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Time Decay and Scattering for Some Nonlinear Wave Equation

By

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§ 1. Introduction

Recently J. M. Chadam proved a global existence and uniqueness theorem to the Cauchy problem for nonlinear Klein-Gordon equations (in three dimensional space) with nonlinear term $G(x, t, u, u_x, u_t)$ [1] and discussed also scattering theory for them [2].

The scattering theory for those equations have been studied by [4], [5], [9], [10], [11], [12], etc. On the other hand, M. Reed in his lecture notes [8] developed an abstract theory of global existence-uniqueness and scattering theory for nonlinear wave equations having a nonlinear Klein-Gordon equation as a specific example

$$\Box u+m^2u=-u^p, \quad x\in \boldsymbol{R}^3$$

in which we are interested.

Our purpose is to prove an abstract existence-uniqueness theorem and to discuss scattering theory for nonlinear wave equations, by extending the method in [8] so that equations of the form

$$\Box u+m^2u+g_1u^\beta+g_2u_t^\gamma=0$$

are included as examples.

First we introduce some notations. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}=\|\cdot\|$ and let the following auxiliary norms on $\mathcal{H}$ be given: for each $j=1, 2$ $\|\cdot\|_{a_j}$ satisfies all the norm conditions except that $\|\phi\|_{a_j}=0$ implies $\phi=0$, and $\|\cdot\|_{b_j}$ satisfies all the norm conditions except that it

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may take the value $+\infty$. Let A be a self-adjoint operator on $\mathcal{H}$ and J_j be nonlinear mappings: $\mathcal{H}\to\mathcal{H}$ which satisfy the following hypotheses: for $\phi\in\mathcal{H}$ and each $j=1, 2$,

(H1) there is a positive constant C such that

$$\|\phi\|_{a_j}\leqq C\|\phi\|\ ,$$

(H2) there are positive constants C and d_j such that

$$\|e^{-iAt}\phi\|_{a_j}\leqq C|t|^{-d_j}\|\phi\|_{b_j}\quad \text{when } |t|\geqq 1$$

and

(H3) there exist positive constants h_j, δ, p (with $p\geqq 1, d_1p>1$) and q (with $q\geqq 1, d_2q>1$) so that for all $\phi, \psi\in\mathcal{H}$ satisfying $\|\phi\|\leqq\delta$, $\|\psi\|\leqq\delta$, there hold

$$\|J_1(\phi)-J_1(\psi)\|\leqq h_1\sum_{\varepsilon_1}(\|\phi\|_{a_1}+\|\psi\|_{a_1})^{p-\varepsilon_1}\|\phi-\psi\|_{a_1}^{\varepsilon_1}\|\phi-\psi\|^{1-\varepsilon_1},$$

$$\|J_2(\phi)-J_2(\psi)\|\leqq h_2\sum_{\varepsilon_2}(\|\phi\|_{a_2}+\|\psi\|_{a_2})^{q-\varepsilon_2}\|\phi-\psi\|_{a_2}^{\varepsilon_2}\|\phi-\psi\|^{1-\varepsilon_2},$$

$$\|J_1(\phi)-J_1(\psi)\|_{b_j}\leqq h_1\sum_{\sigma_j}(\|\phi\|_{a_1}+\|\psi\|_{a_1})^{p-\sigma_j}\|\phi-\psi\|_{a_1}^{\sigma_j}\|\phi-\psi\|^{1-\sigma_j}$$

and

$$\|J_2(\phi)-J_2(\psi)\|_{b_j}\leqq h_2\sum_{\rho_j}(\|\phi\|_{a_2}+\|\psi\|_{a_2})^{q-\rho_j}\|\phi-\psi\|_{a_2}^{\rho_j}\|\phi-\psi\|^{1-\rho_j}$$

where ε_j, σ_j and ρ_j run through some finite sets in $[0, 1]$. Here in case $p=1$ or $q=1$, we assume each h_j can be chosen arbitrarily small if δ is chosen small.

Now we define the scattering states Σ_{scat} with norm $\|\cdot\|_{\text{scat}}$ by

$$\Sigma_{\text{scat}}=\{\phi\in\mathcal{H};\ \|\phi\|_{\text{scat}}<\infty\}$$

where

$$\|\phi\|_{\text{scat}}=|||e^{-itA}\phi|||,\quad |||\cdot|||\quad \text{being defined by}$$

$$|||\psi|||=\sup_{\boldsymbol{R}}(\|\psi(r)\|+\sum_{j=1}^{2}(1+|r|)^{d_j}\|\psi(r)\|_{a_j})$$

for $\psi(r)$, a $\mathcal{H}$-valued function on $\boldsymbol{R}$. We observe that for all $t\in\boldsymbol{R}$

$$\|e^{-itA}\phi\|_{a_j}\leqq C(1+|t|)^{-d_j}(\|\phi\|+\|\phi\|_{b_j}),$$

if $\|\phi\|_{b_j}<\infty$ with $j=1, 2$. For simplicity we often use

p_j as $p_1=p$ and $p_2=q$; w_j as $w_1=\sigma_1$ and $w_2=\rho_1$; and z_j as $z_1=\sigma_2$ and $z_2=\rho_2$.

§ 2. Abstract Theorems

The following lemma will be frequently used.

Lemma 2.1 ([10], [8]). (a) If $a, b>0$ and $\max(a, b)>1$, then

$$\int_{-\infty}^{\infty}(1+|t-s|)^{-a}(1+|s|)^{-b}ds\leqq C(1+|t|)^{-\min(a,b)}$$

(b) If $a, b>0$ and $\max(a, b)>\min(a, b)$, then

$$\sup_{\boldsymbol{R}}(1+|r|)^{\min(a,b)}\int_{t_1}^{t_2}(1+|r-s|)^{-a}(1+|s|)^{-b}ds$$

tends to 0 as $t_1, t_2\to+\infty$ (or $-\infty$).

Our first main theorem is

Theorem 2.2 (*global existence and uniqueness for small data*). *Assume* A, J_j *and the norms* $\|\cdot\|$, $\|\cdot\|_{a_j}$, $\|\cdot\|_{b_j}$ *satisfy* (H1), (H2) *and* (H3) *in* § 1, *and let* $J=J_1+J_2$. *Then there exists an* $\eta_0>0$ *such that for all* $\phi_-\in\Sigma_{\text{scat}}$ *with* $\|\phi_-\|_{\text{scat}}\leqq\eta_0$, *the integral equation*

$$\phi(t)=e^{-itA}\phi_-+\int_{-\infty}^{t}e^{-iA(t-s)}J(\phi(s))\,ds \tag{1}$$

has a solution in $C((-\infty, \infty); \mathcal{H})$ *with* $|||\phi(\cdot)|||\leqq2\eta_0$, *and the solution is unique when* $p, q>1$, $d_1p>d_2$ *and* $d_2q>d_1$.

Proof. (I) Existence. As our method is based on the contraction mapping principle, we introduce a complete metric space $X(\eta, \phi_-)$ with norm $|||\cdot|||$ defined by

$$X(\eta, \phi_-)=\{\psi(t); \psi(t)\in C((-\infty, \infty); \mathcal{H}) \text{ and } |||\psi-e^{-itA}\phi_-|||\leqq\eta\}$$

where $\|\phi_-\|_{\text{scat}}\leqq\eta\leqq\delta$, δ being a constant in (H3). Let $\mathcal{G}$ be an operator on $X(\eta, \phi_-)$ given by

$$(\mathcal{J}\psi)(t)=\int_{-\infty}^{t}e^{-iA(t-s)}J(\psi(s))ds \quad \text{for} \quad \psi(\cdot)\in X(\eta,\phi_{-}).$$

It can easily be seen that $e^{-iA(t-s)}J(\psi(s))$ is a continuous function of s for each fixed t and that

$$|||\psi|||\leqq|||e^{iAt}\phi_{-}|||+\eta\leqq 2\eta .$$

Under the hypotheses (H1), (H2) and (H3) we have the following estimates.

(2)
$$\|J_{j}(\psi(s))\|\leqq\sum_{\varepsilon_j}h_{j}\|\psi(s)\|_{a_j}^{p_j}\|\psi(s)\|^{1-\varepsilon_j}$$
$$\leqq\sum_{\varepsilon_j}h_{j}(2\eta)^{p_j+1-\varepsilon_j}(1+|s|)^{-d_jp_j} \quad \text{for} \quad j=1,2 .$$

and (since $d_jp_j>1$)

$$\|(\mathcal{J}\psi)(t)\|\leqq\int_{-\infty}^{\infty}\|J(\psi(s))\|ds\leqq c_0(h_1,h_2,\eta)<+\infty .$$

(3)
$$\|(\mathcal{J}\psi)(t)\|_{a_1}\leqq\int_{-\infty}^{\infty}\|e^{-iA(t-s)}J(\psi(s))\|_{a_1}ds$$
$$\leqq c_1(h_1,h_2,\eta)(1+|t|)^{-d_1} \quad \text{if} \quad d_2p_2\geqq d_1 .$$

(4)
$$\|(\mathcal{J}\psi)(t)\|_{a_2}\leqq\int_{-\infty}^{\infty}\|e^{-iA(t-s)}J(\psi(s))\|_{a_2}ds$$
$$\leqq c_2(h_1,h_2,\eta)(1+|t|)^{-d_2} \quad \text{if} \quad d_1p_1\geqq d_2 .$$

Here we remark that all the constants $c_j(h_1,h_2,\eta)$ depend on h_1,h_2 and η so that c_j's tend to 0 when $h_1,h_2\to 0$ or $\eta\to 0$. In fact, when $d_2p_2\geqq d_1$

$$\int_{-\infty}^{\infty}\|e^{-iA(t-s)}J(\psi(s))\|_{a_1}ds$$
$$\leqq c\int_{-\infty}^{\infty}(1+|t-s|)^{-d_1}(\|J(\psi(s))\|+\|J(\psi(s))\|_{b_1})ds$$
$$\leqq c\sum_{j=1}^{2}h_j\int_{-\infty}^{\infty}(1+|t-s|)^{-d_1}\|\psi(s)\|_{a_j}^{p_j}$$
$$\times(1+\sum_{\varepsilon_j}\|\psi(s)\|^{1-\varepsilon_j}+\sum_{w_j}\|\psi(s)\|^{1-w_j})ds$$
$$\leqq c\sum_{j=1}^{2}h_j(2\eta)^{p_j}(1+\sum_{\varepsilon_j}(2\eta)^{1-\varepsilon_j}+\sum_{w_j}(2\eta)^{1-w_j})$$
$$\times\int_{-\infty}^{\infty}(1+|t-s|)^{-d_1}(1+|s|)^{-d_jp_j}ds$$

$$\leqq c_2(h_1, h_2, \eta)\,(1+|t|)^{-d_1}.$$

For the inequality (4), a required estimate can be done analogously. It follows $|||\mathcal{J}\psi|||<\infty$ from (2), (3), (4).

Now we define a mapping $M: X(\eta, \phi_-)\to X(\eta, \phi_-)$ by

$$(M\psi)\,(t) = e^{-iAt}\phi_- + (\mathcal{J}\psi)\,(t).$$

We want to show that M is a contraction mapping on $X(\eta, \phi_-)$ provided η is taken to be sufficiently small. Similar calculations to (2), (3), (4) lead to the inequalities:

$$(2') \qquad \|(\mathcal{J}\phi)\,(t) - (\mathcal{J}\psi)\,(t)\| \leqq c_0'(h_1, h_2, \eta)\,|||\phi-\psi|||$$

$$(3') \qquad \|(\mathcal{J}\phi)\,(t) - (\mathcal{J}\psi)\,(t)\|_{a_1} \leqq (1+|t|)^{-d_1} c_1'(h_1, h_2, \eta)\,|||\phi-\psi|||$$

and also

$$(4') \qquad \|(\mathcal{J}\phi)\,(t) - (\mathcal{J}\psi)\,(t)\|_{a_2} \leqq (1+|t|)^{-d_2} c_2'(h_1, h_2, \eta)\,|||\phi-\psi|||.$$

Hence from (2′), (3′) and (4′) we obtain for $\phi, \psi\in X(\eta, \phi_-)$

$$|||M\phi - M\psi||| \leqq \{\sum_{j=0}^{2} c_j'(h_1, h_2, \eta)\}\,|||\phi-\psi|||.$$

Then we take η so small that $\sum_{j=0}^{2} c_j'(h_1, h_2, \eta)<1$ and fix an η, say η_0. Thus M is a contraction mapping on $X(\eta, \phi_-)$ for all positive $\eta\leqq\eta_0$. This guarantees existence of a unique fixed point ϕ in $X(\eta, \phi_-)$, which gives a global solution of (1) with $|||\phi|||\leqq 2\eta_0$ by definition of M.

(II) Global uniqueness. Let $X(T, \alpha, \phi_0)$ be the set

$$\{\phi\in C([0, T]; \mathcal{H}); \phi(0) = \phi_0,\ \sup_{[0,T]} \|\phi(t) - e^{-iAt}\phi_0\| \leqq \alpha\},$$

where α is an arbitrary fixed positive number. First we note that local uniqueness is true for the equation

$$\phi(t) = e^{-iAt}\phi_0 + \int_0^t e^{-iA(t-s)} J(\phi(s))\,ds$$

in $X(T, \alpha, \phi_0)$, since, by virtue of (H1), (H2) and (H3),

$$\|\phi(t) - \psi(t)\| \leqq \int_0^t \|e^{-iA(t-s)}(J(\phi(s)) - J(\psi(s)))\|\,ds$$

$$\leqq \sum_{j=1}^{2} h_j \sum_{\varepsilon_j} \int_0^t (\|\phi\|_{a_j} + \|\psi\|_{a_j})^{p_j-\varepsilon_j}$$

$$\times \|\phi(s)-\psi(s)\|_{a_j}^{\varepsilon_j}\|\phi(s)-\psi(s)\|^{1-\varepsilon_j}ds$$

$$\leqq c\sum_{j=1}^{2} h_j \sum_{\varepsilon_j}\int_0^t (\|\phi\|+\|\psi\|)^{p_j-\varepsilon_j}\|\phi(s)-\psi(s)\|ds \quad \text{for} \quad t>0\,.$$

Now we shall prove global uniqueness. Let $\phi_1(t)$ be another solution of (1). Then by using (2′), (3′) and (4′) we obtain the inequalities:

$$\|\phi(t)-\phi_1(t)\|\leqq\sum_{j=1}^{2} h_j(|||\phi|||+|||\phi_1|||)^{p_j-\varepsilon_j} \times\int_{-\infty}^{t}(1+|s|)^{-d_jp_j}ds\,|||\phi-\phi_1|||, \tag{5}$$

$$\begin{aligned}(1+|t|)^{d_1}\|\phi(t)-\phi_1(t)\|_{a_1} &\leqq c\sum_{j=1}^{2} h_j\Big(\sum_{\varepsilon_j}(|||\phi|||+|||\phi_1|||)^{p_j-\varepsilon_j}+\sum_{w_j}(|||\phi|||+|||\phi_1|||)^{p_j-w_j}\Big)\\ &\times\left[\int_{-\infty}^{t}(1+|t|)^{d_1}(1+|t-s|)^{-d_1}(1+|s|)^{-d_jp_j}ds\right]|||\phi-\phi_1|||\end{aligned} \tag{6}$$

and

$$\begin{aligned}(1+|t|)^{d_2}\|\phi(t)-\phi_1(t)\|_{a_2} &\leqq c\sum_{j=1}^{2} h_j\Big(\sum_{\varepsilon_j}(|||\phi|||+|||\phi_1|||)^{p_j-\varepsilon_j}+\sum_{z_j}(|||\phi|||+|||\phi_1|||)^{p_j-z_j}\Big)\\ &\times\left[\int_{-\infty}^{t}(1+|t|)^{d_2}(1+|t-s|)^{-d_2}(1+|s|)^{-d_jp_j}ds\right]|||\phi-\phi_1|||.\end{aligned} \tag{7}$$

Hence follows

$$|||\phi-\phi_1|||\leqq f(t)\,|||\phi-\phi_1||| \tag{8}$$

from (5), (6), (7) for a continuous function $f(t)$. This function $f(t)$ tends to 0 as $t\to-\infty$ when $d_1p>d_2$, $d_2q>d_1$ and $p, q>1$, which is a consequence of Lemma 2.1 (b).

Combining (8) with local uniqueness yields the required uniqueness. Thus our proof is completed.

Our second main theorem is

Theorem 2.3 (*the scattering operator for small data*). *Let all the hypotheses of Theorem 2.2 be satisfied and let* $\phi(t)$ *be the solution of* (1) *with Cauchy dat* $\phi_-\in\Sigma_{\text{scat}}$ *satisfying* $\|\phi_-\|_{\text{scat}}\leqq\eta_0$. *Then*

(a) *for each* t, $\phi(t)\in\Sigma_{\mathrm{scat}}$ *and*

$$\|\phi(t)-e^{-itA}\phi_-\|\to 0 \quad as\ t\to-\infty,$$

(b) *there exists* $\phi_+\in\Sigma_{\mathrm{scat}}$ *with* $\|\phi_+\|_{\mathrm{scat}}\leqq 2\eta_0$ *such that*

$$\|\phi(t)-e^{-itA}\phi_+\|\to 0 \quad as\ t\to+\infty$$

and in case $p,q>1$, $d_1p>d_2$, $d_2q>d_1$

(c) $\|e^{itA}\phi(t)-\phi_\pm\|_{\mathrm{scat}}\to 0$ *as* $t\to\pm\infty$

(d) $S:\phi_-\to\phi_+$ *is an injective continuous map of the set* $\{\psi\in\Sigma_{\mathrm{scat}};\ \|\psi\|_{\mathrm{scat}}\leqq\eta_0\}$ *into the set* $\{\|\psi\|_{\mathrm{scat}}\leqq 2\eta_0\}$ *in the* $\|\cdot\|_{\mathrm{scat}}$-*topology*.

Proof. (a) and a part of (c). To show $|||e^{-irA}\phi(t)|||<+\infty$ for each t, we do estimations just as in (2′), (3′), (4′).

$$\sup_{r\in\boldsymbol{R}}\|e^{-irA}\phi(t)\|\leqq\sup_{r\in\boldsymbol{R}}\|e^{-irA}e^{-itA}\phi_-\|+\sup_{r\in\boldsymbol{R}}\int_{-\infty}^{t}\|e^{-iA(t-s+r)}J(\phi(s))\|ds$$

$$\leqq\sup_{r\in\boldsymbol{R}}\|e^{-irA}\phi_-\|+\sum_{j=1}^{2}h_j(2\eta_0)^{p_j+1-\varepsilon_j}\int_{-\infty}^{t}(1+|s|)^{-d_jp_j}ds$$

$$\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_1}\|e^{-irA}\phi(t)\|_{a_1}\leqq\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_1}\|e^{-iAr}e^{-iAt}\phi_-\|_{a_1}$$

$$+\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_1}\int_{-\infty}^{t}\|e^{-iA(t-s+r)}J(\phi(s))\|_{a_1}ds$$

$$\leqq\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_1}(1+|t+r|)^{-d_1}\|\phi_-\|_{\mathrm{scat}}$$

$$+c\sum_{j=1}^{2}h_j(2\eta_0)^{p_j}[1+(2\eta_0)^{1-\varepsilon_j}+(2\eta_0)^{1-w_j}]$$

$$\times\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_1}\int_{-\infty}^{t}(1+|t+r-s|)^{-d_1}(1+|s|)^{-d_jp_j}ds$$

$$\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_2}\|e^{-irA}\phi(t)\|_{a_2}\leqq\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_2}(1+|t+r|)^{-d_2}\|\phi_-\|_{\mathrm{scat}}$$

$$+c\sum_{j=1}^{2}h_j(2\eta_0)^{p_j}[1+(2\eta_0)^{1-\varepsilon_j}+(2\eta_0)^{1-w_j}]$$

$$\times\sup_{r\in\boldsymbol{R}}(1+|r|)^{d_2}\int_{-\infty}^{t}(1+|t+r-s|)^{-d_2}(1+|s|)^{-d_jp_j}ds.$$

Thus, using Lemma 2. 1, we conclude

$$\|\phi(t)\|_{\mathrm{scat}}\leqq c_1\|\phi_-\|_{\mathrm{scat}}+c_2<\infty \quad \text{for each } t$$

where c_1 does not depend on t but c_2 does. Similarly, when $d_1p_1>d_2$,

$d_2p_2>d_1,\ p_1, p_2>1$

$$\|e^{-irA}(\phi(t)-e^{-itA}\phi_-)\|=\|\phi(t)-e^{-itA}\phi_-\|$$

$$\leqq\int_{-\infty}^{t}\|J(\phi(s))\|ds\leqq\sum_{j=1}^{2}h_j(2\eta_0)^{p_j+1-\varepsilon_j}\int_{-\infty}^{t}(1+|s|)^{-d_jp_j}ds\to 0$$

$$\text{as}\quad t\to-\infty;$$

$$(1+|r|)^{d_1}\|e^{-iAr}(\phi(t)-e^{-itA}\phi_-)\|_{a_1}$$

$$\leqq(1+|r|)^{d_1}\int_{-\infty}^{t}\|e^{-iA(r-s)}J(\phi(s))\|_{a_1}ds$$

$$\leqq c(1+|r|)^{d_1}\sum_{j=1}^{2}\int_{-\infty}^{t}(1+|r-s|)^{-d_1}(1+|s|)^{-d_jp_j}ds\to 0\ \text{ as } t\to-\infty;$$

Just as above,

$$(1+|r|)^{d_2}\|e^{-iAr}(\phi(t)-e^{-itA}\phi_-)\|_{a_2}\to 0\quad\text{as}\quad t\to-\infty\,.$$

Hence we proved

$$\|\phi(t)-e^{-itA}\phi_-\|\to 0$$

$$\|\phi(t)-e^{-itA}\phi_-\|_{\text{scat}}\to 0$$

when $p_1, p_2>1,\ d_1p_1>d_2,\ d_2p_2>d_1$.

(b) Since $|||\phi|||\leqq 2\eta_0$ by Theorem 2. 2 and since for $t_1>t_2$

$$\|e^{it_1A}\phi(t_1)-e^{it_2A}\phi(t_2)\|\leqq\int_{t_1}^{t_2}\|e^{isA}J(\phi(s))\|ds$$

$$\leqq\sum_{j=1}^{2}h_j(2\eta_0)^{p_j+1-\varepsilon_j}\int_{t_1}^{t_2}(1+|s|)^{-d_jp_j}ds\to 0\quad\text{as}\quad t_1, t_2\to\infty\,,$$

there exists $\lim_{t\to\infty} e^{itA}\phi_-=\phi_+$ in $\mathcal{H}$, and

$$\|e^{itA}\phi(t)-\phi_+\|=\|\phi(t)-e^{-itA}\phi_+\|\to 0$$

as $t\to+\infty$. To show $\|\phi_+\|_{\text{scat}}\leqq 2\eta_0$, letting $t\to+\infty$ (in the $\|\cdot\|$-topology) in the equation

$$e^{itA}\phi(t)=\phi_-+\int_{-\infty}^{t}e^{isA}J(\phi(s))\,ds\,,$$

we observe that

$$(*)\qquad \phi_+=\phi_-+\int_{-\infty}^{\infty}e^{isA}J(\phi(s))\,ds\,.$$

Then the same techniques as in (a) can be applied to (*) to obtain $\|\phi_+\|_{\text{scat}} \leqq 2\eta_0$ for sufficiently small η_0.

Also replacing each $(e^{it_2A}\phi(t_2) - e^{it_1A}\phi(t_1))$ and $\int_{t_1}^{t_2}$ for $(\phi(t) - e^{-itA}\phi_-)$ and $\int_{-\infty}^{t}$ respectively gives

$$\lim_{t\to\infty} \|e^{itA}\phi(t) - \phi_+\|_{\text{scat}} = 0 .$$

Thus (b) and the remaining part of (c) are proved.

Finally we turn to prove (d). That S is injective on $\{\|\psi\|_{\text{scat}} \leqq \eta_0\}$ can be derived in a similar way to the uniqueness proof in Theorem 2.2 as follows. Note that the solution $\phi(t)$ of (1) satisfies

$$\phi(t) = e^{-i(t-\tau)A}\phi(\tau) + \int_\tau^t e^{-i(t-s)A} J(\phi(s))\, ds . \tag{9}$$

Let ϕ_- and ψ_- correspond to ϕ_+ and let $\phi(t)$ and $\psi(t)$ be the corresponding solutions of (1) to Cauchy data ϕ_- and ψ_- respectively. Then by (9) we have

$$\phi(\tau) - \psi(\tau) = e^{i(t-\tau)A}(\phi(t) - \psi(t)) - \int_\tau^t e^{-i(\tau-s)A}(J(\phi(s)) - J(\psi(s)))\, ds .$$

Estimations similar to (5), (6), (7) give

$$\|\phi(\tau) - \psi(\tau)\| \leqq \|\phi(t) - \psi(t)\| + C(\phi,\psi)|||\phi-\psi|||_{[\tau,\infty)} \sum_j \int_\tau^t (1+|s|)^{-d_jp_j} ds ,$$

$$(1+|\tau|)^{d_j}\|\phi(\tau) - \psi(\tau)\|_{a_j} \leqq (1+|\tau|)^{d_j}\|e^{-i\tau A}(e^{-itA}\phi(t) - e^{itA}\psi(t))\|$$

$$+ C(\phi,\psi)|||\phi-\psi|||_{[\tau,\infty)} (1+|\tau|)^{d_j} \int_\tau^t (1+|\tau-s|)^{-d_j}(1+|s|)^{-d_jp_j} ds$$

from which follows

$$|||\phi-\psi|||_{[\tau,\infty)} \leqq \|e^{itA}\phi(t) - e^{itA}\psi(t)\|_{\text{scat}} \tag{10}$$

$$+ C(\phi,\psi)|||\phi-\psi|||_{[\tau,\infty)} \sum_j (1+|\tau|)^{d_j} \int_\tau^t (1+|\tau-s|)^{-d_j}(1+|s|)^{-d_jp_j} ds$$

where $$|||\phi|||_{[\tau,\infty]} = \sup_{\tau\leqq r<\infty} \left(\|\phi(r)\| + \sum_{j=1}^{2}(1+|r|)^{d_j}\|\phi(r)\|_{a_j}\right).$$

Then by Lemma 2.1(b), for any $\varepsilon > 0$ there exists $\tau_0 > 0$ such that

$$|||\phi-\psi|||_{[\tau,\infty)} \leqq \|e^{itA}\phi(t) - e^{itA}\psi(t)\|_{\text{scat}} + \varepsilon|||\phi-\psi|||_{[\tau,\infty)}$$

when $\tau \geqq \tau_0$. Letting $t\to\infty$, by (b) we have

$$|||\phi-\psi|||_{[\tau,\infty)} \leqq \varepsilon |||\phi-\psi|||_{[\tau,\infty)} \quad \text{for} \quad \tau \geqq \tau_0$$

which is false unless $|||\phi-\psi|||_{[\tau,\infty)} \equiv 0$ $(\tau \geqq \tau_0)$. Thus $\phi(t) \equiv \psi(t)$ for $\tau \geqq \tau_0$ Now given $\varepsilon > 0$, choose $\delta > 0$ so that

$$\sup_{\tau_0-\delta \leqq s \leqq \tau_0} \|\phi(s) - \psi(s)\| < \varepsilon .$$

Then

$$0 = \phi(\tau_0) - \psi(\tau_0) = e^{-iA(\tau_0-\tau)}(\phi(\tau) - \psi(\tau)) + \int_\tau^{\tau_0} e^{-iA(\tau_0-s)}(J(\phi(s)) - J(\psi(s)))\,ds .$$

Hence if $\tau_0 - \delta \leqq \tau \leqq \tau_0$,

$$\|\phi(\tau) - \psi(\tau)\| \leqq \int_\tau^{\tau_0} \|J(\phi(s)) - J(\psi(s))\|\,ds \leqq C\delta \sup_{\tau_0-\delta \leqq \tau \leqq \tau_0} \|\phi(\tau) - \psi(\tau)\|$$

where C depends only on η. Thus δ chosen so small as $C\delta < 1$, the inequalities are false unless $\sup_{\tau_0-\delta \leqq \tau \leqq \tau_0} \|\phi(\tau) - \psi(\tau)\| = 0$ which proves $\phi \equiv \psi$ when $\tau \geqq \tau_0 - \delta$. Therefore $\phi \equiv \psi$ in $\boldsymbol{R}$ and so $\phi_- = \psi_-$.

To show the continuity of S on $\{\|\psi\|_{\text{scat}} \leqq \eta_0\}$, we need estimate $\|\phi_+^{(1)} - \phi_+^{(2)}\|_{\text{scat}}$ where $\phi_+^{(j)} = S\phi_-^{(j)}$ for $j = 1, 2$. As is easily seen from estimates (2), (3), (4), it follows that

$$\|\phi_1(t) - \phi_2(t)\| \leqq \|\phi_-^{(1)} - \phi_-^{(2)}\| + C(\eta_0) |||\phi_1 - \phi_2|||,$$

$$(1+|t|)^{d_j} \|\phi_1(t) - \phi_2(t)\|_{a_j} \leqq (1+|t|)^{d_j} \|e^{-iAt}(\phi_-^{(1)} - \phi_-^{(2)})\|_{a_j} + C(\eta_0) |||\phi_1 - \phi_2||| \quad \text{for} \quad j = 1, 2$$

if $p, q > 1$, $d_1 p > d_2$ and $d_2 q > d_1$, where $C(\eta_0) \to 0$ as $|\eta_0| \to 0$. Therefore we have

$$|||\phi_1 - \phi_2||| \leqq \|\phi_-^{(1)} - \phi_-^{(2)}\|_{\text{scat}} + 3C(\eta_0) |||\phi_1 - \phi_2|||$$

or

$$|||\phi_1 - \phi_2||| \leqq \frac{1}{1 - 3C(\eta_0)} \|\phi_-^{(1)} - \phi_-^{(2)}\|_{\text{scat}} .$$

Similarly

$$(1+|r|)^{d_j} \|e^{-iAr}(e^{iAt}\phi_1(t) - e^{iAt}\phi_2(t))\|_{a_j} \leqq (1+|r|)^{d_j} \|e^{-iAr}(\phi_-^{(1)} - \phi_-^{(2)})\|_{a_j} + C(\eta_0) |||\phi_1 - \phi_2|||$$

when $p, q > 1$, $d_2 q > d_1$ and $d_1 p > d_2$. Consequently

$$\|e^{-iAt}\phi_1(t)-e^{iAt}\phi_2(t)\|_{\text{scat}}\leqq\|\phi_-^{(1)}-\phi_-^{(2)}\|_{\text{scat}}+3C(\eta_0)|||\phi_1-\phi_2|||$$

$$\leqq\frac{1}{1-3C(\eta_0)}\|\phi_-^{(1)}-\phi_-^{(2)}\|_{\text{scat}}\leqq 2\|\phi_-^{(1)}-\phi_-^{(2)}\|_{\text{scat}}$$

if $|\eta_0|$ is taken so small as $3C(\eta_0)\leqq 1/2$. Taking $t\to\infty$ we have

$$\|\phi_+^{(1)}-\phi_+^{(2)}\|_{\text{scat}}\leqq 2\|\phi_-^{(1)}-\phi_-^{(2)}\|_{\text{scat}}$$

for $\phi_-^{(j)}\in\{\|\phi\|_{\text{scat}}\leqq\eta_0\}$, which proves continuity of S.

Thus the proof has been completed.

§ 3. Applications

We consider a nonlinear Klein-Gordon equation

(12) $$\Box u+m^2u+g_1u^\beta+g_2u_t^\gamma=0,\quad x\in\mathbf{R}^n$$

where $n=1$ or 3, and g_1, g_2 are the coupling constants. Let

$$A=i\begin{pmatrix}0 & I\\ -B^2 & 0\end{pmatrix}$$

where B^2 is the self-adjoint realization of $m^2I-\Delta$ in $L^2=L^2(\mathbf{R}^n)$, and let $J=J_1+J_2$ where

$$J_1(\phi)=\begin{pmatrix}0\\ -g_1u^\beta\end{pmatrix},\quad J_2(\phi)=\begin{pmatrix}0\\ -g_2u_t^\gamma\end{pmatrix}\quad\text{for}\quad \phi(t)=\begin{pmatrix}u(\cdot,t)\\ u_t(\cdot,t)\end{pmatrix}.$$

Then the equation (12) is written as

$$\phi'(t)=-iA\phi(t)+J(\phi(t))$$

in its standard vector valued form. Further, this equation can be reformulated as an integral equation

(13) $$\phi(t)=e^{-itA}\phi_-+\int_{-\infty}^t e^{-iA(t-s)}J(\phi(s))\,ds$$

provided that $\phi(t)$ has the Cauchy data ϕ_- at $t=-\infty$. Now we define, for $n=3$, the solution space $\mathscr{H}$ by the completion of $D(B^3)\oplus D(B^2)$ with respect to the inner product

$$(\phi,\psi)=(B^3u_1,B^3u_2)+(B^2v_1,B^2v_2)$$

where (,) denotes the usual inner product in L^2 and

$$\phi=\begin{pmatrix}u_1\\v_1\end{pmatrix},\quad \psi=\begin{pmatrix}u_2\\v_2\end{pmatrix}.$$

Moreover we define the auxiliary norms $\|\phi\|_{a_j}$, $\|\phi\|_{b_j}$ in $\mathcal{H}$ for $\phi=\begin{pmatrix}u\\v\end{pmatrix}$ $j=1, 2$ by

$$\|\phi\|_{a_1}=\|u\|_\infty,\quad \|\phi\|_{a_2}=\|v\|_\infty;\ \|\phi\|_{b_1}=\|B^3u\|_1+\|B^2v\|_1,$$

$$\|\phi\|_{b_2}=\|B^3u\|_r+\|B^2v\|_r\quad (1<r<2).$$

The following three lemmas are useful in our applications.

Lemma 3.1 (*Sobolev's inequality*). *Let m and n be positive numbers and let p be defined by*

$$1/p=1/2-m/n.$$

Then for all $u\in C_0(\boldsymbol{R}^n)$, there is a constant C such that

$$\|u\|_p\leqq C\|B^m u\|_2,$$

where we take $p=\infty$ in case $m>n/2$.

Lemma 3.2 (*Nelson's theorem*). *Let $W_{t,a}(x)$, $x\in\boldsymbol{R}^n$, be the functions whose Fourier transforms are*

$$\widehat{W}_{t,a}(y)=(m^2+y^2)^{-a}\exp[-it(m^2+y^2)^{1/2}]$$
$$=\widehat{E}_{t,a}-i\widehat{F}_{t,a}$$

where $E_{t,a}=(m^2-\Delta)^{-a}\cos t(m^2-\Delta)^{1/2}$ and

$$F_{t,a}=(m^2-\Delta)^{-a}\sin t(m^2-\Delta)^{1/2}.$$

Fix $a>(n-1)/4$ and $2\leqq p<\infty$. Then for every $t\neq 0$, $W_{t,a}$ is in $L^p(\boldsymbol{R}^n)$ if and only if $2a-(n+1)/2>-1/p$, and in this case

$$\|W_{t,a}\|_p\simeq t^{n/p-n/2}\ \text{if}\ s(p,a)>-2$$
$$\simeq(\log t)^{1/p}t^{n/p-n/2}\ \text{if}\ s(p,a)=-2, p>2$$
$$\simeq t^{(1-2a)}t^{(n-2)/p}\ \text{if}\ s(p,a)<-2$$

where $s(p,a)=p(2a-(n+2)/2)$ and $f(t)\simeq g(t)$ means that $f(t)/g(t)=0(1)$ and $g(t)/f(t)=0(1)$ as $t\to\infty$. When $p=\infty$, for $t>0$

$$\|W_{t,a}\|_\infty\simeq t^{-n/2}\ \text{if}\ a\geqq(n+2)/4$$

$$\simeq t^{1-2a} \text{ if } (n+1)/4 < a \leqq (n+2)/4$$

and

$W_{t,a}(|x|)$ *is unbounded along the cone* $|x| = t > 0$ *if*

$(n+1)/4 \geqq a > (n-1)/4$.

Lemma 3.3 (*Nirenberg's inequality*). *Let* $D^\alpha f$ *be the* α*-th weak derivative of* f *and let* $|D^i f|_p = \max_{|\alpha|=i} \|D^\alpha f\|_p$. *Suppose* $f \in D(B^a)$ *with* $a \geqq 2$. *Then*

$$|D^j f|_p \leqq \text{const.} |D^k f|_2^\gamma \|f\|_\infty^{1-\gamma}$$

when $2 \leqq p < \infty$, $2 \leqq k \leqq a$, $0 \leqq j < k$, *where* $p^{-1} = j/n + \gamma(1/2 - k/n)$ *for all with* $j/k \leqq \gamma < 1$.

Now in $\boldsymbol{R}^3$ we shall prove

Theorem 3.4. *Let* $\beta \geqq 3$ *and* $\gamma > 7/2$. *If the Cauchy data* ϕ_- *at* $t = -\infty$ *is sufficiently small in the* $\|\cdot\|_{\text{scat}}$*-norm, then* (12) *has a unique global solution* $\phi(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix}$ *and then there exists* ϕ_+ *with* $\|\phi_+\|_{\text{scat}}$ *small such that*

$$\|\phi(t) - e^{-itA}\phi_\pm\| \to 0,$$

and moreover, except for $\beta = 3$,

$$(1 + |t|)^{3/2} \|u(t) - [e^{-itA}\phi_\pm]_1\|_\infty \to 0$$

and

$$(1 + |t|)^{d_2} \|u_t(t) - [e^{-itA}\phi_\pm]_2\|_\infty \to 0$$

as $t \to \pm\infty$, *where* $[\phi]_j$ *denotes the jth component of* ϕ *and* d_2 *is arbitrary but* $3/2(\gamma - 2) < d_2 < 1$.

Moreover the scattering operator $S\colon \phi_- \mapsto \phi_+$ *is injective and continuous in the* $\|\cdot\|$*-topology. Except for* $\beta = 3$, S *is continuous in the* $\|\cdot\|_{\text{scat}}$*-topology.*

When $3 \leqq \gamma \leqq 7/2$, *also the same results are true for* $d_1 = 3(\gamma - 1)/(3r + 2)$ *and* $d_2 = 1/r$, *where* $1 < r < 2$.

Proof. From Lemma 3.1 (H1) follows immediately. To show that

(H3) holds we estimate $\|B^2(u^\alpha - v^\alpha)\|_r$ for $1 \leqq r \leqq 2$ and $\alpha \geqq 3$. We denote $\partial f/\partial x_i$ simply by f_i. Since

$$B^2(u^\alpha - v^\alpha) = (m^2 - \Delta)(u^\alpha - v^\alpha)$$
$$= m^2(u^\alpha - v^\alpha) - \sum_{j=1}^{3}(u^\alpha - v^\alpha)_{jj}$$

and since

$$(u^\alpha - v^\alpha)_{jj} = \alpha(\alpha-1)\left[u^{\alpha-2}(u_j^2 - v_j^2) + v_j^2(u^{\alpha-2} - v^{\alpha-2})\right]$$
$$+ \alpha\left[u^{\alpha-1}(u_{jj} - v_{jj}) + v_{jj}(u^{\alpha-1} - v^{\alpha-1})\right].$$

Noting that $1 \leqq r \leqq 2$ and $\beta \geqq 3$ we obtain

$$\|B^2(u^\alpha - v^\alpha)\|_r \leqq g \sum_\varepsilon (\|u\|_\infty + \|v\|_\infty)^{\alpha - 2/r - \varepsilon} \|u - v\|_\infty^\varepsilon \|B^2(u-v)\|^{1-\varepsilon}$$

where ε runs through $F_r = \{0, 1/2, 2-2/r\}$ and $g = g(\|B^2 u\|_2, \|B^2 v\|_2)$ with $\lim_{a,b\to 0} g(a,b) = 0$. In the hypothesis (H3) if we choose F_1 as the finite set through which both $\varepsilon_1 = \varepsilon_2 (= \varepsilon)$ run, then we have

$$\|J_j(\phi) - J_j(\psi)\| \leqq h_j \sum_\varepsilon (\|\phi\|_{a_j} + \|\psi\|_{a_j})^{\alpha-1-\varepsilon} \|\phi - \psi\|_{a_j}^\varepsilon \|\phi - \psi\|^{1-\varepsilon}. \tag{13}$$

From (H1) and (13) it follows immediately that

$$\|J_j(\phi) - J_j(\psi)\| \leqq h_j' \sum_\varepsilon (\|\phi\|_{a_j} + \|\psi\|_{a_j})^{\alpha-2} \|\phi - \psi\| \tag{13'}$$

where $h_j' = ch_j$. Also we have

$$\|J_j(\phi) - J_j(\psi)\|_{b_1} \leqq h_j \sum_\sigma (\|\phi\|_{a_j} + \|\psi\|_{a_j})^{\alpha-2-\sigma} \|\phi - \psi\|_{a_j}^\sigma \|\phi - \psi\|^{1-\sigma} \tag{14}$$

and

$$\|J_j(\phi) - J_j(\psi)\|_{b_2} \leqq h_j \sum_\rho (\|\phi\|_{a_j} + \|\psi\|_{a_j})^{\alpha-2/r-\rho} \|\phi - \psi\|_{a_j}^\rho \|\phi - \psi\|^{1-\rho} \tag{15}$$

where σ runs through F_2, ρ through F_r and α equals β or γ according as $j=1$ or 2.

Now, although (H2) cannot be verified to our example, we proceed the theorem directly as in Theorems 2.2, 2.3, with the help of (H1), (H3) just obtained above. Thus we define as in the proof of Theorem 2.2

$$X(\eta, \phi_-) = \{\psi \in C(R; \mathscr{H}); |||\psi(t) - e^{-itA}\phi_-||| \leqq \eta, \|\phi_-\|_{\text{scat}} \leqq \eta\}$$

and

$$(M\phi)(t) = e^{-itA}\phi_- + \int_{-\infty}^{t} e^{-i(t-s)A}(J_1(\phi(s)) + J_2(\phi(s)))ds.$$

By (13), (14), (15) and Lemma 3.2, the following estimates are immediate: for $\phi, \psi \in X(\eta, \phi_-)$

$$\|M(\phi) - M(\psi)\|$$

$$\leqq \sum_{\varepsilon} [h_1 \int_{-\infty}^{t} (\|\phi(s)\|_{a_1} + \|\psi(s)\|_{a_1})^{\beta-1-\varepsilon} \|\phi(s) - \psi(s)\|_{a_1}^{\varepsilon} \|\phi(s) - \psi(s)\|^{1-\varepsilon} ds$$

$$+ h_2 \int_{-\infty}^{t} (\|\phi(s)\|_{a_2} + \|\psi(s)\|_{a_2})^{\gamma-1-\varepsilon} \|\phi(s) - \psi(s)\|_{a_2}^{\varepsilon} \|\phi(s) - \psi(s)\|^{1-\varepsilon} ds]$$

$$\leqq \sum_{\varepsilon} [h_1 (4\eta)^{\beta-1-\varepsilon} \int_{-\infty}^{\infty} (1 + |s|)^{(\beta-1)d_1} ds$$

$$+ h_2 (4\eta)^{\gamma-1-\varepsilon} \int_{-\infty}^{\infty} (1 + |s|)^{(\gamma-1)d_2} ds] |||\phi - \psi|||.$$

The integrals in the third member are convergent provided that

$$(16) \qquad (\beta - 1)d_1 > 1, \quad (\gamma - 1)d_2 > 1.$$

Since

$$e^{-iAt} = \begin{pmatrix} \cos tB & B^{-1}\sin tB \\ -B\sin tB & \cos tB \end{pmatrix}$$

and so since

$$M(\phi)(t) - M(\psi)(t) = -\int_{-\infty}^{t} \begin{pmatrix} B^{-1}\sin(t-s)B(g_1 u^\beta + g_2 u_s^\gamma) \\ \cos(t-s)B(g_1 u^\beta + g_2 u_s^\gamma) \end{pmatrix} ds,$$

it follows from the definitions of auxiliary norms that

$$\|M(\phi) - M(\psi)\|_{a_1} \leqq |g_1| \int_{-\infty}^{t} \sup_x |B^{-1}\sin(t-s)B(u^\beta - v^\beta)| ds$$

$$+ |g_2| \int_{-\infty}^{t} \sup_x |B^{-1}\sin(t-s)B(u_s^\gamma - v_s^\gamma)| ds$$

$$\leqq |g_1| \int_{-\infty}^{t} \sup_x |\mathfrak{F}^{-1}[\hat{F}_{t-s,3} \cdot B^2(u^\beta - v^\beta)]| ds$$

$$+|g_2|\int_{-\infty}^{t}\sup_x|\mathcal{F}^{-1}[\hat{F}_{t-s,3}\cdot B^2(u_s^\gamma-v_s^\gamma)]|ds$$

$$\leqq|g_1|\int_{-\infty}^{t}\sup_x|F_{t-s,3}*B^2(u^\beta-v^\beta)|ds$$

$$+|g_2|\int_{-\infty}^{t}\sup_x|F_{t-s,3}*B^2(u_s^\gamma-v_s^\gamma)|ds$$

$$\leqq|g_1|\int_{-\infty}^{t}\|F_{t-s,3}\|_{p/(p-1)}\|B^2(u^\beta-v^\beta)\|_p ds$$

$$+|g_2|\int_{-\infty}^{t}\|F_{t-s,3}\|_{q/(q-1)}\|B^2(u_s^\gamma-v_s^\gamma)\|_q ds$$

Now applying Lemma 3. 2 to the last members of above inequalities, we obtain

$$\|M(\phi)-M(\psi)\|_{a_1}\leqq|||\phi-\psi|||$$

$$\times\{c\sum_{\sigma\in F_p}|g_1|(4\eta)^{\gamma-2/p-\sigma}\int_{-\infty}^{t}(1+|t-s|)^{-(3/p-3/2)}(1+|s|)^{-(\beta-2/p)d_1}ds$$

$$+c\sum_{\sigma'\in F_q}|g_2|(4\eta)^{\gamma-2/q-\sigma'}\int_{-\infty}^{t}(1+|t-s|)^{-(3/q-3/2)}(1+|s|)^{-(\gamma-2/q)d_2}ds\}.$$

Taking Lemma 2. 1 into consideration, we impose the following conditions to the exponents appearing in the above integrals:

(17) $$\max(3/p-3/2,\ (\beta-2/p)d_1)>1$$

$$\min(3/p-3/2,\ (\beta-2/p)d_1)\geqq d_1$$

(18) $$\max(3/q-3/2,\ (\gamma-2/q)d_2)>1,$$

$$\min(3/q-3/2,\ (\gamma-2/q)d_2)\geqq d_1$$

where $1\leqq p, q<2$.

Similarly we have

$$\|M(\phi)-M(\psi)\|_{a_2}\leqq\int_{-\infty}^{t}\|E_{t-s,2}\|_{r/(r-1)}\|B^2(u^\beta-v^\beta)\|_r ds$$

$$+\int_{-\infty}^{t}\|E_{t-s,2}\|_{r/(r-1)}\|B^2(u_s^\gamma-v_s^\gamma)\|_r ds,$$

and so

$$\|M(\phi)-M(\psi)\|_{a_2}\leqq|||\phi-\psi|||$$

$$\times c\sum_{\rho\in F_r}\{|g_1|(4\eta)^{\beta-2/r-\rho}\int_{-\infty}^{t}(1+|t-s|)^{-1/r}(1+|s|)^{-(\beta-2/r)d_1}ds$$

$$+|g_2|(4\eta)^{\gamma-2/r-\rho}\int_{-\infty}^{t}(1+|t-s|)^{-1/r}(1+|s|)^{-(\gamma-2/r)d_2}ds\}$$

with the conditions

(19) $$\max(1/r,\ (\beta-2/r)\,d_1)>1$$

$$\min(1/r,\ (\beta-2/r)\,d_1)\geqq d_2$$

(20) $$\max(1/r,\ (\gamma-2/r)\,d_2)>1$$

$$\min(1/r,\ (\gamma-2/r)\,d_2)\geqq d_2$$

where $1<r<2$.

For $\beta\geqq 3$ and $\gamma>7/2$ we may choose $p=1$, $q=1$ and r arbitrary but $1<r<2$ so that (16) (20) hold, whence $d_1=3/2$ and d_2 arbitrary but $1/2<d_2<1$. For $\beta\geqq 3$ and $7/2\geqq\gamma\geqq 3$ we may choose $p=1$, $d_2=1/r$ with $1<r<4/3$ so that (16), (17), (19) and (20) hold, provided $d_1>1$. Actually (20) follows from $\gamma\geqq 3>r+2/r$ when $1<r<4/3$. Concerning (18) we choose q with

$$3/q-3/2=(\gamma-2/q)\cdot 1/r$$

which implies $3/q-3/2=3(\gamma-1)/(3r+2)$. Since $3/q-3/2>1$ when $1<r<4/3$, we thus obtain

$$d_1=3(\gamma-1)/(3r+2)<3(\gamma-1)/5\,.$$

Consequently we have proved by Theorems 2. 2 and 2. 3 the existence of a global unique solution and of scattering operator for small data. We finally note that continuity of S in $\|\cdot\|$-topology is derived from (13′) just as in [8]. Q.E.D.

In case $n=1$ we define $\mathcal{H}$ by completion of $D(B^2)\oplus D(B^1)$ with respect to the inner product

$$(\phi,\psi)=(B^2u_1,B^2u_2)+(Bv_1,Bv_2)\,.$$

Then the corresponding conditions to (16) $\sim$ (20),

(16′) $$(\beta-1)\,d_1>1,\ \ (\gamma-1)\,d_2>1$$

(17′) $$\max(1/p-1/2,\ (\beta-2/p)\,d_1)>1$$

$$\min(1/p-1/2,\ (\beta-2/p)\,d_1)\geqq d_1$$

(18′) $$\max(1/q-1/2,\ (\gamma-2/q)\,d_2)>1$$

$$\min(1/q-1/2,\ (\gamma-2/q)\,d_2)\geqq d_1$$

(19′) $$\max(1/r-1/2,\ (\beta-2/r)\,d_1)>1$$

$$\min(1/r-1/2,\ (\beta-2/r)\,d_1)\geqq d_2$$

and

(20′) $$\max(1/r-1/2,\ (\gamma-2/r)\,d_2)>1$$

$$\min(1/r-1/2,\ (\gamma-2/r)\,d_2)\geqq d_2\,.$$

should be imposed where $1\leqq p, q\leqq 2$ and $4/3<r\leqq 2$.

Then we have analogously to the proof of Theorem 3. 4.

Theorem 3. 5. *The same claims as Theorem* 3. 4 *hold with* $d_1=1/2$, $d_2<1/4$ *for* $\beta>4$, $\gamma>6$.

For $11/2<\gamma\leqq 6$, β *should be chosen as* $\beta>\gamma-4+2/(\gamma-5)$ *with* $d_1<(\gamma-5)/2$, $d_2<1/4$.

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