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Commutativity up to a Factor: More Results and the Unbounded Case

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Abstract. We give more results on the question of commutativity up to a factor for bounded operators and which has been recently of interest to a number of mathematicians. We also give some generalizations to unbounded operators.

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1. Introduction

The problem of commutativity up to a factor has been of interest recently to many authors thanks to its direct applications to quantum mechanics. Broadly speaking, in some situations two operators A and B do not commute, i.e., $BA \neq AB$ but instead, they satisfy a relation of the form $BA = \lambda AB$ for some complex number λ different from zero.

Brooke, Busch and Pearson proved in [1] the following theorem:

Theorem 1.1. *Let A, B be bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. Then:*

1. *if A or B is self-adjoint, then $\lambda \in \mathbb{R}$;*
2. *if both A and B are self-adjoint, then $\lambda \in \{-1, 1\}$;*
3. *if A and B are self-adjoint and one of them is positive, then $\lambda = 1$.*

Yang and Du [13] improved some results in the previous theorem and using the Fuglede–Putnam theorem they arrived at

Theorem 1.2. *Let A, B be bounded operators such that $AB = \lambda BA \neq 0$, $\lambda \in \mathbb{C}^*$. Then:*

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1. *if A or B is self-adjoint, then $\lambda \in \mathbb{R}$;*
2. *if either A or B is self-adjoint and the other is normal, then $\lambda \in \{-1, 1\}$;*
3. *if A and B are both normal, then $|\lambda| = 1$.*

The natural generalization to Banach algebras was carried out by Schmoeger in [12]. The other natural generalization, i.e., to unbounded operators is, in part, the purpose of this paper. We also note that in the bounded case if A and B are such that $BA = \lambda AB$, then setting $B = I$ (the identity operator) we see that $\lambda = 1$ with no extra assumption on A. This observation means that there is hope of doing more and we in effect can do more, i.e., we can still obtain the same conclusions with different and/or weaker hypotheses. The main tools needed to achieve this aim are the following:

Lemma 1.3 (Embry [3]). *If H and K are commuting normal operators and $AH = KA$, where 0 is not in $W(A)$, then $H = K$.*

Theorem 1.4 (Fuglede–Putnam [4,8]). *If A, N and M are bounded operators such that M and N are normal, then*

$$AN = MA \Longrightarrow AN^* = M^*A,$$

and if N and M are unbounded, then "$=$" is replaced by "$\subset$" in the last displayed equation.

Theorem 1.5 (Mortad [6]). *Assume that N, H and K are unbounded operators having the property: $N = HK = KH$ are normal. Also assume that $D(H) \subset D(K)$. Assume further that A is a bounded operator for which $0 \notin W(A)$ and such that $AH \subset KA$. Then $H = K$.*

The main results in this present paper are as follows: We improve some results obtained in Theorems 1.1 & 1.2. We then generalize them to unbounded operators.

Throughout this paper the numerical range of an operator A defined on a Hilbert space $\mathcal{H}$, i.e., the set $\{\langle Af, f\rangle : f \in \mathcal{H}\}$, will be denoted by $W(A)$.

Finally, we assume the reader is familiar with notions and results about bounded and unbounded linear operators in a Hilbert space. Some general references are [2, 5, 9].

2. Improving the bounded case

We begin with the following improvement of some parts of Theorem 1.1.

Proposition 2.1. *Assume that A and B are two bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. If A or B is normal and the other does not have 0 in its numerical range, then $\lambda = 1$.*

Proof. The proof is based on Lemma 1.3. Since B is normal, λB is normal and it obviously commutes with B. As $0 \notin W(A)$, then Lemma 1.3 gives us $B = \lambda B$ and hence $\lambda = 1$. The proof is very similar if one assumes that A is normal and that $0 \notin W(B)$. □

We have the following corollary which is yet another improvement of the third assertion of Theorem 1.1.

Corollary 2.2. *Let A and B be two bounded operators such that $AB \neq 0$ and $AB = \lambda BA$, $\lambda \in \mathbb{C}^*$. If A or B is normal and the other is strictly positive, then $\lambda = 1$.*

Proof. Assume that A is strictly positive, i.e., $A > 0$, and that B is normal. Hence $0 \notin W(A)$. Since B is normal, the foregoing proposition then applies. □

Remark 2.3. The previous corollary allows us to give a new proof of (3) of Theorem 1.2 which goes as follows (it also uses the Fuglede–Putnam theorem): Assume that A and B are normal, then λB is normal. Whence:

$$AB = \lambda BA \Rightarrow AB = (\lambda B)A \Rightarrow AB^* = \overline{\lambda} B^* A \Rightarrow AB^* B = \overline{\lambda} B^* AB = |\lambda|^2 B^* BA.$$

But B^*B is self-adjoint and positive and A is normal, hence the previous corollary applies and we obtain $|\lambda|^2 = 1$ or $|\lambda| = 1$.

Proposition 2.4. *Assume that A, B and C are bounded operators on a Hilbert space such that $AB = \lambda CA \neq 0$. If B and C are self-adjoint, then $\lambda \in \mathbb{R}$.*

Proof. Since B and C are self-adjoint, B and λC are normal and applying the Fuglede–Putnam theorem gives us $AB = \overline{\lambda} CA$. This, combined with $AB = \lambda CA$ yields $\lambda = \overline{\lambda}$, i.e., $\lambda \in \mathbb{R}$. □

Remark 2.5. The result does not hold in general if only A is assumed to be self-adjoint. First, the method of proof uses the Fuglede–Putnam theorem and we would need in this case a four-operator version of the this well-known theorem which does not exist (cf. [7]). We may also illustrate this more by the following example:

Example 2.6. Take $\lambda \in \mathbb{C}^*$ and consider

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then A is self-adjoint and $AB = \lambda CA$ ($\neq 0$) but λ is arbitrary.

3. The unbounded case

Now we pass to the case where one of the operators is unbounded. We have

Theorem 3.1. *Let A be an unbounded operator and let B be a bounded one. Assume that $BA \subset \lambda AB \neq 0$ where $\lambda \in \mathbb{C}$. Then:*

1. *λ is real if A is self-adjoint;*
2. *$\lambda = 1$ if $0 \notin W(B)$ (the numerical range of B) and if A is normal; hence $\lambda = 1$ if B is strictly positive and A is normal;*
3. *$\lambda \in \{-1, 1\}$ if A is normal and B is self-adjoint.*

Proof. 1. Since $BA \subset \lambda AB$ and since A is self-adjoint (and hence A and λA are normal), the Fuglede–Putnam theorem yields $BA \subset \overline{\lambda} AB$. Now for $f \in D(A) = D(BA) \subset D(\lambda AB) = D(\overline{\lambda} AB)$, one has

$$\lambda ABf = \overline{\lambda} ABf.$$

Hence λ is real as $AB \neq 0$.

2. Let us prove the first part of the assertion. Since A is normal, so is λA. Besides $\lambda AA = A\lambda A = \lambda A^2$. Since $0 \notin W(B)$, Theorem 1.5 yields $\lambda = 1$.

Now we prove the second assertion. We note that B cannot have 0 in its numerical range as B is strictly positive. Since A is self-adjoint, λA is normal and hence Theorem 1.5 gives $A = \lambda A$ which, in its turn, gives $\lambda = 1$.

3. One has

$$BA \subset \lambda AB \Longrightarrow B^2 A \subset \lambda BAB \subset \lambda^2 AB^2.$$

Since B is self-adjoint, B^2 is positive and by 2) of this theorem we obtain that $\lambda^2 = 1$. Thus $\lambda = 1$ or $\lambda = -1$. □

Remark 3.2. The question of whether the result in 3) remains valid for normal B and self-adjoint A is open. Another natural question is whether one can prove that λ lies on the unit circle if A and B are both normal.

Remark 3.3. The relation $AB = \lambda BA$, $\lambda \notin \mathbb{R}$, has no *bounded* self-adjoint operators A and B verifying $AB \neq 0$. However, the relation $AB = \lambda BA$, with $|\lambda| = 1$, has representations by *unbounded* self-adjoint operators A and B (see [10, 11]). Such unbounded operators are the "natural" representations of this relation.

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