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Injective Envelopes of Operator Systems

By

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Abstract

We show the existence and uniqueness of a minimal injective operator system (resp. minimal unital C^*-algebra) "containing" a given operator system V, which will be called the injective (resp. C^*-) envelope of V. This result can be applied to prove the existence of the Šilov boundary in the sense of Arveson, which was left open in [1].

§ 1. Introduction

We will use terminologies in Arveson [1] and Choi-Effros [2], [3] without further explanation, and we will denote the set of all bounded operators on a Hilbert space H by $B(H)$. For a subset S of a unital C^*-algebra A, $C^*(S)$ stands for the C^*-subalgebra of A generated by S and the unit 1. If, in addition, S is self-adjoint, linear, and contains 1, S can be regarded as an operator system in the obvious fashion. In fact consider a faithful $*$-representation $\{\pi, H\}$ of A and identify S with the operator system $\pi(S)\subset B(H)$. This identification is justified since $\pi|_S$: $S\to\pi(S)$ is a unital (=unit-preserving) complete order isomorphism, and will be made throughout the paper.

Let $V\subset B(H)$ be an operator system and let κ: $V\to B(K)$ be a unital complete order injection (i.e. a unital complete order isomorphism of V onto $\kappa(V)\subset B(K)$). Then, although V and $\kappa(V)\subset B(K)$ have the same structure as matrix order unit spaces, generally we can not guess any relation between the C^*-algebras $C^*(V)\subset B(H)$ and $C^*(\kappa(V))\subset B(K)$ generated by them. So it will be an interesting problem to find a minimal C^*-algebra (if it makes sense) which is generated by the operator system which is unitally completely order isomorphic to the given operator system V.

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We will show in the following that such a C^*-algebra exists uniquely and that it is $*$-isomorphic to the quotient C^*-algebra $C^*(V)/J$, where J is the Šilov boundary for V in the sense of Arveson [1, Definition 2. 1. 3]. We call the C^*-algebra the C^*-envelope of V. Thus an operator system determines its C^*-envelope uniquely. Conversely, it may be said that any unital C^*-algebra A is determined by its self-adjoint linear subspace V, containing 1, which has A as its C^*-envelope (or equivalently, which has $\{0\}$ as its Šilov boundary): If κ is a unital complete order isomorphism of V onto an operator system $V_1 \subset B(H_1)$, κ extends uniquely to a $*$-isomorphism $\hat{\kappa}$ of A onto $C^*(V_1)/J_1$ so that $\hat{\kappa}=\pi\circ\kappa$, where J_1 is the Šilov boundary for V_1 and π: $V_1 \hookrightarrow C^*(V_1) \to C^*(V_1)/J_1$ is the canonical map. This fact, which is no other than the uniqueness of the C^*-envelope of V, was proved by Arveson under an additional hypothesis [1, Theorem 2. 2. 5]. (There he does not assume that V is self-adjoint; but without loss of generality, we may assume so.)

To solve the above problem we introduce the injective envelope of an operator system, which generalizes the injective envelope defined for a unital C^*-algebra [4].

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§ 2. Definitions and Preliminalies

Throughout this section $V \subset B(H)$ will denote a fixed operator system.

Definition 2. 1. An *extension* of V is a pair (W, κ) of an operator system W and a unital complete order injection κ: $V \to W$.

Definition 2. 2. (W, κ) is an *injective* (resp. C^*-) *extension* of V iff W is an injective operator system (resp. unital C^*-algebra such that $C^*(\kappa(V))=W$).

Definition 2. 3. (W, κ) is an *essential extension* of V iff, given any operator system Z and any unital completely positive map φ: $W \to Z$, φ is a complete order injection whenever $\varphi\circ\kappa$ is.

Remark. This definition of essential extensions of operator systems is consistent with that of unital C^*-algebras in [4]. In fact, let A and B be unital C^*-algebras and $\kappa: A \to B$ a unital complete order injection. Then, if the extension (B, κ) is essential, κ is a $*$-monomorphism (=one-to-one $*$-homomorphism). (See the proof of Theorem 4.1.)

Definition 2.4. (W, κ) is a *rigid extension* of V iff, given any completely positive map $\varphi: W \to W$, $\varphi \circ \kappa = \kappa$ implies $\varphi = \mathrm{id}_W$ (the identity map on W).

Remark. The essentiality of (W, κ) implies the rigidity of (W, κ), and further provided that (W, κ) is injective, they are equivalent. (See Lemma 3.7 and the remark succeeding to Lemma 3.7.)

Definition 2.5. (W, κ) is an *injective* (resp. C^*-) *envelope* of V iff it is both injective and essential (resp. C^*- and essential) extension of V.

Definition 2.6. Let V and V_1 be operator systems such that there exists a unital complete order isomorphism $\iota: V \to V_1$. Given extensions (W, κ) and (W_1, κ_1) of V and V_1, respectively, $(W, \kappa) \sim (W_1, \kappa_1)$ iff there exists a unital complete order isomorphism $\hat{\iota}: W \to W_1$ such that $\hat{\iota} \circ \kappa = \kappa_1 \circ \iota$.

The injective envelope (resp. C^*-envelope) of V can be regarded as a minimal object in the family of all injective extensions of V or a maximal one in the family of all essential extensions of V (resp. a minimal one in the family of all C^*-extensions of V). (Cf. Lemma 3.6, Theorem 4.1, Corollary 4.2 below.)

We list a few known results which will be used later. A unital complete order isomorphism between unital C^*-algebras is an algebraic $*$-isomorphism [3], so that an operator system is unitally completely order isomorphic to at most one unital C^*-algebra. Any injective operator system W is unitally completely order isomorphic to a unique injective monotone complete C^*-algebra B [3, Theorem 3.1]. Hence W and B are identified as matrix order unit spaces. Henceforth this identification will be made without referring; thus a unital complete order isomorphism between injective operator systems will be regarded as a $*$-isomorphism between C^*-algebras.

§ 3. Minimal Projections on an Injective Operator System

Let $V\subset W\subset B(H)$ be any operator systems with W injective.

Definition 3. 1. A linear map $\varphi: W\to W$ is a *V-projection* on W iff it is unital, completely positive, idempotent ($\varphi^2=\varphi$) and $\varphi|_V=\mathrm{id}_V$.

Definition 3. 2. A seminorm p on W is a *V-seminorm* on W iff $p=\|\psi(\cdot)\|$ for some completely positive map $\psi: W\to W$ with $\psi|_V=\mathrm{id}_V$.

Definition 3. 3. Given V-projections φ, ψ on W (resp. V-seminorms p, q on W), define a partial ordering $\prec$ (resp. $\leqq$) by $\varphi\prec\psi$ (resp. $p\leqq q$) iff $\varphi\circ\psi=\psi\circ\varphi=\varphi$ (resp. $p(x)\leqq q(x)$ for all x in W).

A V-projection (resp. V-seminorm) which is minimal with respect to this partial ordering $\prec$ (resp. $\leqq$) will be called a *minimal V-projection* (resp. *minimal V-seminorm*).

Lemma 3. 4. *Any decreasing net $\{p_i\}$ of V-seminorms on W has a lower bound.*

Proof. We note that the unit ball of $B(W, B(H))$, the Banach space of all bounded linear maps of W into $B(H)$, is compact in the point-σ-weak topology.

Let $\varphi_i: W\to W$ be completely positive maps such that $p_i=\|\varphi_i(\cdot)\|$ and $\varphi_i|_V=\mathrm{id}_V$. The injectivity of W implies the existence of a completely positive idempotent linear map ψ of $B(H)$ onto W. Regarding $\{\varphi_i\}$ as a subset of the unit ball of $B(W, B(H))$, the above remark shows that there are a subnet $\{\varphi_j\}$ of $\{\varphi_i\}$ and a $\varphi_0\in B(W, B(H))$ such that $\varphi_j(x)\to\varphi_0(x)$ σ-weakly for all x in W. Then it is immediately seen that φ_0 is completely positive and $\varphi_0|_V=\mathrm{id}_V$, so that the seminorm $p: x\mapsto\|\psi\circ\varphi_0(x)\|$ is a V-seminorm on W. Moreover we have for all x in W,

$$p(x)=\|\psi\circ\varphi_0(x)\|\leqq\|\varphi_0(x)\|\leqq\limsup\|\varphi_j(x)\|=\lim p_i(x).\ \text{Q.E.D.}$$

This lemma and Zorn's lemma imply the existence of a minimal V-seminorm p_0 on W.

Theorem 3.5. *Let $V \subset W \subset B(H)$ be as above. Then there exists a minimal V-projection on W.*

Proof. Let $\varphi: W \to W$ be a completely positive map such that $p_0(x) = \|\varphi(x)\|$, $x \in W$, and let $\varphi^{(n)} = (\varphi + \varphi^2 + \cdots + \varphi^n)/n$, $n = 1, 2, \cdots$. Then it follows from a reasoning similar to that in Lemma 3.4 that there exist a subnet $\{\varphi^{(n_i)}\}$ of $\{\varphi^{(n)}\}$ and a completely positive map $\varphi_0 \in B(W, B(H))$ such that $\varphi^{(n_i)}(x) \to \varphi_0(x)$ σ-weakly for all x in W. As in Lemma 3.4 take a completely positive idempotent linear map ψ of $B(H)$ onto W. Then

$$\|\psi \circ \varphi_0(x)\| \leqq \|\varphi_0(x)\| \leqq \limsup \|\varphi^{(n_i)}(x)\|$$
$$\leqq \|\varphi(x)\| = p_0(x), \quad x \in W,$$

so the minimality of p_0 implies that $\|\psi \circ \varphi_0(x)\| = p_0(x)$, hence that $\limsup \|\varphi^{(n_i)}(x)\| = \|\varphi(x)\|$. Therefore

$$\|\varphi(x) - \varphi^2(x)\| = \limsup \|\varphi^{(n_i)}(x - \varphi(x))\| = 0, \quad x \in W,$$

so that φ is a V-projection on W.

The proof of the minimality of φ is exactly the same as that of the case where V is a unital C^*-algebra [4, Theorem 3.4]. Q.E.D.

Lemma 3.6. *Let $V \subset W \subset B(H)$ be as above and let φ be a minimal V-projection on W. Then the extension* $(\mathrm{Im}\,\varphi, j)$ *of V, where $j: V \to \mathrm{Im}\,\varphi = \varphi(W)$ is the inclusion map, is rigid.*

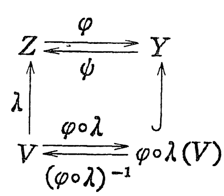
Proof. Let $\psi: \mathrm{Im}\,\varphi \to \mathrm{Im}\,\varphi$ be any completely positive map such that $\psi \circ j = j$. Putting $(\psi \circ \varphi)^{(n)} = (\psi \circ \varphi + \cdots + (\psi \circ \varphi)^n)/n$, an argument similar to above implies the existence of a subnet $\{(\psi \circ \varphi)^{(n_j)}\}$ of $\{(\psi \circ \varphi)^{(n)}\}$ such that $\limsup \|(\psi \circ \varphi)^{(n_j)}(x)\| = \|\varphi(x)\|$, $x \in W$. Hence we have for each $x \in \mathrm{Im}\,\varphi$,

$$\|x - \psi(x)\| = \|\varphi(x - (\psi \circ \varphi)(x))\|$$
$$= \limsup \|(\psi \circ \varphi)^{(n_j)}(x - (\psi \circ \varphi)(x))\| = 0,$$

so that $\psi=\mathrm{id}_{\mathrm{Im}\varphi}$. Q.E.D.

Lemma 3.7. *Let (Z, λ) be an injective extension of an operator system V. Then (Z, λ) is rigid iff it is essential.*

Proof. Necessity: Let Y be any operator system and $\varphi: Z \to Y$ any unital completely positive map such that $\varphi\circ\lambda$ is a complete order injection. Then we must show that φ also is a complete order injection. Since $\lambda\circ(\varphi\circ\lambda)^{-1}$: $\varphi\circ\lambda(V)\to Z$ is completely positive and Z is injective, there is a completely positive map $\psi: Y\to Z$ such that $\psi|_{(\varphi\circ\lambda)(V)}=\lambda\circ(\varphi\circ\lambda)^{-1}$.



Then $\omega=\psi\circ\varphi: Z\to Z$ is a completely positive map such that $\omega\circ\lambda=\lambda$. Hence by the rigidity of (Z, λ), $\omega=\mathrm{id}_Z$, so that φ is a complete order injection.

Sufficiency: Let $\varphi: Z\to Z$ be a completely positive map such that $\varphi\circ\lambda=\lambda$. Put $\psi=(\mathrm{id}_Z+\varphi)/2$. Since $\psi\circ\lambda=\lambda$, the essentiality of (Z, λ) implies that ψ is a complete order injection. We claim that ψ is onto. In fact there exists a completely positive map $(\psi^{-1})^\wedge: Z\to Z$ such that $(\psi^{-1})^\wedge|_{\psi(Z)}=\psi^{-1}$. Then $\psi\circ(\psi^{-1})^\wedge$ is idempotent and is a complete order injection since $\psi\circ(\psi^{-1})^\wedge\circ\lambda=\lambda$, so that $\psi\circ(\psi^{-1})^\wedge=\mathrm{id}_Z$. This shows that ψ is onto. Hence ψ is a unital complete order isomorphism of Z onto itself, so it defines a $*$-automorphism of the C^*-algebra which is unitally completely order isomorphic to Z. From $\psi=(\mathrm{id}_Z+\varphi)/2$ and the extremality of a $*$-automorphism it follows that $\varphi=\mathrm{id}_Z$. Q.E.D.

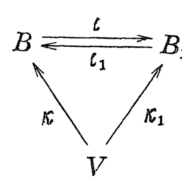
Remark. Let (W, κ) be an essential extension of an operator system V. Then, taking the injective envelope (Z, λ) of W, whose existence will be proved below, and applying the above lemma to the injective and essential extension $(Z, \lambda\circ\kappa)$ of V, it follows readily that (W, κ) is a rigid extension of V.

§ 4. Main Results

Theorem 4.1. *Any operator system $V \subset B(H)$ has a unique injective (resp. C^*-) envelope, where the uniqueness of the extensions is up to the equivalence relation $\sim$ defined in Section 2.*

Proof. By Theorem 3.5 applied to $V \subset W = B(H)$, there exists a minimal V-projection φ on $B(H)$. Let $I_\varphi = \{x \in B(H) : \varphi(x^*x) = \varphi(xx^*) = 0\}$ and $B(H)_\varphi = \mathrm{Im}\,\varphi + I_\varphi$. Then $B(H)_\varphi$ is a unital C^*-subalgebra of $B(H)$, I_φ is a closed two-sided ideal of $B(H)_\varphi$, and the canonical map $\mathrm{Im}\,\varphi \hookrightarrow B(H)_\varphi \to B(H)_\varphi / I_\varphi$ is a unital complete order isomorphism ([4, Theorem 2.1, Lemma 2.4], [3, Theorem 3.1]). Put $B = B(H)_\varphi / I_\varphi$, and let $\kappa : V \hookrightarrow \mathrm{Im}\,\varphi \to B(H)_\varphi / I_\varphi = B$ be the canonical map and A the C^*-subalgebra of B generated by $\kappa(V)$. Then Lemmas 3.6 and 3.7 imply that the extension (B, κ) [resp. (A, κ)] of V is the desired injective (resp. C^*-) envelope of V.

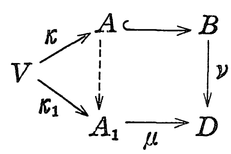
To see the uniqueness of the injective envelope (B, κ) take another injective envelope (B_1, κ_1) of V. The injectivity of B and B_1 implies the existence of completely positive maps $\iota : B \to B_1$ and $\iota_1 : B_1 \to B$ such that $\iota \circ \kappa = \kappa_1$ and $\iota_1 \circ \kappa_1 = \kappa$.



Hence $\iota_1 \circ \iota : B \to B$ is a completely positive map with $\iota_1 \circ \iota \circ \kappa = \kappa$, so that $\iota_1 \circ \iota = \mathrm{id}_B$ by Lemma 3.7. Similarly $\iota \circ \iota_1 = \mathrm{id}_{B_1}$. Hence ι is a $*$-isomorphism of B onto B_1, where we regard B_1 as an injective C^*-algebra, i.e. $(B, \kappa) \sim (B_1, \kappa_1)$. Note also that if V is completely isometric to a C^*-algebra, then the embedding κ (hence κ_1 too) becomes a $*$-monomorphism. Indeed we may then assume that V is a C^*-subalgebra, containing the unit, of $B(H)$. So the map $\kappa : V \to B(H)_\varphi / I_\varphi = B$ is a $*$-monomorphism. (Since any essential extension of V can be embedded in the injective envelope of V by the definition, this shows that if V is a C^*-algebra and (C, λ) is an essential extension of V with C a C^*-algebra, then λ becomes a $*$-

monomorphism.)

The uniqueness of the C^*-envelope follows from that of the injective envelope. Indeed let (A_1, κ_1) be another C^*-envelope of V and (D, μ) the injective envelope of A_1. Then $(D, \mu\circ\kappa_1)$ is the injective envelope of V, so that from the uniqueness of the injective envelope the existence of a $*$-isomorphism ν of B onto D with $\nu\circ\kappa=\mu\circ\kappa_1$ follows.



Since μ is a $*$-monomorphism as noted above,

$$\nu(A)=\nu(C^*(\kappa(V)))=C^*(\nu\circ\kappa(V))=C^*(\mu\circ\kappa_1(V_1))$$
$$=\mu(C^*(\kappa_1(V_1)))=\mu(A_1).$$

Hence we have $(A, \kappa)\sim(A_1, \kappa_1)$. Q.E.D.

Corollary 4.2. *Let $V\subset B(H)$ be an operator system and (A, κ) the C^*-envelope of V. If (B, λ) is a C^*-extension of V, then there is an onto $*$-homomorphism $\pi: B\rightarrow A$ such that $\pi\circ\lambda=\kappa$; hence $(A, \kappa)\sim(B/\mathrm{Ker}\,\pi, q\circ\lambda)$, where $q: B\rightarrow B/\mathrm{Ker}\,\pi$ is the quotient homomorphism.*

Proof. Without loss of generality, we may assume that $V\subset B=C^*(V)\subset B(K)$ for some Hilbert space K and $\lambda: V\rightarrow B$ is the inclusion map. Taking, as in the above proof, a minimal V-projection ψ on $B(K)$ and letting $\rho: B(K)_\psi\rightarrow B(K)_\psi/I_\psi$ be the quotient map, $A_1=C^*(\rho(V))\subset B(K)_\varphi/I_\varphi$ and $\kappa_1=\rho|_V$, we obtain the C^*-envelope (A_1, κ_1) of V. On the other hand, $B\subset B(K)_\psi$ since $B(K)_\psi$ is a C^*-subalgebra of $B(H)$ containing V and $B=C^*(V)$, so that $\rho|_B: B\rightarrow B(K)_\psi/I_\psi$ defines a $*$-homomorphism of B onto A_1 which extends κ_1. Hence the uniqueness of the C^*-envelope completes the proof. Q.E.D.

Remark. The above corollary generalizes Choi-Effros [2, Theorem 4.1] in which $V=A$ and κ is the identity map.

Definition 4.3 (Arveson [1, Definition 2.1.3]). Let A be a linear subspace of a unital C^*-algebra B which contains the unit and generates B as a C^*-algebra. A closed two-sided ideal J of B is called a *boundary ideal* for A if the canonical quotient map $q: B \to B/J$ is completely isometric on A. The boundary ideal which contains every other boundary ideal is called the *Šilov boundary* for A.

We show in the following the existence of the Šilov boundary, which was left open in the general situation [1]. Note first that a completely isometric linear map on A extends uniquely to a completely isometric linear map on the operator system $A+A^*$ [1, Proposition 1.2.8] and that a unital linear map between operator systems is completely isometric iff it is a complete order injection.

Theorem 4.4. *Let A and B be as above. Then there exists the Šilov boundary for A.*

Proof. By the above remark we may assume that $A=A^*$, i.e. A is an operator system, hence that (B, j) is a C^*-extension of A, where $j: A \to B$ is the inclusion map. Let (C, κ) be the C^*-envelope of A (Theorem 4.1). Then there is an onto $*$-homomorphism $\pi: B \to C$ such that $\pi \circ j = \kappa$ (Corollary 4.2). We verify that Ker $\pi = J$, say, is the Šilov boundary for A. Let $\tilde{\pi}: B/J \to C$ be the $*$-isomorphism induced by π and q: $B \to B/J$ the quotient map. Then $\tilde{\pi} \circ q \circ j = \kappa$. Hence $q \circ j = \tilde{\pi}^{-1} \circ \kappa$ is a complete order injection and $(B/J, q \circ j) \sim (C, \kappa)$. Therefore J is a boundary ideal for A.

Let $K \subset B$ be any boundary ideal for A and $q': B \to B/K$ the quotient map. Then $(B/K, q' \circ j)$ is a C^*-extension of A, so again by Corollary 4.2, there is an onto $*$-homomorphism $\rho: B/K \to B/J$ such that $\rho \circ q' \circ j = q \circ j$, i.e. $\rho(x+K) = x+J$ for all x in A. Since ρ is a $*$-homomorphism and B is generated by A,

$$\rho(x+K) = x+J \quad \text{for all } x \text{ in } B.$$

In particular, for each x in K,

$$0+J = \rho(0+K) = \rho(x+K) = x+J,$$

i.e. $K \subset J$. Q.E.D.

Let V, (A, κ), (B, λ) and π be as in Corollary 4. 2. Then, as seen in the above proof, Ker π is the Šilov boundary for $\lambda(V)$. Hence the C^*-envelope of V is described as a C^*-extension (B, λ) of V such that the Šilov boundary for $\lambda(V)$ is $\{0\}$. Moreover the uniqueness of the C^*-envelope in Theorem 4. 1 is restated as follows: Given a unital complete order isomorphism ι of an operator system $V \subset B(H)$ onto another operator system $V_1 \subset B(H_1)$, there exists a unique $*$-isomorphism $\hat{\iota}$ of $C^*(V)/J$ onto $C^*(V_1)/J_1$ such that $\hat{\iota} \circ q|_V = q_1 \circ \iota$, where J (resp. J_1) denotes the Šilov boundary [in $C^*(V)$ (resp. $C^*(V_1)$)] for V (resp. V_1) and $q: C^*(V) \to C^*(V)/J$ [resp. $q_1: C^*(V_1) \to C^*(V_1)/J_1$] means the quotient map. (Compare with [1, Theorem 2. 2. 5].)

We conclude this section with a remark on non-unital complete order isomorphisms. Let V and V_1 be operator systems and suppose that there exists a (not necessarily unital) complete order isomorphism $\varphi: V \to V_1$. We want to prove that the corresponding C^*-envelopes of V and V_1 are $*$-isomorphic (hence so are the injective envelopes of V and V_1, too). We may and shall assume that $V \subset A$ (resp. $V_1 \subset A_1$), where A (resp. A_1) is the C^*-envelope of V (resp. V_1). Put $\varphi(1) = b \in V_1$ (1 denotes the unit of V). Then b, being an order unit for V_1, is a positive invertible element of A_1. In this situation we have the following:

Proposition 4. 5. *There exists a $*$-isomorphism α of A onto A_1 which is uniquely determined by the condition: $\alpha(x) = b^{-1/2}\varphi(x)b^{-1/2}$ for all x in V.*

Proof. It is straightforward to see that $C^*(b^{-1/2}V_1b^{-1/2}) = C^*(V_1) = A_1$ and that $\{0\}$ is the only boundary ideal for $b^{-1/2}V_1b^{-1/2}$. Hence A_1 is the C^*-envelope of $b^{-1/2}V_1b^{-1/2}$. Since $V \ni x \mapsto b^{-1/2}\varphi(x)b^{-1/2} \in b^{-1/2}V_1b^{-1/2}$ is a unital complete order isomorphism, this map extends uniquely to a unital complete order isomorphism, hence a $*$-isomorphism of A onto A_1 (Theorem 4. 1). Q.E.D.

Now we show by an example that two operator systems which are completely order isomorphic need not be unitally completely order isomorphic. Let V be an operator system which is embedded in its C^*-envelope

A as a self-adjoint linear subspace containing 1. Take an element $b \in V$ which is positive and invertible in A. Then $b^{-1/2}Vb^{-1/2} \subset A$ may be regarded as an operator system (note that $1 \in b^{-1/2}Vb^{-1/2}$), the map $V \ni x \mapsto b^{-1/2}xb^{-1/2} \in b^{-1/2}Vb^{-1/2}$ is a complete order isomorphism, and V and $b^{-1/2}Vb^{-1/2}$ have A as their C^*-envelopes (see the above proof). Hence V and $b^{-1/2}Vb^{-1/2}$ are unitally completely order isomorphic iff there exists a $*$-automorphism α of A such that $b^{-1/2}Vb^{-1/2} = \alpha(V)$.

Take as the above V the linear subspace $\{\beta + \gamma x + \delta x^2 : \beta, \gamma, \delta \in \mathbf{C}\}$ of the C^*-algebra $C([0,1])$ of all continuous functions on the unit interval $[0,1]$, where x stands for the function $x \mapsto x$. Then the C^*-envelope A of V is $C([0,1])$, because the Šilov boundary for V in the usual sense is $[0,1]$. Let $b = \beta_0 + \gamma_0 x + \delta_0 x^2 \in V$ be positive and invertible in $C([0,1])$. Since a $*$-automorphism α of A is induced by a self-homeomorphism h of $[0,1]$ so that $\alpha(f) = f \circ h$, $f \in A$, the equality $b^{-1/2}Vb^{-1/2} = \alpha(V)$ is rewritten as

$$\{(\beta + \gamma x + \delta x^2)/(\beta_0 + \gamma_0 x + \delta_0 x^2) : \beta, \gamma, \delta \in \mathbf{C}\}$$
$$= \{\beta + \gamma h(x) + \delta h(x)^2 : \beta, \gamma, \delta \in \mathbf{C}\}.$$

But an easy computation shows that this equality does not hold provided that $\delta_0 \neq 0$.

§ 5. Examples

This section is devoted to give some simple examples of operator systems.

Example 5.1. Let $V \subset B(H)$ be a two-dimensional operator system (i.e. $V = \mathbf{C}1 + \mathbf{C}a$ with $a^* = a$ and $a \notin \mathbf{C}1$). Then V is unitally completely order isomorphic to the commutative C^*-algebra $\mathbf{C}^2$. In fact, let $\lambda_1 < \lambda_2$ be the end-points of the spectrum of a. Then the map

$$\alpha 1 + \beta a \mapsto (\alpha + \beta\lambda_1, \alpha + \beta\lambda_2)$$

defines a unital complete order isomorphism of V onto $\mathbf{C}^2$ since it is isometric and $\mathbf{C}^2$ is commutative [1, Proposition 1.2.2].

Example 5.2. Let $T \in B(H)$ and let $V = \mathbf{C}1 + \mathbf{C}T + \mathbf{C}T^*$. Suppose that the C^*-envelope, say A, of V is commutative and so $A = C(X)$ with

X a compact Hausdorff space. Then X is identified with the Šilov boundary (in the usual sense) for $\mathbf{C}+\mathbf{C}z\subset C(\mathrm{Sp}\,T)$, where $C(\mathrm{Sp}\,T)$ denotes the C^*-algebra of continuous functions on the spectrum $\mathrm{Sp}\,T$ of T and z denotes the function $\lambda\mapsto\lambda$ on $\mathrm{Sp}\,T$. In fact, by Theorem 4. 4, $A\cong C^*(V)/J$ with J the Šilov boundary for V, $C^*(V)/J$ is generated by $q(1)$ and $q(T)$, where $q: C^*(V)\to C^*(V)/J$ is the quotient homomorphism, and further $\mathrm{Sp}\,q(T)\subset\mathrm{Sp}\,T$. Hence the map $\mathbf{C}q(1)+\mathbf{C}q(T)\ni\alpha q(1)+\beta q(T)\mapsto\alpha+\beta z|_{\mathrm{Sp}\,q(T)}\in C(\mathrm{Sp}\,q(T))$ extends uniquely to a $*$-isomorphism of $C^*(V)/J$ onto $C(\mathrm{Sp}\,q(T))$, so that the map $V\ni\alpha 1+\beta T+\gamma T^*\mapsto\alpha+\beta z+\gamma\bar{z}\in C(\mathrm{Sp}\,T)$ is a complete order injection. This shows our assertion and further that $\|\alpha 1+T\|=\sup\{|\alpha+\lambda|:\lambda\in\mathrm{Sp}\,T\}$ for all $\alpha\in\mathbf{C}$. The class of operators satisfying this equality was studied by Saitô [5].

Example 5. 3. Let $V\subset B(H)$ be an operator system. Then the injective envelope of V is $(B(H), j)$, where $j: V\to B(H)$ is the inclusion map, iff $C^*(V)\supset C(H)$, the set of all compact operators on H, and the canonical map $V\hookrightarrow C^*(V)\to C^*(V)/C(H)$ is not a complete order injection.

Now $(B(H), j)$ is the injective envelope of V iff $(C^*(V), j)$ is the C^*-envelope of V [or, what is the same, the Šilov boundary (in $C^*(V)$) for V is $\{0\}$] and $(B(H), j')$ is the injective envelope of $C^*(V)$, where $j': C^*(V)\to B(H)$ is the inclusion map (cf. the proof of Theorem 4. 1). Further, noting that $C(H)$ is the smallest nonzero closed two-sided ideal of $B(H)$, we see that the Šilov boundary for V is $\{0\}$ iff the canonical map $V\hookrightarrow B(H)\to B(H)/C(H)$ is not a complete order injection. Thus we need only show that $(B(H), j')$ is the injective envelope of $C^*(V)=A$ iff $C(H)\subset A$. If $(B(H), j')$ is the injective envelope of A, then $A''=B(H)$ by [4, Corollary 4. 3]. Hence $C(H)\subset A$ or $C(H)\cap A=\{0\}$. The latter is not the case because if $C(H)\cap A=\{0\}$, the seminorm $x\mapsto\inf_{y\in C(H)}\|x+y\|$ is an A-seminorm (in the sense of [4, Definition 3. 3]) different from the norm on $B(H)$ [4, Remark 4. 4]. Hence $C(H)\subset A$. Conversely, if $C(H)\subset A$, then the injective extension $(B(H), j')$ of A is rigid, because the identity map on $B(H)$ is a unique contractive linear map of $B(H)$ into itself which fixes $C(H)$ elementwise, so that $(B(H), j')$ is the injective envelope of A (Lemma 3. 7).

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