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Integral of Differential Forms along the Path of Diffusion Processes

Dedicated to Professor G. Maruyama on his 60th birthday

By

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§ 1. Introduction

Let $(X_t, \mathcal{F}_t, P_x)$ be a Brownian motion on a Riemannian manifold M and α be a differential 1-form on M. In this paper we will be concerned with the integral of α along X_t. This integral is a stochastic version of the ordinary integral of the form α along a smooth curve on M and is defined by using the symmetric integral. We denote by $A(t;\alpha)$, $t\geqq 0$, the integrals of α along X_t. The one-parameter family $A=\{A(t;\alpha); t\geqq 0\}$ of random variables then defines a continuous additive functional of $(X_t, \mathcal{F}_t, P_x)$.

In Section 3 we will show that $A(t;\alpha)$, $t\geqq 0$, can be decomposed into a sum of a local martingale and a bounded variation process which is expressed by the divergence of α. The structure of the local martingale part will be analyzed by using the lifted diffusion $(r_t, \mathcal{F}_t, P_r)$ on $O(M)$ of $(X_t, \mathcal{F}_t, P_x)$ through the Riemannian connection where $O(M)$ is the bundle of orthonormal frames. Next in Section 4, using some results in Section 3 we will give a representation theorem for continuous square integrable martingale additive functionals of $(X_t, \mathcal{F}_t, P_x)$ which was obtained, in some special cases, by a number of authors (cf. [11], [14], [15], [16], [17]). As an application of Theorem 3.1, we discuss in Section 5 the Cameron-Martin formula. An approximation theorem similar to Nakao-Yamato [12] also holds in our case. Using this we will formulate and prove a stochastic version of Stokes' theorem.

M. Yor [18] recently discussed a closely related subject independently in the case that $M=\boldsymbol{C}$.

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§ 2. Definition

Throughout this paper we will be using the following notation and assumptions. Let M be a d-dimensional complete Riemannian manifold[1] and g be its Riemannian metric. Let Δ be the Laplace-Beltrami operator on M. Let $O(M)$ be the set of $(d+1)$-tuples $(x, e_1, e_2, \cdots, e_d)$, where $x \in M$ and $e_1, e_2, \cdots, e_d$ is an orthonormal basis of T_xM. Then $a=(a_{ij}) \in O(d)$ acts on $O(M)$ as

$$R_a(x, e_1, e_2, \cdots, e_d) = (x, \sum_{i=1}^{d} a_{i1}e_i, \sum_{i=1}^{d} a_{i2}e_i, \cdots, \sum_{i=1}^{d} a_{id}e_i).$$

Let $\pi: O(M) \to M$ be given by $\pi(x, e_1, e_2, \cdots, e_d) = x$. Thus we have a bundle of orthonormal frames $(O(M), O(d), M)$. We will denote the bundle by its bundle space $O(M)$ alone. We refer the reader to [1] for the precise definition of the bundle of orthonormal frames. If we take a local coordinate $(x^1, x^2, \cdots, x^d)$ in a coordinate neighborhood U of M, every orthonormal frame $r \in \pi^{-1}(U)$ may be expressed in the form:

$$(2.1) \qquad r=(x, e_1, e_2, \cdots, e_d), \qquad e_i=\sum_{k=1}^{d} e_i^k \frac{\partial}{\partial x^k}, \qquad i=1, 2, \cdots, d,$$

where $e=(e_i^k)$ is such that

$$(2.2) \qquad \sum_{k,l=1}^{d} e_i^k e_j^l g_{kl}(x) = \delta_{ij}, \qquad i, j=1, 2, \cdots, d,$$

or equivalently

$$(2.3) \qquad \sum_{i=1}^{d} e_i^k e_i^l = g^{kl}(x), \qquad k, l=1, 2, \cdots, d,$$

(cf. Ikeda-S. Watanabe [7]). Here

$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \qquad i, j=1, 2, \cdots, d \quad \text{and} \quad (g^{ij}) = (g_{ij})^{-1}.$$

For the sake of brevity, we introduce the following notation:

$$r=(x, e_1, e_2, \cdots, e_d) = [x, e],$$

where $x=(x^1, x^2, \cdots, x^d)$ and $e=(e_i^k)$. Let Γ_{pq}^j be the coefficients of the Riemannian connection associated to the Riemannian metric g, i.e.,

[1] We always assume that M is connected.

$$\Gamma^j_{pq}=\frac{1}{2}\sum_{k=1}^{d}\left(\frac{\partial}{\partial x^p}g_{kq}+\frac{\partial}{\partial x^q}g_{pk}-\frac{\partial}{\partial x^k}g_{pq}\right)g^{kj}.$$

Denote by $\Lambda^1(M)$, $\mathfrak{X}(M)$ the set of all C^2 differential 1-forms and C^2 vector fields respectively. Let $(\Omega, \mathcal{F})$ be a basic measurable space. Now consider a Brownian motion $(X_t, \mathcal{F}_t, P_x)$ on M defined on $(\Omega, \mathcal{F})$. Let $(r_t, \mathcal{F}_t, P_r)$ be the lifted diffusion on $O(M)$ of $(X_t, \mathcal{F}_t, P_x)$, i.e., the horizontal Brownian motion on $O(M)$, (cf. Malliavin [9]). Then it holds that for $r(t)=[X_t, e(t)]$,

$$de^j_i(t)=-\sum_{p,q=1}^{d}\Gamma^j_{pq}(X_t)e^p_i(t)\circ dX^q_t, \qquad i, j=1, 2, \cdots, d, \tag{2.4}$$

where the symbol $\circ$ denotes the symmetric multiplication in the sense of Itô [5].

Let $\{K_n\}$ be an increasing sequence of compact sets with $M=\bigcup_{n=1}^{\infty}K_n$ and $\zeta_n(\omega)$ be the first exit time from K_n. Throughout this paper, we assume the following:

Assumption 2.1. There exists a certain $\mathcal{F}$-measurable set Ω_0 such that $P_x(\Omega_0)=1$ for every $x\in M$ and

$$\lim_{n\to\infty}\zeta_n(\omega)=\infty, \qquad \text{for every } \omega\in\Omega_0.$$

We choose locally finite open coverings $\{W_n\}_{n\in N}$,[2] $\{U_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$ of M satisfying the following conditions:

(i) For any $n\in N$, W_n is a coordinate neighborhood.

(ii) For any $n\in N$,

$$\bar{U}_n\subset V_n\subset \bar{V}_n\subset W_n. \tag{2.5}$$

Let $\{\psi_n\}_{n\in N}$ be a partition of unity subordinate to $\{U_n\}$. Define the sequences $\{\sigma_{n,k}\}$ and $\{\tau_{n,k}\}$ of stopping times by the following relations: for $n=1, 2, \cdots$ and $k=0, 1, \cdots$,

$$\begin{aligned}&\sigma_n=\inf\{t; X_t\notin V_n\},^{3)}\\ &\tau_n=\inf\{t; X_t\in U_n\}, \qquad \sigma_{n,-1}=0,\\ &\sigma_{n,k}=\tau_{n,k}+\sigma_n\circ\theta_{\tau_{n,k}}, \qquad \tau_{n,k}=\sigma_{n,k-1}+\tau_n\circ\theta_{\sigma_{n,k-1}},\end{aligned} \tag{2.6}$$

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[2] N denotes the set of all positive integers.

[3] The infimum of the empty set is understood to be ∞.

where θ_t is the shift operator of $(X_t, \mathcal{F}_t, P_x)$.

We will now define the integral $\int_{X[0,t]} \alpha$ of $\alpha \in A^1(M)$ where X denotes the curve in M defined by

$$[0, t] \ni s \mapsto X_s \in M. \tag{2.7}$$

We begin with the case in which the support of α is contained in a neighborhood U_n. If we take a local coordinate $(x^1, x^2, \cdots, x^d)$ in (W_n, ϕ_n), α can be written in the form

$$\alpha = \sum_{i=1}^{d} \alpha_i dx^i. \tag{2.8}$$

Let $\phi_n(X_t) = (X_t^1, X_t^2, \cdots, X_t^d) \in R^d$. Keeping these considerations in mind, we define the integral

$$\int_{X[0,t]} \alpha \tag{2.9}$$

of $\alpha \in A^1(M)$ by

$$\int_{X[0,t]} \alpha = \sum_{k=0}^{\infty} \sum_{i=1}^{d} \int_{\tau_{n,k} \wedge t}^{\sigma_{n,k} \wedge t} \alpha_i(X_s) \circ dX_s^i. \tag{2.10}$$

We will now consider the general case of C^2 differential 1-form α. In this case, define the integral (2.9) of α by

$$\int_{X[0,t]} \alpha = \sum_{n \in N} \int_{X[0,t]} \psi_n \alpha. \tag{2.11}$$

Then we have

Lemma 2.1. *The integral* $\int_{X[0,t]} \alpha$ *defined by* (2.10) *and* (2.11) *is uniquely defined from* α *and is independent of a particular choice of the locally finite coverings* $\{W_n\}_{n \in N}$, $\{U_n\}_{n \in N}$ *and* $\{V_n\}_{n \in N}$ *and the local coordinate* $(x^1, x^2, \cdots, x^d)$.

Using Itô's formula we can easily check and so the proof is omitted.

Definition 2.1. $\int_{X[0,t]} \alpha$ defined above is called the *integral of* α *along the path* X_t, $0 \leq t < \infty$. We will often denote $\int_{X[0,t]} \alpha$ as $A(t; \alpha, \omega)$ or more simply $A(t; \alpha)$.

Remark 2.1. It is easy to see that the integral $A(t;\alpha,\omega)$ of α is defined as a stochastic process: In fact, a version of $A(t;\alpha,\omega)$ can be chosen so that the mapping

$$[0,\infty)\times\Omega\ni(t,\omega)\mapsto A(t;\alpha,\omega)\in(-\infty,\infty]$$

gives a continuous local quasi-martingale additive functional with respect to $\mathfrak{F}_t$.

The above definition of integral of α along the path X_t, $0\leqq t<\infty$, is extended to the case of the horizontal Brownian motion on $O(M)$ by the following way: Let α be a C^2 differential 1-form on $O(M)$ such that

(i) The support of α is contained in $\pi^{-1}(U_n)$.

(ii) Let $(x^1, x^2, \cdots, x^d)$ be a local coordinate in W_n. Then

$$\alpha=\sum_{i=1}^{d}\alpha_i(r)\,dx^i+\sum_{i,j=1}^{d}\alpha_j^i(r)\,de_i^j\,,\qquad r=[x,e]\in O(M).$$

Then, set

$$\int_{r[0,t]}\alpha=\sum_{k=0}^{\infty}\int_{\tau_{n,k}\wedge t}^{\sigma_{n,k}\wedge t}\Big[\sum_{i=1}^{d}\alpha_i(r_s)\circ dX_s^i+\sum_{i,j=1}^{d}\alpha_j^i(r_s)\circ de_i^j\Big],$$

where $r(t)=[X_t,e(t)]$, $e(t)=(e_i^j(t))$ and $r[0,t]$ is the curve in $O(M)$ defined by

$$[0,t]\ni s\to r_s\in O(M).$$

Next, using the partition $\{\psi_n\}_{n\in\boldsymbol{N}}$ of unity subordinate to $\{U_n\}$, we can define the integral $\int_{r[0,t]}\alpha$ for a general smooth differential form α on $O(M)$.

Remark 2.2. If $\alpha\in A^1(M)$, then we have

$$\text{(2.12)}\qquad \int_{r[0,t]}\pi^*\alpha=\int_{X[0,t]}\alpha\,,\qquad\text{for every } t\geqq 0,\ \text{a.s.},$$

where $\pi^*\alpha$ is the pull-back of α by π.

Remark 2.3. Let α be an exact C^2 differential 1-form, i.e., there exists a function u on M such that $\alpha=du$. Then

$$\text{(2.13)}\qquad A(t;\alpha)=u(X_t)-u(X_0)\,,\qquad\text{for every } t\geqq 0,\ \text{a.s.}.$$

To prove (2.13), take locally finite coverings $\{W_n\}_{n\in N}$, $\{U_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$ mentioned above. Let $\sigma_{n,k}$ and $\tau_{n,k}$ be the sequences of stopping times defined by (2.6). It is enough to show (2.13) in the case when the support of α is contained in U_n. In this case, by the chain rule,

$$\sum_{i=1}^{d}\int_{\tau_{n,k}\wedge t}^{\sigma_{n,k}\wedge t}\alpha_i(X_s)\circ dX_s^i=u(X_{\sigma_{n,k}\wedge t})-u(X_{\tau_{n,k}\wedge t}).$$

Hence

$$A(t;\alpha)=\sum_{k=0}^{\infty}(u(X_{\sigma_{n,k}\wedge t})-u(X_{\tau_{n,k}\wedge t}))$$
$$=u(X_t)-u(X_0),$$

which completes the proof.

§ 3. Some Formulas on Integral of Differential Forms

In this section, we will show some formulas on the integral $A(t;\alpha)$ of α along the path X_t. First we will prepare several lemmas for future use. The solder 1-forms ω^i, $i=1,2,\cdots,d$, on $O(M)$ are defined as follows. For $r=(x,e_1,e_2,\cdots,e_d)\in O(M)$,

$$d\pi\xi=\sum_{i=1}^{d}\omega^i(\xi)e_i, \quad \text{for every} \quad \xi\in T_rO(M), \tag{3.1}$$

where $d\pi$ is the differential of the mapping π. We refer the reader to Bishop-Crittenden [1], pp. 90-93, for the definition and properties of the solder 1-form. Take a local coordinate $(x^1,x^2,\cdots,x^d)$ in a coordinate neighborhood U of M. Then we have

Lemma 3.1. *The solder* 1*-forms* ω^i, $i=1,2,\cdots,d$, *on* $\pi^{-1}U$ *can be expressed in the following form*:

$$\omega^i=\sum_{k=1}^{d}(\sum_{j=1}^{d}g_{kj}(x)e_i^j)dx^k, \quad r=[x,e]\in O(M). \tag{3.2}$$

Proof. For every C^2 function f on M,

$$d\pi(\xi)f=\xi f(\pi(r)), \quad \text{for every } \xi\in T_rO(M).$$

Hence ω^i, $i=1,2,\cdots,d$, can be expressed in the following form:

(3. 3) $$\omega^i=\sum_{k=1}^{d}\omega_k^i(r)\,dx^k.$$

Combining this with (3. 1), we have

$$\frac{\partial}{\partial x^k}f=\sum_{i,j=1}^{d}\omega_k^i(r)\,e_i^j\frac{\partial}{\partial x^j}f\,.$$

Hence $(\omega_j^i(r))=(e_j^i)^{-1}$.[4] Now, using (2. 2), we have

$$\omega_j^i(r)=\sum_{k=1}^{d}g_{jk}(x)\,e_i^k\,,\qquad r=[x,e]\in O(M).$$

This completes the proof.

The following lemma is an easy consequence of Lemma 3. 1.

Lemma 3. 2. *Let* $\alpha\in A^1(M)$. *Then we have*

(i) α *can be written as a differential* 1*-form on* $O(M)$ *in the following form*:

(3. 4) $$\pi^*\alpha=\sum_{i=1}^{d}\bar{\alpha}_i(r)\,\omega^i.$$

(ii) *Take a local coordinate* $(x^1, x^2, \cdots, x^d)$ *in a coordinate neighborhood U. Then* $\bar{\alpha}_i(r)$ *defined by* (3. 4) *can be expressed in the following form*:

(3. 5) $$\bar{\alpha}_i(r)=\sum_{k=1}^{d}\alpha_k(\pi(r))\,e_i^k\,,\qquad r=[x,e]\in\pi^{-1}(U),$$

where

(3. 6) $$\alpha=\sum_{k=1}^{d}\alpha_k(x)\,dx^k.$$

Now take the solder 1-forms $\omega^i, i=1, 2, \cdots, d$, on $O(M)$ and define continuous functionals $B_t^i, i=1, 2, \cdots, d$, of $(r_t, \mathcal{F}_t, P_r)$ by

(3. 7) $$B_t^i=\int_{r[0,t]}\omega^i,\quad \text{for } t\geqq 0 \quad\text{and}\quad i=1,2,\cdots,d\,.$$

Then $B_t=(B_t^1, B_t^2, \cdots, B_t^d)$, $0\leqq t<\infty$, defines a random curve in $\mathbb{R}^d$ which we call the *stochastic development of* X_t, $0\leqq t<\infty$, into $\mathbb{R}^d$. Take a

[4] In general, when we regard (c_i^k) as a matrix $c=(c_{ik})$, we set $c_{ik}=c_i^k$.

local coordinate $(x^1, x^2, \cdots, x^d)$ in a coordinate neighborhood U in M. Then, since

$$(3.8)\qquad dB_t^i=\sum_{k=1}^d \sum_{j=1}^d g_{kj}(X_t)e_i^j(t)\circ dX_t^k, \qquad i=1,2,\cdots,d,$$

by Lemma 3. 1, we obtain the following proposition which was proved by Itô [6].

Proposition 3. 1. (Itô [6]). *The stochastic development* $B_t = (B_t^1, B_t^2, \cdots, B_t^d)$, $0\leqq t<\infty$, *of* X_t *into* $\boldsymbol{R}^d$ *is a d-dimensional* $\mathfrak{F}_t$-*Brownian motion, that is,*

(i) $B_t - B_s$ *is independent of* $\mathfrak{F}_s$, $(t\geqq s)$,

(ii) $B_t - B_s$ *is Gaussian, i.e.,*

$E_x[\exp\{i\langle\lambda, B_t-B_s\rangle\}] = \exp\{-\|\lambda\|^2(t-s)/2\}, \qquad \lambda\in\boldsymbol{R}^d,\ t>s.$[5)]

Lemma 3. 3. *For every* $\alpha=\sum_{i=1}^d \bar{\alpha}_i(r)\omega^i\in\Lambda^1(M)$

$$(3.9)\qquad \int_{X[0,t]}\alpha=\sum_{i=1}^d\int_0^t\bar{\alpha}_i(r_s)\circ dB_s^i, \quad \textit{for every } t\geqq 0, \text{ a.s..}$$

Proof. By Lemma 3. 1 and Lemma 3. 2,

$$dA(t;\alpha) = \sum_{i,j,k=1}^d (\bar{\alpha}_k(r_t)g_{ij}(X_t)e_k^j(t))\circ dX_t^i$$

$$=\sum_{k=1}^d \bar{\alpha}_k(r_t)\circ\{(\sum_{i,j=1}^d g_{ij}(X_t)e_k^j(t))\circ dX_t^i\}.$$

Combining this with (3. 8) and using the chain rule, we can prove (3. 9).

The following lemma is an immediate consequence of Lemma 3. 3.

Lemma 3. 4. (Itô [6]). *Take a local coordinate* $(x^1, x^2, \cdots, x^d)$ *in a coordinate neighborhood* U *in* M. *Then*

$$(3.10)\qquad dX_t^k=\sum_{i=1}^d e_i^k(t)\circ dB_t^i, \qquad k=1,2,\cdots,d,$$

where $r(t)=[X_t, e(t)]$, $X_t=(X_t^1, X_t^2, \cdots, X_t^d)$ *and* $e(t)=(e_j^i(t))$.

5) $\langle a, b\rangle=\sum_{i=1}^d a_ib_i$ and $\|a\|^2=\langle a, a\rangle$ for any $a, b\in\boldsymbol{R}^d$.

We are now in a position to state our main theorem in this section.

Theorem 3. 1. *Let* $B_t=(B_t^1, B_t^2, \cdots, B_t^d)$, $0\leqq t<\infty$, *be the stochastic development of* X_t *into* $\boldsymbol{R}^d$ *and consider a differential* 1-*form* $\alpha\in A^1(M)$. *Then the continuous additive functional* $A(t;\alpha)$ *can be expressed in the following form*:

$$A(t;\alpha)=\sum_{i=1}^{d}\int_0^t \bar{\alpha}_i(r_s)\,dB_s^i-\frac{1}{2}\int_0^t \delta\alpha(X_s)\,ds, \tag{3.11}$$

for every $t\geqq 0$, a.s.,

where $\alpha=\sum_{i=1}^{d}\bar{\alpha}_i(r)\,\omega^i$ *and* δ *is the adjoint operator of the exterior differential operator* d.

Proof. It is enough to show (3. 11) in the case where the support of α is contained in a coordinate neighborhood U. Set $Y_i(t)=\bar{\alpha}_i(r_t)$, $0\leqq t<\infty$, $i=1,2,\cdots,d$. Then, by Lemma 3. 3,

$$dA(t;\alpha)=\sum_{i=1}^{d}\bar{\alpha}_i(r_t)\,dB_t^i+\frac{1}{2}\sum_{i=1}^{d}d\langle M_{Y_i}, B^i\rangle_t, \tag{3.12}$$

where M_{Y_i} is the martingale part of the quasi-martingale $Y_i(t)$, $0\leqq t<\infty$ and $\langle M_{Y_i}, B^i\rangle_t$ is the quadratic variation process corresponding to M_{Y_i} and B_t^i, (cf. Kunita-S. Watanabe [8] and Itô [5]). Let $(x^1, x^2, \cdots, x^d)$ be a coordinate in U. Letting

$$\alpha=\sum_{k=1}^{d}\alpha_k(x)\,dx^k,$$

we have

$$\bar{\alpha}_i(r)=\sum_{k=1}^{d}\alpha_k(x)\,e_i^k, \qquad \text{for } r=[x,e]\in\pi^{-1}(U),$$

by Lemma 3. 2. Hence,

$$\begin{aligned}
&\sum_{i=1}^{d}d\langle M_{Y_i}, B^i\rangle_t\\
&=\sum_{i,j,k=1}^{d}\frac{\partial}{\partial x^j}\alpha_k(X_t)\,e_i^k(t)\sum_{p,q=1}^{d}g_{pq}(X_t)\,e_i^q(t)\,g^{jp}(X_t)\,dt\\
&\quad-\sum_{i,k=1}^{d}\alpha_k(X_t)\sum_{m,n,p,q=1}^{d}\Gamma_{pq}^k(X_t)\,e_i^p(t)\,e_i^n(t)\,g^{mq}(X_t)\,g_{mn}(X_t)\,dt
\end{aligned}$$

$$= \sum_{i,j,k=1}^{d} \frac{\partial}{\partial x^j} \alpha_k (X_t) e_i^k (t) e_i^j (t) dt$$

$$- \sum_{i,k=1}^{d} \alpha_k (X_t) \sum_{p,q=1}^{d} \Gamma_{pq}^k (X_t) e_i^p (t) e_i^q (t) dt .$$

Therefore, using (2. 2) we obtain

$$\text{(3. 13)} \quad \sum_{i=1}^{d} d\langle M_{Y_i}, B^i \rangle_t$$

$$= \sum_{j,k=1}^{d} \frac{\partial}{\partial x^j} \alpha_k (X_t) g^{kj} (X_t) dt - \sum_{k,p,q=1}^{d} \alpha_k (X_t) \Gamma_{pq}^k (X_t) g^{pq} (X_t) dt .$$

Since

$$\delta\alpha(x) = - \sum_{i,j=1}^{d} g^{ij}(x) \left(\frac{\partial \alpha_j}{\partial x^i}(x) - \sum_{m=1}^{d} \Gamma_{ij}^m (x) \alpha_m (x) \right), \qquad x \in U ,$$

(3. 13) implies

$$\sum_{i=1}^{d} d\langle M_{Y_i}, B^i \rangle_t = -\delta\alpha(X_t) dt .$$

Combining this with (3. 12), we can complete the proof.

It is not difficult to see that the first term in the right hand of (3. 11)

$$\sum_{i=1}^{d} \int_0^t \bar{\alpha}_i (r_s) dB_s^i$$

is uniquely determined from $A(t; \alpha)$ and is called the *martingale part of* $A(t; \alpha)$. Throughout this paper we will use $M(t; \alpha)$ to denote the martingale part of $A(t; \alpha)$. It is easy to see that $M(t; \alpha)$ is a local-martingale additive functional of $(X_t, \mathcal{F}_t, P_x)$.

§ 4. Representation Theorem for Continuous Additive Functionals

Let $\mathcal{M}$ be the space of continuous additive functionals such that

$$E_x[A(t)^2] < \infty \text{ and } E_x[A(t)] = 0, \quad \text{for every } x \in M \text{ and } t \geqq 0 .$$

Let $p(t, x, y)$, $t > 0$, $x, y \in M$, be the minimal fundamental solution with respect to dm of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \Delta u , & t > 0 , \quad x \in M , \\ u(0+, x) = f(x), & x \in M , \end{cases}$$

where dm is the volume element corresponding to the Riemannian metric g on M. Set

$$K(t, x, y) = \int_0^t p(s, x, y)\, ds\,, \qquad t>0\,, \quad x, y \in M\,. \tag{4.1}$$

Let $\langle\ ,\ \rangle(x)$ be the inner product on $(T_xM)^*$ where $(T_xM)^*$ is the dual space of T_xM. Fix any $(t, x) \in (0, \infty) \times M$ and define a norm of $\alpha \in A^1(M)$ by

$$\|\alpha\|_{t,x}^2 = \int_M \langle\alpha, \alpha\rangle(y) K(t, x, y)\, m(dy)\,. \tag{4.2}$$

Then if $\|\alpha\|_{t,x}$ and $\|\beta\|_{t,x}$ are finite, the integral

$$(\alpha, \beta)_{t,x} = \int_M \langle\alpha, \beta\rangle(y) K(t, x, y)\, m(dy) \tag{4.3}$$

converges absolutely and define a positive definite symmetric bilinear form on

$$\mathcal{H}_{t,x} = \{\alpha; \alpha \in A^1(M), \|\alpha\|_{t,x} < \infty\}\,. \tag{4.4}$$

Let $\bar{\mathcal{H}}_{t,x}$ be the completion of $\mathcal{H}_{t,x}$ with respect to the norm $\|\ \|_{t,x}$. Then, for any fixed $(t, x) \in (0, \infty) \times M$, the space $\bar{\mathcal{H}}_{t,x}$ is a Hilbert space with inner product $(\ .\ )_{t,x}$. Set

$$\mathcal{H} = \bigcap_{t>0,\, x\in M} \mathcal{H}_{t,x} \quad \text{and} \quad \bar{\mathcal{H}} = \bigcap_{t>0,\, x\in M} \bar{\mathcal{H}}_{t,x}\,.$$

Now let $\{U_n\}$ be the locally finite open covering of M given in Section 2 and $(x^1, x^2, \cdots, x^d)$ be a local coordinate in the coordinate neighborhood (W_n, ϕ_n). Choose an $\alpha \in \bar{\mathcal{H}}$. Then α can be expressed in the following:

$$\alpha = \sum_{i=1}^d \alpha_i(x)\, dx^i, \qquad \text{on} \qquad U_n\,, \tag{4.5}$$

where $\alpha_i(x)$, $i=1, 2, \cdots, d$, are measurable functions on M. Let $\bar{\alpha}_i$, $i=1, 2, \cdots, d$, be the functions on $O(M)$ defined by

$$\bar{\alpha}_i(r) = \sum_{k=1}^d \alpha_k(\pi(r))\, e_i^k \qquad \text{for every } r \in \pi^{-1}(U_n)\,. \tag{4.6}$$

Then, by Lemma 3.1, it follows that

$$\pi^*\alpha = \sum_{i=1}^d \bar{\alpha}_i(r)\, \omega^i, \qquad \text{for every } r \in O(M)\,, \tag{4.7}$$

where ω^i, $i=1, 2, \cdots, d$, are the solder forms. Now, following Motoo-S. Watanabe [11], we can define the stochastic integral:

$$S(t; \alpha) = \sum_{i=1}^{d} \int_0^t \bar{\alpha}_i(r_s)\, dB_s^i \tag{4.8}$$

where $B_t = (B_t^1, B_t^2, \cdots, B_t^d)$ is the stochastic development of X_t into $\boldsymbol{R}^d$, since

$$\sum_{i=1}^{d} \bar{\alpha}_i(r)^2 = \langle \alpha, \alpha \rangle (\pi(r)), \qquad \text{for every } r \in O(M), \tag{4.9}$$

by (2.3) and (4.6). Then we have

$$E_x[S(t; \alpha)^2] = \|\alpha\|_{t,x}^2 \quad \text{and} \quad E_x[S(t; \alpha)] = 0. \tag{4.10}$$

Further if $\alpha \in \mathcal{H}$, then $S(t; \alpha) = M(t; \alpha)$. We then have the following result.

Proposition 4.1. *If* $\alpha \in \bar{\mathcal{H}}$, *then* $S(\cdot; \alpha) \in \mathcal{M}$.

Proof. For any $n \in N$, we define ξ_n by

$$\xi_n = \inf\{t; X_t \notin U_n\}.$$

Then, using (2.4) and (3.8), we can show that, for $0 \leqq s < \xi_n$,

$$\begin{aligned} dB_s^i = & \sum_{k,j=1}^{d} g_{kj}(X_s) e_i^j(s)\, dX_s^k \\ & + \frac{1}{2} \sum_{j,k,p=1}^{d} \frac{\partial}{\partial x^p} g_{kj}(X_s) e_i^j(s) g^{pk}(X_s)\, ds \\ & - \frac{1}{2} \sum_{j,k,p,q=1}^{d} g_{kj}(X_s) \Gamma_{pq}^j(X_s) e_i^p(s) g^{kq}(X_s)\, ds \end{aligned}$$

where $\phi_n(X_t) = (X_t^1, X_t^2, \cdots, X_t^d)$. Hence, if the support of α is contained in U_n, then

$$\begin{aligned} & S(t; \alpha) \\ & = \sum_{i,j,k,m=1}^{d} \int_0^t \alpha_m(X_s) e_i^m(s) g_{kj}(X_s) e_i^j(s)\, dX_s^k \\ & \quad + \frac{1}{2} \sum_{i,j,k,m,p=1}^{d} \int_0^t \alpha_m(X_s) e_i^m(s) \frac{\partial}{\partial x^p} g_{kj}(X_s) e_i^j(s) g^{pk}(X_s)\, ds \\ & \quad - \frac{1}{2} \sum_{i,j,k,m,p,q=1}^{d} \int_0^t \alpha_m(X_s) e_i^m(s) g_{kj}(X_s) \Gamma_{pq}^j(X_s) e_i^p(s) g^{kq}(X_s)\, ds. \end{aligned}$$

Combining this with (2. 3), we have

$$S(t;\alpha)$$

$$=\sum_{k=1}^{d}\int_0^t \alpha_k(X_s)\,dX_s^k-\frac{1}{2}\sum_{j,k=1}^{d}\int_0^t \alpha_j(X_s)\frac{\partial}{\partial x^k}g^{jk}(X_s)\,ds$$

$$-\frac{1}{2}\sum_{j,k,p=1}^{d}\int_0^t \alpha_j(X_s)\Gamma_{pk}^k(X_s)g^{jp}(X_s)\,ds\,.$$

Since

$$\frac{\partial}{\partial x^i}\log\sqrt{G}=\sum_{k=1}^{d}\Gamma_{ik}^k\,,\qquad \text{where } G=\det(g_{ij}),$$

it is easy to see that if the support of α is contained in U_n, then

$$S(t;\alpha)=\sum_{k=1}^{d}\int_0^t \alpha_k(X_s)\,dX_s^k$$

$$-\frac{1}{2}\sum_{j,k=1}^{d}\int_0^t \alpha_j(X_s)\left\{\frac{\partial}{\partial x^k}g^{jk}(X_s)+\left(g^{jk}\frac{\partial}{\partial x^k}\log\sqrt{G}\right)(X_s)\right\}ds\,.$$

Now by using a standard localization argument, it is easy to check that, for any $\alpha\in\bar{\mathcal{H}}$, $\{S(t;\alpha),t\geqq 0\}$ is an additive functional of $(X_t,\mathcal{F}_t,P_x)$ (cf. [11]). Combining this fact with (4. 10) we have the conclusion of the proposition.

The result we want to show is the following

Theorem 4. 1. *For any $A\in\mathcal{M}$, there exists a unique $\alpha\in\bar{\mathcal{H}}$ such that*

(4. 11) $\qquad A(t)=S(t;\alpha),\quad$ *for every* $t\geqq 0$, a.s.,

and further

(4. 12) $\qquad E_x[A(t)^2]=\|\alpha\|_{t,x}^2,\quad$ *for every* $(t,x)\in(0,\infty)\times M$.

Before proceeding to the proof of Theorem 4. 1 we will prepare three lemmas. For any $A\in\mathcal{M}$, we set

$$m(t,x:A)=E_x[A(t)^2],\qquad \text{for } (t,x)\in(0,\infty)\times M.$$

Now consider a subclass of $\mathcal{M}$ given by

$\tilde{\mathcal{M}}=\{A; A\in\mathcal{M}, m(t,x;A)$ is bounded in x for any fixed $t\}$.

Lemma 4.1. *For any $A\in\tilde{\mathcal{M}}$, there exists a unique non-negative Radon measure μ on M such that*

$$m(t,x;A)=\int_M K(t,x,y)\mu(dy), \tag{4.13}$$

$$\textit{for every } (t,x)\in(0,\infty)\times M.$$

Proof. By Lemma 8.3 in Motoo-S. Watanabe [11], if $A\in\tilde{\mathcal{M}}$, $m(t,x;A)$ is a bounded regular characteristic in the sense of [11]. By setting

$$u(\lambda,x;A)=\int_0^\infty e^{-\lambda t}d_t m(t,x;A), \quad \text{for } \lambda>0,$$

and using Theorem 3.4 in [11], it is easy to see that $u(\lambda,x;A)$ is a bounded regular λ-excessive function in the sense of [11]. By using Proposition (2.10) in [2], p. 272 and the remark in [2], p. 266, it is easy to check that there exists a unique non-negative Radon measure μ satisfying

$$u(\lambda,x;A)=\int_M g_\lambda(x,y)\mu(dy), \quad \text{for every } \lambda>0, \tag{4.14}$$

where

$$g_\lambda(x,y)=\int_0^\infty e^{-\lambda t}p(t,x,y)dt.$$

Then (4.13) follows from (4.14).

We want to apply the above lemma to the following situation. For $A\in\mathcal{M}$ and $\beta\in\mathcal{H}$ we set

$$\Phi(\beta,A)(t,x)=E_x[A(t)S(t;\beta)], \quad \text{for } (t,x)\in(0,\infty)\times M,$$

and

$$\Phi_\pm(\beta,A)(t,x)=E_x[(A(t)\pm S(t;\beta))^2],$$

$$\text{for } (t,x)\in(0,\infty)\times M.$$

If $A\in\widetilde{\mathcal{M}}$ and $\langle\beta,\beta\rangle(x)$ is bounded on M, then

$$A(t)\pm S(t;\beta)\in\widetilde{\mathcal{M}}.$$

Hence, if $A\in\widetilde{\mathcal{M}}$, $\beta\in\bar{\mathcal{H}}$ and $\langle\beta,\beta\rangle(x)$ is bounded on M, then there exists a unique non-negative Radon measure $\mu_\pm(\cdot;\beta)$ such that

$$\Phi_\pm(\beta, A)(t,x)=\int_M K(t,x,y)\,\mu_\pm(dy;\beta),$$

$$\text{for every } (t,x)\in(0,\infty)\times M.$$

Furthermore there exists a unique signed measure $\mu(\cdot;\beta)$ satisfying the following

$$\Phi(\beta, A)(t,x)=\int_M K(t,x,y)\,\mu(dy;\beta),$$

(4.15) $$\text{for every } (t,x)\in(0,\infty)\times M,$$

$$\mu(\cdot;\beta)=\frac{1}{4}(\mu_+(\cdot;\beta)-\mu_-(\cdot;\beta)).$$

Remark 4.1. Let $\beta\in\mathcal{H}$ and $A\in\mathcal{M}$. We further assume that $\langle\beta,\beta\rangle(x)$ is bounded on M. It is not hard to show that if β vanishes identically in an open set E, then $\mu(\cdot;\beta)$ has no mass inside E.

Next we will prove the following:

Lemma 4.2. *Fix $(t,x)\in(0,\infty)\times M$. Then, for any $A\in\mathcal{M}$, there exists a unique $\alpha=\alpha_{t,x}\in\bar{\mathcal{H}}_{t,x}$ such that*

(4.16) $$\Phi(\beta, A)(t,x)=(\alpha,\beta)_{t,x},\quad \textit{for every } \beta\in\mathcal{H}.$$

Proof. Since

$$E_x[M(t;\beta)^2]=\|\beta\|_{t,x}^2,\quad \text{for every } \beta\in\mathcal{H}_{t,x},$$

by Theorem 3.1, we have

(4.17) $$|\Phi(\beta, A)(t,x)|\leq\|\beta\|_{t,x}\sqrt{m(t,x;A)},$$

by Schwarz' inequality. It is easy to see that, for $\lambda_i\in\mathbb{R}^1$ and $\beta_i\in\mathcal{H}$, $i=1,2$, (4.18) $$\Phi(\lambda_1\beta_1+\lambda_2\beta_2, A)(t, x)=\sum_{i=1}^{2}\lambda_i\Phi(\beta_i, A)(t, x).$$

Hence, by Riesz' representation theorem, there exists a unique $\alpha\in\mathcal{H}_{t,x}$ such that

$$\Phi(\beta, A)(t, x)=(\alpha, \beta)_{t,x}, \quad \text{for every } \beta\in\mathcal{H},$$

which completes the proof.

Now using a minor modification of arguments in Tanaka [15], we will prove the following.

Lemma 4.3. *For any $A\in\tilde{\mathcal{M}}$, there exists a unique $\alpha\in\bar{\mathcal{H}}$ such that*

(4.19) $$A(t)=S(t;\alpha), \quad \textit{for every } t\geqq 0, \text{ a.s.},$$

and further

(4.20) $$E_x[A(t)^2]=\|\alpha\|_{t,x}^2, \quad \textit{for every } (t, x)\in(0, \infty)\times M.$$

Proof. First fix any positive integer n and take a local coordinate $(x^1, x^2, \cdots, x^d)$ in the coordinate neighborhood (U_n, ϕ_n). Let γ^i, $i=1, 2, \cdots, d$, be the differential 1-forms defined by

$$\gamma^i=I_{U_n}(x)\,dx^i, \quad \text{for } i=1, 2, \cdots, d.$$ [6]

Let μ be the measure associated with A by the relation in (4.13). Take an open subset E of U_n such that $\mu(\partial E)=0$ and let $\beta^i(E)$, $i=1, 2, \cdots, d$, be the differential 1-forms defined by

$$\beta^i(E)=I_E(x)\,dx^i, \quad \text{for } i=1, 2, \cdots, d.$$

Then, it is easy to see that, for $i=1, 2, \cdots, d$,

$$\gamma^i, \beta^i(E)\in\bar{\mathcal{H}} \quad \text{and} \quad S(t;\gamma^i), S(t;\beta^i(E))\in\tilde{\mathcal{M}}.$$

Since

$$\begin{aligned}&\Phi_+(\beta^i(E), A)(t, x)+\Phi_-(\beta^i(E), A)(t, x)\\&\quad=2m(t, x; A)+2\int_M K(t, x, y)\langle\beta^i(E), \beta^i(E)\rangle(y)\,m(dy),\end{aligned}$$

[6] I_A means the indicator function of a set A.

we have

$$\mu_+(\cdot\,;\beta^i(E))+\mu_-(\cdot\,;\beta^i(E))=2\mu(\cdot)+2\langle\beta^i(E),\beta^i(E)\rangle m(\cdot),$$
$$i=1,2,\cdots,d.$$

Hence

$$\mu_+(\partial E;\beta^i(E))=\mu_-(\partial E;\beta^i(E))=0\,,\qquad i=1,2,\cdots,d.$$

From this and (4. 15), $\mu(\partial E;\beta^i(E))=0$ follows. Combining this with Remark 4. 1 we have

$$(4.\,21)\qquad \begin{aligned}&\mu(\cdot\,;\beta^i(E))=0 &&\text{on } M\backslash E,\\ &\mu(\cdot\,;\gamma^i-\beta^i(E))=0 &&\text{on } E.\end{aligned}$$

Since the identity

$$\mu(\cdot\,;\beta^i(E))+\mu(\cdot\,;\gamma^i-\beta^i(E))=\mu(\cdot\,;\gamma^i)$$

follows from (4. 18), (4. 21) implies

$$(4.\,22)\qquad \mu(\cdot\,;\beta^i(E))=I_E\mu(\cdot\,;\gamma^i).$$

Combining this with (4. 15) we have

$$(4.\,23)\qquad \Phi(\beta^i(E),A)(t,x)=\int_E K(t,x,y)\,\mu(dy\,;\gamma^i),$$
$$\text{for every } (t,x)\in(0,\infty)\times M\,.$$

By using (4. 17) it is easy to see that (4. 23) must hold for any Borel set E in U_n. Now consider a differential 1-form:

$$\beta=\beta_i(x)\,dx^i\quad\text{and}\quad \beta_i(x)=\sum_{k=1}^m I_{E_k}(x)$$

where E_k, $k=1,2,\cdots,m$, are Borel sets in U_n. Then, from the fact mentioned above, we have

$$(4.\,24)\qquad \Phi(\beta,A)(t,x)=\int_M K(t,x,y)\beta_i(y)\,\mu(dy\,;\gamma^i).$$

Now fix any $(t,x)\in(0,\infty)\times M$. Let $\alpha_{t,x}$ be the differential 1-form associated with A by the relation in (4. 16). Then the restriction $\alpha_{t,x|U_n}$ of $\alpha_{t,x}$ to U_n can be written in the following:

$$(\alpha_{t,x|U_n})_y=\sum_{k=1}^d(\alpha_{t,x})_k(y)\,dy^k,\qquad\text{for } y\in U_n\,.$$

Comparing (4. 16) with (4. 24) we have,

$$\sum_{k=1}^{d} (\alpha_{t,x})_k (y) g^{ki}(y) m(dy)_{|U_n} = \mu(dy; \gamma^i)$$

$$\text{for } y \in U_n \quad \text{and} \quad i=1, 2, \cdots, d.$$

Hence the functions $(\alpha_{t,x})_k(y)$, $k=1, 2, \cdots, d$, on U_n are independent of (t, x).

Now using a standard localization argument we come to the important conclusion that the differential 1-form $\alpha_{t,x}$ is independent of (t, x). From now on we denote the differential 1-form mentioned above by α. Further, from (4. 16) it follows that for every $\beta \in \mathcal{H}$

(4. 25) $\quad E_x[\langle A - S(\cdot\,;\alpha), M(\cdot\,;\beta)\rangle_t] = 0$, for every $(t, x) \in (0, \infty) \times M$.

By Theorem 12. 2 in Motoo-S. Watanabe [11], it follows that $\mathcal{M}$ = the minimal subspace of $\mathcal{M}$ which contains $\{M(\cdot\,;\beta); \beta \in \mathcal{H}\}$. Hence from (4. 25), we have

$$A(t) = S(t; \alpha),$$

which completes the proof of the first assertion in the lemma. (4. 20) is an immediate consequence of (4. 19). This completes the proof.

Now we turn to the proof of theorem 4. 1.

Proof of Theorem 4. 1. By Theorem 5. 5 in Motoo-S. Watanabe [11] there exists an increasing sequence $\{m_n(t, x)\}$ of bounded regular characteristic such as

(i) $0 \leqq f_n \leqq 1$,

(ii) $m_n(t, x) = E_x[(\sqrt{f_n} \cdot A)_t^2]$,

where $(\sqrt{f_n} \cdot A)_t$ is the stochastic integral of $\sqrt{f_n}$ with respect to A. Hence, by Lemma 4. 3, there exists a sequence $\{\alpha_n\}$ of differential 1-forms satisfying the following:

(a) $\alpha_n \in \bar{\mathcal{H}}$, for every $n \in \boldsymbol{N}$.

(b) $m_n(t, x) = (\alpha_n, \alpha_n)_{t,x}$, for every $(t, x) \in (0, \infty) \times M$.

(c) Set $g_n(x) = \langle \alpha_n, \alpha_n \rangle(x)$, $x \in M$. Then the sequence $\{g_n\}$ is increasing.

By (c), there is a function g such as

$$g(x) = \lim_{n\to\infty} g_n(x).$$

Also it is easy to see that for any $(t, x) \in (0, \infty) \times M$

$$m(t, x; A) = \int_M K(t, x, y) g(y) m(dy),$$

by using the monotone convergence theorem. Hence, for any $A \in \mathcal{M}$ and $\beta \in \mathcal{H}$, there exists a function $g(x; \beta)$ such that

$$\Phi(\beta; A) = \int_M K(t, x, y) g(y; \beta) m(dy).$$

Now let E be an open set in M. It is easy to see that if β vanishes identically in E, then the measure $g(x; \beta) m(dx)$ has no mass inside E. Then using an argument similar to the proof of Lemma 4. 3, we come to the conclusion that for any $A \in \mathcal{M}$, there exists an $\alpha \in \bar{\mathcal{H}}$ satisfying

$$\Phi(\beta; A) = (\alpha, \beta)_{t,x}, \quad \text{for every } \beta \in \mathcal{H} \text{ and } (t, x) \in (0, \infty) \times M.$$

By using an argument similar to the final step of the proof of Lemma 4. 3, we have

$$A(t) = S(t; \alpha).$$

It is easy to see that (4. 12) follows from (4. 11). This completes the proof of Theorem 4. 1.

§ 5. Brownian Motion with Drift

Throughout this section we assume the following:

Assumption 5. 1. M is a compact Riemannian manifold.

Then, by using Theorem 3. 1, it is easy to see that for any $\alpha \in A^1(M)$, $M(t; \alpha)$ is a martingale with respect to $\mathcal{F}_t$. Setting

$$(5.1) \qquad N(t; \alpha) = \exp\left\{A(t; \alpha) + \frac{1}{2}\int_0^t (\delta\alpha - \|\alpha\|^2)(X_s)\, ds\right\}^{7)},$$

we can easily conclude that for any $\alpha \in A^1(M)$, $N(t; \alpha)$ is a martingale

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7) $\|\alpha\|^2(x) = \langle \alpha, \alpha \rangle(x)$ for $\alpha \in A^1(M)$.

with respect to $\mathcal{F}_t$. Define a family of measures $\tilde{P}_x, x\in M$, by

$$\tilde{P}_x(B)=E_x[N(t;\alpha);B], \quad \text{for any } B\in\mathcal{F}_t. \tag{5.2}$$

Then, it is well known that $\tilde{X}=(X_t, \mathcal{F}_t, \tilde{P}_x)$ is the diffusion process with the infinitesimal generator

$$L=\frac{1}{2}\Delta+b^{\alpha}, \tag{5.3}$$

where b^{α} is a unique vector field such that at each point $x\in M$,

$$\alpha_x(v)=g_x(b_x^{\alpha}, v_x), \quad \text{for every } v\in\mathfrak{X}(M). \tag{5.4}$$

Now we have

Theorem 5.1. *Let $\alpha\in A^1(M)$. Then the following three statements are equivalent.*

(i) $\delta\alpha=0$, *i.e., α can be expressed in the following form*:

$$\alpha=\delta\beta+\alpha_1, \tag{5.5}$$

where β is a differential 2-form and α_1 is a harmonic 1-form.

(ii) *$A(t;\alpha)$ is a martingale with respect to $\mathcal{F}_t$.*

(iii) *The Riemannian volume $m(dx)$ on M is invariant under each $\tilde{T}_t$ where $\tilde{T}_t$ is the semi-group of $\tilde{X}$.*

Proof. The implication (i) ⇒ (ii) is clear by Theorem 3.1. If (ii) is holds, then, by Theorem 3.1,

$$E_x\left[\int_0^t \delta\alpha(X_s)\,ds\right]=0, \quad \text{for every } x\in M \text{ and } t>0.$$

Since $\delta\alpha$ is a continuous function on M, we have

$$\delta\alpha(x)=0, \quad \text{for every } x\in M,$$

that is, $\delta\alpha=0$. It is well known, (de Rham [3]), that $\delta\alpha=0$ implies (5.5). Hence (ii) implies (i).

The statement (iii) holds if and only if

$$\int_M Lu(x)\,m(dx)=0, \quad \text{for every } u\in C^{\infty}(M), \tag{5.6}$$

where $C^{\infty}(M)$ is the space of all C^{∞} functions on M, (cf. Nelson [13]). By (5.4),

$$\int_M (b^\alpha u)(x)\, m(dx) = \int_M \langle \alpha, du \rangle (x)\, m(dx), \qquad u \in C^\infty(M).$$

Hence, since

$$\int_M \Delta u(x)\, m(dx) = 0, \qquad \text{for every } u \in C^\infty(M),$$

(5. 6) is equivalent to

$$\int_M \langle \alpha, du \rangle (x)\, m(dx) = 0, \qquad \text{for every } u \in C^\infty(M).$$

Therefore (5. 6) holds if and only if $\delta\alpha = 0$, (cf. de Rham [3]). This completes the proof.

§ 6. Approximation Theorem

Let $\{W_n\}_{n\in N}$, $\{U_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$ be the locally finite open coverings of M given in Section 2. We further assume that $\{W_n\}_{n\in N}$ satisfies the following condition: For any $n\in N$ and $x, y \in W_n$, there exists a unique minimal geodesic $\gamma_{x,y}$ such that

$$(6.1) \qquad \gamma_{x,y}(0) = x, \quad \gamma_{x,y}(1) = y \quad \text{and} \quad \{\gamma_{x,y}(s) : 0 \leqq s \leqq 1\} \subset W_n,$$

(cf. Helgason [4]). Let $\delta(m) = \{0 = t_0^{(m)} < t_1^{(m)} < \cdots\}$ be a subdivision of $[0, \infty)$ and $\tilde{\delta}(m) = \{0 = s_0^{(m)} < s_1^{(m)} < \cdots\}$ be the refinement of $\delta(m)$ obtained by adding $\{\sigma_{n,k}\}$ and $\{\tau_{n,k}\}$ where $\{\sigma_{n,k}\}$ and $\{\tau_{n,k}\}$ are the sequences of stopping times defined by (2. 6). Let $X_m(t)$ be the broken geodesic such that the restriction $X_m | [s_k^{(m)}, s_{k+1}^{(m)}]$ of X_m to $[s_k^{(m)}, s_{k+1}^{(m)}]$ is the minimal geodesic joining $X_{s_k^{(m)}}$ and $X_{s_{k+1}^{(m)}}$. Consider the curve $X_m[0, t]$ in M defined by

$$[0, t] \ni s \mapsto X_m(s) \in M.$$

Then, since $X_m[0, t]$ is piecewise smooth, the integral $\int_{X_m[0,t]} \alpha$ of α along the curve $X_m[0, t]$ is well defined. Setting, for every positive integer N,

$$\eta_N = \inf\{t;\, X_t \notin \bigcup_{n=1}^{N} U_n\},$$

we have

Theorem 6. 1. *Let* $\alpha\in\Lambda^1(M)$. *If*

$$\lim_{m\to\infty}\sup_k (t_{k+1}^{(m)}-t_k^{(m)})=0\,,$$

then, for any positive integer N,

$$\lim_{m\to\infty} E_x\Big[\sup_{0\leqq t\leqq T\wedge\eta_N}\Big|\int_{X_m[0,t]}\alpha-\int_{X[0,t]}\alpha\Big|^2\Big]=0\,, \tag{6.2}$$

for every $T>0$.

Before the proceeding to the proof, we give a remark. As we mentioned in Section 2, if α is exact, i.e., there exists a function u on M such that $\alpha=du$,

$$\int_{X[0,t]}\alpha=u(X_t)-u(X_0)\,.$$

Also

$$\int_{X_m[0,t]}\alpha=u(X_m(t))-u(X_m(0))\,.$$

Hence, in this case, we can easily prove (6. 2).

Proof of Theorem 6. 1. Let $\{\psi_n\}_{n\in\boldsymbol{N}}$ be a partition of unity subordinate to $\{U_n\}$ and $(x^1, x^2, \cdots, x^d)$ be a local coordinate in the coordinate neighborhood (W_n, ϕ_n). Setting

$$I_{nk}(t)=\int_{\tau_{n,k}\wedge t}^{\sigma_{n,k}\wedge t}\Big\{\sum_{i=1}^d(\psi_n\alpha)_i(X_s)\circ dX_s^i-\sum_{i=1}^d(\psi_n\alpha)_i(X_m([s]))\,dX_s^i \\ -\frac{1}{2}\sum_{i,j=1}^d\Big(g^{ij}\frac{\partial(\psi_n\alpha)_i}{\partial x^j}\Big)(X_m([s]))\,ds\Big\}$$

and

$$J_{nk}(t)=\int_{\tau_{n,k}\wedge t}^{\sigma_{n,k}\wedge t}\Big\{\sum_{i=1}^d(\psi_n\alpha)_i(X_m(s))\,dX_m^i(s) \\ -\sum_{i=1}^d(\psi_n\alpha)_i(X_m([s]))\,dX_m^i(s) \\ -\frac{1}{2}\sum_{i,j=1}^d\Big(g^{ij}\frac{\partial(\psi_n\alpha)_i}{\partial x^j}\Big)(X_m([s]))\,ds\Big\},$$

where $[s]=\max\{s_k^{(m)};\, s_k^{(m)}<s\}$, $(\psi_n\alpha)=\sum_{i=1}^d(\psi_n\alpha)_i\,dx^i$ and $\phi_n(X_m(t))=(X_m^1(t), X_m^2(t), \cdots, X_m^d(t))$, we have

$$|\int_{X_m[0,t]}\alpha-\int_{X[0,t]}\alpha|\leqq|\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}I_{nk}(t)|+|\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}J_{nk}(t)|.$$

Hence

$$\begin{aligned}&E_x\Big[\sup_{0\leqq t\leqq T\wedge\eta_N}|\int_{X_m[0,t]}\alpha-\int_{X[0,t]}\alpha|^2\Big]\\ &\leqq 2\{E_x[\sup_{0\leqq t\leqq T\wedge\eta_N}|\sum_{n=1}^{N}\sum_{k=0}^{\infty}I_{nk}(t)|^2]\\ &\qquad+E_x[\sup_{0\leqq t\leqq T\wedge\eta_N}|\sum_{n=1}^{N}\sum_{k=0}^{\infty}J_{nk}(t)|^2]\}.\end{aligned}\tag{6.3}$$

Therefore there exists a positive constant C depending only on N such that the left hand side of (6. 3) is smaller than

$$2C\sum_{n=1}^{N}\{E_x[\sup_{0\leqq t\leqq T\wedge\eta_N}|\sum_{k=0}^{\infty}I_{nk}(t)|^2]+E_x[\sup_{0\leqq t\leqq T\wedge\eta_N}|\sum_{k=0}^{\infty}J_{nk}(t)|^2]\}.$$

Hence to complete the proof it is sufficient to show (6. 2) in the case where the support of α is contained in U_n for some $n\in\boldsymbol{N}$. Now using a modification of the technique used in Nakao-Yamato [12], we can complete the proof and so the details of the proof is omitted.

Remark 6. 1. Let α be a C^2 differential 1-form on M and b^α be the vector field defined by (5. 4). By Theorem 6. 1, the integral of α along the path may be considered as the measurement of the amount of work done in travelling the path in the force field determined by the vector field b^α.

§ 7. An Application of Approximation Theorem

In this section, we will give a stochastic version of Stokes' theorem. Y. Takahashi[8] did this for the special case that $M=\boldsymbol{R}^d$. In the rest of this paper, we will assume the following:

Assumption 7. 1. M is a simply connected, complete Riemannian

[8] Private communication.

manifold with non-positive sectional curvature.

Under this condition, there exists a coordinate system $(x^1, x^2, \cdots, x^d)$ valid on the whole M, (cf. [10]). Consider a piecewise smooth curve ϕ: $[0, t] \to M$, such that $X(0) = \phi(0)$. Let $\gamma(u; x, y)$, $0 \leqq u \leqq 1$, be the minimal geodesic joining x and y and c be the mapping defined by

$$[0, t] \times [0, 1] \ni (s, u) \mapsto c(s, u) = \gamma(u; X(s), \phi(s)) \in M.$$

Denote by $S(t; X, \phi)$ the chain on M determined by the mapping c. For any C^2 differential 2-form β, we define the integral $\int_{S(t; X, \phi)} \beta$ of β on $S(t; X, \phi)$ by

$$(7.1) \quad \int_{S(t; X, \phi)} \beta = \sum_{i<j} \int_0^1 du \sum_{k=1}^d \int_0^t \Big\{ \Big[\beta_{ij}(X_s) \times \Big(\frac{\partial \gamma^j}{\partial u} \frac{\partial \gamma^i}{\partial x^k} - \frac{\partial \gamma^i}{\partial u} \frac{\partial \gamma^j}{\partial x^k} \Big) (u; X_s, \phi(s)) \Big] \circ dX_s^k + \Big[\beta_{ij}(X_s) \Big(\frac{\partial \gamma^j}{\partial u} \frac{\partial \gamma^i}{\partial x^k} - \frac{\partial \gamma^i}{\partial u} \frac{\partial \gamma^j}{\partial x^k} \Big) (u; X_s, \phi(s)) \Big] \frac{d\phi^k}{ds}(s) ds \Big\},$$

where $\phi(s) = (\phi^1(s), \phi^2(s), \cdots, \phi^d(s))$ and

$$\beta = \sum_{i<j} \beta_{ij} dx^i \wedge dx^j.$$

By using Itô's formula it is easy to see that the integral $\int_{S(t; X, \phi)} \beta$ defined by (7.1) is independent of a particular choice of the coordinate system $(x^1, x^2, \cdots, x^d)$ in M.

Let $\delta(m) = \{0 = t_0^{(m)} < t_1^{(m)} < \cdots\}$ be a subdivision of $[0, \infty)$ and X_m be the polygonal geodesic approximation of X given in Section 5. Denote by $S_m(t; X, \phi)$ the chain determined by the mapping c_m: $[0, t] \times [0, 1] \ni (s, u) \mapsto \gamma(u; X_m(s), \phi(s)) \in M$. Then we have the following

Lemma 7.1. *If*

$$\lim_{m \to \infty} \sup_k |t_{k+1}^{(m)} - t_k^{(m)}| = 0,$$

then, for every positive T,

$$\lim_{m\to\infty} E_x[\sup_{0\leq s\leq T} | \int_{S(t;X,\phi)} \beta - \int_{S_m(t;X,\phi)} \beta|^2] = 0 .$$

Using the same method as in Section 6, we can show this and so the proof is omitted.

Now we can state the following

Theorem 7.1. *Let* $\alpha \in A^1(M)$. *Then*

$$\int_{S(t;X,\phi)} d\alpha = \int_{\partial S(t;X,\phi)} \alpha , \quad \text{a.s.} . \tag{7.2}$$

Proof. By Stokes' theorem,

$$\int_{S_m(t;X,\phi)} d\alpha = \int_{\partial S_m(t;X,\phi)} \alpha . \tag{7.3}$$

Combining Theorem 6.1, Lemma 7.1 and (7.3), we can show (7.2).

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