\le t_0$.

Proof. Put

$$f(\nu, t, \zeta', \lambda, s, t, \eta) = P(\nu^{-1}r\xi^{0\prime} + r\zeta' - irt\eta' - i(s+\gamma_0)\vartheta', \nu^{-1/\rho}r^{1/\rho}\lambda - irt\eta_n),$$

where $0<\nu\leq\nu_0$, $r\geq r_0$, $\zeta'\in\boldsymbol{R}^{n-1}$, $|\zeta'|\leq\varepsilon$, $\mathrm{Re}\, s\geq 0$, $\mathrm{Re}\, t\geq 0$, $|s|\leq s_0$, $|t|\leq t_0$, $\lambda\in\boldsymbol{R}$, $|\lambda|\geq 1$ and $\eta\in\widetilde{M}$. Then we have

$$\begin{aligned}(\nu r^{-1})^m f(\nu, r, \zeta', \lambda, s, t, \eta) &= (\nu r^{-1})^{m_\rho(\xi^{0\prime},0)} P_{\rho,(\xi^{0\prime},0)}(r\zeta' - irt\eta' - i(s+\gamma_0)\vartheta', -irt\eta_n; \lambda) \\ &+ \sum_{n_h > m_\rho(\xi^{0\prime},0)} (\nu r^{-1})^{n_h} P_{\rho,(\xi^{0\prime},0),h}(r\zeta' - irt\eta' - i(s+\gamma_0)\vartheta', -irt\eta_n; \lambda),\end{aligned}$$

$$\deg_\rho P_{\rho,(\xi^{0\prime},0),h}(\eta; \lambda) \leq n_h .$$

Since $0<\rho<l_1$,

$$P_{\rho,(\xi^{0\prime},0)}(\eta; \lambda) = (P_{m'})_{\xi^{0\prime}}(\eta')\lambda^{m'}.$$

Since the degree of $P_{\rho,(\xi^{0\prime},0),h}(\eta; \lambda)$ with respect to λ is not greater than m', it follows that

$$\begin{aligned}&\nu^{m-m_\rho(\xi^{0\prime},0)} r^{-m+m'(\rho-1)/\rho} \lambda^{-m'} f(\nu, r, \zeta', \lambda, s, t, \eta) \\ &= (P^0_{m'})_{\xi^{0\prime}}(\zeta' - it\eta' - ir^{-1}(s+\gamma_0)\vartheta') + o(1) \quad \text{as } \nu, r^{-1}\to 0 ,\end{aligned}$$

i.e. for any positive number δ there exist r_0, ν_0 (>0) such that

$$\begin{aligned}&|\nu^{m-m_\rho(\xi^{0\prime},0)} r^{-m+m'(\rho-1)/\rho} \lambda^{-m'} f(\nu, r, \zeta', \lambda, s, t, \eta) \\ &\quad - (P^0_{m'})_{\xi^{0\prime}}(\zeta' - it\eta' - ir^{-1}(s+\gamma_0)\vartheta')| < \delta\end{aligned}$$

when $0<\nu\leq\nu_0$, $r\geq r_0$, $|\zeta'|\leq\varepsilon$, $|s|\leq s_0$, $|\lambda|\geq 1$, $|t|\leq t_0$ and $\eta\in\widetilde{M}$. So we can apply the same argument as in Lemma 3. 7 in [7] to $f(\nu, r, \zeta', \lambda, s, t, \eta)$ and we obtain the lemma. Q.E.D.

Lemma 2. 11. *Let* $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$ *and* M *be a compact set in* $\dot{\Gamma}_{\xi^{0\prime}}$. *There exists a neighborhood* U *of* $\xi^{0\prime}$ such that

$$M\subset\dot{\Gamma}_{\xi'} \quad \textit{for } \xi'\in U .$$

Remark. From the proof of Lemma 2. 11 it follows that

$$\bigcup_{\xi_n\in\boldsymbol{R}} \Gamma(P_{(\xi^{0\prime},\xi_n)}) \supset \Gamma((P_{(0,1)})_{(\xi^{0\prime},0)}) .$$

Proof. Assume that there exists a sequence $\{\xi^j, \eta^{j\prime}\}_{j=1,2,\ldots}$ such that $|\xi^{j\prime}-\xi^{0\prime}|<1/j$, $\xi_n^j\in\boldsymbol{R}$, $\eta^{j\prime}\in M$ and $P_{\xi^j}^0(-i\eta^{j\prime}, 0)=0$. Then from the inner semi-continuity of $\dot{\Gamma}_\xi$ (or Γ_ξ) it follows that $|\xi_n^j|\to\infty$ as $j\to\infty$. Let $\widetilde{M}$ be a compact set in $\Gamma((P_{(0,1)})_{(\xi^{0\prime},0)})$ such that the interior of $\widetilde{M}$ includes $M\times\{0\}$. Lemma 2.10 implies that there exist a neighborhood U of $\xi^{0\prime}$ and λ_0, t_0 (>0) such that

$$P^0(\xi'-it\eta', \lambda-it\eta_n)\neq 0$$

when $\eta\in\widetilde{M}$, $\xi'\in U$, $\lambda\in\boldsymbol{R}$, $|\lambda|\geq\lambda_0$ and $0<t\leq t_0$, which leads us to a contradiction, using Rouché's theorem. Q.E.D.

§ 3. Proof of Theorem 1.1

Let $P(\xi)$ be written in the form

$$P(\xi)=\prod_{j=1}^{q} p_j(\xi)^{\nu_j},$$

where the $p_j(\xi)$ are irreducible polynomials. We assume that $\prod_{j=1}^{q'} p_j(\xi', \lambda)^{\nu_j}=0$ has roots $\lambda_1^+(\xi'), \cdots, \lambda_l^+(\xi')$ when $\xi'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\Gamma_0$, i.e. $\prod_{j=q'+1}^{q} p_j(\xi', \lambda)^{\nu_j}=0$ does not have roots with positive imaginary part when $\xi'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\Gamma_0$. Then put

$$\dot{\Gamma}_{\xi^{0\prime}}=\bigcap_{j=1}^{q'}\{\bigcap_{\xi_n^0\in\boldsymbol{R}}\dot{\Gamma}((p_j)_{\xi^0})\cap\dot{\Gamma}(((p_j)_{(0,1)})_{(\xi^{0\prime},0)})\}.$$

We note that

$$((p_j)_{(0,1)})_{(\xi^{0\prime},0)}(\eta)=((p_j)_{(0,1)})_{(\xi^{0\prime},\xi_n)}(\eta) \quad \text{for all } \xi_n\in\boldsymbol{R}.$$

The following lemma is obvious.

Lemma 3.1. $\oint B_j(\xi', \lambda)\lambda^{k-1}P_+(\xi', \lambda)^{-1}d\lambda$ *is a polynomial of* ξ' *and* $\sigma^k(\xi')$, $1\leq k\leq l$, *when* $P_+(\xi', \lambda)$ *is well-defined.*

From Lemma 2.8 we have the following

Lemma 3.2. *Let* $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$. *For any compact set* K *in* $\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$ *there exists* ν_K (>0) *such that* $R(\nu^{-1}\xi^{0\prime}+\eta')$ *is well-defined for* $\eta'\in K$ *and* $0<\nu\leq\nu_K$ *and*

$$\nu^{h_{\xi^{0\prime}}} R(\nu^{-1}\xi^{0\prime}+\eta')=\sum_{j=0}^{\infty}\nu^{j/L}R_{\xi^{0\prime},j}(\eta'),$$

$$R_{\xi^{0\prime},0}(\eta')\equiv R_{\xi^{0\prime}}(\eta')\not\equiv 0,$$

whose convergence is uniform in $(\eta',\nu)\in K\times\{0\leq\nu\leq\nu_K\}$, *where* $h_{\xi^{0\prime}}$ *is a rational number and L is a positive integer. Moreover the* $R_{\xi^{0\prime},j}(\eta')$ *are holomorphic in* $\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$.

Remark. $R_{\xi^{0\prime},0}(\eta')\equiv R_{\xi^{0\prime}}(\eta')$ is the localization of $R(\xi')$ at $\xi^{0\prime}$. Moreover this lemma for $\xi^{0\prime}=0$ implies that $R(\xi')$ is holomorphic in $\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}$ (see [3]).

The following lemma is also obtained by Lemma 2. 9.

Lemma 3. 3. *Let* $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$. *For any compact set* K *in* $\boldsymbol{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}}$, *there exist* ν_K *and* r_K (>0) *such that* $R(\nu^{-1}r\xi^{0\prime}+r\eta')$ *is well-defined when* $r_K\eta'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$, $\alpha\eta'\in K$ *for some* $\alpha\in\boldsymbol{C}$ $(|\alpha|=1)$, $0<\nu\leq\nu_K$ *and* $r\geq r_K$. *We have*

$$(\nu r^{-1})^{h_{\xi^{0\prime}}}R(\nu^{-1}r\xi^{0\prime}+r\eta')=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}r^{h_j(\xi^{0\prime})-i}\nu^{j/L}R^{i}_{\xi^{0\prime},j}(\eta'),$$

$$R^{0}_{\xi^{0\prime},j}(\eta')\not\equiv 0\quad \textit{if } R_{\xi^{0\prime},j}(\eta')\not\equiv 0,$$

whose convergence is uniform in $\{(\eta',\nu,r);r_K\eta'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}},$ $\alpha\eta'\in K$ *for some* $\alpha\in\boldsymbol{C}$ $(|\alpha|=1)$, $0\leq\nu\leq\nu_K$ *and* $r\geq r_K\}$, *where the* $h_j(\xi^{0\prime})$ *are rational numbers. Moreover the* $R^{i}_{\xi^{0\prime},j}(\eta')$ *are holomorphic in* $\boldsymbol{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}}$ *and homogeneous and*

$$R_{\xi^{0\prime},j}(r\eta')=r^{h_j(\xi^{0\prime})+j/L}\sum_{i=0}^{\infty}r^{-i}R^{i}_{\xi^{0\prime},j}(\eta'),$$

whose convergence is uniform in $\{(\eta',r);r_K\eta'\in\boldsymbol{R}^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}},$ $\alpha\eta'\in K$ *for some* $\alpha\in\boldsymbol{C}$ $(|\alpha|=1)$ *and* $r\geq r_K\}$.

Remark. The principal part $(R_{\xi^{0\prime}})^0(\eta')$ of $R_{\xi^{0\prime}}(\eta')$ is equal to $R^{0}_{\xi^{0\prime},0}(\eta')$. Moreover this lemma for $\xi^{0\prime}=0$ implies Lemma 3. 2 in [3].

In the above two lemmas we can replace $R(\xi')$ by $R_{jk}(\xi')$ or $P_{\pm}(\xi',\lambda)$ with obvious modifications.

Lemma 3.4. *Let $\xi^0 \in \boldsymbol{R}^n$. There exist the localizations $P_{\pm\xi^0}(\xi)$ and $(P^0_\pm)_{\xi^0}(\xi)$ of $P_\pm(\xi)$ and $P^0_\pm(\xi)$ at ξ^0, respectively, and*

$$P^0_{\pm\xi^0}(\xi) \equiv (P^0_\pm)_{\xi^0}(\xi) = (P_{\pm\xi^0})^0(\xi). \tag{3.1}$$

Proof. The existence of $P_{\pm\xi^0}(\xi)$, $(P^0_\pm)_{\xi^0}(\xi)$ and $(P_{\pm\xi^0})^0(\xi)$ follows from Lemmas 3.2 and 3.3 and the above remark. It easily follows that $P_{\xi^0}(\xi) = P_{+\xi^0}(\xi) P_{-\xi^0}(\xi)$, $(P_{\xi^0})^0(\xi) = (P^0)_{\xi^0}(\xi) = (P^0_+)_{\xi^0}(\xi)(P^0_-)_{\xi^0}(\xi)$ and $\deg^\dagger (P_{\pm\xi^0})^0(\xi) \leq \deg (P^0_\pm)_{\xi^0}(\xi)$. This implies (3.1). Q.E.D.

Let us denote by $\Gamma(R_{\xi^{0\prime}})$ the component of the set $\{\eta' \in \dot{\Gamma}_{\xi^{0\prime}};\ (R_{\xi^{0\prime}})^0(-i\eta') \neq 0\}$ which contains $\vartheta'^{\dagger\dagger}$. Then we have the following

Lemma 3.5 ([8]). *Let $\xi^{0\prime} \in \boldsymbol{R}^{n-1}$. $\Gamma(R_{\xi^{0\prime}})$ is an open convex cone and*

$$R_{\xi^{0\prime}}(\xi') \neq 0 \qquad \textit{for } \xi' \in \boldsymbol{R}^{n-1} - i\gamma_1\vartheta' - i\Gamma(R_{\xi^{0\prime}}),$$
$$(R_{\xi^{0\prime}})^0(\xi') \neq 0 \qquad \textit{for } \xi' \in \boldsymbol{R}^{n-1} - i\Gamma(R_{\xi^{0\prime}}).$$

Let us denote by $\Gamma(P_{+\xi^0})$ the component of the set $\{\eta \in \dot{\Gamma}_{\xi^{0\prime}} \times \boldsymbol{R};\ P^0_{+\xi^0}(-i\eta) \neq 0\}$ which contains ϑ. Then we have also the following

Lemma 3.6. *Let $\xi^0 \in \boldsymbol{R}^n$. $\Gamma(P_{+\xi^0})$ is an open convex cone and*

$$P_{+\xi^0}(\xi) \neq 0 \qquad \textit{for } \xi \in \boldsymbol{R}^n - i\gamma_0\vartheta - i\Gamma(P_{+\xi^0}),$$
$$P^0_{+\xi^0}(\xi) \neq 0 \qquad \textit{for } \xi \in \boldsymbol{R}^n - i\Gamma(P_{+\xi^0}),$$
$$\Gamma(P_{+\xi^0}) \supset \Gamma_{\xi^0}.$$

In our case we can prove Lemma 3.2 in [8].

Lemma 3.7. *Let $\xi^{0\prime} \in \boldsymbol{R}^{n-1}$ and let M be a compact set in $\dot{\Gamma}_{\xi^{0\prime}}$. Then there exist a conic neighborhood Δ_1 $(\subset \boldsymbol{R}^{n-1})$ of $\xi^{0\prime}$ and positive numbers C, t_0 such that $P_+(\zeta', \lambda)$ is holomorphic in $(\zeta', \lambda) \in \Lambda \times \boldsymbol{C}$, where*

$$\Lambda = \{\zeta' = \xi' - it|\xi'|\eta' - i\gamma_0\vartheta';\ \xi' \in \Delta_1,\ |\xi'| \geq C,\ \eta' \in M \ \textit{and}\ 0 < t \leq t_0\}.$$

† $\deg p^0(\xi)$ denotes the degree of homogeneity of p^0.

†† $(R_{\xi^{0\prime}})^0(-i\vartheta') \neq 0$ was shown in [7].

Therefore $R(\zeta')$ *and* $R_{jk}(\zeta')$ *are also holomorphic in* Λ.

Proof. The lemma is trivial for $\xi^{0\prime}=0$. So we assume that $\xi^{0\prime}\in\boldsymbol{R}^{n-1}\setminus\{0\}$. Let $\lambda(\zeta')$ be a root of $p_j(\zeta',\lambda)=0$. We can assume that $\lambda(\zeta')$ is continuous in Λ when C and t_0 are suitably chosen. In fact, there exist a conic neighborhood Δ_1 ($\subset\boldsymbol{R}^{n-1}$) of $\xi^{0\prime}$ and C, t_0 (>0) such that

$$p_{j(0,1)}(\xi-it|\xi|\eta-i\gamma_0\vartheta)\neq 0 \text{ if } \xi'\in\Delta_1,\ |\xi'|\geq C,\ \eta'\in M \text{ and } 0<t\leq t_0.$$

For $p_{j(0,1)}(\xi)$ is independent of ξ_n and $M\subset\dot{\Gamma}_{\xi^{0\prime}}\subset\dot{\Gamma}((p_{j(0,1)})_{(\xi^{0\prime},0)})$. The argument in Section 2 shows that $\lim_{\nu\to 0}\nu\lambda(\nu^{-1}\xi^{0\prime}-i\eta'-i\gamma_0\vartheta')=\mu_0$ exists if $\lim_{\nu\to 0}|\mathrm{Im}\,\nu\lambda(\nu^{-1}\xi^{0\prime}-i\eta'-i\gamma_0\vartheta')|=0$, where $\eta'\in M$. Moreover from Lemmas 2.4 and 2.7 it follows that μ_0 is a real root of $\partial^q/\partial\xi_1^q p_j^0(\xi^{0\prime},\lambda)=0$. Now let us assume that μ_0 is a real multiple root of $\partial^q/\partial\xi_1^q p_j^0(\xi^{0\prime},\lambda)=0$. We can assume without loss of generality that M is small so that $\{(\eta',\eta_n);\ \eta'\in M \text{ and } \eta_n^1\leq\eta_n\leq\eta_n^2\}\subset\Gamma(p_{j(\xi^{0\prime},\mu_0)})$ for some η_n^1, $\eta_n^2\in\boldsymbol{R}$. Then it follows that there exist a conic neighborhood $\tilde{\Delta}$ of $(\xi^{0\prime},\mu_0)$ and C, t_0 (>0) such that

$$p_j(\xi'-it|\xi'|\eta'-i\gamma_0\vartheta',\lambda-it|\xi'|\eta_n)\neq 0$$
$$\text{if } (\xi',\lambda)\in\tilde{\Delta},\ |\xi'|\geq C,\ \eta'\in M,\ \eta_n^1\leq\eta_n\leq\eta_n^2 \text{ and } 0<t\leq t_0.$$

This implies that

$$\mathrm{Im}\,\lambda(\xi'-it|\xi'|\eta'-i\gamma_0\vartheta')\notin[-it|\xi'|\eta_n^2,\ -t|\xi'|\eta_n^1]$$

for $\xi'\in\Delta_1$, $|\xi'|\geq C$, $\eta'\in M$ and $0<t\leq t_0$, modifying Δ_1, C and t_0, if necessary (see Lemma 3.2 in [8]). If $\vartheta'\notin M$, we choose a continuous curve $\eta'(\theta)$ in $\dot{\Gamma}_{\xi^{0\prime}}$ such that $\eta'(0)=\vartheta'$ and $\eta'(1)\in M$ and we repeat the above argument for each small neighborhood of $\eta'(\theta)$, $0\leq\theta\leq 1$. This proves the lemma (see Lemma 3.2 in [8]). Q.E.D.

Put

$$t_j\equiv t_j(\xi^{0\prime})=h_{\xi^{0\prime}}+h_j(\xi^{0\prime}),$$

where $h_{\xi^{0\prime}}$ and $h_j(\xi^{0\prime})$ are defined in Lemma 3.3. Then it is easy to see that $t_j(\xi^{0\prime})$ is an integer and that $t_j(\xi^{0\prime})\leq t_0(0)$ †. Put

$$t\equiv t(\xi^{0\prime})=\max t_j(\xi^{0\prime}),\ \omega\equiv\omega(\xi^{0\prime})=\min\{j;\, t(\xi^{0\prime})=t_j(\xi^{0\prime})\}.$$

It easily follows that $t(\xi^{0\prime})=t_0(0)$. From Lemma 3. 3 we have the following

Lemma 3. 8. *Let $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$. The principal part $R^0(\xi')$ of $R(\xi')$ is well-defined and there exists the localization $(R^0)_{\xi^{0\prime}}(\eta')$ of $R^0(\xi')$ at $\xi^{0\prime}$. Moreover for any compact set K in $\boldsymbol{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}}$ there exists ν_K (>0) such that*

$$\nu^{h_{\xi^{0\prime}}-t_0(0)-\omega(\xi^{0\prime})/L}R^0(\xi^{0\prime}+\nu\eta') = \sum_{t_j(\xi^{0\prime})=t_0(0)}\nu^{(j-\omega(\xi^{0\prime}))/L}R^0_{\xi^0,j}(\eta'), \tag{3. 2}$$

whose convergence is uniform in $\{(\eta',\nu);\,\eta'\in\boldsymbol{R}^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}},\ \alpha\eta'\in K$ for some $\alpha\in\boldsymbol{C}$ $(|\alpha|=1)$ and $0\leq\nu\leq\nu_K\}$, and

$$(R^0)_{\xi^{0\prime}}(\eta')=R^0_{\xi^{0\prime},\omega(\xi^{0\prime})}(\eta').$$

Let $\Gamma((R^0)_{\xi^{0\prime}})$ be the component of the set $\{\eta'\in\dot{\Gamma}_{\xi^{0\prime}};\,(R^0)_{\xi^{0\prime}}(-i\eta')\neq0\}$ which contains ϑ' ††.

Lemma 3. 9 ([8]). *Let $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$. $\Gamma((R^0)_{\xi^{0\prime}})$ is an open convex cone and*

$$(R^0)_{\xi^{0\prime}}(\xi')\neq0 \text{ for } \xi'\in\boldsymbol{R}^{n-1}-i\Gamma((R^0)_{\xi^{0\prime}}).$$

Lemma 3. 10 ([8]). *Let $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$. For any compact set M in $\Gamma((R^0)_{\xi^{0\prime}})$, there exist a conic neighborhood Δ_1 $(\subset\boldsymbol{R}^{n-1})$ of $\xi^{0\prime}$ and positive numbers C, t_0 such that*

$$R(\xi'-it|\xi'|\eta'-i\gamma_1\vartheta')\neq0 \text{ if } \eta'\in M,\ \xi'\in\Delta_1,\ |\xi'|\geq C \text{ and } 0<t\leq t_0.$$

Let $\tilde{\xi}^0\in\boldsymbol{R}^{n+1}$ and put

† $R(r\eta')=r^{t_0(0)}\sum_{i=0}^{\infty}r^{-i}R^i_{\xi^{0\prime},0}(\eta')$.

†† $(R^0)_{\xi^{0\prime}}(-i\vartheta')\neq0$ was shown in [8].

(3.3) $$\Gamma_{\tilde{\xi}^0}=\{\tilde{\eta}\in \mathbb{R}^{n+1};\ (\eta',\eta_{n+1})\in\Gamma(P_{(\xi^{0\prime},\xi^0_{n+1})})\} \cap(\Gamma(P_{+\xi^0})\times\mathbb{R})\cap(\Gamma(R_{\xi^{0\prime}})\times\mathbb{R}^2),$$

(3.4) $$\Gamma^0_{\tilde{\xi}^0}=\{\tilde{\eta}\in \mathbb{R}^{n+1};\ (\eta',\eta_{n+1})\in\Gamma(P_{(\xi^{0\prime},\xi^0_{n+1})})\} \cap(\Gamma(P_{+\xi^0})\times\mathbb{R})\cap(\Gamma((R^0)_{\xi^{0\prime}})\times\mathbb{R}^2).$$

Then Theorem 1.1 can be proved by the same arguments as in [7], [8]. (1.2) follows from Lemma 4.1.

§4. Some Remarks and Examples

Lemma 4.1 ([8]). *Let* $\xi^{0\prime}\in\mathbb{R}^{n-1}$. $\Gamma((R^0)_{\xi^{0\prime}})\subset\Gamma(R_{\xi^{0\prime}})$.

Let us prove the inner semi-continuity of $\Gamma((R^0)_{\xi'})$ and, therefore, $\Gamma^0_{\tilde{\xi}}$.

Lemma 4.2. *Let* $\xi^{0\prime}\in\mathbb{R}^{n-1}$ *and let* M *be a compact set in* $\Gamma((R^0)_{\xi^{0\prime}})$. *Then there exist a neighborhood* U *of* $\xi^{0\prime}$ *and positive number* t_0 *such that* $R^0(\xi')$ *is holomorphic in* $U-iD$ *and* $R^0(\xi')\neq 0$ *for* $\xi'\in U-iD$, *where* $D=\{t\eta';\ \eta'\in\mathring{M}$ *and* $0<t\leq t_0\}$†.

Proof. We can assume without loss of generality that $P(\xi)$ is irreducible. Since $M\subset\mathring{\Gamma}((P_{(0,1)})_{(\xi^{0\prime},0)})$, it follows that there exist a neighborhood U of $\xi^{0\prime}$ and t_0, ν_0 (>0) such that

$$P_{(0,1)}(\nu^{-1}\xi)=P_{m'}(\nu^{-1}\xi')\neq 0$$

if $\xi'\in U-iD-i\nu_1\gamma_0\vartheta'$, $0<\nu\leq\nu_1\leq\nu_0$. Let K be a compact set in $U-iD$. Then there exists ν_K (>0) such that $\nu_K\leq\nu_0$ and $K\subset U-iD-i\nu_K\gamma_0\vartheta'$. Let $\lambda_j^{\mp}(\xi';\nu)$ be a root of $P(\nu^{-1}\xi',\nu^{-1}\lambda)=0$ such that $\lambda_j^{\mp}(\xi';\nu)=\nu\lambda_j^{\mp}(\nu^{-1}\xi')$ when $\xi'\in K$ and $0<|\nu|\leq\nu_K$. In fact, since $P_{m'}(\nu^{-1}\xi')\neq 0$ for $\xi'\in K$ and $0<|\nu|\leq\nu_K$, modifying ν_K if necessary, the above statement is meaningful. Moreover we can assume that $\lambda_j^{\mp}(\xi';\nu)$ is continuous when $\xi'\in K$ and $0\leq|\nu|\leq\nu_K$. Since $\lambda_j^{\pm}(\xi';0)$ is a root of $P^0(\xi',\lambda)=0$, the same argument as in Lemma 3.7 gives

† $\mathring{M}$ denotes the interior of M.

$$\{\lambda_j^+(\xi';0)\}\cap\{\lambda_j^-(\xi';0)\}=\emptyset \quad \text{for } \xi'\in K,$$

modifying U and t_0 if necessary. Therefore it follows from cntinuity of $\lambda_j^\pm(\xi';\nu)$ that

$$\text{(4. 1)} \qquad \{\lambda_j^+(\xi';\nu)\}\cap\{\lambda_j^-(\xi';\nu)\}=\emptyset \quad \text{for } \xi'\in K \text{ and } |\nu|\leq\nu_K,$$

modifying ν_K if necessary. Put

$$P_+(\xi',\lambda;\nu)=\prod_{j=1}^{l}(\lambda-\lambda_j^+(\xi';\nu))=\lambda^l+b_1^+(\xi';\nu)\lambda^{l-1}+\cdots+b_l^+(\xi';\nu),$$

$$P_+(\xi',\lambda)=\prod_{j=1}^{l}(\lambda-\lambda_j^+(\xi'))=\lambda^l+a_1^+(\xi')\lambda^{l-1}+\cdots+a_l^+(\xi').$$

(4. 1) implies that the $b_j^+(\xi';\nu)$ are holomorphic in $\{(\xi',\nu);\xi'\in K$ and $|\nu|\leq\nu_K\}$. Moreover we have $a_j^+(\nu^{-1}\xi')=\nu^{-j}b_j^+(\xi';\nu)$. Therefore we have

$$b_j^+(\xi';\nu)=a_{j0}^+(\xi')+\nu a_{j1}^+(\xi')+\nu^2a_{j2}^+(\xi')+\cdots,$$

whose convergence is uniform in $\{(\xi',\mu);\xi'\in K$ and $|\nu|\leq\nu_K\}$. $a_{jk}^+(\xi')$ is holomorphic in $U-iD$ and homogeneous of degree $j-k$. So $R^0(\xi')$ is well-defined and holomorphic in $U-iD$. (3. 2) and the above result yields us

$$R^0(\xi')\neq 0 \quad \text{for } \xi'\in U-iD,$$

using the same argument as in the proof of Lemma 3. 7 in [8].

Q.E.D.

Theorem 4. 3. *Let $\xi^{0\prime}\in\boldsymbol{R}^{n-1}$ and let M be a compact set in $\Gamma((R^0)_{\xi^{0\prime}})$. There exists a neighborhood U of $\xi^{0\prime}$ such that*

$$M\subset\Gamma((R^0)_{\xi'}) \quad \textit{for } \xi'\in U.$$

Proof. It is obvious that $M\subset\dot{\Gamma}_{\xi'}$ for $\xi'\in U$, shrinking U. Now assume that there exist $\xi^{1\prime}\in U$ and $\eta^{0\prime}\in M$ such that $(R^0)_{\xi^{1\prime}}(-i\eta^{0\prime})=0$, where U is sufficiently small. Since $(R^0)_{\xi^{1\prime}}(-i\eta')\not\equiv 0$, there exists $\zeta^{0\prime}\in\boldsymbol{C}^{n-1}$ such that $\xi^{1\prime}-i(\eta^{0\prime}+\mu\zeta^{0\prime})\in U-iM$ for $|\mu|\leq 1$ and $(R^0)_{\xi^{1\prime}}(-i(\eta^{0\prime}+\zeta^{0\prime}))\neq 0$. Therefore it follows that there exist ε, δ (>0) such that

$$|(R^0)_{\xi^{1\prime}}(-i(\eta^{0\prime}+\mu\zeta^{0\prime}))|\geq 2\varepsilon \quad \text{for } |\mu|=\delta.$$

On the other hand from (3. 2) we have

$$|t^{h_{\xi^{1\prime}}-t_0(0)-\omega(\xi^{1\prime})/L}R^0(\xi^{1\prime}-it(\eta^{0\prime}+\mu\zeta^{0\prime}))$$
$$-(R^0)_{\xi^{1\prime}}(-i(\eta^{0\prime}+\mu\zeta^{0\prime}))|<\varepsilon \quad \text{for } |\mu|=\delta \text{ and } 0<t\leq t_1 \ (\leq t_0),$$

where t_0 and t_1 are suitably chosen. Rouché's theorem implies that $R^0(\xi^{1\prime}-it(\eta^{0\prime}+\mu\zeta^{0\prime}))$ has zeros within $|\mu|<\delta$ for $0<t\leq t_1$, which is a contradiction to Lemma 4. 2. Q.E.D.

Theorem 4. 3 yields us the following

Theorem 4. 4. $\bigcup_{\xi\in\boldsymbol{R}^{n+1}\setminus\{0\}} K_\xi^0\times\{\tilde{\xi}\}$ *is closed in* $T^*X\setminus 0$.

In Section 2 the developments of $\sigma^k(\nu^{-1}\xi^{0\prime}+\eta')$ and $\sigma^k(\nu^{-1}r\xi^{0\prime}+r\eta')$ was given. However we can similarly obtain the developments $f(\nu^{-1}\xi^{0\prime}+\eta')$ and $f(\nu^{-1}r\xi^{0\prime}+r\eta')$, where

$$f(\xi')=(2\pi i)^{-1}\int_{\mathscr{C}^+} g(\xi',\lambda)P(\xi',\lambda)^{-1}d\lambda$$

and $g(\xi',\lambda)$ is a polynomial of (ξ',λ) and $\mathscr{C}^+$ encloses only the roots $\lambda_1^+(\xi'),\cdots,\lambda_l^+(\xi')$ of $P(\xi',\lambda)=0$. This will be useful for hyperbolic systems.

Next let us consider some examples.

Example 4. 5. Put $n=4$ and

$$P(\xi)=(\xi_1^2-\xi_2^2-\xi_3^2-\xi_4^2+a\xi_3)(\xi_1^2-\xi_4^2), \qquad a>0,$$
$$B_1(\xi)=1,\ B_2(\xi)=(-\xi_1-i\xi_3)\xi_4-\xi_4^2.$$

Then we have $R(\xi')=i\xi_3+\sqrt[+]{\xi_1^2-\xi_2^2-\xi_3^2+a\xi_3}$. It is obvious that $\{P,B_1,B_2\}$ satisfies the condition (A). We can show that $\bigcup_{\xi\in\boldsymbol{R}^5\setminus\{0\}} K_\xi\times\{\tilde{\xi}\}$ is not closed in $T^*X\setminus 0$ and that

$$\bigcup_{\xi\in\boldsymbol{R}^5\setminus\{0\}}\bigcup_{j=0}^{\infty}\operatorname{supp}\tilde{F}_{\xi,j}\times\{\tilde{\xi}\}=\bigcup_{\xi\in\boldsymbol{R}^5\setminus\{0\}}K_\xi\times\{\tilde{\xi}\}$$
$$\subsetneq WF(\tilde{F})\subset WF_A(\tilde{F})\subset\bigcup_{\xi\in\boldsymbol{R}^5\setminus\{0\}}K_\xi^0\times\{\tilde{\xi}\}$$

(see [9]). Moreover we have

$$\overline{\text{ch}}[WF(\tilde{F})|_{\tilde{\xi}^0}]=\overline{\text{ch}}[WF_A(\tilde{F})|_{\tilde{\xi}^0}]=K_{\tilde{\xi}^0}^0 \qquad \text{for } \tilde{\xi}^0\neq 0.$$

Example 4. 6. Put $n=3$ and

$$P(\xi)=((\xi_1-\xi_2)^2-\xi_3^2+a)((2\xi_1-\xi_2)^2-\xi_3^2),$$

$$B_1(\xi)=1, \quad B_2(\xi)=\xi_3.$$

Then $R(\xi')=-1$ and $\{P, B_1, B_2\}$ satisfies the condition (A). We note that $(\xi_1-\xi_2)^2-\xi_3^2+a$ is irreducible when $a\neq 0$. It is easy to see that

$$WF(\tilde{F})|_{(1,1,-1,1)}=\{\tilde{z}\in X; \tilde{z}=\alpha(2,-1,0,-1)+\beta(2,-1,1,0)$$
$$+\gamma(1,-1,0,0), \alpha, \beta>0 \text{ and } \gamma\geq 0\} \text{ when } a\neq 0,$$
$$WF(\tilde{F})|_{(1,1,-1,1)}=\{\tilde{z}\in X; \tilde{z}=\alpha(2,-1,0,-1)+\beta(2,-1,1,0)$$
$$\text{and } \alpha, \beta>0\} \text{ when } a=0.$$

This shows that so called lateral wave appears when $a\neq 0$.

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