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A Remark on the Automorphism Pseudo-Group of a Differential Equation of Immersion Type

By

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Introduction

Let Γ be a pseudo-group on a manifold Q such that $\mathcal{L}_\Gamma$ is N-regular (Definition 3. 1). It seems to be important to know whether the orbit system $\mathcal{L}_\Gamma(k, x, f)$ is Γ-automorphic or not (Definition 4. 1).

If $\dim N \geqq \dim Q \geqq 1$, it is known that, for a sufficiently large integer k, $\mathcal{L}_\Gamma(k, x, f)$ is Γ-automorphic (Theorem 6. 1 in [2]). But in case $1 \leqq \dim N < \dim Q$, we do not know whether the same assertion holds or not.

In the present paper, we consider the problem for the automorphism pseudo-group $\mathcal{A}(E)$ of some kind of a differential equation E and seek for a necessary and sufficient condition for the orbit system $\mathcal{L}_{\mathcal{A}(E)}(k, x, f)$ to be $\mathcal{A}(E)$-automorphic for a sufficiently large k (Theorem 6. 1).

Through this paper, we assume the differentiability of class C^∞. For a pseudo-group Γ, we always assume that any element of Γ is close to the identity. For the completeness of a pseudo-group Γ, refer to Definition 5. 2 in [2].

§ 1. Admissibility of a Reduced Pair

1. Let $J^l(N, Q)$, $l \geqq 0$, denote the space of l-jets of local maps of N to Q and $\tilde{J}^l(N, Q)$ denote the open submanifold of $J^l(N, Q)$ which consists of l-jets of local submersions or local immersions if $\dim N \geqq \dim Q$ or $\dim N < \dim Q$, respectively.

By a differential equation E at $j^k_x(f) \in \tilde{J}^k(N, Q)$, we mean a family of functions locally defined at $j^k_x(f)$. We denote by $\mathcal{S}(E)$ or $\mathcal{A}(E)$ the solution space or the automorphism pseudo-group of E, respectively, and

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for a neighbourhood $\mathcal{U}^l$ of $j_x^l(f)$, $\mathscr{S}(E)|\mathcal{U}^l$ or $\mathscr{A}(E)|\mathcal{U}^l$ means the restriction of $\mathscr{S}(E)$ or $\mathscr{A}(E)$ to $\mathcal{U}^l$, respectively. Let (Q, Q', π) be a fibred manifold and let E or E' denote a differential equation at $j_x^\alpha(f) \in \tilde{J}^\alpha(N, Q)$ or $j_x^{\alpha'}(\pi\circ f) \in \tilde{J}^{\alpha'}(N, Q')$, respectively. Through this paper, we fix this f.

Definition 1.1. The pair (E, E') is said to be k-reduced pair (resp. weakly k-reduced pair) if the following conditions 1) and 2) are satisfied (resp. if the condition 1) is satisfied):

1) There are neighbourhoods $\mathcal{U}^k$ of $j_x^k(f)$ and $\mathcal{U}'^k$ of $j_x^k(\pi\circ f)$ such that i〉 for any $s\in\mathscr{S}(E)|\mathcal{U}^k, \pi\circ s\in\mathscr{S}(E')|\mathcal{U}'^k$, ii〉 for any $s' \in\mathscr{S}(E')|\mathcal{U}'^k$, there exists an $s\in\mathscr{S}(E)|\mathcal{U}^k$ with $\pi\circ s=s'$ and iii〉 for $s: N\supset\mathcal{U}\to Q$ and $t: N\supset\mathcal{V}\to Q\in\mathscr{S}(E)|\mathcal{U}^k$, if $\pi\circ s=\pi\circ t$, then $s=t$ on $\mathcal{U}\cap\mathcal{V}$.

2) For any $\phi\in\mathscr{A}(E)|\mathcal{U}^k$, there exists a local diffeomorphism φ of Q' with $\varphi\circ\pi=\pi\circ\phi$.

Let ξ be a map of $\mathcal{U}'^k$ to $\mathbb{R}^r$ where $r=\dim Q-\dim Q'$.

Definition 1.2. A (weakly) k-reduced pair (E, E') is said to be of type ξ if any element $s\in\mathscr{S}(E)|\mathcal{U}^k$ is expressed as $s=(s', \xi(j^k(s')))$, $s'\in\mathscr{S}(E')|\mathcal{U}'^k$.

2. Let E' be a differential equation at $j_x^\alpha(f')\in\tilde{J}^\alpha(N, Q')$ and ξ be a map of a neighbourhood $\mathcal{U}'^k$ of $j_x^k(f')$ to $\mathbb{R}^m$. We set $S_{x,z'}^k=\{j_x^k(s)\mid s'\in\mathscr{S}(E'), s'(x)=z'\}$.

Definition 2.1. ξ is said to be E'-admissible at x_0 if there exists a neighbourhood $\mathcal{U}'$ of $(x_0, f'(x_0))$ such that, to any $\phi\in\mathscr{A}(E')$, there correspond an open subset $O^\phi\subset\mathbb{R}^m$ and a map $\eta^\phi: \mathcal{U}'\times O^\phi\to\mathbb{R}^m$ with $\eta_{x,z'}^\phi\circ\xi=\xi\circ\phi^{(k)}$ on $S_{x,z'}^k\cap\mathcal{U}'^k\cap D(\phi^{(k)})$, where $\eta_{x,z'}^\phi(v)=\eta^\phi(x, z', v)$ and $D(\phi^{(k)})=$ the domain of $\phi^{(k)}\cap(\phi^{(k)})^{-1}(\mathcal{U}'^k)$.

§ 2. Reducibility of a Regular Differential Equation

3. Given a sheaf of vector fields $\mathscr{L}$ on a manifold Q, then $\mathscr{L}$ is

prolonged to a sheaf of vector fields $\mathcal{L}^{(k)}$ on $\tilde{J}^k(N,Q)$. Let ${}^0\mathcal{L}_p^{(k)}$ denote the isotropy of the stalk $\mathcal{L}_p^{(k)}$ and set $D_p^{(k)}=\mathcal{L}_p^{(k)}/{}^0\mathcal{L}_p^{(k)}$.

Definition 3.1. $\mathcal{L}$ is said to be N-regular if, for any integer k, the correspondence $\tilde{J}^k(N,Q)\ni p\to D_p^{(k)}\subset T_p(\tilde{J}^k(N,Q))$ defines an involutive distribution.

Let Γ be a pseudo-group on Q. Then we can give the sheaf of germs of local vector fields X on Q such that the local 1-parameter group of local transformations generated by X is contained in Γ, which we denote by $\mathcal{L}_\Gamma$. Assume that $\mathcal{L}_\Gamma$ is N-regular. Let $\{\theta_j^k\}_{j=1}^{m_k}$ be a fundamental system of differential invariants of $\mathcal{L}_\Gamma$ at $j_x^k(f)\in\tilde{J}^k(N,Q)$. We set $\Theta^k=(\theta_1^k,\cdots,\theta_{m_k}^k)$. Then Θ^k is a submersion of a neighbourhood $\mathcal{U}^k$ of $j_x^k(f)$ to an open subset $W\subset\mathbb{R}^{m_k}$. We set $p^h\Theta^K(j_x^{h+k}(s))=j_x^h(\Theta^k(j^k(s)))$. Then $p^h\Theta^k$ is a map of a neighbourhood $\mathcal{U}^{h+k}$ of $j_x^{h+k}(f)\in\tilde{J}^{h+k}(N,Q)$ to a neighbourhood W^h of $j_x^h(\Theta^k(j^k(f)))\in J^h(N,\mathbb{R}^{m_k})$. For any function F defined on an open subset of W^h, we put $F(\Theta^k)=p^h\Theta^{k*}F$. Then $F(\Theta^k)$ is a differential invariant of $\mathcal{L}_\Gamma$ at $j_x^{h+k}(f)$. $F(\Theta^k)$ is called a Γ-differential invariant of type k at $j_x^{h+k}(f)$. A family $\{F_j(\Theta^k)\}_{j=1}^r$ of linearly independent Γ-differential invariants of type k at $j_x^{h+k}(f)$ is called a Γ-family of type (k,r) at $j_x^{h+k}(f)$ if the differential equation E generated by $\{F_j(\Theta^k)\}_{j=1}^r$ possesses a solution and if $\mathcal{A}(E)$ is equal to Γ on a neighbourhood $\mathcal{U}$ of $f(x)$. In general, for any differential equation E, we set $S^k(E)=\bigcup_{s\in\mathcal{S}(E)}\operatorname{Im}j^k(s)$.

Definition 3.2. A differential equation E at $j_x^\alpha(f)\in\tilde{J}^\alpha(N,Q)$ is said to be l-regular at x if the following conditions are satisfied:

1) $\mathcal{L}_{\mathcal{A}(E)}$ is N-regular.

2) For each integer $\tilde{l}\geq l$, there exists an integer $\tilde{k}$ and an $\mathcal{A}(E)$-family $\tilde{\mathcal{H}}=\{\tilde{\mathfrak{H}}_j\}_{j=1}^{\tilde{r}}$ of type $(\tilde{l},\tilde{r})$ at $j_x^{\tilde{l}+\tilde{k}}(f)$ such that $p^{\tilde{l}+\tilde{k}-\alpha}(E)$, the standard prolongation of E, is generated by $\tilde{\mathcal{H}}$.

3) For each integer $k\geq l$, there is a neighbourhood $\mathcal{U}^k$ of $j_x^k(f)$ such that $S^k(E)\cap\mathcal{U}^k$ is a regular submanifold of $\tilde{J}^k(N,Q)$.

4. Let $\mathcal{L}$ be a weak Lie algebra sheaf on Q which is N-regular.

For such a sheaf $\mathcal{L}$, we can consider the orbit system $\mathcal{L}(k, x, f)$ for any positive integer k and a local submersion or immersion of a neighbourhood of $x \in N$ to Q. Namely let $\{\theta_j^k\}_{j=1}^{m_k}$ be a fundamental system of differential invariants of a weak Lie algebra sheaf $\mathcal{L}$ at $j_x^k(f)$. We set $\lambda_j(x) = \theta_j^k(j_x^k(f))$. Then $\mathcal{L}(k, x, f)$ is generated by $\theta_j^k - \lambda_j (1 \leqq j \leqq m_k)$ as a differential equation.

Definition 4. 1. $\mathcal{L}_\Gamma(k, x, f)$ is said to be Γ-automorphic if any solution of $\mathcal{L}_\Gamma(k, x, f)$ is of the form $\phi \circ f$, $\phi \in \Gamma$.

Proposition 4. 1. *If* $\dim N \geqq \dim Q$, *and if* Γ *is complete at* $(f(x), 1)$, *then for a sufficiently large integer* k, $\mathcal{L}_\Gamma(k, x, f)$ *is* Γ-*automorphic.*

For the proof, refer to [2].

Definition 4. 2. Γ is said to be k-automorphic at (x, f) if the orbit system $\mathcal{L}(k, x, f)$ is Γ-automorphic.

Proposition 4. 1 means that, if $\dim N \geqq \dim Q$ and if Γ is complete at $(f(x), 1)$, Γ is k-automorphic at (x, f) for a sufficiently large integer k. (As for the definition of completeness, refer to [2].)

5. Let (Q, Q', π) be a fibred manifold and E or E' be a differential equation at $j_x^\alpha(f) \in \tilde{J}^\alpha(N, Q)$ or $j_x^{\alpha'}(\pi \circ f) \in \tilde{J}^{\alpha'}(N, Q')$, respectively. Suppose E and E' are l-regular at x. Then, for $k \geqq l$, $\mathcal{L}_{\mathcal{A}(E)}$ or $\mathcal{L}_{\mathcal{A}(E')}$ induces an involutive distribution D_E^k or $D_{E'}^k$ on a neighbourhood of $j_x^k(f)$ or of $j_x^k(\pi \circ f)$, respectively. Assume that $\dim N \geqq \dim Q'$.

Lemma 5. 1. *Suppose that* (E, E') *is a* k-*reduced pair for a sufficiently large* k *and that* $\dim \pi_*^k D_E^k = \dim D_{E'}^k$. *We denote by* $\mathcal{A}(E)'_k$ *the pseudo-group generated by* $\{g' : \exists g \in \mathcal{A}(E) \mid \mathcal{U}^k, \pi \circ g = g' \circ \pi\}$. *If* $\mathcal{A}(E')$ *and* $\mathcal{A}(E)'_k$ *are complete at* $(\pi \circ f(x), 1)$, *then* $(\mathcal{L}_{\mathcal{A}(E)}(k, x, f), \mathcal{L}_{\mathcal{A}(E')}(k, x, \pi \circ f))$ *is a weakly* k-*reduced pair.*

Proof. Since (E, E') is a k-reduced pair, we see that $(\pi^k)_* D_E^k = D_{E'}^k$ for a sufficiently large k. On the other hand, since $\mathcal{A}(E')$ and $\mathcal{A}(E)'_k$ are complete at $(\pi\circ f(x_0), 1)$, by Lemma 7. 1 in [2], $\mathcal{A}(E)'_k|\mathcal{U}'^k = \mathcal{A}(E')|\mathcal{U}'^k$ for a neighbourhood $\mathcal{U}'^k$ of $j^k_{x_0}(\pi\circ f)$. Since, by Proposition 4. 1, $\mathcal{A}(E')$ is k-automorphic, $\mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f)$ is $\mathcal{A}(E)'_k$-automorphic. Since π maps $\mathcal{S}(\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f))|\mathcal{U}^k$ into $\mathcal{S}(\mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f))|\mathcal{U}'^k$ and since $\mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f)$ is $\mathcal{A}(E)'_k$-automorphic, $\mathcal{S}(\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f))|\mathcal{U}^k$ is transfered onto $\mathcal{S}(\mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f))|\mathcal{U}'^k$. Since (E, E') is a k-reduced pair, the pair $(\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f), \mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f))$ is a weakly k-reduced pair. The proof is completed.

§ 3. Main Theorem

6. Let Γ be a pseudo-group on Q where $\mathcal{L}_\Gamma$ is N-regular and let f be an immersion or a submersion of a neighbourhood of $x_0 \in N$ to Q.

Theorem 6. 1. *Let (Q, Q', π) be a fibred manifold and let E or E' be a differential equation at $j^{\alpha}_{x_0}(f)$ or $j^{\alpha'}_{x_0}(\pi\circ f)$, respectively. Assume that*

1) *E and E' are l-regular at x_0 and $\mathcal{A}(E')$ is complete at $(\pi\circ f(x_0), 1)$.*

2) *For a sufficiently large integer k, (E, E') is a k-reduced pair of type ξ and $\mathcal{A}(E)'_k$ is complete at $(\pi\circ f(x_0), 1)$, where $\xi: \tilde{J}^k(N, Q') \supset \mathcal{U}'^k \to \mathbf{R}^r$, $r = \dim Q - \dim Q'$.*

3) *$\bigcup_{s\in\mathcal{S}(E)|\mathcal{U}^k} \mathrm{Im}\, s$ contains an open neighbourhood of $f(x_0)$ for a sufficiently small neighbourhood $\mathcal{U}^k$.*

4) *There exists a neighbourhood $\mathcal{U}'$ of $(x_0, \pi\circ f(x_0)) \in N\times Q'$ such that $\bigcap_{(x,z')\in\mathcal{U}'} \xi(S^k_{x,z'} \cap \mathcal{U}'^k)$ contains an open neighbourhood of $\xi(j^k_{x_0}(\pi\circ f))$.*

5) *$\dim N \geqq \dim Q'$.*

Then ξ is E'-admissible at x_0 if and only if $\mathcal{A}(E)$ is k-automorphic at (x_0, f).

Proof. Suppose ξ is E'-admissible at x_0. Let $\phi' \in \mathcal{A}(E')|\mathcal{U}'^k$. We may assume that $(\mathcal{U}^k, \mathcal{U}'^k, \pi^k)$ is a fibred manifold and $\mathcal{S}(E)|\mathcal{U}^k$, $\mathcal{S}(E')|\mathcal{U}'^k$ and $\mathcal{A}(E)|\mathcal{U}^k$ satisfy the conditions 1) and 2) of Definition 1. 1.

For any $s=(s', \xi(j^k(s')))\in\mathscr{S}(E)|\mathcal{U}^k$, we set $s^{\phi'}=(\phi'\circ s', \xi(j^k(\phi'\circ s')))$. Then since $s'\in\mathscr{S}(E')|\mathcal{U}'^k$, $s^{\phi'}\in\mathscr{S}(E)|\mathcal{U}^k$. Since ξ is E'-admissible at x_0, $s^{\phi'}(x)=(\phi'\circ s'(x), \eta^{\phi'}_{x,s'(x)}\circ\xi(j^k_x(s')))$. Therefore for any two s and $t\in\mathscr{S}(E)|\mathcal{U}^k$, if $s(x)=t(x)$, then we have $s^{\phi'}(x)=t^{\phi'}(x)$. By the condition 3), if we set $\phi\circ s(x)=s^{\phi'}(x)$, ϕ is a local diffeomorphism of Q to Q. It is now clear that $\phi\in\mathcal{A}(E)|\mathcal{U}^k$ and $\pi\circ\phi=\phi'\circ\pi$. Let $\varphi\in\mathcal{A}(E)|\mathcal{U}^k$ such that $\pi\circ\varphi=\phi'\circ\pi$. Then $\varphi\circ s\in\mathscr{S}(E)|\mathcal{U}^k$ and $\pi\circ\varphi\circ s=\phi'\circ\pi\circ s=\phi'\circ s'$. Therefore $\varphi\circ s$ is of the form $(\phi'\circ s', \xi(j^k(\phi'\circ s')))$. By the condition 3), this implies $\varphi=\phi$ on the intersection of their domains.

Now let $\phi\in\mathcal{A}(E)|\mathcal{U}^k$. Then by the k-reducibility of the pair (E, E'), we have a local transformation ϕ' on Q' such that $\phi'\circ\pi=\pi\circ\phi$. For any $s'\in\mathscr{S}(E')|\mathcal{U}'^k$, we have $s\in\mathscr{S}(E)|\mathcal{U}^k$ such that $s'=\pi\circ s$. Then $\phi'\circ s'=\phi'\circ\pi\circ s=\pi\circ\phi\circ s\in\mathscr{S}(E')|\mathcal{U}'^k$. Therefore $\phi'\in\mathcal{A}(E')|\mathcal{U}'^k$.

Thus we see that $\dim\pi^k_* D^k_E=\dim D^k_{E'}$. On the other hand, since $\dim N\geqq\dim Q'$ and $\pi\circ f$ is a local submersion, by Proposition 4. 1, $\mathcal{A}(E')$ is k-automorphic at $(x_0, \pi\circ f)$ for a sufficiently large k. Therefore by Lemma 5. 1, the pair $(\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f), \mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f))$ is a weakly k-reduced pair. Since $\mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi\circ f)$ is $\mathcal{A}(E')$-automorphic, it is easy to see that $\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f)$ is $\mathcal{A}(E)$-automorphic. That is to say, $\mathcal{A}(E)$ is k-automorphic at (x_0, f).

Conversely, for a sufficiently large k, we assume that $\mathcal{A}(E)$ is k-automorphic at (x_0, f). Let $\phi\in\mathcal{A}(E)|\mathcal{U}^k$. Then by the k-reducibility of the pair (E, E'), there is a $\phi'\in\mathcal{A}(E')|\mathcal{U}'^k$ such that $\pi\circ\phi=\phi'\circ\pi$. Conversely let $\phi'\in\mathcal{A}(E')|\mathcal{U}'^k$. Then $\phi'\circ\pi\circ f\in\mathscr{S}(E')|\mathcal{U}'^k$. Therefore we have $s\in\mathscr{S}(E)|\mathcal{U}^k$ such that $\pi\circ s=\phi'\circ\pi\circ f$. Since $\mathcal{A}(E)$ is k-automorphic at (x_0, f), we have $\phi\in\mathcal{A}(E)|\mathcal{U}^k$ such that $s=\phi\circ f$. Then we get $\pi\circ\phi\circ f=\phi'\circ f'$, where $f'=\pi\circ f$. Since the pair (E, E') is of type ξ, for any $s\in\mathscr{S}(E)|\mathcal{U}^k$, we have $s=(s', \xi(j^k(s'))$ where $s'\in\mathscr{S}(E')|\mathcal{U}'^k$. Therefore the equality $\pi\circ\phi\circ f=\phi'\circ f'$ implies $\phi\circ f=(\phi'\circ f', \xi(j^k(\phi'\circ f')))$. On the other hand, there exists $\varphi'\in\mathcal{A}(E')|\mathcal{U}'^k$ such that $\phi\circ s=(\varphi'\circ s', \xi(j^k(\varphi'\circ s')))$ for any $s\in\mathscr{S}(E)|\mathcal{U}^k$. Since f' is a submersion, we get $\phi'=\varphi'$. Therefore, for any $s\in\mathscr{S}(E)|\mathcal{U}^k$, we have $\phi\circ s=(\phi'\circ s', \xi(j^k(\phi'\circ s')))$. By the condition 3), we can easily see that $\pi\circ\phi=\phi'\circ\pi$. It is now clear that, if there are ϕ and $\varphi\in\mathcal{A}(E)|\mathcal{U}^k$ such that $\pi\circ\phi=\pi\circ\varphi=\phi'\circ\pi$, then $\phi=\varphi$ on the intersection of their domains.

Now we shall show that ξ is E'-admissible at x_0. Let $\phi' \in \mathcal{A}(E')|\mathcal{U}'^k$. Then as is stated above, we have an element $\phi \in \mathcal{A}(E)|\mathcal{U}^k$ such that $\pi \circ \phi = \phi' \circ \pi$. Since the pair (E, E') is of type ξ, any $s \in \mathcal{S}(E)|\mathcal{U}^k$ possesses the form $s = (s', \xi(j^k(s')))$, $s' \in \mathcal{S}(E')|\mathcal{U}^k$. Therefore we have $\phi \circ s = (\phi' \circ s', \xi(j^k(\phi' \circ s')))$. Assume that $\xi(j^k_x(s')) = \xi(j^k_x(t'))$, where $s', t' \in \mathcal{S}(E')|\mathcal{U}'^k$ and $s'(x) = t'(x) = z'$. Then we get $s(x) = t(x)$, where $s = (s', \xi(j^k(s')))$ and $t = (t', \xi(j^k(t')))$. Consequently we have $\phi \circ s(x) = \phi \circ t(x)$ and thus we get $\xi(j^k_x(\phi' \circ s')) = \xi(j^k_x(\phi' \circ t'))$. That is to say, if $\xi(j^k_x(s')) = \xi(j^k_x(t'))$ for $j^k_x(s'), j^k_x(t') \in S^k_{x,z'}$, then we have $\xi(j^k_x(\phi' \circ s')) = \xi(j^k_x(\phi' \circ t'))$ for $\phi' \in \mathcal{A}(E')|\mathcal{U}'^k$. If we set $\eta^{\phi'}_{x,z'} \circ \xi(j^k_x(s')) = \xi(j^k_x(\phi' \circ s'))$, by the condition 4), $\eta^{\phi'}_{x,z'}$ is a local map of a neighbourhood $\mathcal{O}$ of $\xi(j^k_{x_0}(\pi \circ f))$ $\mathbb{R}^r$ to $\mathbb{R}^r$ for any $(x, z') \in \mathcal{U}'$. If we set $\eta^{\phi'}(x, z', v) = \eta^{\phi'}_{x,z'}(v)$, it is easy to see that $\eta^{\phi'}$ is an analytic map of $\mathcal{U}' \times \mathcal{O}$ to $\mathbb{R}^r$. This proves that ξ is E'-admissible at x_0. The proof of Theorem 6. 1 is completed.

Corollary 6. 1. *Let (Q, Q', π) be a fibred manifold and E or E' be a differential equation at $j^{\alpha}_{x_0}(f)$ or $j^{\alpha'}_{x_0}(\pi \circ f)$, respectively. Assume that*

1) *E and E' are l-regular at x_0 and $\mathcal{A}(E)$ (resp. $\mathcal{A}(E')$) is complete at $(f(x_0), 1)$ (resp. $(\pi \circ f(x_0), 1)$).*

2) *For a sufficiently large integer k, (E, E') is a k-reduced pair of type ξ and $\mathcal{A}(E)'_k$ is complete at $(\pi \circ f(x_0), 1)$.*

3) *There exist neighbourhoods $\mathcal{U}'$ of $(x_0, \pi \circ f(x_0))$ and $\mathcal{U}'^k$ of $j^k_{x_0}(\pi \circ f)$ such that $\bigcap_{(x,z') \in \mathcal{U}'} \xi(S^k_{x,z'} \cap \mathcal{U}'^k)$ contains an open neighbourhood of $\xi(j^k_{x_0}(\pi \circ f))$.*

4) $\dim N \geqq \dim Q$.

Then the pair $(\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f), \mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi \circ f))$ is a weakly k-reduced pair.

Proof. In this case, the conditions 3) and 5) in Theorem 6. 1 are clearly satisfied. Furthermore by Proposition 4. 1, $\mathcal{A}(E)$ is k-automorphic at (x_0, f). Therefore by Theorem 6. 1, ξ is E'-admissible at x_0. Then as was stated in the proof of Theorem 6. 1, we have $\dim \pi^k_* D^k_E = \dim D^k_{E'}$. By Lemma 5. 1, we see that the pair $(\mathcal{L}_{\mathcal{A}(E)}(k, x_0, f), \mathcal{L}_{\mathcal{A}(E')}(k, x_0, \pi \circ f))$

is a weakly k-reduced pair. The proof is completed.

§ 4. Medolaghi-Vessiot's Example

7. We denote by E the differential equation at $j^1_{x_0}(f)\in\tilde{J}^1(\boldsymbol{R}^2,\boldsymbol{R}^3)$ generated by $\frac{1}{z_2}\cdot\frac{\partial z_1}{\partial x_1}-\alpha^1(x_1,x_2)\,(=F_1)$, $\frac{1}{z_2}\cdot\frac{\partial z_1}{\partial x_2}-\alpha^2(x_1,x_2)\,(=F_2)$, $\frac{1}{z_2}\cdot\left(\frac{\partial z_2}{\partial x_1}-z_3\cdot\frac{\partial z_1}{\partial x_1}\right)-\beta^1(x_1,x_2)\,(=F_3)$ and $\frac{1}{z_2}\cdot\left(\frac{\partial z_2}{\partial x_2}-z_3\cdot\frac{\partial z_1}{\partial x_2}\right)-\beta^2(x_1,x_2)$ $(=F_4)$ where $\{x_1,x_2\}$ (resp. $\{z_1,z_2,z_3\}$) is the canonical coordinate system on $\boldsymbol{R}^2$ (resp. $\boldsymbol{R}^3$) and $z_2\circ f(x_0)\neq 0$, $\frac{\partial(z_1\circ f)}{\partial x_1}(x_0)\neq 0$, $\alpha^1(x_0)\neq 0$. On the other hand, we denote by E' the differential equation at $j^1_{x_0}(\pi\circ f)$ $\tilde{J}^1(\boldsymbol{R}^2,\boldsymbol{R})$ generated by $\frac{\partial z}{\partial x_2}-\frac{\alpha^2}{\alpha^1}\cdot\frac{\partial z}{\partial x_1}$ where $\{z\}$ is the canonical coordinate on $\boldsymbol{R}$ and the projection π of $\boldsymbol{R}^3$ onto $\boldsymbol{R}$ is defined by $\pi(z_1,z_2,z_3)=z_1$. Then it is easy to see that the pair (E,E') is a weakly 1-reduced pair.

We define a map $\xi=(\xi_1,\xi_2)$ of a neighbourhood of $j^2_{x_0}(\pi\circ f)$ to $\boldsymbol{R}^2$ by $\xi_1(j^2_x(s))=p_1(j^1_x(s))/\alpha^1(x)$ and $\xi_2(j^2_x(s))=p_{11}(j^2_x(s))/\alpha^1(x)\cdot p_1(j^1_x(s))-\frac{\partial\alpha^1}{\partial x_1}(x)/(\alpha^1(x))^2-\beta(x)/\alpha^1(x)$, where $\{x_1,x_2,z,p_1,p_2,p_{11},p_{12},p_{22}\}$ is the canonical system on $J^2(\boldsymbol{R}^2,\boldsymbol{R})$. Then it is easily proved that the pair (E,E') is of type ξ, considering (E,E') as a weakly 2-reduced pair.

We denote by $\mathcal{L}$ the sheaf on $\boldsymbol{R}^3$ of local vector fields of the following form:

$$\eta(z_1)\frac{\partial}{\partial z_1}+\eta'(z_1)\cdot z_2\cdot\frac{\partial}{\partial z_2}+\eta''(z_1)\cdot z_2\cdot\frac{\partial}{\partial z_3}$$

where η is any local function with one variable. If we set $\boldsymbol{R}^3_*=\{(z_1,z_2,z_3)\in\boldsymbol{R}^3;z_2\neq 0\}$, $\mathcal{L}$ is a Lie algebra sheaf on $\boldsymbol{R}^3_*$ which is $\boldsymbol{R}^2$-regular. Let Γ be the pseudo-group on $\boldsymbol{R}^3_*$ such that $\mathcal{L}_\Gamma=\mathcal{L}$ and Γ is complete at $(p,1)$ where p is any point of $\boldsymbol{R}^3_*$. Then since F_1, F_2, F_3 and F_4 are differential invariants of $\mathcal{L}_\Gamma$, we have $\mathcal{A}(E)\supset\Gamma$ on a neighbourhood of $j^1_{x_0}(f)$.

Next let $\mathcal{L}'$ be the sheaf of all local vector fields on $\boldsymbol{R}$ and let Γ' be the pseudo-group of all local transformations on $\boldsymbol{R}$. Then it is easy to see that $\mathcal{A}(E')=\Gamma'$ on a neighbourhood of $j^1_{x_0}(\pi\circ f)$. Moreover since $\{p_2/p_1\}$ is a fundamental system of differential invariants of $\mathcal{L}'$ at

$j^1_{x_0}(\pi\circ f)$, E' is $\mathcal{A}(E')$-automorphic.

Now we can eassily see that ξ is E'-admissible at x_0.

On the other hand, since Γ is transitive on $\boldsymbol{R}^3_*$, the condition 3) of Theorem 6.1 is clearly satisfied. Now ξ is defined by

$$\xi_1(j^2_x(s)) = p_1(j^1_x(s))/\alpha^1(x),$$

$$\xi_2(j^2_x(s)) = p_{11}(j^2_x(s))/\alpha^1(x)\cdot p_1(j^1_x(s)) - \frac{\partial\alpha^1}{\partial x_1}(x)/(\alpha^1(x))^2 - \beta^1(x)/\alpha^1(x).$$

We prove that, for any constants a and b, there is a solution s of E' such that $p_1(j^1_x(s)) = a$ and $p_{11}(j^2_x(s)) = b$. Since E' is generated by the local vector field $\dfrac{\partial}{\partial x_2} - \dfrac{\alpha^2}{\alpha^1}\cdot\dfrac{\partial}{\partial x_1}$, if s is a particular solution of E' with $s(x) = z'$, then $\mathcal{S}(E') \supseteq \{\varphi\circ s : \varphi$ is any function locally defined at $z'\}$. Then it is easy to see that, for some φ, we have $p_1(j^1_x(\varphi\circ s)) = a$ and $p_{11}(j^2_x(\varphi\circ s)) = b$. Therefore $\xi(S^2_{x,z'}\cap\mathcal{U}'^2)$ is open in $\boldsymbol{R}^2$ for a neighbourhood $\mathcal{U}'^2$ of $j^2_{x_0}(\pi\circ f)$. Furthermore if $\dfrac{\partial\alpha^1}{\partial x_1}/(\alpha^1)^2 + \beta^1/\alpha^1$ is constant, we can easily see that, for a neighbourhood $\mathcal{U}'$ of $(x_0, \pi\circ f(x))$, $\bigcap_{(x,z')\in\mathcal{U}'}\xi(S^2_{x,z'}\cap\mathcal{U}'^2)$ is open. In this case by Theorem 6.1, $\mathcal{A}(E)$ is k-automorphic at (x_0, f) for some k. We have seen that $\mathcal{A}(E)|\mathcal{U}^2$ is mapped by π bijectively to $\mathcal{A}(E')|\mathcal{U}'^2$. On the other hand, $\Gamma|\mathcal{U}^2$ is mapped by π bijectively to $\mathcal{A}(E')|\mathcal{U}'^2$, because Γ consists of all local transformation ϕ on $\boldsymbol{R}^3_*$ defined by

$$\begin{cases} z_1\circ\phi(z_1, z_2, z_3) = X(z_1), \\ z_2\circ\phi(z_1, z_2, z_3) = z_2\cdot X'(z_1), \\ z_3\circ\phi(z_1, z_2, z_3) = z_3 + z_2\cdot\dfrac{X''(z_1)}{X'(z_1)} \end{cases}$$

where X is any local transformation on $\boldsymbol{R}$. Since we have $\mathcal{A}(E) \supset \Gamma$ on a neighbourhood of $j^2_x(f)$, we see that $\mathcal{A}(E) = \Gamma$ on a neighbourhood of $j^2_x(f)$.

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