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Finiteness Theorems on Weakly 1-complete Manifolds

By

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Let X be a complex manifold of (complex) dimension n and $\pi: B \to X$ be a holomorphic vector bundle over X. We consider the vector space of C^∞ $\bar{\partial}$ closed B-valued (p, q) forms modulo C^∞ $\bar{\partial}$ exact B-valued forms, which we denote by $H^{p,q}(X, B)$. It is interesting and sometimes useful to know whether $H^{p,q}(X, B)$ is finite dimensional or not. Specifically, when X is noncompact, the finite dimensionality of $H^{p,q}(X, B)$ is closely related to the function theoretic properties of X. The purpose of this article is to prove the following statement which was conjectured by S. Nakano:

If X is weakly 1-complete and B is positive outside a compact subset of X and of rank 1, $H^{n,q}(X, B)$ is finite dimensional for $q \geq 1$.

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§ 1. Notations

Let us fix the notations. We denote by X a connected paracompact complex manifold of dimension n. We denote for a subset K of X, the interior of K, the boundary of K and the closure of K by Int K, ∂K and $\bar{K}$, respectively. For two subsets K_1 and K_2 of X, we mean by $K_1 \Subset K_2$ that $\bar{K}_1$ is compact and contained in Int K_2. Let $\pi: B \to X$ be a holomorphic line bundle on X, and $\{U_i\}$ be an open covering of X consisting of coordinate neighbourhoods U_i with holomorphic coordinates $(z_i^1, \cdots, z_i^n)$, over which $\pi: B \to X$ is trivial, namely $\pi^{-1}(U_i) = U_i \times \boldsymbol{C}$, and $(x, \zeta_i) \in U_i \times \boldsymbol{C}$ and $(x, \zeta_j) \in U_j \times \boldsymbol{C}$ represent the same point of B if and only if

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(1) $$\zeta_i = e_{ij}(x)\zeta_j$$

where $\{e_{ij}\}$ is a system of transition functions for B.

A *B-valued differential form* f on X is a system $\{f_i\}$ ofdiff erential forms defined on U_i, satisfying $f_i = e_{ij} f_j$ in $U_i \cap U_j$.

We denote by $C^{p,q}(X)$ ($C^{p,q}(X, B)$) the space of differential forms (resp. B-valued forms) of class C^∞ and of type (p, q) on X, and by $C_0^{p,q}(X)$ ($C_0^{p,q}(X, B)$) the space of the forms in $C^{p,q}(X)$ (resp. $C^{p,q}(X, B)$) with compact supports. For a subset K of X, we denote by $C^\infty(\overline{K})$ the space of functions on $\overline{K}$ which are restrictions of C^∞ functions defined on a neighbourhood of $\overline{K}$. $C^{p,q}(\overline{K}, B)$ is defined similarly.

We fix a hermitian metric ds^2 in X, and a hermitian metric $\{a_i\}$ along the fibers of B. Here a_i is a positive function such that

(2) $$a_i |e_{ij}|^2 = a_j \text{ in } U_i \cap U_j\,.$$

For $f, g \in C^{p,q}(X, B)$, we set

(3) $$a_i f_i \wedge * \bar{g}_i = \langle f, g \rangle dv$$

where $*$ is the star operator and dv is the volume element with respect to the metric ds^2. $\langle f, g \rangle$ does not depend on i and is a function defined on X. We have $\langle f, f \rangle \geq 0$. If either f or $g \in C_0^{p,q}(X, B)$, then

(4) $$(f, g)_\Psi = \int_X \langle f, g \rangle e^{-\Psi} dv$$

is defined for any real valued function Ψ of class C^∞.

We have the operator

(5) $$\bar{\partial}: C^{p,q}(X, B) \to C^{p,q+1}(X, B)$$

defined by $(\bar{\partial} f)_i = \bar{\partial} f_i$. We form the formal adjoint of $\bar{\partial}$ with respect to the inner product $(f, g)_\Psi$ in $C_0^{p,q}(X, B)$, and denote it by ϑ_Ψ.

We denote by $L^{p,q}(X, B, \Psi)$ the space of measurable B-valued forms f of type (p, q), square integrable in the sense that $(f, f)_\Psi < \infty$. It is a Hilbert space with respect to the inner product $(f, g)_\Psi$. We define

(6) $$\|f\|^2 = (f, f)_0$$

(7) $$(f, g) = (f, g)_0$$

(8) $$\vartheta = \vartheta_0$$

(9) $$L^{p,q}(X, B) = L^{p,q}(X, B, 0).$$

We also denote by $\bar{\partial}$ the smallest closed extension of

(10) $$\bar{\partial}: L^{p,q}(X, B, \Psi) \to L^{p,q+1}(X, B, \Psi).$$

In general, given two Hilbert space H_1 and H_2, and a closed linear operator $T: H_1 \to H_2$ with dense domain, we denote its domain, range and nullity by D_T, R_T and N_T, respectively. We denote the adjoint of T by T^*. In the case when $H_1 = L^{p,q-1}(X, B, \Psi)$, $H_2 = L^{p,q}(X, B, \Psi)$ and $T = \bar{\partial}$, we let $D_{\bar{\partial}} = D_{\bar{\partial}}^{p,q-1}$, $R_{\bar{\partial}} = R_{\bar{\partial}}^{p,q}$ and $N_{\bar{\partial}} = N_{\bar{\partial}}^{p,q-1}$. We define $R_{\bar{\partial}}^{p,0}$ to be 0. $D_{\bar{\partial}^*}^{p,q}$, $R_{\bar{\partial}^*}^{p,q-1}$ and $N_{\bar{\partial}^*}^{p,q}$ are defined similarly.

Definition 1.1.

(11) $$'H^{p,q}(X, B, \Psi) = N_{\bar{\partial}}^{p,q} / \overline{R_{\bar{\partial}}^{p,q}},$$

where we denote by $\overline{R_{\bar{\partial}}^{p,q}}$ the closure of $R_{\bar{\partial}}^{p,q}$ in $L^{p,q}(X, B, \Psi)$.

Note that $'H^{p,q}(X, B, \Psi)$ is a Hilbert space. We define

(12) $$'H^{p,q}(X, B) = 'H^{p,q}(X, B, 0).$$

For a differential form ξ on X, exterior multiplication of ξ to $f \in C^{p,q}(X, B)$ is defined by

(13) $$(e(\xi) \wedge f)_i = \xi \wedge f_i.$$

Let ω be the fundamental form of the hermitian metric ds^2 on X. We define

(14) $$L = e(\omega)$$

(15) $$\Lambda = (-1)^{p+q} {*L*} \quad \text{on} \quad C^{p,q}(X, B).$$

§ 2. Weak Finiteness Theorem

Definition 2.1. X is called weakly 1-complete if there exists a C^∞ plurisubharmonic function Ψ on X such that for any $c \in R$, where R denotes the real numbers,

$$X_c := \{x; \Psi(x) < c\} \Supset X.$$

We call such Ψ an exhaustion function of X. Note that if X is weakly 1-complete X has a countable base, so by Sard's theorem there is a nowhere dense subset $A \subset R$ such that if $c \in R - A$, $\partial\{x; \Psi(x) \leq c\}$

is a smooth manifold of real dimension $2n-1$.

Proposition 2. 2. *If X is weakly 1-complete, then for any compact subset K of X, there exists an exhaustion function Ψ such that $\{x; \Psi(x)=0\} \Supset K$ and $\partial\{x; \Psi(x)=0\}$ is smooth.*

Proof) For any exhaustion function Φ there exists a $c \in R$ such that $\{x; \Phi(x) \leq c\} \Supset K$ and $\partial\{x; \Phi(x) \leq c\}$ is smooth. We define $\lambda: R \rightarrow R$ as follows

$$\lambda(t)=0 \quad \text{if} \quad t \leq c, \tag{16}$$

$$\lambda(t)=\exp\left(-\frac{1}{(t-c)^2}+t-c\right) \quad \text{if} \quad t \geq c. \tag{17}$$

Then $\lambda(t)$ is a C^∞ plurisubharmonic function vanishing in a neighbourhood of $\bar{K}$, and $\partial\{x; \lambda(\Phi(x))=0\}$ is smooth, so we may take $\Psi := \lambda(\Phi)$.

q.e.d.

Since X is a paracompact manifold of class C^∞ it has a hermitian metric ds^2. Let $\{a_i\}$ be a metric along the fibers of B.

Definition 2. 3. A holomorphic line bundle $\pi: B \rightarrow X$ is said to be positive on a subset $Y \subset X$, if there exists a coordinate cover $\{U_i\}$ of X such that $\pi^{-1}(U_i)$ are trivial and the metric $\{a_i\}$ along the fibers of B can be so chosen that

$$\left(\frac{\partial^2 \log(a_i^{-1})}{\partial z_i^\alpha \partial \bar{z}_i^\beta}\right) > 0 \quad \text{on} \quad U_i \cap Y \quad \text{for any } i. \tag{18}$$

From now on, we let X be a weakly 1-complete manifold and $\pi: B \rightarrow X$ be a holomorphic line bundle which is positive on the complement of a compact subset K of X, and an exhaustion function Ψ is so fixed that $K \Subset \{x; \Psi(x)=0\}$ and $\partial\{x; \Psi(x)=0\}$ is smooth. For convenience we denote $\mathrm{Int}\{x; \Psi(x)=0\}$ by X_0.

Lemma 2. 4. *For any finite number of B-valued forms $f^1, \cdots, f^m$ which are measurable and locally square integrable there exist a hermitian metric ds^2 and a metric $\{a_i\}$ along the fibers of B such that*

1) *ds^2 is a complete hermitian metric*

2) $ds^2=\sum_{\alpha,\beta}\frac{\partial^2\log(a_i^{-1})}{\partial z_i^\alpha\partial\bar{z}_i^\beta}(dz_i^\alpha,d\bar{z}_i^\beta)$ on $X-K_1$

3) $(f^l,f^l)<\infty$, *for* $1\leq l\leq m$,

for some compact set K_1 *with* $K\Subset K_1\Subset X_0$.

Proof) By the hypothesis on B there exist ds_0^2 and $\{a_i^0\}$ which satisfy 2) for some K_1 with $K\Subset K_1\Subset X_0$. We define ds^2 and $\{a_i\}$ by

$$ds^2=\sum_{\alpha,\beta}\frac{\partial^2\log(a_i^{-1})}{\partial z_i^\alpha\partial\bar{z}_i^\beta}(dz_i^\alpha,d\bar{z}_i^\beta)\quad\text{on}\quad X-K_1,\tag{19}$$

$$ds^2=ds_0^2\quad\text{on}\quad K_1\tag{20}$$

$$a_i=a_i^0e^{-\tau(\Psi)},\tag{21}$$

where τ is a real valued function of class C^∞ such that

$$\tau(t)\geq0,\tag{22}$$

$$\tau'(t)\geq0,\quad\text{and}\tag{23}$$

$$\tau''(t)\geq0\quad\text{for any}\quad t\in R.\tag{24}$$

We set

$$g_{i,\alpha\beta}=\frac{\partial^2\log((a_i^0)^{-1})}{\partial z_i^\alpha\partial\bar{z}_i^\beta}\tag{25}$$

$$\Gamma_{i,\alpha\beta}=\frac{\partial^2\log(a_i^{-1})}{\partial z_i^\alpha\partial\bar{z}_i^\beta}\tag{26}$$

$$a_i^0f_i\wedge*\bar{f}_i=a^0(f)\,dv_0\tag{27}$$

$$a_if_i\wedge\star\bar{f}_i=a(f)\,dv=A(f)\,dv_0,\tag{28}$$

where $*$, $\star$, and dv_0, dv are the star operators and the volume forms of ds_0^2 and ds^2, respectively. We have by a direct calculation

$$A(f)\leq e^{-\tau(\Psi)}\frac{\det(\Gamma_{i,\alpha\beta})}{\det(g_{i,\alpha\beta})}a^0(f)\quad\text{on}\quad X-K_1\tag{29}$$

$$\frac{\det(\Gamma_{i,\alpha\beta})}{\det(g_{i,\alpha\beta})}\leq(1+\tau'(\Psi))v^n+(1+\tau'(\Psi)v)^{n-1}\tau''(\Psi)u\tag{30}$$

where v and u are non-negative continuous functions independent of τ. We choose a non-decreasing continuous function $\rho(t)$ such that

$$\int_X e^{-\rho(\Psi)} a^0(f^l)\, dv_0 < \infty, \quad \text{for} \quad 1 \leq l \leq m. \tag{31}$$

We use the following lemma due to S. Nakano.

Sublemma 2.5 ([6], Lemma). *Given a real valued, continuous and strictly increasing function $\lambda(t)$ on $0 \leq t < \infty$ with $\lambda(0) = 0$, $\lambda(t) \to \infty$ for $t \to \infty$, we can find a C^∞ function $\tau(t)$ on $-\infty < t < \infty$ such that*

$$\tau'(t) \geq 0 \quad \text{and} \quad \tau''(t) \geq 0, \quad \text{for any } t, \tag{32}$$

$$\tau(t) \geq \lambda(t) \quad \text{for } t \geq c, \quad \text{and} \tag{33}$$

$$\tau'(t) \leq K\tau(t)^2 \quad \text{and} \tag{34}$$

$$\tau''(t) \leq K\tau(t)^3 \quad \text{for} \quad t \geq c',$$

for some constants c, c' and $K > 0$.

We apply it to

$$\lambda(t) = \max\{v(t), u(t)\} + 2\rho(t) + t^2 \tag{35}$$

and take $\tau(t)$ as in Lemma 2.5. Since

$$\int_0^\infty \sqrt{\tau''(t)}\, dt = \infty \tag{36}$$

ds^2 is complete ([5], Proposition 1). By (30) it follows that

$$\int_{X-K_1} \exp(-\tau(\Psi)) \frac{\det(\Gamma_{i,\alpha\beta})}{\det(g_{i,\alpha\beta})} a^0(f)\, dv_0 \tag{37}$$

$$\leq \sup_{X-K_1} \exp\left(-\frac{\tau(\Psi)}{2}\right) \frac{\det(\Gamma_{i,\alpha\beta})}{\det(g_{i,\alpha\beta})}$$

$$\times \int_{X-K_1} \exp(-\rho(\Psi))\, a^0(f)\, dv_0, \quad \text{for } 1 \leq l \leq m.$$

It is clear that for such τ the metrics ds^2 and $\{a_i\}$ satisfy 1), 2) and 3). q.e.d.

Theorem 2.6. *Assume that ds^2 and $\{a_i\}$ satisfy* 1) *and* 2) *of Lemma* 2.4, *then if $p+q>n$*

(38) $R^{p,q}$ *is closed*

and

(39) $\dim_C {}'H^{p,q}(X, B) < \infty$.

Proof) We use the following lemma which is theorem 1. 1. 3 of [2].

Lemma 2. 7. *Let* H_i $(i=1, 2, 3)$ *be three Hilbert spaces and* $T: H_1 \to H_2$ *and* $S: H_2 \to H_3$ *be closed linear operators with dense domains such that* $ST=0$. *Assume that for any sequence* $\{f_\nu\}$ *with* $f_\nu \in H_2 \cap D_S \cap D_T$, $\|f_\nu\|_{H_2} \leq 1$, $\lim_{\nu\to\infty} \|Sf_\nu\|_{H_3}=0$, *and* $\lim_{\nu\to\infty} \|Tf_\nu\|_{H_1}=0$, *we can choose a strongly convergent subsequence of* $\{f_\nu\}$. *Then* R_T *is closed and* N_S/R_T *is a finite dimensional vector space.*

According to Lemma 2. 7, in order to prove (38) and (39) it suffices to show that for any sequence $\{f_\nu\}$ such that $f_\nu \in D_{\bar\partial}^{p,q} \cap D_{\bar\partial^*}^{p,q}$, $\|f_\nu\| \leq 1$, $\lim_{\nu\to\infty} \|\bar\partial f_\nu\|=0$, and $\lim_{\nu\to\infty} \|\bar\partial^* f_\nu\|=0$, we can choose a strongly convergent subsequence of $\{f_\nu\}$.

Since ds^2 is complete, $C_0^{p,q}(X, B)$ is a dense subset of $D_{\bar\partial}^{p,q} \cap D_{\bar\partial^*}^{p,q}$ with respect to the graph norm,

(40) $$\{(\bar\partial f, \bar\partial f) + (\bar\partial^* f, \bar\partial^* f) + (f, f)\}^{1/2}$$

([8], Theorem 1. 1). Hence we may assume

(41) $$f_\nu \in C_0^{p,q}(X, B).$$

Therefore, we have

(42) $$(\bar\partial f_\nu, \bar\partial f_\nu) + (\bar\partial^* f_\nu, \bar\partial^* f_\nu) + (f_\nu, f_\nu) = ((\bar\partial\vartheta + \vartheta\bar\partial) f_\nu, f_\nu) + (f_\nu, f_\nu).$$

Hence by the assumption

(43) $$((\bar\partial\vartheta + \vartheta\bar\partial) f_\nu, f_\nu) + (f_\nu, f_\nu)$$

is bounded above, which combined with ellipticity of $\bar\partial\vartheta + \vartheta\bar\partial$ means that $(f_\nu)_i$ and their first order derivatives are bounded. Here the derivatives are taken with respect to the coordinate of U_i. Combining this with

Rellich's lemma, it follows that $\{f_\nu\}$ has a subsequence $\{f_{\nu_j}\}$ which is strongly convergent on compact subsets. We use the following estimate which is proved later (See Lemma 3. 3 in Section 3.)

$$(44)\qquad \int_{X-K_2}\langle f, f\rangle dv \leq C\left\{\|\bar{\partial} f\|^2+\|\vartheta f\|^2+\int_{K_2}\langle f, f\rangle dv\right\},$$

if $p+q>n$ and $f\in D_{\bar{\partial}}^{p,q}\cap D_{\vartheta}^{p,q}$, where C is a constant and $K_1 \Subset K_2 \Subset X_0$. By this estimate we conclude that $\{f_{\nu_j}\}$ converges strongly on X.

q.e.d.

§ 3. The Basic Estimate

In what follows ds^2 and $\{a_i\}$ are assumed to satisfy 1) and 2) of Lemma 2. 4.

For an integer ν, we define

$$(45)\qquad (f, g)_\nu=(f, g)_{\nu\Psi}, \quad \text{and}$$

$$(46)\qquad \|f\|_\nu^2=(f, f)_\nu, \quad \text{for} \quad f, g\in L^{p,q}(X, B, \nu\Psi).$$

We let ϑ_ν be the formal adjoint of $\bar{\partial}$ with respect to the inner product $(f, g)_\nu$ and we define

$$(47)\qquad \Box_\nu=\bar{\partial}\vartheta_\nu+\vartheta_\nu\bar{\partial}.$$

Since ds^2 is complete ϑ_ν is equal to the adjoint of $\bar{\partial}$. ([8], Theorem 1. 1).

We use a well known formula in differential geometry in the following form.

Lemma 3. 1.

$$(48)\qquad \Box_\nu-*^{-1}\Box_\nu*=e(\chi_\nu)\Lambda-\Lambda e(\chi_\nu), \quad on \quad X-K_1$$

where $e(\chi_\nu)=L+e(\nu\sqrt{-1}\,\partial\bar{\partial}\Psi)$.

Proof) We let

$$(49)\qquad b_i=e^{-\nu\Psi}a_i,$$

$$(50)\qquad D=d+\partial\log b_i, \quad \text{and}$$

$$(51)\qquad D'=\partial+\partial\log b_i.$$

Then we have

(52) $$e(\chi_\nu)=\sqrt{-1}\,D^2, \quad \text{and}$$

(53) $$D^2=(\bar\partial+D')(\bar\partial+D')=\bar\partial D'+D'\bar\partial\,.$$

Letting δ' be the formal adjoint of $\bar\partial: C_0^{p,q}(X)\to C_0^{p,q+1}(X)$ we have

(54) $$\sqrt{-1}\,(\bar\partial\Lambda-\Lambda\bar\partial)=\delta' \quad \text{on} \quad X-K_1\,.$$

We have

(55) $$\sqrt{-1}\,(D'\Lambda-\Lambda D')=\vartheta_\nu \quad \text{on} \quad X-K_1\,.$$

Hence

(56) $$\begin{aligned} &e(\chi_\nu)\Lambda-\Lambda e(\chi_\nu)\\ &=\sqrt{-1}\,\{(\bar\partial D'+D'\bar\partial)\Lambda-\Lambda(\bar\partial D'+D'\bar\partial)\}\\ &=\sqrt{-1}\,\{(\bar\partial(D'\Lambda-\Lambda D')+(D'\Lambda-\Lambda D')\bar\partial\\ &\quad -D'(\bar\partial\Lambda-\Lambda\bar\partial)-(\bar\partial\Lambda-\Lambda\bar\partial)D'\}\\ &=\bar\partial\vartheta_\nu+\vartheta_\nu\bar\partial-(D'\delta'+\delta'D')\\ &=\square_\nu-*^{-1}\square_\nu* \quad \text{on} \quad X-K_1\,, \end{aligned}$$ q.e.d.

Lemma 3.2.

(57) $$\|f\|^2\leq\|\bar\partial f\|^2+\|\vartheta f\|^2$$

if $f\in C_0^{p,q}(X-K_1, B)$ *and* $p+q>n$, *and*

(58) $$\|f\|_\nu^2\leq\|\bar\partial f\|_\nu^2+\|\vartheta_\nu f\|_\nu^2$$

if $f\in C_0^{n,q}(X-K_1, B)$ *and* $q\geq 1$.

Proof) We prove (58) first. If $f\in C_0^{n,q}(X-K_1, B)$ and $q\geq 1$, we have

(59) $$\begin{aligned} \|\bar\partial f\|_\nu^2+\|\vartheta_\nu f\|_\nu^2&=(\square_\nu f, f)_\nu\\ &\geq((e(\chi_\nu)\Lambda-\Lambda e(\chi_\nu))f, f)_\nu\\ &=((L\Lambda-\Lambda L)f, f)_\nu\\ &\quad+\nu(e(\sqrt{-1}\,\partial\bar\partial\Psi)\Lambda-\Lambda e(\sqrt{-1}\,\partial\bar\partial\Psi))f, f)_\nu\\ &=q(f, f)_\nu+\nu(e(\sqrt{-1}\,\partial\bar\partial\Psi)\Lambda f, f)_\nu\geq q(f, f)_\nu\,. \end{aligned}$$

For more details, see [5]. However we note that what is proved there is that if the dual of B is positive and φ is a $(0, n-q)$ form with support contained in a coordinate neighbourhood U, we have

(60) $$((\Lambda e(\sqrt{-1}\partial\bar{\partial}\Psi) - e(\sqrt{-1}\partial\bar{\partial}\Psi)\Lambda)\varphi, \varphi) \geq 0$$

so it seems not sufficient to establish (60) for the elements of $C_0^{0,n-q}(X, B)$. But his argument also implies

(61) $$\langle \Lambda e(\sqrt{-1}\partial\bar{\partial}\Psi)\varphi, \varphi\rangle \geq 0$$

for $\varphi \in C^{0,n-q}(X, B)$. Hence rewriting (61) we obtain (58).

The proof of (57) is similar as above. q.e.d.

Lemma 3. 3. *there is a constant* C *and a compact set* K_2 *with* $K_1 \Subset K_2 \Subset X_0$ *such that*

(62) $$\int_{X-K_2} \langle f, f\rangle e^{-\nu\Psi} dv \leq C\left\{\|\bar{\partial} f\|_\nu^2 + \|\vartheta_\nu f\|_\nu^2 + \int_{K_2} \langle f, f\rangle dv\right\}$$

if $\nu \geq 0$, $f \in D_{\bar{\partial}}^{n,q} \cap D_{\vartheta_\nu}^{n,q}$ and $q \geq 1$, and

(63) $$\int_{X-K_2} \langle f, f\rangle dv \leq C\left\{\|\bar{\partial} f\|^2 + \|\vartheta f\|^2 + \int_{K_2} \langle f, f\rangle dv\right\}$$

if $f \in D_{\bar{\partial}}^{p,q} \cap D_{\vartheta}^{p,q}$ and $p+q > n$.

Proof) Since ds^2 is complete we may assume $f \in C_0^{p,q}(X, B)$. We prove (63). The proof of (62) is similar.

Let χ be a C^∞ function on X such that for a compact set K_2 with $K_1 \Subset K_2 \Subset X_0$,

(64) $$\chi = 1 \quad \text{in} \quad X - K_2, \quad \text{and}$$

(65) $$\chi = 0 \quad \text{in a neighbourhood of } K_1.$$

We have $\chi f \in C_0^{n,q}(X - K_1, B)$, so we can apply Lemma 3. 2, getting

(66) $$\|\chi f\|_\nu^2 \leq \|\bar{\partial}(\chi f)\|_\nu^2 + \|\vartheta_\nu(\chi f)\|_\nu^2.$$

We estimate the both sides of this inequality. The left hand side:

(67) $$\int_{X-K_2} \langle f, f\rangle e^{-\nu\Psi} dv \leq \|\chi f\|_\nu^2.$$

The right hand side:

(68) $$\|\bar{\partial}(\chi f)\|_\nu^2 + \|\vartheta_\nu(\chi f)\|_\nu^2$$
$$= \|\bar{\partial}\chi\Lambda f + \chi\bar{\partial} f\|_\nu^2 + \|-*a_i^{-1} e^{\nu\Psi}\bar{\partial} * a_i e^{-\nu\Psi}(\chi f)\|_\nu^2$$

$$= \|\bar{\partial}\chi\Lambda f + \chi\bar{\partial}f\|_\nu^2 + \|\chi\vartheta_\nu f - *(\partial\chi\Lambda * f)\|_\nu^2$$

$$\leq \|\bar{\partial}\chi\Lambda f\|_\nu^2 + 2\,\mathrm{Re}(\bar{\partial}\chi\Lambda f, \chi\bar{\partial}f)_\nu + \|\chi\bar{\partial}f\|_\nu^2 + \|\chi\vartheta_\nu f\|_\nu^2$$

$$- 2\,\mathrm{Re}(\chi\vartheta_\nu f, *(\partial\chi\Lambda * f))_\nu + \|*(\partial\chi\Lambda * f)\|_\nu^2$$

$$\leq \int_{K_2} \langle\bar{\partial}\chi, \bar{\partial}\chi\rangle dv \cdot \int_{K_2} \langle f, f\rangle dv + \|\bar{\partial}\chi\Lambda f\|_\nu^2 + \|\chi\bar{\partial}f\|_\nu^2$$

$$+ \|\chi\bar{\partial}f\|_\nu^2 + \|\chi\vartheta_\nu f\|_\nu^2 + \|\chi\vartheta_\nu f\|_\nu^2 + \|*(\partial\chi\Lambda * f)\|_\nu^2$$

$$+ \int_{K_2} \langle\bar{\partial}\chi, \bar{\partial}\chi\rangle dv \cdot \int_{K_2} \langle f, f\rangle dv$$

$$\leq 2\left(\int_{K_2} \{\langle\bar{\partial}\chi, \bar{\partial}\chi\rangle + \langle\partial\chi, \partial\chi\rangle\}\, dv \cdot \int_{K_2} \langle f, f\rangle dv\right)$$

$$+ 2 \sup_{x\in X} \chi(x) \cdot (\|\bar{\partial}f\|_\nu^2 + \|\vartheta_\nu f\|_\nu^2).$$

Therefore, if

$$C \geq 2 \max\left(\int_{K_2} \{\langle\bar{\partial}\chi, \bar{\partial}\chi\rangle + \langle\partial\chi, \partial\chi\rangle\}\, dv,\ \sup_{x\in X} \chi(x)\right) \tag{69}$$

we have

$$\int_{X-K_2} \langle f, f\rangle e^{-\nu\Psi} dv \leq C\left(\|\bar{\partial}f\|_\nu^2 + \|\vartheta_\nu f\|_\nu^2 + \int_{K_2} \langle f, f\rangle dv\right) \tag{70}$$

q.e.d.

§ 4. The Main Theorem

Definition 4.1. We denote by $\mathcal{H}^{p,q}(X_0, B)$ the set of elements $h \in L^{p,q}(X_0, B)$ with $\bar{\partial}h = 0$ and $\bar{\partial}^* h = 0$, where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}: L^{p,q-1}(X_0, B) \to L^{p,q}(X_0, B)$.

Proposition 4.2. *There exist ν_0 and C_0 such that for any $\nu \geq \nu_0$,*

$$\|f\|_\nu^2 \leq C_0(\|\bar{\partial}f\|_\nu^2 + \|\vartheta_\nu f\|_\nu^2), \text{ where} \tag{71}$$

provided

$$f \in L^{n,q}(X, B, \nu\Psi) \cap D_{\bar{\partial}}^{n,q} \cap D_{\vartheta_\nu}^{n,q} \quad (q \geq 1), \tag{72}$$

and

$$\int_{X_0} \langle f, h\rangle dv = 0 \quad \text{for any } h \in \mathcal{H}^{n,q}(X_0, B). \tag{73}$$

Proof) If the proposition were false, we may assume that there is a sequence $\{f_\nu\}$ such that

(74) $$\|f_\nu\|_\nu^2=1$$

(75) $$\|\bar{\partial}f_\nu\|_\nu^2+\|\vartheta_\nu f_\nu\|_\nu^2\leq\frac{1}{\nu}$$

(76) $$f_\nu\in L^{n,q}(X, B, \nu\Psi)\cap D_{\bar{\partial}}^{n,q}\cap D_{\vartheta_\nu}^{n,q}$$

and

(77) $$\int_{X_0}\langle f_\nu, h\rangle dv=0 \quad \text{for any} \quad h\in\mathcal{H}^{n,q}(X_0, B).$$

Let $g_\nu=e^{-\nu\Psi}f_\nu$, then we have

(78) $$\vartheta g_\nu=e^{-\nu\Psi}f_\nu$$

so that

(79) $$\|\vartheta g_\nu\|_{-\nu}=\|\vartheta_\nu f_\nu\|_\nu$$

hence

(80) $$\lim_{\nu\to\infty}\|\vartheta g_\nu\|_{-\nu}=\lim_{\nu\to\infty}\|\vartheta_\nu f_\nu\|_\nu=0$$

by (75). Since $\|\vartheta g_\nu\|\leq\|\vartheta g_\nu\|_{-\nu}$, we have $\lim_{\nu\to\infty}\|\vartheta g_\nu\|=0$. By (74), we have

(81) $$\|g_\nu\|_{-\nu}=\|f_\nu\|_\nu=1$$

hence $\|g_\nu\|\leq 1$. Therefore choosing a subsequence, we may assume that $\{g_\nu\}$ has a weak limit g in $L^{n,q}(X, B)$. It is easily verified that

(82) $$\|g\|_{-\nu}\leq\inf_{\mu\geq\nu}\sup_{\mu'\geq\nu}\|g_{\mu'}\|_{-\mu'}=1\,,$$

for any $\nu\geq 1$. Thus we have $\operatorname{supp} g\Subset X_0$. Therefore,

(83) $$\bar{\partial}^*(g|_{X_0})=0\,.$$

(See Appendix.) By (75) $\bar{\partial}g=0$, and g satisfies (77). By (74) and (75), it may be assumed that $\{g_\nu\}$ is strongly convergent on K_2, and the limit is not zero on K_2 by Lemma 3. 3. Hence we conclude that $g|_{X_0}\neq 0$. This contradiction completes the proof. q.e.d.

Definition 4. 3.

$$H^{p,q}_{loc}(X, B)=\frac{\{f;\, f\in L^{p,q}_{loc}(X, B),\, \bar{\partial}f=0\}}{L^{p,q}_{loc}(X, B)\cap\{\bar{\partial}g;\, g\in L^{p,q-1}(X, B)\}} \tag{84}$$

where we denote by $L^{p,q}_{loc}(X, B)$ the space of B-valued (p, q) forms which are measurable and square integrable on compact subsets of X.

By the Dolbeault's theorem (see [2], Theorem 2. 2. 4 and Theorem 2. 2. 5), there is a natural isomorphism between the spaces $H^{p,q}_{loc}(X, B)$ and $H^{p,q}(X, B)$.

We define $'H^{p,q}(X_0, B)$ with respect to ds^2 and $\{a_i\}$.

Proposition 4. 4. *The natural map*

$$\rho\colon H^{n,q}_{loc}(X, B)\to{}'H^{n,q}(X_0, B) \tag{85}$$

is injective for $q\geq 1$.

Proof) We show that if $f\in L^{n,q}_{loc}(X, B)$, $\bar{\partial}f=0$, and there exists a sequence $\{g_\nu\}$ with $g_\nu\in L^{n,q-1}(X_0, B)$, $\bar{\partial}g_\nu\in L^{n,q}(X_0, B)$ and

$$\int_{X_0}\langle f-\bar{\partial}g_\nu,\, f-\bar{\partial}g_\nu\rangle dv<\frac{1}{\nu} \tag{86}$$

then there exists $g\in L^{n,q-1}_{loc}(X, B)$ such that $\bar{\partial}g=f$.

We replace the exhaustion function Ψ by $\tilde{\Psi}=\lambda(\Psi)$ where λ is a convex increasing C^∞ function with $\lambda(0)=0$ and $\lambda(t)>0$ if $t>0$ which increases so rapidly that

$$\int_X\langle f, f\rangle e^{-\tilde{\Psi}}dv<\infty\,. \tag{87}$$

For $\nu\geq 1$, we have $f\in L^{n,q}(X, B, \nu\tilde{\Psi})$.

By (86) it follows that

$$\int_{X_0}\langle f, h\rangle dv=0 \quad \text{for any } h\in\mathcal{H}^{n,q}(X_0, B)\,. \tag{88}$$

Therefore, combining Proposition 4. 2 with Hörmander's theorem ([2], Theorem 1. 1. 4) we conclude that for some $\nu\geq 1$ there exists $g\in L^{n,q-1}(X, B, \nu\tilde{\Psi})$ such that $\bar{\partial}g=f$. q.e.d.

We fix $c>0$. X_c is a weakly 1-complete manifold with an exhaustion function

(89) $$\Psi_c = \frac{1}{c-\Psi}\,.$$

We choose a hermitian metric $d\sigma^2$ and a metric $\{b_i\}$ along the fibers of B such that

i) $d\sigma^2$ is complete

ii) $$d\sigma^2 = \sum_{\alpha,\beta} \frac{\partial^2 \log (b_i^{-1})}{\partial z_i^\alpha \partial \bar{z}_i^\beta} (dz_i^\alpha, d\bar{z}_i^\beta) \quad \text{on} \quad X - K_1\,.$$

Moreover we may assume that

iii) $$\int_{X_0} b_i f_i \wedge \star \bar{f}_i < \infty, \qquad \text{for} \quad f \in C^{p,q}(X, B)\,,$$

where $\star$ is the star operator for $d\sigma^2$. This follows from Lemma 2. 4 and that $C^{p,q}(\overline{X}_c, B)$ is a finitely generated $C^\infty(\overline{X}_c)$ module. We define $'H^{p,q}(X_c, B)$ with respect to $d\sigma^2$ and $\{b_i\}$.

Theorem 4. 5.

(90) $$\dim_C H^{n,q}(X, B) < \infty \qquad \text{for} \quad q \geq 1\,.$$

Proof) Since $'H^{n,q}(X_c, B)$ is finite dimensional it suffices to show that the natural map (induced by the restriction of forms)

$$r\colon H^{n,q}(X, B) \to {}'H^{n,q}(X_c, B)$$

is injective. (Note that r is well defined by the choice of $d\sigma^2$ and $\{b_i\}$. We consider the following diagram.

$$\begin{array}{ccccc} & \overset{\xi}{\nearrow} & H^{n,q}_{loc}(X, B) & \overset{\rho}{\searrow} & \\ H^{n,q}(X, B) & & \xrightarrow{\quad \rho' \quad} & & {}'H^{n,q}(X_0, B) \\ & \underset{r}{\searrow} & {}'H^{n,q}(X_c, B) & \underset{r'}{\nearrow} & \end{array}$$

where ρ', r' and ξ are natural homomorphisms. Since ρ is injective and ξ is an isomorphism, ρ' is injective, hence r is injective. q.e.d.

Theorem 4. 6. *If* $g\in L^{n,q}(X_0, B)$ $(q\geq 0)$ *and* $\bar\partial g=0$, *then for any* $\varepsilon>0$ *there exists* $f\in L^{n,q}_{loc}(X, B)$ *such that*

$$\int_{X_0}\langle f-g, f-g\rangle dv<\varepsilon \tag{91}$$

and $\bar\partial f=0$.

Proof) By the Hahn-Banach's theorem it suffices to show that if $u\in L^{n,q}(X_0, B)$ and

$$\int_{X_0}\langle f, u\rangle dv=0 \tag{92}$$

for any $f\in L^{n,q}_{loc}(X, B)$ with $\bar\partial f=0$, then we have

$$\int_{X_0}\langle g, u\rangle dv=0 \tag{93}$$

if $g\in L^{n,q}(X_0, B)$ and $\bar\partial g=0$.

We define $\mathring{u}$ by $\mathring{u}=u$ on X_0 and $\mathring{u}=0$ on $X-X_0$. Since $\mathring{u}$ is orthogonal to $N^{n,q}_{\bar\partial}$ for any ν, we have $\mathring{u}\in \overline{R^{n,q}_{\vartheta_\nu}}$. $R^{n,q}_{\vartheta_\nu}=\overline{R^{n,q}_{\vartheta_\nu}}$ is equivalent to $R^{n,q+1}_{\bar\partial}=\overline{R^{n,q+1}_{\bar\partial}}$ (See [2], Theorem 1. 1. 1). $R^{n,q+1}_{\bar\partial}=\overline{R^{n,q+1}_{\bar\partial}}$ is proved for $\nu\geq 0$ similarly as Theorem 2. 6. Hence, by Proposition 4. 2 there exists ν_0 such that

$$\mathring{u}=\vartheta_\nu v_\nu, \quad \text{for some } v_\nu\in L^{n,q+1}(X, B, \nu\Psi), \tag{94}$$
$$\text{with } \|v_\nu\|^2_\nu\leq C_0\|\mathring{u}\|^2, \quad \text{for } \nu\geq\nu_0 .$$

We set

$$w_\nu=e^{-\nu\Psi}v , \tag{95}$$

then as in the proof of Proposition 4. 2, $\{w_\nu\}$ has a subsequence which is weakly convergent in $L^{n,q+1}(X, B)$. Let the weak limit be w, then as in the proof of Proposition 4. 2, $\vartheta w=\mathring{u}$ and $\operatorname{supp} w\Subset \overline{X}_0$, hence $\bar\partial^*(w|_{X_0})=u$. Therefore, if $g\in L^{n,q}(X_0, B)$ and $\bar\partial g=0$, we have

$$\int_{X_0}\langle g, u\rangle dv = \int_{X_0}\langle\bar\partial g, w\rangle dv=0 . \tag{96}$$

q.e.d.

Theorem 4. 6. *The natural map*

$$\rho_d\colon H^{n,q}(X, B)\to H^{n,q}(X_d, B) \tag{97}$$

is an isomorphism if $q \geq 1$ *and* $d > 0$.

Proof) We consider the following diagram

$$\begin{array}{ccc} H^{n,q}(X, B) & \xrightarrow{\quad \rho' \quad} & 'H^{n,q}(X_0, B) \\ & {\scriptstyle \rho_d} \searrow \quad H^{n,q}(X_d, B) \quad \nearrow {\scriptstyle \rho''} & \end{array}$$

Since ρ' is injective ρ_d is injective. To show the subjectivity of ρ_d, we have only to show that $\operatorname{Im} \rho' = \operatorname{Im} \rho''$, where $\operatorname{Im} \rho'$ denotes the image of ρ'.

By Theorem 4. 6 (and by the Dolbeault's theorem), $\operatorname{Im} \rho'$ is dense in $\operatorname{Im} \rho''$. Since $\operatorname{Im} \rho''$ is a finite dimensional subspace of the Hilbert space $'H^{n,q}(X_0, B)$, we have

$$\operatorname{Im} \rho' = \operatorname{Im} \rho'' . \tag{98}$$

Thus ρ_d is an isomorphism. q.e.d.

§ 5. Application to Analytic Geometry

Let M be a complex manifold and S be a nonsingular divisor on M with a proper and smooth holomorphic map $p: S \to D$ onto a Stein manifold D.

Proposition 5. 1.

Assumption:

1) $[S]|_{p^{-1}(x)}$ *is a negative line bundle for any* $x \in D$.

2) *There is a compact subset* $K \Subset D$ *such that* Ω_S^n *is negative on* $p^{-1}(D-K)$. *Here* $n = \dim S$ *and* Ω_S^n *is the sheaf of holomorphic n-forms on* S.

Conclusion: S *is contractible to* D *in* M, *namely, there is a neighbourhood* V *of* S, *an analytic space* U *containing* D *as a closed analytic subset, and a proper surjective holomorphic map* $\varpi: V \to U$ *such that* $\varpi|_S = p$ *and* $\varpi|_{V-S}: V-S \to U-D$ *is biholomorphic.*

Proof) By assumption 1), there is a neighbourhood $V^{(2)}$ and a C^∞

plurisubharmonic function $\Psi^{(2)}$ such that

$$(99) \qquad \Psi^{(2)}(x)=0 \quad \text{if} \quad x\in S,$$

and $\Psi^{(2)}|_{V^{(2)}-S}$ is strictly plurisubharmonic with positive values, (see [1], 4). By 1) and 2) $p^{-1}(D-K)$ is contractible to $D-K$ in a neighbourhood of $p^{-1}(D-K)$, (see [1], Theorem 1). Let the contraction be

$$(100) \qquad \begin{array}{ccc} \varpi': V^* & \longrightarrow & U^* \\ \cup & & \cup \\ p^{-1}(D-L) & \longrightarrow & D-K. \end{array}$$

Since D is a Stein manifold D is properly embedded into some C^N. Let ψ be the restriction of $\sum_{i=1}^{N}|z_i|^2$ to D. Applying Richberg's theorem ([7], Satz 3. 3) to $D-K$ and U^*, we obtain a neighbourhood U^{**} of $D-K$ and a C^∞ strongly pseudoconvex function φ on U^{**} such that $\varphi|_{D-K}=\psi$.

Let $c=\sup_{x\in K}\psi(x)$, $d_1>d_2>c$, $\varepsilon>0$ $V^{(1)}=\{x; x\in V^{(2)}, \Psi^{(2)}(x)<\varepsilon\}$, and V be the union of connected components of $V^{(1)}-\{x; x\in\varpi'^{-1}(U^{**}), \psi\circ\varpi'(x)\geq d_1\}$ that meet $\{x; x\in S, \varphi\circ p(x)<d_2\}$, then V is weakly 1-complete for sufficiently small ε.

By 1) and 2), $\Omega_V^{n+1}\otimes[S]|_V^{-1}$ is positive outside a compact subset. Thus, by Theorem 4. 5, $H^1(V, [S]|_V^{-1})$ is finite dimensional. Therefore, for any compact subset Q in $\{x; x\in S, \varphi\circ p(x)<d_2\}$ we can choose an analytic polyhedron P such that $Q\subset P\Subset\{x; x\in S, \varphi\circ p(x)<d_2\}$ and the functions defining P are holomorphic functions on $\{x; x\in S, \varphi\circ p(x)<d_2\}$ which are restrictions of holomorphic functions on V. Since ∂V_c is strongly pseudoconvex outside S for almost all $c>0$, this proves that V is holomorphically convex, ([3], Satz 3. 4). It is clear that $V-S$ does not contain a compact analytic subset whose dimension is greater than 1. Consequently, S is contractible to D in M. q.e.d.

Theorem 5. 2. *Let X be a weakly 1-complete manifold. If there is a holomorphic line bundle $\pi: B\to X$ which is positive outside a compact subset of X, then there is a meromorphic map $\iota: X_c\to \boldsymbol{P}^N$, where $c>0$ and N is a natural number depending on c, such that there exists a compact analytic set $A\subset X$ such that $\iota|_{X_0-A}: X_c-A\to\boldsymbol{P}^N$ is a holomorphic imbedding of X_c-A as a locally closed analytic* *subset of* $\boldsymbol{P}^N$.

Proof) Similar as Kodaira's embedding theorem ([4]).

Appendix

We let the notations be as before.

Lemma *If* $g \in L^{p,q+1}(X, B)$, $\vartheta g \in L^{p,q}(X, B)$ *and* $\operatorname{supp} g \subset \overline{X}_0$ *then* $g|_{X_0} \in D_{\bar\partial^*}$ *where* $\bar\partial^*$ *is the adjoint of* $\bar\partial^*$: $L^{p,q}(X_0, B) \to L^{p,q+1}(X_0, B)$

Proof) Since $\overline{X}_0$ is compact and ∂X_0 is smooth, there is a sequence $\{g_\nu\}$ such that $g_\nu \in C_0^{p,q+1}(X, B)$, $\operatorname{supp} g_\nu \subset X_0$, $\lim_{\nu\to\infty} \|g_\nu - g\| = 0$, and $\lim_{\nu\to\infty} \|\vartheta g_\nu - \vartheta g\| = 0$ (cf. [2], Proposition 1. 2. 3).

Thus, if $u \in L^{p,q}(X_0, B)$ and $\bar\partial u \in L^{p,q+1}(X_0, B)$,

(101)
$$\int_{X_0} \langle u, \vartheta g \rangle dv = \lim_{\nu\to\infty} \int_{X_0} \langle u, \vartheta g_\nu \rangle dv$$
$$= \lim_{\nu\to\infty} \int_{X_0} \langle \bar\partial u, g_\nu \rangle dv = \int_{X_0} \langle \bar\partial u, g \rangle dv \,.$$

Hence $g|_{X_0} \in D_{\bar\partial^*}$ and $\bar\partial^*(g|_{X_0}) = \vartheta g|_{X_0}$. q.e.d.

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Added in Proof: The author proved in a forthcoming paper 'On $H^{p,q}(X,B)$ of weakly 1-complete maniforlds' the finte dimensionality of $H^{p,q}(X,B)$ and bijectivity of $H^{p,q}(X,B) \to H^{p,q}(X_d,B)$ in the case of $p+q>\dim X$.