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Liouville Type Theorem for a System $\{P(D), B_j(D), j=1, \ldots, p\}$ of Differential Operators with Constant Coefficients in a Half Space

By

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§ 1. Introduction

Concerning the behavior at infinity of solutions of a partial differential equation $Pu=f$, theorems of following three type are known. (1) The *theorem of Liouville type* claims that *if the function* $u(x)$ *is a solution of* $Pu=0$ *such that* $u(x)=O(|x|^d)$ *as* $|x|\to\infty$ (*or* $\varliminf_{R\to\infty} R^{-d}\int_{R<|x|<2R}|u(x)|^2dx=0$) *for some real* d *independent of* u, *then* $u(x)$ *must vanish identically* (see, for example, Agmon [1], Hörmander [7], Littman [9], Murata [10], Rellich [12] and so on). (2) The *theorem of Rellich type* claims that *if* Pu *has compact support, then* u *has compact support* (see, for example, Rellich [12], Agmon [2], Littman [8, 9], Murata [11], Hörmander [7], Trèves [16] and so on). (3) The *theorem of Sommerfeld type gives conditions at infinity which derive the unique solution of* $Pu=f$ (see, for example, Grušin [4], Agmon and Hörmander [3], Vainberg [15] and so on). Recently the study on (1) and (2) has been completed by Hörmander [7] in the constant coefficient and the whole space case and the study on (3) has been remarkably promoted by Agmon and Hörmander [3].

The purpose of this paper is to study the problem of type (1) for a system $\{P(D), B_j(D), j=1, \cdots, p\}$ of differential operators with constant coefficients in a half space (boundary value problem) and to give almost corresponding results to those obtained by Hörmander [7] in the whole space case. In order to state results more precisely, let us first of all introduce certain notations. Let $\boldsymbol{R}^{n+1}$

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denote the n-dimensional Euclidean space, Ξ^{n+1} its dual space and write (x, y) for the coordinate $(x_1, \cdots, x_n, y)$ in $\boldsymbol{R}^{n+1}$ and (ξ, λ) for the dual coordinate $(\xi_1, \cdots, \xi_n, \lambda)$. We denote by $\boldsymbol{R}^{n+1}_+$ the half-space $\{(x, y) \in \boldsymbol{R}^{n+1}; y > 0\}$. For differentiation we use the symbol $D = i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n, \partial/\partial y)$, $D_x = i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $D_y = i^{-1}\partial/\partial y$. We denote by $\mathcal{S}(\overline{\boldsymbol{R}^{n+1}_+})$ the space of restrictions to $\boldsymbol{R}^{n+1}_+$ of all elements in $\mathcal{S}(\boldsymbol{R}^{n+1})$ and denote by $\mathcal{S}(\overline{\boldsymbol{R}^{n+1}_+})$ the space of all temperate distributions in $\boldsymbol{R}^{n+1}_+$ (see Hörmander [6]). For a positive number δ we put $C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n)) = \{u \in \mathcal{S}'(\overline{\boldsymbol{R}^{n+1}_+}); \langle u(\cdot, y), \phi(\cdot)\rangle$ is a C^∞ function of y in $[0, \delta)$ for any $\phi(x) \in \mathcal{S}(\boldsymbol{R}^n)\}$. Let $\sigma(y)$ be a $C_0^\infty((-\delta, \delta))$ function with $\sigma(y) = 1$ for $y \in [-\delta/2, \delta/2]$. Put

$$\langle u, v\rangle = \int_0^\infty \langle \sigma(y)u(x, y), v(x, y)\rangle_x dy + \langle (1-\sigma(y))u(x, y), v(x, y)\rangle$$

for $u \in C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))$ and $v \in \mathcal{S}(\overline{\boldsymbol{R}^{n+1}_+})$, where $\langle\ \rangle_x$ denotes the duality between $\mathcal{S}'(\boldsymbol{R}^n_x)$ and $\mathcal{S}(\boldsymbol{R}^n_x)$ and $\langle\ \rangle$ denotes the duality between $\mathcal{S}'(\boldsymbol{R}^{n+1})$ and $\mathcal{S}(\boldsymbol{R}^{n+1})$. Let

$$P(D) = P(D_x, D_y) = \sum_{j=0}^m a_j(D_x) D_y^j$$

be a differential operator with constant coefficients and $B_j(D)$, $j = 1, \cdots, p$, be some other differential operators with constant coefficients of order r_j where $a_j(D_x)$ is a differential operator in D_x with constant coefficients. We consider solutions $u(x, y) \in C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))$ of the following equations:

(1.1) $$P(D)u = 0 \qquad \text{in } \boldsymbol{R}^{n+1}_+,$$

(1.2) $$B_j(D)u|_{y=0} = 0\,, \qquad j = 1, \cdots, p\,, \quad \text{in } \boldsymbol{R}^n_x\,.$$

We make two assumptions when $m \geqq 1$. The following is an assumption about the number p of boundary conditions:

(A-1) The number of roots with positive imaginary part of the equation $P(\xi, \lambda) = 0$ in λ is less than or equal to p whenever $\xi \in \Xi^n$.

Write $P(\xi, \lambda) = \prod_{j=1}^k P_j(\xi, \lambda)^{k_j}$ and $\tilde{P}(\xi, \lambda) = \prod_{j=1}^k P_j(\xi, \lambda)$ where all $P_j(\xi, \lambda)$ are irreducible polynomials. Let us denote by $Q(\xi)$ the resultant of $\tilde{P}(\xi, \lambda)$ and $(\partial\tilde{P}/\partial\lambda)(\xi, \lambda)$. Put $A_Q = \{\xi \in \Xi^n; Q(\xi) = 0\}$ and $A_{a_m} = \{\xi \in \Xi^n; a_m(\xi) = 0\}$. Note that A_Q and A_{a_m} are empty or real analytic sets in this case. We decompose $\Xi^n - (A_Q \cup A_{a_m})$ into open connected components V_j, that is,

$$\Xi^n - (A_Q \cup A_{a_m}) = \bigcup_{\text{finite}} V_j$$

where $V_j \cap V_{j'} = \emptyset$ if $j \neq j'$. Write $V_j = V$ for the sake of simplicity. Let us denote by $\lambda_j(\xi)$, $j = 1, \cdots, m$, the roots of the equation $P(\xi, \lambda) = 0$ in λ when $\xi \in V$. We have that the imaginary part of $\lambda_j(\xi)$ (denoting them by $\operatorname{Im} \lambda_j(\xi)$) is a real analytic function of ξ. Without loss of generality, we may assume that $\operatorname{Im} \lambda_j(\xi)$, $j = 1, \cdots, \mu$, $(\mu \geqq 0)$ do not vanish identically in V and $\operatorname{Im} \lambda_j(\xi)$, $j = \mu + 1, \cdots, m$, vanish identically in V. Put

$$A_{V,\mathrm{Im}} = \{\xi \in V;\ \operatorname{Im} \lambda_t(\xi) = 0 \qquad \text{for some } t \in \{1, \cdots, \mu\}\}.$$

It is obvious that $A_{V,\mathrm{Im}}$ is either empty or a real analytic set. $V - A_{V,\mathrm{Im}}$ may be decomposed into open connected components $W_{V,j}$, that is,

$$V - A_{V,\mathrm{Im}} = \cup W_{V,j}$$

where $W_{V,j} \cap W_{V,j'} = \emptyset$ if $j \neq j'$. Write $W_{V,j} = W$ for the sake of simplicity. Thus we have

$$\Xi^n - (A_Q \cup A_{a_m} \cup (\bigcup_j A_{V_j,\mathrm{Im}})) = \cup W .$$

Moreover, when $\xi \in W$, the roots of the equation $P(\xi, \lambda) = 0$ in λ have constant multiplicity and split into three classes: real roots, those with positive imaginary part and those with negative imaginary part. We denote those by $\lambda_j^0(\xi)$, $j = 1, \cdots, a$, $\lambda_j^+(\xi)$, $j = 1, \cdots, b$, and $\lambda_j^-(\xi)$, $j = 1, \cdots, c$, where $\operatorname{Im} \lambda_j^0(\xi) = 0$, $\operatorname{Im} \lambda_j^+(\xi) > 0$ and $\operatorname{Im} \lambda_j^-(\xi) < 0$. Thus we have

$$P(\xi, \lambda) = a_m(\xi) \prod_{j=1}^a (\lambda - \lambda_j^0(\xi))^{\alpha_j} \cdot \prod_{j=1}^b (\lambda - \lambda_j^+(\xi))^{\beta_j} \cdot \prod_{j=1}^c (\lambda - \lambda_j^-(\xi))^{\gamma_j} , \quad \xi \in W.$$

Put

$$P^0(\xi, \lambda) = \prod_{j=1}^a (\lambda - \lambda_j^0(\xi))^{\alpha_j} = \sum_{j=0}^{\tilde{a}} a_j^0(\xi) \lambda^j ,$$

$$P^+(\xi, \lambda) = \prod_{j=1}^b (\lambda - \lambda_j^+(\xi))^{\beta_j} = \sum_{j=0}^{\tilde{b}} a_j^+(\xi) \lambda^j ,$$

$$P^-(\xi, \lambda) = a_m(\xi) \prod_{j=1}^c (\lambda - \lambda_j^-(\xi))^{\gamma_j} = \sum_{j=0}^{\tilde{c}} a_j^-(\xi) \lambda^j, \qquad \xi \in W,$$

where $\tilde{a} = \sum_{j=1}^a \alpha_j$, $\tilde{b} = \sum_{j=1}^b \beta_j$, $\tilde{c} = \sum_{j=1}^c \gamma_j$. Note that $\tilde{b} \leqq p$ under the assumption (A-1). Put

$$L_{W,\sigma}^+(\xi) = \det((2\pi i)^{-1} \int_{\gamma(\xi)} B_{\sigma_j}(\xi, \lambda) \lambda^{k-1} (P^+(\xi, \lambda))^{-1} d\lambda)_{j,k=1,\cdots,\tilde{b}} .$$

where $\gamma(\xi)$ is a simple closed curve in the complex upper half λ-plane which surrounds all $\lambda_j^+(\xi)$, $j = 1, \cdots, b$ when $\xi \in W$ and we denote by $\sigma = (\sigma_1, \cdots, \sigma_{\tilde{b}})$ a set consisting of $\tilde{b}$ elements of $\{1, \cdots, p\}$. When $\tilde{b} > 0$ and $m \geqq 1$ we make the following assumption on linear independence of boundary conditions:

(A-2) $L^{+}_{W,\sigma}(\xi)$ does not vanish identically in W for some $\sigma\subset\{1,\cdots,p\}$.

Main Theorem. *Let u be a solution of the equations* (1.1) *and* (1.2) *which belongs to* $C^{\infty}([0,\delta);\mathcal{S}'(\boldsymbol{R}^n))\cap L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$ *for some positive number δ. We make the assumptions* (A-1) *and* (A-2) *when $m\geqq 1$. Then there exist an open cone Γ in $\boldsymbol{R}^{n+1}$ and a natural number N such that if u satisfies the condition:*

$$\varliminf_{R\to\infty} R^{-N}\int_{\Gamma_R}|u(x,y)|^2dxdy=0 \tag{1.3}$$

then $u=0$. Here Γ and N are independent of u, $\Gamma_R=\{(x,y)\in\Gamma;\, y\geqq 0,\, R<|(x,y)|<2R\}$ and $|(x,y)|=(\sum_{j=1}^n x_j^2+y^2)^{1/2}$.

Moreover, in the case where $m\geqq 1$, if at least one of (A-1) *and* (A-2) *is not fulfilled, there exists a solution of the equations* (1.1) *and* (1.2) *which belongs to* $\mathcal{S}(\overline{\boldsymbol{R}^{n+1}_+})$.

Remark. In the case where $m=0$ and $A_{a_0}=\{\xi\in\Xi^n;\, P(\xi,\lambda)=a_0(\xi)=0\}$ is empty, if u satisfies equations (1.1) and (1.2) and belongs to $C^{\infty}([0,\delta);\mathcal{S}'(\boldsymbol{R}^n))$ then $u=0$. In the case where $m\geqq 1$ and the system $\{P(\xi,\lambda), B_j(\xi,\lambda), j=1,\cdots,p\}$ satisfies the following conditions:

(i) $\{\xi\in\Xi^n;\, a_m(\xi)=a_{m-1}(\xi)=\cdots=a_0(\xi)=0\}$ is empty,

(ii) for each $\xi^0\in\Xi^n$ all roots of the equation $P(\xi^0,\lambda)=0$ have negative imaginary part or the degree d of $P^0(\xi^0,\lambda)P^+(\xi^0,\lambda)$ is equal to or less than p and

$$\det\Big((2\pi i)^{-1}\int_{\gamma(\xi^0)} B_{\sigma_j}(\xi^0,\lambda)\lambda^{k-1}(P^0(\xi^0,\lambda)P^+(\xi^0,\lambda))^{-1}d\lambda\Big)_{j,k=1,\cdots,d}\neq 0$$

for some $\sigma=(\sigma_1,\cdots,\sigma_d)\subset\{1,\cdots,p\}$, if u satisfies the equations (1.1) and (1.2) and belongs to $C^{\infty}([0,\delta);\mathcal{S}'(\boldsymbol{R}^n))\cap L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$, then $u=0$. Here denoting by τ^0_j, $j=1,\cdots,\mu$, and τ^+_j, $j=1,\cdots,\nu$ the roots whose imaginary parts are zero and positive, respectively, of the equation $P(\xi^0,\lambda)=0$ in λ, we wrote $P^0(\xi^0,\lambda)=\prod_{j=1}^{\mu}(\lambda-\tau^0_j)$ and $P^+(\xi^0,\lambda)=\prod_{j=1}^{\nu}(\lambda-\tau^+_j)$, and $\sigma=(\sigma_1,\cdots,\sigma_d)$ $(d=a+b)$ is a subset consisting of d elements of $\{1,\cdots,p\}$. Thus in the statement of the Main Theorem we put $\Gamma=\emptyset$ and interpret that the condition (1.3) is satisfied automatically under the situation which is stated above.

We will state more details on Γ and N in the proof of the Main Theorem.

On the other hand, we can show that the system $\{P(\xi,\lambda), B_j(\xi,\lambda), j=1,\cdots,p\}$ does not satisfy the condition stated in above Remark, there exists a solution $u\in\mathcal{S}'(\overline{\boldsymbol{R}^{n+1}_+})\cap C^{\infty}(\overline{\boldsymbol{R}^{n+1}_+})$ of the equations (1.1) and (1.2) with

$$\int_{\substack{|(x,y)|<R \\ y\geqq 0}} |u(x, y)|^2 dxdy = 0(R^{N'}) .$$

In general case we have $N'\geqq N$ and $N=N'$ for certain class of systems of differential operators with constant coefficients containing $\{P(D)=\Delta+k, B(D)=1\}$ for which the result was given in Rellich [12] or Agmon [1] where Δ is the Laplacian operator and k is a positive number. But we can not show that $N'=N$ in general case.

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§2. A Condition in Order That the Support of $\hat{u}(\xi, y)$ Is Contained in a Real Analytic Set of Higher Codimension

When $m=0$, put $A_0=\Xi^n$. When $m\geqq 1$, let us denote by A_1 an open set contained in W, by $A_{r+1}(1\leqq r\leqq n-1)$ a real analytic manifold which is defined by $\xi''=\mu(\xi')$ where $\xi'\in\Omega\subset\Xi^{n-r}$ and $\mu(\xi')$ is a real analytic function in Ω. A_{n+1} denotes a set consisting of finite many points in Ξ^n. Further, when $1\leqq r\leqq n+1$, we assume that A_r is contained in X_e where

$$\begin{aligned} X_m &= \{\xi\in\Xi^n;\ a_m(\xi)\neq 0\} , \\ X_e &= \{\xi\in\Xi^n;\ a_m(\xi)=\cdots=a_{e+1}(\xi)=0,\ a_e(\xi)\neq 0 \quad e=1,\cdots,m-1\} , \\ X_0 &= \{\xi\in\Xi^n;\ a_m(\xi)=\cdots=a_1(\xi)=0\} . \end{aligned}$$

Let u be a solution of the equations (1.1) and (1.2) which belongs to $C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))$ such that the support of $\phi(\xi)\hat{u}(\xi, y)$[1)] is contained in $A_r\times R^1_+ (0\leqq r\leqq n+1)$ for some $\phi\in C_0^\infty(\Xi^n)$. In this section we study what conditions imply that there is a real analytic set $B\subset A_r$ such that $\operatorname{codim} B>\operatorname{codim} A_r$ and $\operatorname{supp}\phi(\xi)\hat{u}(\xi, y)\subset B\times\overline{\boldsymbol{R}^1_+}$. We need the following lemma.

Lemma 2.1. *Let $f(\xi, y)$ belong to $C^\infty([0, \sigma); \mathcal{S}(\Xi^n))$. If the support of f is contained in the plane $\xi'=0$, then f has the form*

$$f(\xi, y)=\sum_{|\alpha|\leqq q} f_\alpha(\xi', y)\otimes D^{\alpha}_{\xi''}\delta$$

1) We denote by $\hat{u}(\xi, y)$ the partial Fourier transform of $u(x, y)$ with respect to x.

where $f_\alpha \in C^\infty([0, \sigma); \mathcal{S}'(\Xi^{n-r}_{\xi'}))$, δ *is the Dirac measure at the origin in* $\Xi^r_{\xi''}$, $\alpha = (\alpha_{n-r+1}, \cdots, \alpha_n)$ *and* $D_{\xi''} = (D_{\xi_{n-r+1}}, \cdots, D_{\xi_n})$. *Here we wrote* $\xi' = (\xi_1, \cdots, \xi_{n-r})$ *and* $\xi'' = (\xi_{n-r+1}, \cdots, \xi_n)$.

Proof. From a theorem due to Seeley [13] it follows that f has the form

$$f(\xi, y) = \sum_{|\alpha| \leq q} D^\alpha g_\alpha(\xi, y) \qquad \text{when } y \geqq 0$$

where the $g_\alpha(\xi, y)$ are continuous functions of polynomial growth, $F(\xi, y) = \sum_{|\alpha| \leq q} D^\alpha g_\alpha(\xi, y) \in C^\infty((-\infty, \sigma); \mathcal{S}'(\Xi^n))$ and the support of $F(\xi, y)$ is contained in the plane $\xi'' = 0$. Thus, by a slight modification of the proof of the fact that every distribution having the point x_0 as support has the form $f = \sum_{|\alpha| \leq p} a_\alpha D^\alpha \delta(x - x_0)$, we have that for $\phi(\xi', y) \in \mathcal{S}(\Xi^{n-r}_{\xi'} \times \boldsymbol{R}^1_y)$ and $\rho(\xi'') \in C_0^\infty(\Xi^r_{\xi''})$

$$(F, \phi \otimes \rho) = \sum_{|\alpha| \leq q} (F, \phi \otimes (h(\xi'')(\xi'')^\alpha/\alpha!)(D^\alpha \rho)(0))$$

where $h(\xi'') \in C_0^\infty(\Xi^r_{\xi''})$ equals to 1 in a neighborhood of the origin in $\Xi^r_{\xi''}$. Since

$$\begin{aligned}&(F, \phi \otimes h(\xi'')^\alpha/\alpha!) \\ &= \sum_{|\delta| \leq q} \int (-D_{\xi', y})^\beta \phi(\xi', y) d\xi' dy [\int (D_{\xi''})^\gamma [h(\xi'')(\xi'')^\alpha/\alpha!] g_\delta(\xi, y) d\xi'' \\ &= \langle \{\sum_{|\delta| \leq q} D^\beta_{\xi', y} [\int -(D_{\xi''})^\gamma [h(\xi'')(\xi'')^\alpha/\alpha!] g_\delta(\xi, y) d\xi'']\}, \phi(\xi', y) \rangle,\end{aligned}$$

if we put

$$F_\alpha(\xi', y) = \sum_{|\delta| \leq q} D^\beta_{\xi', y} [\int (-D_{\xi''})^\gamma [h(\xi'')(\xi'')^\alpha/\alpha!] g_\delta(\xi, y) d\xi''],$$

we have

$$F = \sum_{|\alpha| \leq q} F(\xi', y) \otimes D_{\xi''} \delta,$$

where $D_{\xi', y} = (D_{\xi_1}, \cdots, D_{\xi_{n-r}}, D_y)$ and $D_{\xi''} = (D_{\xi_{n-r+1}}, \cdots, D_{\xi_n})$. Since

$$\int (-D_{\xi''})^\gamma [h(\xi'')(\xi'')^\alpha/\alpha!] g_\delta(\xi, y) d\xi''$$

is a continuous function in (ξ', y) of polynomial growth, $F_\alpha(\xi', y)$ belongs to $\mathcal{S}'(\Xi^{n-r}_{\xi'} \times \boldsymbol{R}^1_y)$. Further, choosing $\rho \in C_0^\infty(\Xi^r_{\xi''})$ with $(D^\alpha_{\xi''} \rho)(0) = 1$ and $(D^\alpha_{\xi''} \rho)(0) = 0$ for $\beta \neq \alpha$ and $|\beta| \leqq q$, we have, when $y < \sigma$,

$$\langle F_\alpha(\xi', y), \phi(\xi')\rangle_{\xi'} = \langle \sum_{|\alpha|\leqslant q} F_\alpha(\xi', y)\otimes D_{\xi''}\delta, \phi(\xi')\otimes\rho(\xi'')\rangle$$
$$= \langle F(\xi', y), \phi(\xi')\otimes\rho(\xi'')\rangle$$

for any $\phi\in\mathcal{S}(\Xi^{n-r}_{\xi'})$, which shows that $F_\alpha(\xi', y)\in C^\infty((-\infty, \sigma); \mathcal{S}'(\Xi^{n-r}_{\xi'}))$. Q.E.D.

First of all we consider the case where the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in A_0 or A_r $(2\leqslant r\leqslant n+1)$ which is contained in X_0. Let us denote by A such an A_r. When the codimension of A is positive, that is, $A_r=A$ $(2\leqq r\leqq n+1)$ and $A\subset X_0$, we denote by $v(\xi, y)$ the composition of $\phi(\xi)u(\xi, y)$ and the map $\xi\mapsto(\xi', \xi''+\mu(\xi'))$ (defined arbitrarily for $\xi'\notin\Omega$). It is obvious that the suupport of $v(\xi, y)$ is contained in the plane $\xi''=0$. Thus, by Lemma 2.1, we can write v as a finite sum:

$$v(\xi, y) = \sum_{|\alpha|\leqq s} v_\alpha(\xi', y)\otimes D^{\alpha}_{\xi''}\delta\,,$$

where δ is the Dirac measure at 0 in $\Xi^{r}_{\xi''}$, $\alpha=(\alpha_{n-r+2}, \cdots, \alpha_n)$ and $v_\alpha(\xi', y)$ $C^\infty([0, \delta); \mathcal{S}'(\Xi^{n-r+1}_{\xi'}\times\overline{\boldsymbol{R}^1_+}))$. Let us fix α $(|\alpha|=s)$ and let ψ be a $C^\infty_0(\Xi^{r-1}_{\xi''})$ function with $(D^\alpha\psi)(0)=1$ and $(D^\beta\psi)(0)=0$ for $\beta\neq\alpha$ and $|\beta|\leqq$s. Hence we have

$$\langle P(\xi', \mu(\xi'), D_y)v_\alpha(\xi', y), \chi(\xi')\rho(y)\rangle$$
$$= \langle P(\xi, D_y)\phi(\xi)\hat{u}(\xi, y), \chi(\xi')\psi(\xi''-\mu(\xi'))\rho(y)\rangle = 0$$

for any $\chi(\xi')\in C^\infty_0(\Omega)$ and $\rho(y)\in\mathcal{S}(\overline{\boldsymbol{R}^1_+})$. This shows that

$$\langle a_0(\xi', \mu(\xi'))v_\alpha(\xi', y), w(\xi', y)\rangle = 0\,, \tag{2.1}$$

for any $w(\xi', y)\in\mathcal{S}_0(\Omega\times\overline{\boldsymbol{R}^1_+})$. Here we wrote for any open set Ω in $\Xi^k\mathcal{S}_0(\Omega\times\overline{\boldsymbol{R}^1_+})=\{\phi\in C^\infty(\Xi^k\times\boldsymbol{R}^1_+);$ There is a $\tilde{\phi}\in\mathcal{S}(\Xi^k\times\boldsymbol{R}^1)$ with $\operatorname{supp}\tilde{\phi}\subset\Omega\times\boldsymbol{R}^1$ such that $\phi=\tilde{\phi}\,|_{y>0}\}$. (2.1) shows the support of $v_\alpha(\xi', y)$ $(|\alpha|=s)$ is contained in $\{\xi'\in\Omega;\ a_0(\xi', \mu(\xi'))=0\}\times\overline{\boldsymbol{R}^1_+}$. Since

$$0 = \langle P(\xi, D_y)\phi(\xi)\hat{u}(\xi, y), \chi(\xi')\psi(\xi''-\mu(\xi'))\rho(y)\rangle$$
$$= \langle a_0(\xi', \mu(\xi'))v_\alpha(\xi', y), \chi(\xi')\rho(y)\rangle$$

for any $\chi(\xi')\in C^\infty_0(\{\xi'\in\Omega;\ a_0(\xi', \mu(\xi'))\neq 0\})$ where $\psi\in C^\infty_0(\Xi^{r-1}_{\xi''})$ with $(D^\alpha\psi)(0)=1(|\alpha|=s-1)$ and $(D^\beta\psi)(0)=0$ for $\beta\neq\alpha$ and $|\beta|\leqq s$, we have that the support of $v_\alpha(\xi', y)$ $(|\alpha|=s-1)$ is contained in $\{\xi'\in\Omega;\ a_0(\xi', \mu(\xi'))=0\}\times\overline{\boldsymbol{R}^1_+}$. By repeating the argument we conclude that the support of $v_\alpha(\xi', y)$ is contained in $\{\xi'\in\Omega;\ a_0(\xi', \mu(\xi'))=0\}\times\overline{\boldsymbol{R}^1_+}$ for all α $(|\alpha|\leqq s)$, which shows that the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $\{(\xi', \mu(\xi')); \xi'\in B_{r-1}\}\times \boldsymbol{R}^1_+$ where $B_{r-1}=\{\xi'\in\Omega; a_0(\xi', \mu(\xi'))=0\}$. Consider the case: $m=0$, that is, $A=A_0$. Since $\langle a_0(\xi)\phi(\xi)\hat{u}(\xi, y), v(\xi, y)\rangle=0$ for any $v\in\mathcal{S}_0(\Xi^n\times\boldsymbol{R}^1_+)$, we have that the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $B_0\times\overline{\boldsymbol{R}^1_+}$, where $B_0=\{\xi\in\Xi^n; a_0(\xi)=0\}$. When B_r $(0\leqq r\leqq n)$ is empty, it is obvious that $\phi(\xi)\hat{u}(\xi, y)=0$. On the other hand, when B_r $(0\leqq r\leqq n)$ is not empty, we have, using a theorem due to Hörmander [see Theorem $\mathrm{A_p}$-3 in Appendix], the following

Lemma 2.2. *Let $u\in C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))\cap \boldsymbol{L}^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$ be a solution of the equations* (1.1) *and* (1.2). *Assume that* supp $\phi(\xi)\hat{u}(\xi, y)\subset A_r$ *$(r=0$ or $2\leqq r\leqq n+1)$ for some $\phi(\xi)\in C^\infty(\Xi^n)$, A_r is contained in X_0 when $2\leqslant r\leqslant n+1$ and that B_r $(0\leqq r\leqq n)$ is not empty. Put N_r = the codimension of B_r in Ξ^{n-r} $(0\leqq r\leqq n)$. Set*

$$\Gamma_{r,R} = \{(x, y)\in\Gamma_r; y\geqq 0, R<|(x, y)|<2R\}$$

where Γ_r is an open cone in $\boldsymbol{R}^{n+1}$ which for every analytic manifold M_r and $\xi_0\in M_r$ contains $(n(\xi_0), 0)$. Here when $r=0$, M_0 is contained B_0 and when $1\leqq r\leqq n$, M_r is contained in $\{(\xi', \mu(\xi')); \xi'\in B_r\}$ and $n(\xi_0)$ denotes some normal of M_r at ξ_0 in $\boldsymbol{R}^n_x$. If u satisfies the condition:

$$\varliminf_{R\to\infty} R^{-(N_r+r)}\int_{\Gamma_{r,R}} |u(x, y)|^2 dxdy = 0,$$

then $\overline{\mathcal{F}}[\phi\hat{u}]=0$. Here $\overline{\mathcal{F}}$ denotes the inverse partial Fourier transform with respect to ξ.

Remark. (i) The codimension of $\{(\xi', \mu(\xi')); \xi'\in B_r\}$ is N_r+r in Ξ^n. (ii) When $r=0$, Lemma 2.2 shows the Main Theorem.

Proposition 2.3. *Assume that the set $\{\xi\in\Xi^n; a_m(\xi)=a_{m-1}(\xi)=\cdots=a_0(\xi)=0\}$ is not empty. Let Γ be an open cone in $\boldsymbol{R}^{n+1}$ and N' an integer such that every $w\in\mathcal{S}'(\overline{\boldsymbol{R}^{n+1}_+})\cap C^\infty(\overline{\boldsymbol{R}^{n+1}_+})$ which is a solution of the equations* (1.1) *and* (1.2) *with the condition:*

$$\varliminf_{R\to\infty} R^{-N'}\int_{\Gamma_R} |w(x, y)|^2 dxdy = 0 \tag{2.2}$$

is equal to 0. *If $M\subset\Xi^n$ is a C^∞ manifold where $a_m(\xi)=\cdots=a_0(\xi)=0$ and if $\xi_0\in M$, then it follows that the closure of Γ contains $(n(\xi_0), 0)\neq 0$ and that $N'\leqq$* codim *M. Here $n(\xi_0)$ denotes some nomal of M at ξ_0 in Ξ^n.*

Proof. We may assume that M is defined by $\xi''=\phi(\xi')$ where $\xi'\in\omega\subset\Xi^{n-l}$, $\phi\in C^\infty(\omega)$ and $\xi_0=(0, \phi(0))$. Here l denotes the codimension of M. For $\chi\in C_0^\infty(\omega)$ we put

$$w(x, y) = \int \exp\{i(x'\cdot\xi'+\phi(\xi')\cdot x'')\}\chi(\xi')d\xi'\circ y^h\rho(y),$$

where $h=\operatorname{Max}\{r_j+1;\ 1\leqq j\leqq p\}$ and $\rho\in C_0^\infty((-2, 2))$ with $\rho(y)=1$ for $-y\in[1, 1]$. Since

$$D_y^k(y^h\rho(y))|_{y=0} = 0,\ 0\leqq k\leqq \operatorname{Max}\{r_j;\ 1\leqq j\leqq p\}\ ,$$
$$a_j(\xi', \phi(\xi')) = 0,\ 0\leqq j\leqq m,\ \xi'\in\operatorname{supp}\chi\ ,$$

we have

$$P(D)w(x, y) = \sum_{j=0}^{m}\int a_j(\xi', \phi(\xi'))\exp\{i(x'\cdot\xi'+\phi(\xi')\cdot x')\}\chi(\xi')d\xi'\cdot D_y^j(y^h\rho(y))=0$$

$$B_j(D)w(x, y)|_{y=0} = \sum_{k=0}^{r_j}\int b_j(\xi', \phi(\xi'))\exp\{i(x'\cdot\xi'+\phi(\xi')\cdot x'')\}$$
$$\cdot\chi(\xi')d\xi'\cdot D_y^k(y^h\rho(y))|_{y=0} = 0 \qquad j = 1, \cdots, p.$$

Moreover, the condition (2.2) follows from (A_p-1) for w if codim $M<N'$, which gives a contradiction. If $\bar{\Gamma}$ contains no normal which has the form $(n(\xi_0), 0)$ at ξ_0 and if supp χ is sufficiently close to ξ_0 the condition (2.2) follows from (A_p-2) for any N, which completes the proof.

Remark. Proposition 2.3 shows that Lemma 2.2 is very precise.

Next we consider the case where the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in A_{r+1} ($r=0, 1, \cdots, n$) where A_{r+1} is contained in X_e for some $e\neq 0$ (i.e. $1\leqq e\leqq m$) if $1\leqq r\leqq n$. We need the following fact due to Wakabayashi [16, Lemma 2.10].

Lemma 2.4. *Put*

$$P(\xi, \lambda) = \lambda^j+a_1(\xi)\lambda^{j-1}+\cdots+a_j(\xi)$$

where $a_k(\xi)$ is real analytic in a connected open set V $(\subset\Xi^n)$. Then there exists a real analytic function $D(\xi)$ $(\not\equiv 0)$ in V such that the roots of $p(\xi, \lambda)=0$ in λ have constant multiplicities for ξ and are real analytic functions of ξ in each connected component of $\{\xi\in V;\ D(\xi)\neq 0\}$.

When $1\leqq r\leqq n$, by assumption we have $A_{r+1}\subset X_e$ $(e\neq 0)$ and $P(\xi', \mu(\xi'), \lambda)=\sum_{j=0}^{e}a_j(\xi', \mu(\xi'))\lambda^j$, $a_e(\xi', \mu(\xi'))\neq 0$ for $\xi'\in\Omega$. So it follows from Lemma 2.4 that there exists a real analytic function $D(\xi')$ $(\not\equiv 0)$ in Ω such that the roots of

$P(\xi', \mu(\xi'), \lambda)=0$ in λ have constant multiplicities and are real analytic functions of ξ' in each connected component of $\{\xi'\in\Omega;\ D(\xi')\neq 0\}$. Thus denoting by V_Ω each connected component of $\{\xi'\in\Omega;\ D(\xi')\neq 0\}$, we have that the imaginary parts of the roots of the equation $P(\xi', \mu(\xi'), \lambda)=0$ in λ are identically zero or real analytic functions of ξ' in V. Denote the latter by $\operatorname{Im}\lambda_j(\xi')$, $j=1, \cdots, k$, and put

$$A_{V_\Omega, \mathrm{Im}} = \{\xi'\in V_\Omega;\ \operatorname{Im}\lambda_j(\xi') = 0 \text{ for some } j\in\{1, \cdots, k\}\}\ .$$

It is obvious that $A_{V_\Omega, \mathrm{Im}}$ is empty or a real analytic set. $V_\Omega - A_{V_\Omega, \mathrm{Im}}$ may be decomposed into connected components $\{W_{\Omega,j}\}$. We write $W_{\Omega,j}=W_\Omega$ for the sake of simplicity. Thus when $\xi'\in W_\Omega$, the roots of the equation $P(\xi', \mu(\xi'),\lambda)=0$ in λ have constant multiplicity and split into three classes: real roots, those with positive imaginary part and those with negative imaginary part. We denote those by $\lambda_j^0(\xi'), j=1, \cdots, a,\ \lambda_j^+(\xi'), j=1, \cdots, b$ and $\lambda_j^-(\xi'),\ j=1, \cdots, c$ where $\operatorname{Im}\lambda_j^0(\xi')=0$, $\operatorname{Im}\lambda_j^+(\xi')>0$ and $\operatorname{Im}\lambda_j^-(\xi')<0$, and then we have

$$\begin{aligned} &P(\xi', \mu(\xi'), \lambda) = \textstyle\sum_{j=0}^{e} a_j(\xi', \mu(\xi'))\lambda^j \\ &= a_e(\xi', \mu(\xi'))\cdot\textstyle\prod_{j=1}^{a}(\lambda-\lambda_j^0(\xi'))^{\alpha_j}\cdot\prod_{j=1}^{b}(\lambda-\lambda_j^+(\xi'))^{\beta_j}\cdot\prod_{j=1}^{c}(\lambda-\lambda_j^-(\xi'))^{\gamma_j}\,, \\ &a_e(\xi', \mu(\xi'))\neq 0,\ \xi'\in W_\Omega. \end{aligned}$$

Put

$$\begin{aligned} P^0(\xi', \lambda) &= \textstyle\prod_{j=1}^{a}(\lambda-\lambda_j^0(\xi'))^{\alpha_j} = \sum_{j=0}^{\tilde a} a_j^0(\xi')\lambda^j\,, \\ P^+(\xi', \lambda) &= \textstyle\prod_{j=1}^{b}(\lambda-\lambda_j^+(\xi'))^{\beta_j} = \sum_{j=0}^{\tilde b} a_j^+(\xi')\lambda^j\,, \\ P^-(\xi', \lambda) &= a_e(\xi', \mu(\xi'))\cdot\textstyle\prod_{j=1}^{c}(\lambda-\lambda_j^-(\xi')^{\gamma_j} = \sum_{j=0}^{\tilde c} a_j^-(\xi')\lambda^j\,, \end{aligned}$$

where $\tilde a=\sum_{j=1}^{a}\alpha_j$, $\tilde b=\sum_{j=1}^{b}\beta_j$, $\tilde c=\sum_{j=1}^{c}\gamma_j$.

Let Z be any open set such that $Z\cap A_{r+1}$ is contained in $A_W=\{(\xi', \mu(\xi');\ \xi'\in W_\Omega\}$ when $1\leqq r\leqq n$. Let $\phi_1(\xi)$ be a $C_0^\infty(Z)$ function and $v(\xi, y)$ be the composition of $\phi(\xi)\phi_1(\xi)\hat u(\xi, y)$ and the map $\xi\mapsto(\xi', \xi''+\mu(\xi'))$ (defined arbitrarily for $\xi'\notin W_\Omega$). Since $\operatorname{supp}\phi(\xi)\phi_1(\xi)\hat u(\xi, y)\subset A_{W_\Omega}\times\overline{R_+^1}$, the support of $v(\xi, y)$ is contained in the plane $\{\xi\in\Xi^n;\ \xi''=0\}\times\overline{\boldsymbol{R}_+^1}$ and we can write v as a finite sum:

$$v(\xi, y) = \textstyle\sum_{|\alpha|\leqq s} v_\alpha(\xi', y)\otimes D_{\xi''}\delta\,,$$

using Lemma 2.1. Let us fix α ($|\alpha|=s$). Let ψ be a $C_0^\infty(\Xi_{\xi''}^r)$ function with $(D^\alpha\psi)(0)=1$ and $(D^\beta\psi)(0)=0$ for $\beta\neq\alpha$ and $|\beta|\leqq s$. We have

$$(2.3)\qquad \langle P(\xi', \mu(\xi'), D_y)v_\alpha(\xi', y), \chi(\xi')\rho(y)\rangle$$
$$= \langle P(\xi, D_y)\phi(\xi)\phi_1(\xi)\hat{u}(\xi, y), \chi(\xi')\psi(\xi''-\mu(\xi'))\rho(y)\rangle = 0\,,$$
$$(2.4)\qquad \langle B_j(\xi', \mu(\xi'), D_y)v_\alpha(\xi', y)|_{y=0}, \chi(\xi')\rangle$$
$$= \langle B_j(\xi, D_y)\phi(\xi)\phi_1(\xi)\hat{u}(\xi, y)|_{y=0}, \chi(\xi')\psi(\xi''-\mu(\xi'))\rangle = 0$$
$$j = 1, \cdots, p\,,$$

for any $\chi(\xi')\in C_0^\infty(W_\Omega)$ and $\rho(y)\in \mathcal{S}(\overline{\boldsymbol{R}^1_+})$. In order to simplify the notation,

$$\hat{v}(\xi', y) = \begin{cases} \phi(\xi)\hat{u}(\xi, y) & \text{when } r=0 \\ v_\alpha(\xi', y) & \text{when } 1\leqq r\leqq n \end{cases}$$

$$W = \begin{cases} A_1 & \text{when } r=0 \\ W_\Omega & \text{when } 1\leqq r\leqq n\,, \end{cases}$$

$$P^0(\xi', D_y) = \begin{cases} P^0(\xi, D_y) & \text{when } r=0 \\ P^0(\xi', D_y) & \text{when } 1\leqq r\leqq n \end{cases}$$

$$P^\pm(\xi', D_y) = \begin{cases} P^\pm(\xi, D_y) & \text{when } r=0 \\ P^\pm(\xi', D_y) & \text{when } 1\leqq r\leqq n \end{cases}$$

$$B_j(\xi', D_y) = \begin{cases} B_j(\xi, D_y) & \text{when } r=0 \\ B_j(\xi', \mu(\xi'), D_y) & \text{when } 1\leqq r\leqq n\,. \end{cases}$$

Lemma 2.5. *Let $\hat{v}$, W, P^0 and $P^\pm$ be as in the above statement. Then*

$$(2.5)\qquad \langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y), w(\xi', y)\rangle = 0$$

for any $w(\xi', y)\in \mathcal{S}_0(W\times \overline{\boldsymbol{R}^1_+})$.

Proof. Repeating integration by parts, we have

$$(2.6)\qquad 0 = \langle P(\xi', \mu(\xi'), D_y)\hat{v}(\xi', y)\chi(\xi')\overline{\rho(y)}\rangle$$
$$= i\textstyle\sum_{k=1}^{\tilde{c}-1}\langle\sum_{j=k+1}^{\tilde{c}} a_j^-(\xi')D_y^{j-1-k}P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y)|_{y=0},$$
$$\chi(\xi')\overline{D_y^k\rho(y)}|_{y=0}\rangle+\langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y),$$
$$\overline{\bar{P}^-(\xi', D_y)\rho(y)\chi(\xi')}\rangle\,.$$

Here we wrote

$$\bar{P}^-(\xi', D_y) = \textstyle\sum_{j=0}^{\tilde{c}} \overline{a_j^-(\xi')}D_y^j\,.$$

Since the roots of the equation $\bar{P}^-(\xi', \lambda)=0$ in λ have all positive imaginary part when $\xi\in W$, we can define the Lopatinski determinant of the system $\{\bar{P}^-(\xi' D_y), D_y^{j-1}, j=1, \cdots, \tilde{c}\}$ by

$$L^-(\xi') = \det((2\pi i)^{-1}\oint \lambda^{j+k-2}(\bar{P}^-(\xi', \lambda))^{-1}d\lambda)_{j,k=1,\cdots,\tilde{c}} = 1\,.$$

Since $L^-(\xi')=1\neq 0$ in W, for any $w(\xi', y)\in\mathcal{S}_0(W\times\overline{\boldsymbol{R}^1_+})$ where exists a $\mathcal{S}_0(W\times\overline{R^1_+})$ function $(z(\xi', y)$ such that

$$\bar{P}^-(\xi', D_y)z(\xi', y) = w(\xi', y),\ D_y^j z(\xi', y)|_{y=0} = 0, \qquad j=0, \cdots, \tilde{c}-1\,.$$

Q.E.D.

Lemma 2.6. *Let $\hat{v}$, W, P^0 and P^+ be as in Lemma* 2.5. *If $\hat{v}$ satisfies the formula*:

$$\begin{aligned}(2.7)\qquad &\langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y),\ w(\xi', y)\rangle \\ &= \langle \hat{v}(\xi', y),\ P^0(\xi', -D_y)P^+(\xi'-D_y)w(\xi', y)\rangle\,,\end{aligned}$$

for any $w\in\mathcal{S}_0(U\times\overline{\boldsymbol{R}^1_+})$ then

$$\langle \hat{v}(\xi, y),\ \chi(\xi')\rho(y)\rangle = 0$$

for any $\chi(\xi')\in C_0^\infty(U)$ and $\rho\in C^\infty_{(0)}(\overline{\boldsymbol{R}^1_+})$. Here U is any open subset of W.

Proof. Let $\tilde{U}$ be any open subset of U such that the closure of $\tilde{U}$ is contained in U and compact. It is sufficient to prove that

$$(2.8)\qquad \langle \hat{v}(\xi', y),\ \chi(\xi')\rho(y)\rangle = 0\,,$$

for any $\chi(\xi')\in C_0^\infty(\tilde{U})$ and $\rho(y)\in C^\infty_{(0)}(\overline{\boldsymbol{R}^1_+})$. From (2.7) and Lemma 2.5 we have

$$(2.9)\qquad \langle \hat{v}(\xi', y),\ P^0(\xi', -D_y)P^+(\xi', -D_y)w(\xi', y)\rangle = 0\,,$$

for any $w(\xi', y)\in\mathcal{S}_0(U\times\overline{\boldsymbol{R}^1_+})$. Since we can choose γ so large that $P^0(\xi', \lambda-i\gamma)P^+(\xi', \lambda-i\gamma)\neq 0$ when $\xi'\in\tilde{U}$ and λ is real, we can put

$$w(\xi', y) = \chi(\xi')\cdot\int \exp\{-i(\lambda-i\gamma)\cdot y\}\bar{\mathcal{F}}[\rho_0](\lambda)[P^0(\xi', \lambda-i\gamma)P^+(\xi', \lambda-i\gamma)]^{-1}d\lambda,^{2)}$$
$$y\geqq 0\,,$$

for any $\chi(\xi')\in C_0^\infty(\tilde{U})$ and $\rho\in C^\infty_{(0)}(\overline{\boldsymbol{R}^1_+})$ where ρ_0 is compactly supported C^∞ extension of ρ to $y<0$ (see Seeley [13]), and then $w(\xi', y)$ belongs to $\mathcal{S}_0(\tilde{U}\times\overline{\boldsymbol{R}^1_+})$. We obtain, using (2.9),

$$\langle \hat{v}(\xi', y),\ e^{-\gamma y}\chi(\xi')\rho(y)\rangle = 0$$

for any $\chi(\xi')\in C_0^\infty(\tilde{U})$ and $\rho(y)\in C^\infty_{(0)}(\overline{\boldsymbol{R}^1_+})$. Q.E.D.

2) Here $\bar{\mathcal{F}}$ denotes the inverse Fourier transform with respect to y.

Next lemma which is inspired by an idea due to Hörmander [7] plays an essential role in this section.

Lemma 2.7. *Let* $\hat{v}$, W, P^0 *and* P^+ *be as in Lemma 2.5. Put* $M_j = \{(\xi', \mu(\xi'), \lambda_j^0(\xi'); \xi' \in W\}, j=1, \cdots, a$. *Let* δ_j, $j=1, \cdots, a$, *be non-negative integer such that* $\delta_j \leqq \alpha_j$. *Assume that* $\theta_j = (\theta_1^j, \cdots, \theta_n^j, \theta_{n+1}^j) \in \overline{R_+^{n+1}}$ *is a normal of* M_j *at* $(\xi_0', \mu(\xi_0'), \lambda_j^0(\xi_0'))$ $(\xi_0' \in W$ *and* $|\theta_j|=1)$ *and* $\varepsilon>0$. *If* u *belongs to* $L^2_{\mathrm{loc}}(\overline{R_+^{n+1}})$ *and satisfies the condition*:

$$(2.10) \qquad \lim_{R\to\infty} R^{-(2(\alpha_j-\delta_j)+r+1)} \int_{\substack{|(x,y)/R-\theta_j|<\varepsilon \\ y\geqq 0}} |u(x, y)|^2 dxdy = 0 \qquad \text{for } \delta_j \geqq 1,$$

then there exists a small neighborhood ω *of* ξ_0 *contained in* W *such that*

$$(2.11) \qquad \Big\langle \sum_{i=0}^{d} \sum_{k=0}^{i-1} \frac{(i-1-k)!}{(i-1-k-\nu)!} b_i(\xi') \lambda_j^0(\xi')^{i-1-k-\nu} D_y^k Q(\xi', D_y) v(\xi', y)\,|_{y=0},\ \chi(\xi')\Big\rangle = 0$$

for any $\chi(\xi') \in C_0^\infty(\omega)$ *and* $0 \leqq \nu \leqq \delta_j - 1$ *where we interpret that* $\lambda_j^0(\xi')^{i-1-k-\nu} \equiv 0$ *if* $i-k-1-\nu<0$. *Here* u *is as in the first place of this section and*

$$d = \sum_{j=1}^{a} \delta_j,\ \prod_{j=1}^{a} (\lambda - \lambda_j^0(\xi'))^{\delta_j} = \sum_{j=0}^{d} b_j(\xi') \lambda^j,\ b_d(\xi') = 1,$$

$$Q(\xi', \lambda) = \prod_{j=1}^{a} (\lambda - \lambda_j^0(\xi'))^{\alpha_j - \delta_j} \cdot P^+(\xi', \lambda), \qquad \xi' \in W.$$

Proof. Let us write $\alpha_j = \alpha$, $\delta_j = \delta$, $\theta_k^j = \theta_k$ and $\lambda_j^0(\xi') = \lambda(\xi')$. Let ω_1 be an open neighborhood of ξ_0' with $\omega_1 \subset\subset W$[3]. Put

$$m = \mathrm{Max}\{\sup_{\bar\omega_1}(|\partial\lambda_i^0/\partial\xi_h(\xi')|), \sup_{\bar\omega_1}(|\partial\mu_j/\partial\xi_h(\xi')|);\ h=1, \cdots, r, j=n-r+1, \cdots, n, i=1, \cdots, a\}.$$

Let ε_1 be a positive number determined later on and $\theta'' = (\theta_{n-r+1}^j, \cdots, \theta_n^j)$. If $\theta_{n+1}=0$, we choose $\rho \in C_0^\infty(R^1)$ so that $\mathrm{supp}\,\rho \subset \{y \in R^1; |y|<\varepsilon_1\}$, $\rho(0)=1$ and $(D_y^\nu)(0)=0$ for $\nu \geqslant 1$. If $\theta_{n+1}>0$, we choose $\rho \in C_0^\infty(R^1)$ so that $\overline{R_+^1} \cap \mathrm{supp}(D_y^\nu \rho) \subset \{y \in R^1; |y-\theta_{n+1}|<\varepsilon_1\}$ for $\nu \geqq 1$ and $\rho(0)=1$. In the latter case, we choose ε_1 so small that $\theta_{n+1}-\varepsilon_1>0$, and then $(D_y^\nu \rho)(0)=0$ for $\nu \geqq 1$. Let ε_2 be some positive number determined later on such that

$$(2.12) \qquad \varepsilon_2^2 - 2[(n-r)(r+1)m^2 + \textstyle\sum_{j=n-r+1}^{n+1} |\theta_j|^2]\varepsilon_1^2 = 2c > 0.$$

Then using the inequality:

3) The notation: $\omega_1 \subset\subset W$, means that the closure of ω_1 is compact and contained in W.

$$|a+b|^2>2^{-1}|a|^2-|b|^2,$$

and the formula:

$$\theta_h+\sum_{j=n-r+1}^{n}\partial\mu_j/\partial\xi_h(\xi_0')\theta_j+\partial\lambda/\partial\xi_h(\xi_0')\theta_{n+1}=0, \qquad h=1,\cdots,r,$$

we have

$$\begin{aligned}&\sum_{h=1}^{n-r}(x_h+\sum_{j=n-r+1}^{n}\partial\mu_j/\partial\xi_h(\xi')x_j+\partial\lambda/\partial\xi_h(\xi')y)^2\\ &\geqq 2^{-1}|(x,y)|^2\{\sum_{h=1}^{n-r}|x_h/|(x,y)-\theta_h|^2-|\cdots|\}, \qquad \xi'\in\omega_1.\end{aligned}$$

Here

$$\begin{aligned}|\cdots|^2\leqq 2\sum_{h=1}^{n-r}\{&\sum_{j=n-r+1}^{n}|\partial\mu_j/\partial\xi_h(\xi')|^2|x_j/|(x,y)|-\theta_j|^2\\ &+\sum_{j=n-r+1}^{n}|\partial\mu_j/\partial\xi_h(\xi')-\partial\mu_j/\partial\xi_h(\xi_0')|^2|\theta_j|^2\\ &+|\partial\lambda(\xi')/\partial\xi_h|^2|y/|(x,y)|-\theta_{n+1}|^2+|\partial\lambda/\partial\xi_h(\xi')-\partial\lambda/\partial\xi_h(\xi_0')|^2|\theta_{n+1}|^2\}.\end{aligned}$$

If we choose ω $(\subset\subset\omega_1)$ so that when $\xi'\in\omega$

$$\begin{aligned}|\partial\lambda/\partial\xi_h(\xi')-\partial\lambda/\partial\xi_h(\xi_0')|<\varepsilon_1,\ |\partial\mu_j/\partial\xi_h(\xi')-\partial\mu_j/\partial\xi_h(\xi_0')|<\varepsilon_1,\\ j=n-r+1,\cdots,n,\end{aligned}$$

and put

$$\begin{aligned}V''&=\{(x'',y)\in R^{r+1};\ |(x'',y)/|(x,y)|-(\theta',\theta_{n+1})|<\varepsilon_1\}\\ V&=\{(x,y)\in R^{n+1}|;\ |x'/|(x,y)|-\theta'|<\varepsilon_2,\ (x'',y)\in V''\},\end{aligned}$$

we have

$$|\cdots|^2\leqq 2(n-r)((r+1)m^2+\sum_{j=n-r+1}^{n+1}|\theta_j|^2)\varepsilon_1^2, \qquad (x'',y)\in V''.$$

Thus, we have, using (2.12),

$$\sum_{h=1}^{n-r}(x_h+\sum_{j=n-r+1}^{n}(\partial\mu_j/\partial\xi_h)(\xi')x_j+(\partial\lambda/\partial\xi_h(\xi')y)^2\geqq c|(x,y)|^2,$$

when $(x'',y)\in V''$, $(x,y)\notin V$ and $\xi'\in\omega$. It is obvious that V is a conic neighborhood of θ in R^{r+1}. If we choose ε_1 and ε_2 so that ε_1 and ε_2 satisfy (2.12) and $0<\varepsilon_1,\ \varepsilon_2<\varepsilon/2\sqrt{n+1}$, then

$$\int\exp\{-i(x'\cdot\xi'+x''\cdot\mu(\xi')+y\cdot\lambda(\xi'))\}\chi(\xi')d\xi'(D_{x''}^{\beta}\sigma)(x''/R)(D_y^{\gamma}\rho)(y/R)$$

is rapidly decreasing for $\gamma\geqq 1$, $\chi\in C_0^\infty(\omega)$ and $\sigma\in C_0^\infty(\{x''\in \boldsymbol{R}^r;\ |x''-\theta''|<\varepsilon_1\})$ when $|(x,y)/R-\theta|>\varepsilon/2$. When $r\geqq 1$, let σ be a $C_0^\infty(\boldsymbol{R}^r)$ function such that $\operatorname{supp}\sigma\subset\{x''\in\boldsymbol{R}^r;\ |x''-\theta''|<\varepsilon_1\}$ and

$$(1/2\pi)^r\int(-x'')^{\alpha}\sigma(x'')dx''=1, \tag{2.13}$$

(2.14) $$(1/2\pi)^r \int (-x'')^\beta \sigma(x'')dx'' = 0, \qquad \beta \neq \alpha,\ |\beta| \leqq s.$$

Repeating integration by parts, we have

(2.15) $$\begin{aligned} 0 &= \langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y),\ \chi(\xi')\rho(y/R)(-iy)^\nu e^{-i\lambda(\xi')y}\rangle \\ &= \textstyle\sum_{j=1}^{d} \sum_{k=0}^{j-1} i\langle b_j(\xi') \dfrac{(j-1-k)!}{(j-1-k-\nu)!} \lambda(\xi')^{j-1-k-\nu} D_y^k Q(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0},\ \chi(\xi')\rangle \\ &\quad + \langle Q(\xi', D_y)\hat{v}(\xi', y),\ \textstyle\prod_{h=1}^{a}(-D_y - \lambda_h^0(\xi'))^{\delta_h}[\chi(\xi')\rho(y/R)(-iy)^\nu e^{-i\lambda(\xi')y}]\rangle . \end{aligned}$$

Since

$$\begin{aligned} &\textstyle\prod_{h=1}^{a}(-D_y - \lambda_h^0(\xi'))^{\delta_h}[\chi(\xi')\rho(y/R)e^{-i\lambda(\xi')y}(-iy)^\nu] \\ &\quad = \chi(\xi') \textstyle\prod_{h\neq j}(-D_y - \lambda_h^0(\xi'))^{\delta_h}[R^{-\delta+\nu} \sum_{0\leqq l\leqq \nu} c_l(-iy/R)^{\nu-l}(D_y^{\delta-l}\rho) \\ &\quad\quad \cdot(y/R)(1/R)^{\nu-l} e^{-i\lambda(\xi')y}] \end{aligned}$$

and

$$(D^{\delta-l}\rho)(0) = 0\,, \qquad \delta - l \geqq 1\,,$$

where c_l is some constant, we have

(2.16) $$\begin{aligned} &\langle Q(\xi', D_y)\hat{v}(\xi', y),\ \textstyle\prod_{h=1}^{a}(-D_y - \lambda_h^0(\xi'))^{\delta_h}[\chi(\xi')\rho(y/R)(-iy)^\nu e^{-i\lambda(\xi')y}]\rangle \\ &\quad = \langle \hat{v}(\xi', y),\ P^0(\xi', -D_y)P^+(\xi', -D_y)[\chi(\xi')\rho(y/R)(-iy)^\nu e^{-i\lambda(\xi')y}]\rangle . \end{aligned}$$

When $r \geqq 1$, noting that $\hat{v} = v_\alpha$ and writing

$$\begin{aligned} \hat{\sigma}(\xi'') &= (1/2\pi)^r \int e^{-ix''\xi''}\sigma(x'')dx''\,, \\ P^0(\xi', -D_y)P^+(\xi', -D_y) &= T(\xi', -D_y)\,, \\ (\partial/\partial\lambda)^j T(\xi', \lambda) &= T^{(j)}(\xi', \lambda)\,, \end{aligned}$$

we have, using (2.13) and (2.14),

$$\begin{aligned} &\langle \hat{v}(\xi', y),\ T(\xi', -D_y)\chi(\xi')e^{-i\lambda(\xi')y}(-iy)^\nu\rho(y/R)\rangle \cdot R^{|\alpha|} \\ &\quad = \textstyle\sum_{\gamma\geqq\alpha_j}(\gamma!)^{-1}\langle v_\alpha(\xi', y),\ T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi')D_y^\gamma[(-iy)^\nu\rho(y/R)e^{-i\lambda(\xi')y}]\rangle \cdot R^{|\alpha|} \\ &\quad = \textstyle\sum_{\gamma\geqq\alpha_j}(\gamma!)^{-1}\sum_{0\leqq l\leqq\nu}\langle v_\alpha(\xi', y),\ c_l T^{(\gamma)}(\xi', \lambda(\xi'))\,\chi(\xi')\left(\dfrac{-iy}{R}\right)^{\nu-l}(D_y^{\gamma-l}) \\ &\quad\quad \cdot(y/R)(1/R)^{\gamma-\nu}e^{-i\lambda(\xi')y}\rangle \cdot R^{|\alpha|} \\ &\quad = \textstyle\sum_{|\beta|\leqq s}\sum_{\gamma\geqq\alpha_j}(\gamma!)^{-1}\sum_{0\leqq l\leqq\nu}\langle v_\beta(\xi', y)\otimes D^\beta\delta,\ c_l T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi')\cdot \\ &\quad\quad \cdot(-iy/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)(1/R)^{\gamma-\nu}e^{-i\lambda(\xi')y}\hat{\sigma}(-R\xi'')\rangle \end{aligned}$$

$$= \sum_{\gamma \geqq \alpha_j} (\gamma !)^{-1} \sum_{0<l<\nu} c_l \langle \phi(\xi)\phi_1(\xi)\hat{u}(\xi, y), T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi')\hat{\sigma}(-R(\xi'' - \mu(\xi'))e^{-i\lambda(\xi')y}(-iy/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)(1/R)^{\gamma-\nu}\rangle = 0\,,$$

where c_l are some constants. Therefore, since $|\alpha| \geqq 0$, it is sufficient to show that

$$\text{(2.17)} \quad \varliminf_{R\to\infty} \langle \phi(\xi)\phi_1(\xi)\hat{u}(\xi, y), T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi')\sigma(-R(\xi''-\mu(\xi'))e^{-i\lambda(\xi')y} \cdot(-iy/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)(1/R)^{\gamma-\nu}\rangle = 0\,.$$

When $r=0$, it is sufficient to show that

$$\text{(2.17)}' \quad \varliminf_{R\to\infty} \langle \phi(\xi)\hat{u}(\xi, y), T^{(\gamma)}(\xi, \lambda(\xi))\chi(\xi)e^{-i\lambda(\xi)y}(-iy/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)(1/R)^{\gamma-\nu}\rangle = 0.$$

Note that $\gamma-l\geqq 1$, $\gamma-l\geqq\alpha_j-\nu\geqq\alpha_j-(\delta_j-1)\geqq 1$. Let us show (2.17) and (2.17)$'$. We denote by $\varliminf_{R\to\infty} I$ the left hand-side of (2.17) and (2.17)$'$. Put

$$(-iy)^{\nu-l}(D_y^{\gamma-l}\rho)(y) = \tilde{\rho}(y)$$

$$\bar{\mathcal{F}}[\tilde{\rho}](\lambda) = (1/2\pi)\int_{-\infty}^{+\infty} \exp\{i\lambda y\}\tilde{\rho}(y)dy\,,$$

$$u_0(x, y) = \begin{cases} \mathcal{F}_\xi^{-1}[\phi\hat{u}](x, y)^{4)} & y\geqq 0 \\ 0 & y<0 \end{cases}$$

where $\phi=\phi(\xi)\phi_1(\xi)$ when $1\leqq r\leqq n$. Define v by $\mathcal{F}[v]^{5)}=\phi(\xi)\psi(\xi, \lambda)\mathcal{F}[u_0](\xi, \lambda)$, where $\psi\in C_0^\infty(\Xi^{n+1})$ is equal to 1 in a neighborhood of $(\xi_0', \mu(\xi_0'), \lambda_j^0(\xi_0'))$. Put $I=I_1+I_2$ where

$$I_1 = (1/R)^{\gamma-\nu-1}\langle \phi(\xi)\psi(\xi, \lambda)\mathcal{F}[u_0](\xi, \lambda), T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi') \cdot\hat{\sigma}(-R(\xi''-\mu(\xi'))\bar{\mathcal{F}}[\tilde{\rho}](R(\lambda-\lambda(\xi'))\rangle$$

$$I_2 = (1/R)^{\gamma-\nu-1}\langle \phi(\xi)(1-\psi(\xi, \lambda))\mathcal{F}[u_0](\xi, \lambda), T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi') \cdot\hat{\sigma}-R(\xi''-\mu(\xi'))\bar{\mathcal{F}}[\tilde{\rho}]R(\lambda-\lambda(\xi'))\rangle\,.$$

Since we can choose $\omega(\subset\subset\omega_1)$ and ψ so that there exists a positive constant c such that

$$|\lambda-\lambda(\xi')|\geqq c \quad \text{when} \quad (\xi, \lambda)\in \text{supp}[1-\psi(\xi, \lambda)]\cap\omega\times\Xi_{\xi'',\lambda}^{r+1}\,,$$

we obtain, using the fact that $\bar{\mathcal{F}}[\tilde{\rho}]\in\mathcal{S}$,

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4) $\mathcal{F}_\xi^{-1}[\phi\hat{u}](x, y)=(1/2\pi)^n\int e^{ix\cdot\xi}\phi(\xi)\hat{u}(\xi, y)d\xi.$

5) $\mathcal{F}[v](\xi, \lambda)=\int_{R^n}\int_{-\infty}^{+\infty} e^{-i(x\cdot\xi+y\cdot\lambda)}v(x, y)dxdy.$

$$|D_\lambda^j \overline{\mathscr{F}}[\tilde{\rho}](R(\lambda-\lambda(\xi')))| \leqq C_N(1+CR(1+|\lambda|))^{-N}, \qquad j=0, 1, 2, \cdots,$$

for any N and some positive constants C_N and C when $(\xi, \lambda) \in \operatorname{supp}[1-\psi(\xi, \lambda)] \cap \omega \times \boldsymbol{\Xi}_{\xi'',\lambda}^{r+1}$. Therefore we have

$$\lim_{R\to\infty} I_2 = 0 .$$

On the other hand, since $\phi(\xi)\psi(\xi, \lambda)\mathscr{F}[u_0](\xi, \lambda) \in \mathscr{E}'(\boldsymbol{\Xi}^{n+1})$, we have

$$|v(x, y)| \leqq C(1+|x|+|y|)^k , \tag{2.18}$$

for some positive constants C and k, and $v \in \boldsymbol{L}^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}})$. Setting

$$J(x, y) = \int \exp\{-i(x'\cdot\xi' + x''\cdot\mu(\xi') + y\cdot\lambda(\xi'))\} T^{(\gamma)}(\xi', \mu(\xi'))\chi(\xi')d\xi' ,$$

we have

$$\iint T^{(\gamma)}(\xi', \lambda(\xi'))\chi(\xi')\hat{\sigma}(-R(\xi''-\mu(\xi')))\overline{\mathscr{F}}[\tilde{\rho}](R(\lambda-\lambda(\xi')))e^{-i(x\cdot\xi+y\cdot\lambda)}d\xi d\lambda$$
$$= R^{-(r+1)}J(x, y)(-iy/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)\sigma(x'/R) .$$

We obtain, using Cauchy-Schwarz's inequality,

$$\begin{aligned} |I_1| &< (1/R)^{(\gamma-\nu)+r}\Big(\iint_{|(x,y)/R-\theta|<\varepsilon/2} |v(x, y)|^2 dxdy\Big)^{1/2} \\ &\quad \cdot\Big(\iint_{|(x,y)/R-\theta|<\varepsilon/2} |J(x, y)|^2 |(y/R)^{\nu-l}(D_y^{\gamma-l})\rho(y/R)\sigma(x''/R)|^2 dxdy\Big)^{1/2} \\ &\quad + (1/R)^{\gamma-\nu} \iint_{|(x,y)/R-\theta|\geqq\varepsilon/2} |v(x, y)|\,|J(x, y)(y/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)\sigma(x''/R)|\,dxdy \\ &= I_3(R)+I_4(R) . \end{aligned} \tag{2.19}$$

It follows from Theorem A_p-1 and Lemma A_p-2 that

$$\iint_{|(x,y)/R-\theta|<2\varepsilon} |J(x, y)|^2 |(y/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)\sigma(x''/R)|^2 dxdy \leqq CR^{r+1} ,$$

$$\varliminf_{R\to\infty} R^{-k} \iint_{|(x,y)/R-\theta|<\varepsilon/2} |v(x, y)|^2 dxdy \leqq C \varliminf_{R\to\infty} R^{-k} \iint_{\substack{|(x,y)/R-\theta|<\varepsilon \\ y\geqq 0}} |u(x, y)|^2 dxdy$$

for every $k \in \boldsymbol{R}^1$. Hence we have, using the fact: $\gamma-\nu \geqq \alpha_j - \delta_j + 1$,

$$I_3(R) \leqq C\Big\{R^{-(2(\alpha_j-\delta_j)+r+1)} \iint_{\substack{|(x,y)/R-\theta|<\varepsilon/2 \\ y\geqq 0}} |v(x, y)|^2 dxdy\Big\}^{1/2} ,$$

and then it follows from (2.10) that $\lim_{R\to\infty} I_3(R)=0$. On the other hand, using (2.18), we have

$$|v(x,y)J(x,y)(-iy/R)^{\nu-l}(D_y^{\gamma-l}\rho)(y/R)\sigma(x''/R)| \leqq CR^{-1}(1+|x|+|y|)^{-(n+2)}$$

when $|(x,y)/R-\theta| \geqq \varepsilon/2$. Therefore we have $\lim_{R\to\infty} I(R)=0$. Q.E.D.

For later reference, we now present some facts concerning elementary algebra.

Lemma 2.8. 1) *Put* $f_j(\lambda)=a_m\lambda^j+a_{m-1}\lambda^{j-1}+\cdots+a_{m-j}$ *and* $f_j^{\alpha}(\lambda)=(d/d\lambda)^{\alpha}f_j$. *Then we have*

$$(2.20)\quad \begin{vmatrix} f_0(\lambda_1), & f_1(\lambda_1), & \cdots\cdots, & f_{q-1}(\lambda_1) \\ \vdots & \vdots & & \vdots \\ f_0^{\alpha_1-1}(\lambda_1), & f_1^{\alpha_1-1}(\lambda_1), & \cdots, & f_{q-1}^{\alpha_1-1}(\lambda_1) \\ \vdots & \vdots & & \vdots \\ f_0(\lambda_k), & f_1(\lambda_k), & \cdots\cdots, & f_{q-1}(\lambda_k) \\ \vdots & \vdots & & \vdots \\ f_0^{\alpha_k-1}(\lambda_k), & f_1^{\alpha_k-1}(\lambda_k), & \cdots, & f_{q-1}^{\alpha_k-1}(\lambda_k) \end{vmatrix}$$

$$= (-1)^{\sum_{i=1}^k \sum_{j=i+1}^k \alpha_j\alpha_i} \{\prod_{j=1}^q (j-1)!/\prod_{i=2}^k (\alpha_i \prod_{j=1}^{i-1} \alpha_j)!\}$$
$$\cdot a_m^q \prod_{i=1}^k \prod_{j=i+1}^k (\lambda_i-\lambda_j)^{\alpha_j\alpha_i}.$$

Here $q=\sum_{i=1}^k \alpha_i$, $0<q\leqq m$ *and* $a_m\neq 0$.

2) (*cf. Hörmander* [5]) *Let* $k(\lambda)=\lambda^{\mu}+a_{\mu-1}\lambda^{\mu-1}+\cdots+a_0$ *be a polynomial with constant coefficients of order* μ. *Let* $q_\nu(\lambda)$, $\nu=1,\cdots,\mu$, *be some other polynomials with constant coefficients. Let* λ_j, $j=1,\cdots,l$, *be the roots of the equation* $k(\lambda)=0$ *with multiplicity* α_j. *Then we have*

$$(2.21)\quad \det((2\pi i)^{-1}\oint q_\nu(\lambda)\lambda^{\sigma-1}(k(\lambda))^{-1}d\lambda)_{\nu,\sigma=1,\cdots,\mu}$$

$$= \frac{\begin{vmatrix} q_1(\lambda_1), & q_1'(\lambda_1), & q_1^{(\alpha_1-1)}(\lambda_1), & \cdots, & q_1(\lambda_l), & \cdots, & q_1^{(\alpha_l-1)}(\lambda_l) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ q_\mu(\lambda_1), & q_\mu'(\lambda_1), & q_\mu^{(\alpha_1-1)}(\lambda_1), & \cdots, & q_\mu(\lambda_l), & \cdots, & q_\mu^{(\alpha_l-1)}(\lambda_l) \end{vmatrix}}{\prod_{j=1}^l \prod_{s<\alpha_j} s! \prod_{1\leqq k<j\leqq l}(\lambda_j-\lambda_k)^{\alpha_j\alpha_k}}$$

Lemma 2.9. *Let* $k(\lambda)$, $q_\nu(\lambda)$, $\nu=1,\cdots,\mu$ *be as in Lemma* 2.8. *Assume that*

$$(2.22)\quad \det((2\pi i)^{-1}\oint q_\nu(\lambda)\lambda^{\sigma-1}(k(\lambda))^{-1}d\lambda)_{\nu,\sigma=1,\cdots,\mu}\neq 0.$$

Put $q_\nu(\lambda)=Q_\nu(\lambda)k(\lambda)+q_\nu'(\lambda)$, $\nu=1,\cdots,\mu$, *where* $\deg q_\nu'(\lambda)\leqq\mu-1$. *Then the system*

$$[\{\lambda^{\nu-1}k(\lambda),\ \nu=1,\cdots,r\},\ \{q_\nu'(\lambda),\ \nu=1,\cdots,\mu\}]$$

forms a base of polynomials of order less than $r+\mu$ *for any* r (>0).

Proof. It is sufficient to show that $[\{\lambda^{\nu-1}k(\lambda),\ \nu=1,\cdots,r\},\ \{q'_\nu(\lambda),\ \nu=1,\cdots,\mu\}]$ are linearly independent over the complex number field C. Assume that

$$\sum_{j=1}^{r}\alpha_j\lambda^{j-1}k(\lambda)+\sum_{j=1}^{\mu}\beta_j q'_j(\lambda)\equiv 0$$

for some $\alpha_j,\ \beta_j\in C$. Then we have

$$\sum_{j=1}^{\mu}\beta_j(2\pi i)^{-1}\oint q_j(\lambda)\lambda^{l-1}(k(\lambda))^{-1}d\lambda=0\,,\qquad l=1,\cdots,\mu\,.$$

From (2.22) it follows that $\beta_j=0$ for all j. Thus we have

$$\left(\sum_{j=1}^{r}\alpha_j\lambda^{j-1}\right)k(\lambda)\equiv 0\,.$$

Since $k(\lambda)\not\equiv 0$, we have that $\alpha_j=0$ for all j. Q.E.D.

Lemma 2.10 (*Green's formula*). *Let* U *be an open set in* Ξ^k *and* U' *be any open subset of* U *such that* $U'\subset\subset U$. *Let* $\lambda_j(\xi)$, $j=1,\cdots,a$, *be* C^∞ *functions in* U *such that* $\lambda_j(\xi)\neq\lambda_{j'}(\xi)$ *if* $j\neq j'$ *when* $\xi\in U$ *and let* α_j, $j=1,\cdots,a$, *be natural numbers. Set*

$$P(\xi,\lambda)=\prod_{j=1}^{a}(\lambda-\lambda_j(\xi))^{\alpha_j}=\sum_{j=0}^{m}p_j(\xi)\lambda^j\,,\qquad \xi\in U\,.$$

Let β_j, $j=1,\cdots,a$, *be non-negative integers such that* $\beta_j\leqq\alpha_j$, $j=1,\cdots,a$. *Put*

$$Q(\xi,\lambda)=\prod_{j=1}^{a}(\lambda-\lambda_j(\xi))^{\alpha_j-\beta_j},\ m'=\sum_{j=1}^{a}\alpha_j-\beta_j\,,$$
$$\prod_{j=1}^{a}(\lambda-\lambda_j(\xi))^{\beta_j}=\sum_{j=0}^{m-m'}q_j(\xi)\lambda^j\,.$$

Let $B_j(\xi,\lambda)$, $j=1,\cdots,m'$, *be polynomials in* λ *of order* r_j *whose coefficients are* C^∞ *functions of* ξ *in* U. *If*

$$\det\left((2\pi i)^{-1}\oint B_j(\xi,\lambda)\lambda^{k-1}(Q(\xi,\lambda))^{-1}d\lambda\right)_{j,k=1,\cdots,m'}\neq 0\ \ in\ \ U\,,$$

then there exist ordinary differential operators $C_j(\xi,D_y)$, $j=0,\cdots,t-m$, $E_j(\xi,D_y)$, $j=1,\cdots,m'$, $F_{\nu,j}(\xi,D_y)$, $\nu=0,\cdots,\beta_j-1$, $j=1,\cdots,a$, *whose coefficients are* C^∞ *functions of* ξ *in* U' *such that*

$$\begin{aligned}(2.23)\quad &\langle P(\xi,D_y)u(\xi,y),\ v(\xi,y)\rangle-\langle u(\xi,y),\ P(\xi,-D_y)v(\xi,y)\rangle\\ &=\sum_{j=0}^{t-m}\langle D_y^jP(\xi,D_y)u(\xi,y)|_{y=0},\ C_j(\xi,D_y)v(\xi,y)|_{y=0}\rangle\\ &\quad+\sum_{j=1}^{m'}\langle B_j(\xi,D_y)u(\xi,y)|_{y=0},\ E_j(\xi,D_y)v(\xi,y)|_{y=0}\rangle\\ &\quad+\sum_{j=1}^{a}\sum_{\nu=0}^{\beta_j-1}\Big\langle\sum_{i=0}^{m-m'}\sum_{k=0}^{i-1}\frac{(i-1-k)!}{(i-1-k-\nu)!}q_i(\xi)\lambda_j(\xi)^{i-1-k-\nu}\\ &\quad\cdot D_y^kQ(\xi,D_y)u(\xi,y)|_{y=0},\ F_{\nu,j}(\xi,D_y)v(\xi,y)|_{y=0}\Big\rangle\end{aligned}$$

for any $u(\xi, y)\in C^\infty([0, \delta); \mathcal{S}'(\Xi^k))$ *and* $v(\xi, y)\in\mathcal{S}_0(U'\times\overline{\boldsymbol{R}^1_+})$. *Here we interpret that* $\lambda_j(\xi)^{i-1-k-\nu}=0$ *if* $i-1-k-\nu<0$, $F_{\nu,j}(\xi, D_y)\equiv 0$ *if* $\beta_j=0$, $C_j(\xi, D_y)\equiv 0$ *if* $t<m$, $E_j(\xi, D_y)\equiv 0$ *if* $m'=0$, *and* $t=\mathrm{Max}\,\{r_j;\ 1\leqq j\leqq m'\}$.

Proof. Since $\overline{U'}$ is compact, it follows from Lemmas 2.8 and 2.9 that there exist C^∞ functions in U': $a^l_{\nu,j}(\xi)$, $l=1,\cdots,m-m'$, $\nu=0,\cdots,\beta_j-1$, $j=1,\cdots,a$, such that

$$(2.24)\quad \lambda^{l-1}=\sum_{j=1}^{a}\sum_{\nu=0}^{\beta_j-1}a^l_{\nu,j}(\xi)\Big(\sum_{i=0}^{m-m'}\sum_{k=0}^{i-1}\frac{(i-1-k)!}{(i-1-k-\nu)!}q_i(\xi)\lambda_j(\xi)^{i-1-k-\nu}\lambda^k\Big)$$
$$\xi\in U',\ l=1,\cdots,m-m'.$$

(2.24) and repeated integrations by parts show that there exist ordinary differential operators $G_{\lambda,j}(\xi, D_y)$, $\nu=0,\cdots,\beta_j-1$ $j=1,\cdots,a$, whose coefficients are C^∞ functions of ξ in U' such that

$$(2.25)\quad \langle P(\xi, D_y)u(\xi, y), v(\xi, y)\rangle$$
$$=\sum_{j=1}^{a}\sum_{\nu=0}^{\beta_j-1}\Big\langle\sum_{i=0}^{m-m'}\sum_{k=0}^{i=1}\frac{(i-1-k)!}{(i-1-k-\nu)!}q_i(\xi)\lambda_j(\xi)^{i-1-k-\nu}\cdot$$
$$\cdot D^k_yQ(\xi, D_y)u(\xi, y)|_{y=0},\ G_{\nu,j}(\xi, D_y)v(\xi, y)|_{y=0}\Big\rangle$$
$$+\langle Q(\xi, D_y)u(\xi, y), \sum_{j=1}^{a}(-D_y-\lambda_j(\xi))^{\beta_j}v(\xi, y)\rangle.$$

We write, using Euclidean algorithm,

$$(2.26)\qquad B_j(\xi, \lambda)=R_j(\xi, \lambda)P(\xi, \lambda)+S_j(\xi, \lambda)Q(\xi, \lambda)+B'_j(\xi, \lambda),$$
$$j=1,\cdots,m',\ \xi\in U',$$

where $\deg R_j(\xi, \lambda)\leqq r-m$, $\deg S_j(\xi, \lambda)\leqq m-m'-1$, $\deg B'_j(\xi, \lambda)\leqq m'-1$. Since, by assumptions we have

$$\det\Big((2\pi i)^{-1}\oint B'_j(\xi, \lambda)\lambda^{k-l}(Q(\xi, \lambda))^{-1}d\lambda\Big)_{j,k=1,\cdots,m'}\neq 0,\qquad \xi\in\overline{U'},$$

it follows from Lemma 2.9 that there exist functions $b_{j,k}(\xi)$ $j, k=1,\cdots,m'$ which are infinitely differentiable in U' such that

$$(2.27)\qquad \lambda^{j-1}=\sum_{k=1}^{m'}b_{j,k}(\xi)B'_j(\xi, \lambda),\qquad j=1,\cdots,m',\ \xi\in U'.$$

Thus, repeating integrations by parts, we have from (2.27) that there exist ordinary differential operators $E_j(\xi, D_y)$, $j=1,\cdots,m'$, such that

$$\begin{aligned}&\langle Q(\xi, D_y)u(\xi, y), \textstyle\sum_{j=1}^{a}(-D_y-\lambda_j(\xi))^{\beta_j}v(\xi, y)\rangle \\ &\quad = \textstyle\sum_{j=1}^{m'}\langle B_j'(\xi, -D_y)u(\xi, y)|_{y=0}, E_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle \\ &\qquad + \langle u(\xi, y), P(\xi, D_y)v(\xi, y)\rangle .\end{aligned}$$

It follows from (2.26) that

$$\begin{aligned}&\textstyle\sum_{j=1}^{m'}\langle B_j'(\xi, D_y)u(\xi, y)|_{y=0}, E_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle \\ &\quad = \textstyle\sum_{j=1}^{m'}\langle [B_j(\xi, D_y)-R_j(\xi, D_y)P(\xi, D_y)-S_j(\xi, D_y)Q(\xi, D_y)] \\ &\qquad \cdot u(\xi, y)|_{y=0}, E_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle .\end{aligned}$$

Since $\deg R_j(\xi, \lambda)\leqq t-m$, there exist ordinary differential operators $C_j(\xi, D_y)$ $j=0, \cdots, t-m$ whose coefficients are C^∞ functions of ξ in U' such that

$$\begin{aligned}&-\textstyle\sum_{j=1}^{m'}\langle R_j(\xi, D_y)P(\xi, D_y)u(\xi, y)|_{y=0}, E_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle \\ &\quad = \textstyle\sum_{j=0}^{t-m}\langle D_y^jP(\xi, D_y)u(\xi, u)|_{y=0}, C_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle .\end{aligned}$$

Since $\deg S_j(\xi, \lambda)\leqslant m-m'-1$, it follows from (2.24) that there exist ordinary differential operator $H_{\nu,j}(\xi, D_y)$, $\nu=0, \cdots, \beta_j-1, j=1, \cdots, a$, whose coefficients are C^∞ functions of ξ in U' such that

$$\begin{aligned}&-\textstyle\sum_{j=1}^{m'}\langle S_j(\xi, D_y)Q(\xi, D_y)u(\xi, y)|_{y=0}, E_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle \\ &\quad = \textstyle\sum_{j=1}^{a}\sum_{\nu=0}^{\beta_j-1}\langle\sum_{i=0}^{m-m'}\sum_{k=0}^{i-1}\dfrac{(i-1-k)!}{(i-1-k-\nu)!}q_i(\xi)\lambda_j(\xi)^{i-1-k-\nu} \\ &\qquad \cdot D_y^kQ(\xi, D_y)u(\xi, y)|_{y=0}, H_{\nu,j}(\xi, D_y)v(\xi, y)|_{y=0}\rangle .\end{aligned}$$

Thus, we have

$$\begin{aligned}&\langle P(\xi, D_y)u(\xi, y), v(\xi, y)\rangle-\langle u(\xi, y), P(\xi, -D_y)v(\xi, y)\rangle \\ &\quad = \textstyle\sum_{j=0}^{t-m}\langle D_y^jP(\xi, D_y)u(\xi, y)|_{y=0}, C_j(\xi, D_y)v(\xi, y)|_{y=0}\rangle \\ &\qquad +\textstyle\sum_{j=1}^{a}\sum_{\nu=0}^{\beta_j-1}\langle\sum_{i=0}^{m-m'}\sum_{k=0}^{i-1}\dfrac{(i-1-k)!}{(i-1-k-\nu)!}q_i(\xi)\lambda_j(\xi)^{i-1-k-\nu} \\ &\qquad \circ D_y^kQ(\xi, D_y)u(\xi, y)|_{y=0}, (G_{\nu,j}(\xi, D_y)+H_{\nu,j}(\xi, D_y))v(\xi, y)|_{y=0}\rangle .\end{aligned}$$

This shows the lemma.

Lemma 2.11. *If the system* $\{P(\xi, \lambda); B_j(\xi, \lambda), j=1, \cdots, p\}$ *satisfies the conditions* (i) *and* (ii) *stated in Remark of Section* 1 *for each* $\xi\in W$, *then* $\hat{v}=0$. *Here* $\hat{v}$ *is as in Lemma* 2.5.

Proof. By assumption, we have $\tilde{a}+\tilde{b}\leqq p$ in W. Let $\sigma=(\sigma_1, \cdots, \sigma_{\tilde{a}+\tilde{b}})$ be any subset consisting of $\tilde{a}+\tilde{b}$ elements of $\{1, \cdots, p\}$. Put

$$L_\sigma(\xi') = \det\left((2\pi i)^{-1}\int_{\gamma(\xi')} B_{\sigma_j}(\xi', \lambda)\lambda^{k-1}(P^0(\xi', \lambda)P^+(\xi, \lambda))^{-1}d\lambda\right)_{j,k=1,\cdots,\tilde{a}+\tilde{b}}, \quad \xi' \in W,$$

where $\gamma(\xi')$ is a simple closed curve in the complex λ-plane which surrounds all $\lambda_j^0(\xi'), j=1, \cdots, a$ and $\lambda_j^+(\xi'), j=1, \cdots, b$. Since $L_\sigma(\xi')$ vanishes identically in W or a real analytic function of ξ' in W, putting $A_{L_\sigma}=\{\xi' \in W; L_\sigma(\xi')=0\}$, we have that $A_{L_\sigma}=W$, A_{L_σ} is a real analytic set in W or A_{L_σ} is empty. By assumption we have that $\bigcap_\sigma A_{L_\sigma}$ is empty where the intersection is taken over all $\sigma \subset \{1, \cdots, p\}$. Let $L_\sigma(\xi')$ not vanish identically in W and U be any openset whose closure is compact and contained in $W-A_{L_\sigma}$. It is sufficient to show that

$$\langle \hat{v}(\xi', y), w(\xi', y)\rangle = 0 \tag{2.28}$$

for any $w(\xi', y) \in \mathcal{S}_0(U \times \overline{\boldsymbol{R}_+^1})$. It follows from Lemma 2.10 that there exist ordinary differential operators $C_j(\xi', D_y), j=0, \cdots, t-(\tilde{a}+\tilde{b})$, $E_j(\xi', D_y), j=1, \cdots, \tilde{a}+\tilde{b}$, whose coefficients are C^∞ functions of ξ' in U such that

$$\begin{aligned} &\langle P^0(\xi', D_y)\underset{\sim}{P}^+(\xi', D_y)\hat{v}(\xi', y), w(\xi', y)\rangle \\ &= \textstyle\sum_{j=0}^{t-(\tilde{a}+\tilde{b})}\langle D_y^j P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y)|_{y=0}, C_j(\xi', D_y)w(\xi' y)|_{y=0}\rangle \\ &\quad + \textstyle\sum_{j=1}^{\tilde{a}+\tilde{b}}\langle B_{\sigma_j}(\xi', D_y)\hat{v}(\xi', y)|_{y=0}, E_j(\xi', D_y)w(\xi', y)|_{y=0}\rangle \\ &\quad + \langle \hat{v}(\xi', y), P^0(\xi', -D_y)P^+(\xi', -D_y)w(\xi', y)\rangle \end{aligned} \tag{2.29}$$

for any $w(\xi', y) \in \mathcal{S}_0(U \times \overline{\boldsymbol{R}_+^1})$. Here $t = \mathrm{Max}\{r_{\sigma_j}; 1 \leqq j \leqq \tilde{a}+\tilde{b}\}$. Since it follows from (1.2), (2.4) and Lemma 2.5 that

$$\begin{aligned} &\langle P^0(\xi' D_y)P^+(\xi', D_y)\hat{v}(\xi', y), w(\xi', y)\rangle \\ &= \langle \hat{v}(\xi', y), P^0(\xi', -D_y)P^+(\xi', -D_y)w(\xi', y)\rangle \end{aligned}$$

for any $w(\xi', y) \in \mathcal{S}_0(U \times \overline{\boldsymbol{R}_+^1})$, (2.28) follows from Lemma 2.6.

Now we shall prove the assertion: *there exist an open cone Γ, natural number N and real analytic set B contained in W such that if $u \in \boldsymbol{L}^2_{\mathrm{loc}}(\overline{\boldsymbol{R}_+^{n+1}}) \cap C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))$ satisfies the condition*:

$$\lim_{R\to\infty} R^{-N}\int_{\Gamma_R} |u(x, y)|^2 dx dy = 0,$$

then the support of $\hat{v}(\xi', y)$ is contained in $B \times \overline{\boldsymbol{R}_+^1}$, in the following cases.

Case 1. $\tilde{a}=0$ and $\tilde{b}>0$ in W.

Since we have $\tilde{b} \leqq p$ by the assumption (A−1), we put

$$L_\sigma(\xi') = \det((2\pi i)^{-1} \oint B_{\sigma_j}(\xi', \lambda)\lambda^{k-1}(P^+(\xi', \lambda))^{-1}d\lambda)_{j,k=1,\cdots,\tilde{b}}, \qquad \xi' \in W,$$
$$A_\sigma = \{\xi' \in W; L_\sigma(\xi') = 0\}$$

for $\sigma = (\sigma_1, \cdots, \sigma_{\tilde{b}}) \subset \{1, \cdots, p\}$. Put $B = \bigcap_\sigma A$ where the intersection is taken over all $\sigma \subset \{1, \cdots, p\}$. When $r = 0$, we have that B is empty or a real analytic set in W by the assumption (A-2). But when $1 \leqq r \leqq n$, one of the following three cases may occur: (I) B is empty, (II) B is a real analytic set in W, (III) $B = W$. For the case (III) we have the following by Theorem A_p-3.

Lemma 2.12. *Let u be as in the first place of this section and $\hat{v}$ be as in Lemma 2.5. Assume that the hypotheses of Case 1 are fulfilled in W, $1 \leqq r \leqq n$ and $W = B$ (Case III). Set*

$$\Gamma_R^{(III)} = \{(x, y) \in \Gamma^{(III)}; y \geqq 0, R < |(x, y)| < 2R\},$$

where $\Gamma^{(III)}$ is an open cone in $\boldsymbol{R}^{n+1}$ which contains $(n(\xi'), 0)$ for every $\xi' \in W$. Here $n(\xi')$ denotes some normal of $\{(\xi', \mu(\xi')); \xi' \in W\}$ at $(\xi', \mu(\xi'))$. If u belongs to $L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+}) \cap C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))$ and satisfies the condition:

$$\varliminf_{R\to\infty} R^{-r} \int_{\Gamma_R^{(III)}} |u(x, y)|^2 dxdy = 0$$

then $\hat{v} = 0$. Here r is the codimension of $\{(\xi', \mu(\xi')); \xi' \in W\}$ in Ξ^n.

Proof. Put $u_0 = u$ when $y \geqq 0$ and $u_0 = 0$ when $y < 0$. Since the support of $\phi(\xi)\phi_1(\xi)\hat{u}_0(\xi, y)$ is contained in $\{(\xi', \mu(\xi')); \xi' \in W\} \times \overline{\boldsymbol{R}^1_+}$, the support of $\phi(\xi)\phi_1(\xi)\mathcal{F}[u_0](\xi, \lambda)$ is contained in $\{(\xi'\mu(\xi')); \xi' \in W\} \times \Xi^1_\lambda$. Let $\rho(\lambda)$ be any $C_0^\infty(\Xi^1_\lambda)$ function. It follows from Lemma A_p-2

$$\varliminf_{R\to\infty} R^{-N} \int_{\Gamma'_R} |\mathcal{F}^{-1}[\phi\phi_1\mathcal{F}[u_0]](x, y)|^2 dxdy^{6)}$$
$$\leqq C \varliminf_{R\to\infty} R^{-N} \int_{\Gamma_R^{(III)}} |u(x, y)|^2 dxdy,$$

where $\Gamma'_R = \{(x, y) \in \Gamma'; R < |(x, y)| < 2R\}$ and Γ' is another cone which satisfies the same condition as $\Gamma^{(III)}$. Then we have, using Theorem A_p-3 that $\phi(\xi)\phi_1(\xi)\rho(\lambda)\mathcal{F}[u_0](\xi, \lambda) = 0$. This shows that $\phi(\xi)\phi_1(\xi)\hat{u}(\xi, y) = 0$. Therefore, we have $\hat{v} = 0$.

Proposition 2.13. *Suppose that the hypotheses of Case 1 are fulfilled,*

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6) $\mathcal{F}^{-1}$ denotes the inverse Fourier transform with respect to (ξ, λ).

$1 \leqq r \leqq n$ and $W=B$. Let Γ be an open cone in $\boldsymbol{R}^{n+1}$ and N an integer such that if $w \in C^{\infty}(\overline{\boldsymbol{R}_{+}^{n+1}}) \cap \mathcal{S}'(\overline{\boldsymbol{R}_{+}^{n+1}})$ satisfies the equations (1.1) *and* (1.2) *and the support of $\hat{w}(\xi, y)$ is contained in $\{(\xi', \mu(\xi')); \xi' \in W\} \times \overline{\boldsymbol{R}_{+}^{1}}$ with*

$$\varliminf_{R \to \infty} R^{-N} \int_{\Gamma_R} |w(x, y)|^2 dxdy = 0 \tag{2.30}$$

then $w=0$. Then the closure Γ contains $(n(\xi'), 0)$ for each $\xi' \in W$ and $N \leqq r$. Here $n(\xi')$ is as in Lemma 2.12.

Proof. Since $W=B$, $L_{\sigma}(\xi')$ vanishes identically in W for all $\sigma \subset \{1, \cdots, p\}$. So we have, using Lemma 2.8, that there exist C^{∞} functions $C_{j\nu}(\xi')$, $\nu = 0, \cdots, \beta_j - 1, j = 1, \cdots, b$, of ξ' in W such that if we put

$$\hat{w}(\xi', y) = [\textstyle\sum_{j=1}^{b} \sum_{\nu=0}^{\beta_j-1} C_{j\nu}(\xi')(iy)^{\nu} e^{i\lambda_j^+(\xi')y}]\psi(\xi'),$$

we have that $\hat{w}(\xi', y)$ does not vanish identically in $W \times \boldsymbol{R}_{+}^{1}$ and satisfies the equations:

$$P(\xi', \mu(\xi'), D_y)\hat{w}(\xi', y) = 0, \qquad y>0,$$
$$B_j(\xi', \mu(\xi'), D_y)w(\xi', y)|_{y=0} = 0, \qquad j = 1, \cdots, p.$$

Here $\psi(\xi') \in C_0^{\infty}(W)$. Hence, putting

$$w(x, y) = \int \exp\{i(x' \cdot \xi' + x'' \cdot \mu(\xi'))\} \hat{w}(\xi', y) d\xi',$$

we have that $w(x, y)$ does not vanish identically in $\boldsymbol{R}_{+}^{n+1}$, belongs to $\mathcal{S}'(\overline{\boldsymbol{R}_{+}^{n+1}}) \cap C^{\infty}(\overline{R_{+}^{n+1}})$ and satisfies the equations (1.1) and (1.2). (2.30) follows from Theorem A_p-1 if $N > r$, which gives a contradiction. If $\bar{\Gamma}$ contains no $(n(\xi_0'), 0)$ for some ξ_0' and if supp ψ is sufficiently close to ξ_0' the condition (2.30) follows from Theorem A_p-1 for any N, which gives a contradiction.

Lemma 2.14. *Let u be as in the first place of this section and $\hat{v}$ be as in Lemma* 2.5. *Assume that the hypotheses of Case* 1 *are fulfilled in W.* (1) *If B is empty, then $\hat{v}=0$.* (2) *Suppose that B is a real analytic set in W. If u belongs to $C^{\infty}([0, \delta); \mathcal{S}'(\boldsymbol{R}^n)) \cap \boldsymbol{L}_{\mathrm{loc}}^2(\overline{\boldsymbol{R}_{+}^{n+1}})$ and satisfies the condition*;

$$\varliminf_{R \to \infty} R^{-(N_r+r)} \int_{\Gamma_R^{(II)}} |u(x, y)|^2 dxdy = 0,$$

then $\hat{v}=0$. Here $\Gamma^{(II)}$ is an open cone in $\boldsymbol{R}^{n+1}$ which contains $(n(\xi'), 0)$ for every analytic manifold $C \subset B$ and $\xi' \in C$, when $n(\xi')$ denotes some normal of $\{(\xi', \mu(\xi')); \xi' \in C\}$ at $(\xi', \mu(\xi'))$, $\Gamma_R^{(II)} = \{(x, y) \in \Gamma^{(II)}; y \geqq 0, R < |(x, y)| < 2R\}$

and N_r is the codimension of B in $\Xi_{\xi'}^{n-r}$.

Remark. $N_r+r\geqq 1$ for any $r\in\{0, \cdots, n+1\}$.

Proof. In the same way as in Lemma 2.11, we have that the support of $\nu(\xi', y)$ is contained in $B\times \boldsymbol{R}_+^1$. If $B=\phi$, then $\hat{\nu}=0$. This shows (1). Since the codimension of the analytic set $\{(\xi', \mu(\xi')); \xi'\in B\}\times \Xi_\lambda^1$ is equal to N_r+r, we have the assertion (2) in the same way as in Lemma 2.12.

Proposition 2.15. *Suppose that B is a real analytic set. Let Γ be an open cone in $\boldsymbol{R}^{n+1}$ and N an integer such that if $w\in C^\infty(\overline{\boldsymbol{R}_+^{n+1}})\cap \mathcal{S}'(\overline{\boldsymbol{R}_+^{n+1}})$ satisfies the equations* (1.1) *and* (1.2) *and the support of $\hat{w}(\xi, y)$ is contained in $\{(\xi,' \mu(\xi'))$; $\xi'\in B\}\times \overline{\boldsymbol{R}_+^{n+1}}$ with*

$$\varliminf_{R\to\infty} R^{-N}\int_{\Gamma_R} |w(x, y)|^2dxdy = 0$$

then $w=0$. If C is a C^∞ manifold contained in B, then the closure of Γ contains $(n(\xi'), 0)$ for each $\xi'\in C$ and $N\leqq$ Codim *$C+r$. Here $n(\xi')$ is as in Lemma* 2.14 *and* Codim *C denotes the codimension C in $\Xi_{\xi'}^{n-r}$.*

Proof. We may assume that C is defined by $\eta''=\nu(\eta')$ where $\eta'\in\omega \subset\Xi_{\xi'}^{n-r-k}$ (k is codimension of C in $\Xi_{\xi'}^{n-r}$), ν is a C^∞ function in ω, $\xi'=(\eta', \eta'')$ and $(0, \nu(0))=\xi_0'$. Put $L=(f_{jk}(\eta'))_{j=1,\cdots,p,k=1,\cdots\tilde{b}}$ where

$$f_{jk}(\eta') = (2\pi i)^{-1}\oint B_j(\eta', \nu(\eta'), \lambda)\lambda^{k-1}(P^+(\eta', \nu(\eta'), \lambda))^{-1}d\lambda\,.$$

Since all minor of L of order $\tilde{b}$ vanish identically in ω, we have that the rank l of L is less than $\tilde{b}$ in ω. When $l>0$, without loss of generality, we may assume that

$$\Delta(\eta') = \det(f_{jk}(\eta'))_{j,k=1,\cdots,l}$$

does not vanish identically in ω. When $l>0$, we put

$$\hat{w}(\eta', y) = [-\textstyle\sum_{j,k=1}^l \Delta_{jk}(\eta')f_{jl+1}(\eta')\{(2\pi i)^{-1}\displaystyle\oint e^{iy\lambda}\lambda^{k-1}(P^+(\eta', \zeta(\eta'), \lambda))^{-1}d\lambda\}$$
$$+\Delta(\eta')(2\pi i)^{-1}\oint e^{iy\lambda}\lambda^{l}(P^+(\eta', \zeta(\eta'), \lambda))^{-1}d\lambda]\psi(\eta')$$

where $\psi(\eta')$ is a $C_0^\infty(\omega-\{\eta'\in\omega; \Delta(\eta')=0\})$ function and $\Delta_{jk}(\eta')$ is the (j, k) cofactor of $(f_{jk}(\eta'))_{j,k=1,\cdots,l}$ and $\zeta(\eta')=(\nu(\eta'), \mu(\eta', \nu(\eta')))$. When $l=0$, we put

$$\hat{w}(\xi', y) = (2\pi i)^{-1} \int e^{iy\lambda_+}(P^+(\eta', \zeta(\eta'), \lambda))^{-1} d\lambda \psi(\eta')$$

where $\psi(\eta')$ is a $C_0^\infty(\omega)$ function. It is obvious that we have

$P(\eta', \zeta(\eta'), D_y)\hat{w}(\eta', y)=0,\ y>0,\ B_j(\eta', \zeta(\eta'), D_y)\hat{w}(\eta', y)|_{y=0}=0,\quad j=1, \cdots, p.$

Thus, putting

$$w(x, y) = \int e^{i(x'\cdot\eta' + x''\cdot\zeta(\eta'))}\hat{w}(\eta', y) d\eta',\ x'' = (x_{n-r-k+1}, \cdots, x_n),$$

we have that $w(x, y)$ satisfies the equations (1.1) and (1.2). Since $|\hat{w}(\eta', y)| \leqq C_1 \cdot |\psi(\eta')| e^{-C_2 y}$ when $y \geqq 0$ for some positive constants C_1 and C_2, we have

$$\int_{\substack{|(x, y)|<R \\ y\geqq 0}} |w(x, y)|^2 dx dy \leqq C R^{r+k}$$

for some positive constant C. So if $N > \text{Codim}\, C + r \geqq N_r + r$, then

$$\varliminf_{R\to\infty} R^{-N} \int_{\Gamma_R} |w(x, y)|^2 dx dy = 0 .$$

This gives a contradiction. Thus, we have $N \leqq \text{Codim}\, C + r$. If $\bar{\Gamma}$ does not contain any $(n(\xi_0'), 0)$ and if supp $\psi(\eta')$ is sufficiently close to 0

$$\varliminf_{R\to\infty} R^{-N} \int_{\Gamma_R} |w(x, y)|^2 dx dy = 0 ,$$

for any N. This gives a contradiction. Q.E.D.

Next we consider the case when $\tilde{a} > 0$. Put

$$L^{(p', \delta, \sigma)} = \det\Big((2\pi i)^{-1} \oint B_{\sigma_j}(\xi', \lambda)\lambda^{k-1}(\textstyle\prod_{j+1}^{a}(\lambda - \lambda_j^0(\xi'))^{\alpha_j - \delta_j} \cdot P^+(\xi', \lambda))^{-1} d\lambda\Big)_{j,k=1,\cdots,p'}, \qquad \xi' \in W,$$

if $\tilde{a} > 0$ where $\delta = (\delta_1, \cdots, \delta_a)$ $(0 \leqq \delta_j \leqq \alpha_j)$, $\tilde{b}+1 \leqq p' \leqq p$ and $\sigma = (\sigma_1, \cdots, \sigma_{p'})$ $(\subset \{1, \cdots, p\})$.

Case 2. $\tilde{a} > 0$ and there exist p', δ, σ, having the properties:
(I) $\tilde{b}+1 \leqq p' \leqq p$,
(II) $\sum_{j=1}^{a}(\alpha_j - \delta_j) + \tilde{b} = p'$,
(III) if $p' < p$, $L^{(p'', \delta', \sigma')}(\xi')$ vanishes identically in W for any $p''(p'+1 \leqq p'' \leqq p)$, $\sigma' = (\sigma_1', \cdots, \sigma_{p''}')(\subset \{1, \cdots, p)\}$ and $\delta' = (\delta_1', \cdots, \delta_a')$ with $0 \leqq \delta_j' \leqq \alpha_j$,

$j=1, \cdots, a$, and $\sum_{j=1}^{a}(\alpha_j-\delta_j)+\tilde{b}=p''$, such that $L^{(p',\delta,\sigma)}(\xi')$ does not vanish identically in W.

Remark. When $r=0$, the hypotheses of Case 2 imply the assumption (A-2).

Put $B=\{\xi'\in W;\ L^{(p',\delta,\sigma)}(\xi')=0\}$. It is obvious that B is empty or a real analytic set in W.

Lemma 2.16. *Let $\hat{v}$ be as in Lemma 2.5. If the hypotheses of Case 2 are fulfilled in W with $\sum_{j=1}^{a}\alpha_j+\tilde{b}=p'$, then the support of $\hat{v}$ is contained in $B\times\overline{\boldsymbol{R}_+^1}$.*

Remark. If $B=\phi$, then $\hat{v}=0$.

Proof. Let U be any open set in W whose closure is compact and contained in $W-B$. It is sufficient to show that

$$\langle \hat{v}(\xi', y), w(\xi', y)\rangle = 0\,, \tag{2.31}$$

for any $w(\xi', y)\in\mathcal{S}_0(U\times\overline{\boldsymbol{R}_+^1})$. We obtain, using Lemmas 2.5 and 2.10, (1.2) and (2.4), that

$$\begin{aligned}&\langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y), w(\xi', y)\rangle\\ &\quad= \langle \hat{v}(\xi', y), P^0(\xi', -D_y)P^+(\xi', -D_y)w(\xi', y)\rangle\end{aligned}$$

for any $w(\xi', y)\in\mathcal{S}_0(U\times\overline{\boldsymbol{R}_+^1})$. Therefore, (2.31) follows from Lemma 2.6.

Lemma 2.17. *Suppose that the hypotheses of case 2 are fulfilled with $\delta_j>0$ for at least one j, say, $\delta_j>0$, $j=1, \cdots, k$. Set*

$$\Gamma_{j,R} = \{(x, y)\in\Gamma_j;\ y\geqq 0,\ R<|(x, y)|<2R\}\,,$$

where Γ_j is some open connected cone which for every $\xi'\in W-B$ contains some normal of $M_j=\{(\xi', \mu(\xi'), \lambda_j^0(\xi'));\ \xi'\in W-B\}$ at $(\xi', \mu(\xi'), \lambda_j^0(\xi'))$. If u belongs to $\boldsymbol{L}_{\mathrm{loc}}^2(\overline{\boldsymbol{R}_+^{n+1}})$ with

$$\varliminf_{R\to\infty} R^{-\{2(\alpha_j-\delta_j)+r+1\}}\int_{\Gamma_{j,R}}|u(x, y)|^2 dxdy = 0\,, \qquad j=1, \cdots, k, \tag{2.32}$$

then the support of $\hat{v}(\xi', y)$ is contained in $B\times\overline{\boldsymbol{R}_+^1}$. Here u and $\hat{v}$ are as in the first place of this section and Lemma 2.5, respectively.

Proof. Let U be an open set in W whose closure is compact and contained in $W-B$. It is sufficient to show that

$$\langle \hat{v}(\xi', y), w(\xi', y)\rangle = 0\,, \tag{2.33}$$

for any $w(\xi', y)\in \mathcal{S}_0(U\times \overline{\boldsymbol{R}^1_+})$. It follows from Lemma 2.10 that there exist ordinary differential operators $C_j(\xi', D_y)$, $j=0, \cdots, t-(\tilde{a}+\tilde{b})$, $E_j(\xi', D_y)$, $j=1, \cdots, p$, $F_{\nu,j}(\xi', D_y)$, $\nu=0, \cdots, \delta_j-1$, $j=1, \cdots, k$, whose coefficients are C^∞ functions of ξ' in U such that

$$\begin{aligned}
&\langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y), w(\xi', y)\rangle \\
&= \textstyle\sum_{j=0}^{t-(\tilde{a}+\tilde{b})}\langle D_y^j P^0(\xi, D_y)P^+(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0}, C_j(\xi', D_y)w(\xi', y)\,|_{y=0}\rangle \\
&\quad+\textstyle\sum_{j=1}^{p'}\langle B_{\sigma_j}(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0}, E_j(\xi', D_y)w(\xi', y)\,|_{y=0}\rangle \\
&\quad+\textstyle\sum_{j=1}^{k}\sum_{\nu=0}^{\delta_j-1}\langle\sum_{i=0}^{d-1}\sum_{k=0}^{i-1}\dfrac{(i-1-k)!}{(i-1-k-\nu)!}b_i(\xi')\lambda_j^0(\xi')^{i-1-k-\nu} \\
&\quad\cdot D_j^k Q(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0}, F_{\nu,j}(\xi', D_y)w(\xi', y)\,|_{y=0}\rangle \\
&\quad+\langle \hat{v}(\xi', y), P^0(\xi', -D_y)P^+(\xi', -D_y)w(\xi', y)\rangle
\end{aligned}$$

for any $w(\xi', y)\in \mathcal{S}_0(U\times \overline{\boldsymbol{R}^1_+})$. Here $t=\mathrm{Max}\,\{r_j;\ 1\leqq j\leqq p\}$, $d=\sum_{j=1}^{a}\delta_j$, $\prod_{j=1}^{a}(\lambda-\lambda_j^0(\xi'))^{\delta_j}=\sum_{j=0}^{d}b_j(\xi')\lambda^j$, $Q(\xi')=\sum_{j=1}^{a}(\lambda-\lambda_j^0(\xi'))^{\alpha_j-\gamma_j}\cdot P^+(\xi', \lambda)$. Since it follows from (2.32) and Lemma 2.7 that

$$\begin{aligned}
&\langle\textstyle\sum_{i=0}^{d-1}\sum_{k=0}^{i-1}\dfrac{(i-1-k)!}{(i-1-k-\nu)!}b_i(\xi')\lambda_j^0(\xi')^{i-1-k-\nu}D_y^k Q(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0}, \\
&F_{\nu,j}(\xi', D_y)w(\xi', y)\,|_{y=0}\rangle = 0\,,
\end{aligned}$$

for all $\nu=0, \cdots, \delta_j-1$, $j=1, \cdots, k$, and it follows from (1.2), (2.4) and Lemma 2.5 that

$$\begin{aligned}
&\langle B_{\sigma_j}(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0}, E_j(\xi', D_y)w(\xi', y)\,|_{y=0}\rangle = 0, \qquad j=1, \cdots, p, \\
&\langle D_y^j P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y)\,|_{y=0}, C_j(\xi', D_y)w(\xi_,, y)\,|_{y=0}\rangle = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad j=1, \cdots, t-(\tilde{a}+\tilde{b}),
\end{aligned}$$

we have

$$\langle P^0(\xi', D_y)P^+(\xi', D_y)\hat{v}(\xi', y), w(\xi', y)\rangle = \langle \hat{v}(\xi', y), P^0(\xi', -D_y)P^+(\xi', -D_y)w(\xi', y)\rangle,$$

for any $w(\xi', y)\in \mathcal{S}_0(U\times \overline{\boldsymbol{R}^1_+})$. Thus, (2.33) follows from Lemma 2.6.

Make the same assumption as in Lemma 2.17. Let $n_j(\xi')$ denote some normal of M_j at ξ' and $\theta\in \boldsymbol{R}^{n+1}_+$ be a normal of M_1 at $\xi'_0\in W-B$. Let D be a subset of $\{2, \cdots, a\}$ and let ϕ be a $C_0^\infty(\{(x, y)\in \boldsymbol{R}^{n+1}_+;\ |(x, y)-\theta|<2\varepsilon\})$ with $\phi=1$ on $|(x, y)-\theta|\leqq\varepsilon$ for sufficiently small ε. Assume that $\theta=n_1(\xi'_0)=n_j(\xi'_0)$ for some $n_j(\xi'_0)$ when $j\in D$ and $\theta\neq n_j(\xi'_0)$ for all $n_j(\xi'_0)$ when $j\notin D$. Choose a small neighborhood ω of ξ_0' so that $\{n_j(\xi'), \xi'\in\omega\} \subset \{(tx, ty); t\neq 0, |(x, y)-\theta|<\varepsilon\}$ when $j\in D$ and $\{n_j(\xi'); \xi'\in\omega\} \cap \{(tx, ty); t\neq 0, |(x, y)-\theta|<\varepsilon\}$ is empty when $j\notin D$. Put

$$Q(\xi', \lambda) = (\lambda-\lambda_1^0(\xi'))^{(\alpha_1-\delta_1)+1}\circ \prod_{j=2}^{a}(\lambda-\lambda_j^0(\xi'))^{\alpha_j-\delta_j}\circ P^+(\xi', \lambda)$$
$$= \prod_{j=1}^{a+b}(\lambda-\tau_j(\xi'))^{\mu_j},\ \xi'\in W,$$

where $\mu_1=\alpha_j-\delta_1+1$, $\mu_j=\alpha_1-\delta_j$, $j=2, \cdots, a$, $\mu_j+a=\beta_j$, $j=1, \cdots, b$, $\tau_j(\xi')=\lambda_j^0(\xi')$, $j=1, \cdots, a$, and $\tau_{j+a}(\xi')=\lambda+(\xi')$, $j=1, \cdots, b$. Put

$$L=\begin{bmatrix} B_1(\xi, \tau_1(\xi')), \cdots, B_1^{(\mu_1-1)}(\xi', \tau_1(\xi')), B_1(\xi', \tau_2(\xi')), \cdots, B_1^{(\mu_1-1)}(\xi', \tau_2(\xi')), \cdots \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ B_p(\xi', \tau_1(\xi')),\cdots, B_p^{(\mu_1-1)}(\xi', \tau_1(\xi')), B_p(\xi', \tau_2(\xi')), \cdots, B_p^{(\mu_1-1)}(\xi', \tau_2(\xi')), \cdots \end{bmatrix}.$$

It follows from the hypotheses of Case 2 and Lemma 2.8 that the rank of L is equal to p' in $W-B$ and that

$$\Delta(\xi') = \det\begin{bmatrix} B_{\sigma_1}(\xi', \tau_1(\xi')), \cdots, B_{\sigma_1}^{(\mu_2-2)}(\xi', \tau_1(\xi')), B_{\sigma_1}(\xi', \tau_2(\xi')), \cdots \\ \vdots \qquad \vdots \qquad \vdots \\ B_{\sigma_{p'}}(\xi', \tau_1(\xi')), \cdots, B_{\sigma_{p'}}^{(\mu_2-2)}(\xi', \tau_1(\xi')), B_{\sigma_{p'}}(\xi', \tau_2(\xi')), \cdots \end{bmatrix}$$

does not vanish in $W-B$. So there exist real analytic functions $C_{js}(\xi')$ $s=1, \cdots, \mu_j-1$, $j=1, \cdots, a+b$ in $W-B$ such that $C_{1\mu_1-1}(\xi')\neq 0$ and

$$\hat{w}(\xi', y) = \sum_{j=1}^{a+b}\sum_{s=0}^{\mu_j-1} C_{js}(\xi')(iy)^s e^{i\tau_j(\xi')y}\circ\psi(\xi')$$

satisfies the equations:

$$P(\xi', D_y)w(\xi', y) = 0,\ y>0 \quad\text{and}\quad B_j(\xi', D_y)w(\xi', y)|_{y=0} = 0, \quad j=1, \cdots, p,$$

where $\psi(\xi')\in C_0^\infty(\omega)$ and $P(\xi', D_y)=P^0(\xi', D_y)P^+(\xi', D_y)P^-(\xi', D_y)$. So setting

$$w(x, y) = \int \hat{w}(\xi', y)\exp\{i(x'\circ\xi'+x''\circ\mu(\xi'))\}d\xi',$$

we have that $w(x, y)$ satisfies the equations (1.1) and (1.2) and that $w(x, y)$ belongs to $\mathcal{S}'(\boldsymbol{R}_+^{n+1})\cap C^\infty(\boldsymbol{R}_+^{n+1})$. Put

$$N' = \operatorname{Max}\{s;\ C_{js}(\xi') \text{ does not vanish identically in } \omega \text{ when } j\in D\}.$$

Note that if $\{2, \cdots, k\}\cap D$ is not empty we may assume that $N'\geqq\alpha_1-\delta_1\geqq\alpha_j-\delta_j$ for $j\in\{2, \cdots, k\}\cap D$. Put $N'=\mu_{j_k}-\nu_{j_k}$, $k=1, \cdots, t$ and $C_{1,R}=\{(x, y)\in C_1; y\geqq 0, R<|(x, y)|<2R\}$ where C_1 is a small conic neighborhood of $n_1(\xi_0')$. Since supp $\phi\subset\boldsymbol{R}_+^{n+1}$, it follows from Theorem Ap-1 and Theorem Ap-4 due to Agmon-Hörmander [3 Theorem 3.1] that

$$\begin{aligned}&\lim_{R\to\infty}\int_{C_{1,R}}|w(x,y)|^2dxdy/R^{r+1+2N'}\\&\geqq\lim_{R\to\infty}\int|w(x,y)|^2\phi((x,y)/R)dxdy/R^{r+1+2N'}\\&\geqq\lim_{R\to\infty}\int\phi((x,y)/R)\cdot|\textstyle\sum_{k=1}^t F_k(x,y)|^2dxdy/R^{r+1},\end{aligned}$$

where

$$F_k(x,y)=\int\exp\{i(x'\cdot\xi'+x''\cdot\mu(\xi')+y\cdot\tau_{j_k}(\xi'))\}C_{j_k,\mu_{j_k}-\nu_{j_k}}(\xi')\psi(\xi')d\xi'.$$

Since $\tau_{j_k}(\xi')\neq\tau_{j_{k'}}(\xi')$ if $j_k\neq j_{k'}$, we have, using Corollary Ap-5,

$$\lim_{R\to\infty}\int F_k(x,y)\overline{F_{k'}(x,y)}\phi((x,y)/R)dxdy/R^{r+1}=0,\qquad k\neq k'.$$

Thus, we have, using Theorem Ap-4,

$$\begin{aligned}&\lim_{R\to\infty}\int_{C_{1,R}}|w(x,y)|^2dxdy/R^{r+1+2N'}\\&>C\textstyle\sum_{k=1}^t\displaystyle\int|C_{j_k,\mu_{j_k}-\nu_{j_k}}(\xi')\psi(\xi')|^2d\xi'>0.\end{aligned}$$

This shows that

$$\int_{C_{1,R}}|w(x,y)|^2dxdy\geqq CR^{(2N'+r+1)}.$$

In the same way, we have, using Theorems Ap-1 and Ap-4,

$$\int_{C_{1,R}}|w(x,y)|^2dxdy\leqq CR^{(2N'+r+1)}.$$

Summing up, we have proved.

Proposition 2.18. *Make the same assumption as in Lemma* 2.17. *Let* $\theta_j\in\boldsymbol{R}_+^{n+1}$ $(j=1,\cdots,k)$ *be normal of* M_j *at* $\xi_0'\in W-B$ *and* C_j *a small conic neighborhood of* θ_j. *Then there exists a solution* $w(x,y)\in C^\infty(\overline{\boldsymbol{R}_+^{n+1}})\cap\mathcal{S}'(\overline{\boldsymbol{R}_+^{n+1}})$ *of the equations* (1.1) *and* (1.2) *and a natural number* N_j *such that* $N_j\geqslant\alpha_j-\delta_j$ *and* $w(x,y)$ *satisfies for some positive constants* d_1 *and* d_2

$$d_1R^{(2N_j+r+1)}\leqq\int_{C_{j,R}}|w(x,y)|^2dxdy\leqq d_2R^{(2N_j+r+1)}$$

where $C_{j,R}=\{(x,y)\in C_j;\ y\geqq0,\ R<|(x,y)|<2R\}$.

Remark. For example, if D is empty, then $N_1=\alpha_1-\delta_1$.

Case 3. $\tilde{a}>0$ and $\tilde{b}=0$ and $L^{(p',\delta,\sigma)}(\xi')$ vanishes identically for all p' $(1\leqq p'\leqq p)$, $\delta=(\delta_1, \cdots, \delta_a)$ $(0\leqq\delta_j\leqq\alpha_j)$ and $\sigma=(\sigma_1, \cdots, \sigma_{p'})$ $(\subset\{1, \cdots, p\})$ in W.

Lemma 2.19. *Let u and $\hat{v}$ be as in the first place of this section and Lemma 2.5, respectively. Suppose that the hypotheses of Case 3 are fulfilled. If u belongs to $L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$ and satisfies the condition:*

$$\varliminf_{R\to\infty} R^{-(r+1)}\int_{\Gamma_{j,R}} |u(x, y)|^2 dxdy = 0\,, \qquad j=1, \cdots, a,$$

then $\hat{v}=0$. Here Γ_j and $\Gamma_{j,R}$ are as in Lemma 2.17.

Proof. In the same way as in Lemma 2.17, the assertion follows from Lemmas 2.5, 2.6, 2.7 and 2.10 and formulas (1.2) and (2.4).

Proposition 2.20. *Suppose that the hypotheses of Case 3 are fulfilled. Let Γ be an open cone in $\boldsymbol{R}^{n+1}$ and N is an integer such that if $u\in\mathcal{S}'(\overline{\boldsymbol{R}^{n+1}_+})\cap C^\infty(\overline{\boldsymbol{R}^{n+1}_+})$ satisfies the equations (1.1) and (1.2) with*

$$\varliminf_{R\to\infty} R^{-N}\int_{\Gamma_R} |u(x, y)|^2 dxdy = 0$$

and the support of $\hat{u}(\xi, y)$ is contained in $\{(\xi', \mu(\xi')); \xi'\in W\}\times\boldsymbol{R}^1_+$ then $u=0$. If $(\xi'_0, \mu(\xi'_0), \lambda^0_j(\xi'_0))\in M_j$, then it follows that the closure of Γ contains some normal $\neq 0$ of M_j at $(\xi'_0, \mu(\xi_0), \lambda^0_j(\xi'_0))$ and that $N\leqq r+1$.

Proof. Let $\psi(\xi')$ be a $C^\infty_0(W)$ function. It follows from the hypotheses of Case 3 that

$$B_j(\xi', \lambda^0_j(\xi')) = \oint B_j(\xi', \lambda)(\lambda-\lambda^0_j(\xi'))^{-1}d\lambda = L^{(1,\delta,j)}(\xi') = 0\,, \qquad j=1, \cdots, p,$$

where $\delta=(\alpha_1, \cdots, \alpha_{j-1}, \alpha_j-1, \alpha_{j+1}, \cdots, \alpha_a)$. So, putting

$$v(x, y) = \int \exp\{i(x'\cdot\xi'+x''\cdot\mu(\xi')+y\cdot\lambda^0_j(\xi'))\}\psi(\xi')d\xi',$$

we have the assertion in the same way as in Proposition 2.13.

Case 4. $\tilde{a}>0$ and $\tilde{b}>0$ and $L^{(p',\delta,\sigma)}(\xi')$ vanishes identically for all p' $(1\leqq p'\leqq p)$, $\delta=(\delta_1, \cdots, \delta_a)$ $(0\leqq\delta_j\leqq\alpha_j)$ and $\sigma=(\sigma_1, \cdots, \sigma_{p'})$ $(\subset\{1, \cdots, p\})$.

Let B be as in Case 1. We have the followings in the same way.

Lemma 2.21. *Let u and $\hat{v}$ be as in the first place of this section and Lemma*

2.5, respectively. Suppose that the hypotheses of Case 4 are fulfilled, $1 \leqq r \leqq n$ *and* $B=W$. *If u belongs to* $L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$ *with*

$$\lim_{\overline{R\to\infty}} R^{-r} \int_{\Gamma_R^{(III)}} |u(x, y)|^2 dxdy = 0\,,$$

then $\hat{v}=0$. *Here* $\Gamma_R^{(III)}$ *is as in Lemma* 2.12.

Proposition 2.22. *Suppose that* $1 \leqq r \leqq n$, *the hypotheses of case* 4 *are fulfilled and* $B=W$. *Let* Γ *be an open cone in* $\boldsymbol{R}^{n+1}$ *and N an integer such that if* $w \in C^\infty(\overline{\boldsymbol{R}^{n+1}_+}) \cap \mathcal{S}'(\overline{\boldsymbol{R}^{n+1}_+})$ *satisfies the equations* (1.1) *and* (1.2) *and support of* $\hat{w}(\xi, y)$ *is contained in* $\{(\xi', \mu(\xi'));\ \xi' \in W\} \times \overline{\boldsymbol{R}^1_+}$ *with*

$$\lim_{\overline{R\to\infty}} R^{-N} \int_{\Gamma_R} |w(x, y)|^2 dxdy = 0,$$

then $w=0$. *Then the closure* Γ *contains* $(n(\xi'), 0)$ *for each* $\xi' \in W$ *and* $N \leqq r$. *Here* $n(\xi')$ *is as in Lemma* 2.12.

When $r=0$, the assumption (A-2) implies that B is empty or a real analytic set. Noting this, we have the followings in the same way.

Lemma 2.23. *Let u and* $\hat{v}$ *be as in the first place of this section and Lemma* 2.5, *respectively.* (1) *Suppose that the hypotheses of Case* 4 *are fulfilled and B is empty. If u belongs to* $L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$ *with*

$$\lim_{\overline{R\to\infty}} R^{-(r+1)} \int_{\Gamma_{j,R}} |u(x, y)|^2 dxdy = 0 \qquad j=1, \cdots, a,$$

then $\hat{v}=0$. (2) *Suppose that the hypotheses of Case* 4 *are fulfilled and B is a real anlaytic set in W. If u belongs to* $L^2_{\mathrm{loc}}(\overline{\boldsymbol{R}^{n+1}_+})$ *with*

$$\lim_{\overline{R\to\infty}} R^{-(r+1)} \int_{\Gamma_{j,R}} |u(x, y)|^2 dxdy = 0\,, \qquad j = 1, \cdots, a\,,$$

$$\lim_{\overline{R\to\infty}} R^{-(N_r+r)} \int_{\Gamma_R^{(II)}} |u(x, y)|^2 dxdy = 0\,,$$

then $\hat{v}=0$. *Here* $\Gamma_{j,R}$ *is as in Lemma* 2.17 *and* $\Gamma_R^{(II)}$ *and* N_r *are as in Lemma* 2.14.

Proposition 2.24. *Suppose that the hypotheses of Case* 4 *are fulfilled and B is a real analytic set. Let* Γ *be an open cone in* $\boldsymbol{R}^{n+1}$ *and N an integer such that if* $w \in \mathcal{S}'(\overline{\boldsymbol{R}^{n+1}_+}) \cap C^\infty(\overline{\boldsymbol{R}^{n+1}_+})$ *is a solution of the equations* (1.1) *and* (1.2) *with*

$$\lim_{R\to\infty} R^{-N}\int_{\Gamma_R} |w(x, y)|^2 dxdy = 0$$

and the support of $\hat{w}(\xi, y)$ *is contained in* $\{(\xi', \mu(\xi'));\ \xi'\in W\}\times\overline{\boldsymbol{R}^1_+}$, *then* $w=0$. *If* $C\subset B$ *is a* C^∞ *manifold, then the closure of* Γ *contains* $(n(\xi'), 0)$ *for each* $\xi'\in C$ *and some normal* $\neq 0$ *of* M_j *at* $(\xi', \mu(\xi'), \lambda_j^0(\xi'))$ *for each* $\xi'\in W-B$. *Moreover, we have* $N\leqq r+1$. *Here* $n(\xi')$ *is as in Lemma* 2.14.

§3. Proof of the Main Theorem

Now we prove the Main Theorem. When $m=0$, it follows immediately from Lemma 2.2 ($r=0$). We may assume that $m\geqq 1$. Let u be a soluiton of the equations (1.1) and (1.2) which belongs to $C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))\cap L^2_{loc}(\overline{\boldsymbol{R}^{n+1}_+})$ and satisfies the condition (1.3). For a while let W be as in Section 1. When the hypotheses of Case 1 are fulfilled in W, we write $W=X_{0,W}$ and choose N and Γ so that $N\leqq N_1+1$ and $\Gamma\supset\Gamma^{(\mathrm{II})}$. Here if $B=\emptyset$, we interpret that $N_1=\infty$ and $\Gamma^{(\mathrm{II})}=\emptyset$. When the hypotheses of Case 2 are fulfilled in W, we denote by $X_{0,W}$ each connected component of $\{\xi\in W,\ L^{(p',\delta,\sigma)}(\xi)\neq 0\}$ and choose N and Γ so that $(\alpha_j-\delta_j)+1\geqslant N, j=1, \cdots, k$ and $\Gamma\supset\cup\{\Gamma_j,\ 1\leqslant j\leqslant k\}$. Here if the hypotheses of Lemma 2.16 are fulfilled, we interpret that $\cup_{j=1}^k\Gamma_j=\emptyset$ and $(\alpha_j-\delta_j)+1=\infty$. When the hypotheses of case 3 are fulfilled in W, we write $W=X_{0,W}$ and choose N and Γ so that $N=1$ and $\Gamma\supset\cup_{j=1}^a\Gamma_j$. When the hypotheses of Case 4 are fulfilled in W, we write $W=X_{0,W}$ and choose N and Γ so that $N=1$ and $\Gamma\supset\cup_{j=1}^a\Gamma_j\cup\Gamma^{(\mathrm{II})}$. Here if $B=\emptyset$, we interpret that $\Gamma^{(\mathrm{II})}=\phi$. Put $Y_0=\cup_W X_{0,W}$. Thus, using the notation in Section 1, we have

$$\Xi^n = Y_0\cup A_Q\cup A_{a_m}\cup(\cup_j A_{V_j\mathrm{I,m}})\cup E = Y_0\cup Y_1,$$

where $E=\cup\{\xi\in W;\ L^{(p',\delta,\sigma)}(\xi)=0\}$ and it is obvious that Y_0 is open. Let U be any open set such that U is contained in $W\cap Y_0$ for some W. Let ϕ be any $C_0^\infty(U)$ function. It follows from Section 2 that $\phi(\xi)\hat{u}(\xi, y)=0$. Thus we have that the support of $\hat{u}(\xi, y)$ is contained in $Y_1\times\overline{\boldsymbol{R}^1_+}$. Suppose that $\{\xi\in W;\ L^{(p',\delta,\sigma)}(\xi)=0\}$ is not empty for some W. Let U be any small open set contained in W and let ϕ be any $C_0^\infty(\{\xi\in U;\ (\partial/\partial\xi_n)\,L^{(p',\delta,\sigma)}(\xi)\neq 0\})$. $A_2=\{\xi\in U;\ L^{(p',\delta,\sigma)}(\xi)=0,\ (\partial/\partial\xi_n)L^{(p',\delta,\sigma)}(\xi)\neq 0\}$ is an analytic manifold of codimension 1 or empty and the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $A_2\times\overline{\boldsymbol{R}^1_+}$. If A_2 is not empty, A_2 may be defined by $\xi_n=\mu(\xi')$ where $\xi'\in\omega\subset\Xi^{n-1}$, μ is a real analytic function in ω and $\xi'=(\xi_1, \cdots, \xi_{n-1})$. We can denote by $\lambda_j^0(\xi', \lambda(\xi'))=\lambda_j^0(\xi'), j=1, \cdots, a$, $\lambda_j^+(\xi', \mu(\xi'))=\lambda_j^+(\xi'), j=1, \cdots, b$, and $\lambda_j^-(\xi', \mu(\xi'))=\lambda_j^-(\xi')$,

$j=1, \cdots, c$, the roots of the equation $P(\xi', \mu(\xi'), \lambda)=0$ in λ when $\xi' \in \omega$ and we have $a>0$. When the hypotheses of Case 2 are fulfilled in ω, we choose N and Γ so that $N \leqq (\alpha_j - \delta_j)+2, j=1, \cdots, k$, and $\Gamma \supset \cup \{\Gamma_j; 1 \leqq j \leqq k\}$. Then we have that the support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $\{(\xi', \mu(\xi'); \xi' \in \{\xi' \in \omega; L^{(p'', \delta', \sigma')}(\xi')=0\}$ by Lemma 2.17. When the hypotheses of Case 3 are fulfilled in ω, we choose N and Γ so that $N \leqq 2$ and $\Gamma \supset \subset \} \Gamma_j; 1 \leqq j \leqq a\}$. Then we have $\phi(\xi)\hat{u}(\xi, y)=0$ by Lemma 2.19. If the hypotheses of Case 4 are fulfilled in ω, we choose N and Γ so that $N \leqslant 2$ and $\Gamma \supset \cup \{\Gamma_j; 1 \leqslant j \leqslant a\} \cup \Gamma^{(\mathrm{II})}$. Here if B stated in Lemma 2.23 is empty we interpret that $\Gamma^{(\mathrm{II})}=\emptyset$. Then we have that $\phi(\xi)\hat{u}(\xi, y)=0$ by Lemma 2.23. In the same way, we have the support of $\hat{u}(\xi, y)$ is contained in $A_Q \cup A_{a_m} \cup (\bigcup_j A_{V_j, \mathrm{Im}})$. Let U be any small open set in V stated in Section 1. Let ϕ be any $C_0^\infty(\{\phi \in U; (\partial/\partial\xi_n)\mathrm{Im}\, \lambda_1(\xi) \neq 0, \mathrm{Im}\, \lambda_j(\xi) \neq 0, 2 \leqslant j \leqslant k\})$ function. The support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $\{\xi \in U; \mathrm{Im}\, \lambda_1(\xi)=0, (\partial/\partial\xi_n)\, \mathrm{Im}\, \lambda_1(\xi) \neq 0, \mathrm{Im}\, \lambda_j(\xi) \neq 0, 2 \leqq j \leqq k\}$ which is a real analytic manifold of codimension 1 or empty. In the same way, we choose N and Γ so that $\phi(\xi)\hat{u}(\xi, y)=0$. By repeating the argument, we have that the support of $\hat{u}(\xi, y)$ is contained in $A_Q \cup A_{a_m}$. Let U be any small open set contained in $\{\xi \in \Xi^n; (\partial/\partial\xi_n)Q(\xi) \neq 0, a_m(\xi) \neq 0\}$. Let ϕ be any $C_0^\infty(U)$ function. The support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $\{\xi \in U; Q(\xi)=0, (\partial/\partial\xi_n)Q(\xi) \neq 0, a_m(\xi) \neq 0\}$ which is empty or a real analytic manifold of codimension 1. In the same way, we have that $\phi(\xi)\hat{u}(\xi, y)=0$. We repeat above reasoning on $\{\xi \in \Xi^n; a_m(\xi) \neq 0\}$, and then we have that the support of $\hat{u}(\xi, y)$ is contained in A_{a_m}. Let U be any small open set contained in $\{\xi \in \Xi^n; (\partial/\partial\xi_n)a_m(\xi) \neq 0, a_{m-1}(\xi) \neq 0\}$. Let ϕ be any $C_0^\infty(U)$ function. The support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $\{\xi \in \Xi^n; a_m(\xi)=0, (\partial/\partial\xi_n)a_m(\xi) \neq 0, a_{m-1}(\xi) \neq 0\}$ which is a real analytic manifold of codim 1 or empty and contained in X_{m-1}. In the same way, we have, using the results of section 2, that $\phi(\xi)\hat{u}(\xi, y)=0$. By repeating the argument we conclude that the support of $\hat{u}(\xi, y)$ is contained in X_{m-2}. Repeated arguments imply that the support of $\hat{u}(\xi, y)$ is contained in X_0. Therefore, we choose N and Γ so that $\hat{u}(\xi, y)=0$ by Lemma 2.2.

Finally we show the last statement of the Main Theorem. First of all, we suppose that the assumption (A-1) is not fulfilled, that is, there exists a $\xi^0 \in \Xi^n$ such that the number of roots with positive imaginary parts of the equation $P(\xi^0, \lambda)=0$ in λ is greater than p. Since the union of all W stated in Section 1 is dense in Ξ^n, any open neighborhood U of ξ^0 intersects some W. Thus, $\tilde{b}>p$ for some W. Put

$$A(\xi) = (f_{j,k}(\xi))_{\substack{k=1,\cdots,\tilde{b} \\ j=1,\cdots,p}}, \qquad \xi \in W,$$

where

$$f_{j,k}(\xi) = (2\pi i)^{-1} \oint B_j(\xi, \lambda)\lambda^{k-1}(P^+(\xi, \lambda))^{-1} d\lambda .$$

Since $\tilde{b}>p$, $A(\xi)$ has rank r $(0 \leqq r \leqq p)$ in W, that is, some minor $\Delta(\xi)$ of $A(\xi)$ of order r does not vanish identically in W when $1 \leqq r \leqq p$ and every $(r+1), \cdots, p$-rowed minor of $A(\xi)$ vanish identically in W when $1 \leqq r \leqq p-1$. When $r \neq 0$, we may assume without loss of generality that

$$\Delta(\xi) = \det (f_{j,k}(\xi))_{j,k=1,\cdots,r} .$$

When $1 \leqq r \leqq p$, we put

$$G(\xi, y) = [-\textstyle\sum_{j,k=1}^{r} \Delta_{j,k}(\xi) f_{j,k}(\xi) \{(2\pi i)^{-1} \oint e^{-y\lambda} \lambda^{k-1} (P^+(\xi, \lambda))^{-1} d\lambda\}$$

$$+ \Delta(\xi)(2\pi i)^{-1} \oint e^{iy\lambda} \lambda^r (P^+(\xi, \lambda))^{-1} d\lambda] \phi(\xi) ,$$

where $\Delta_{j,k}(\xi)$ is the (j, k) cofactor of $(f_{j,k}(\xi))_{j,k=1,\cdots,r}$ and $\phi(\xi)$ is a $C_0^\infty(\{\xi \in W; \Delta(\xi) \neq 0\})$ function. It is obvious that we have

(5.1) $$P(\xi, D_y) G(\xi, y) = 0 , \qquad y>0 ,$$

(5.2) $$B_j(\xi, D_y) G(\xi, y)|_{y=0} = 0 , \qquad j = 1, \cdots, p .$$

When $r=0$, we put

$$G(\xi, y) = (2\pi i)^{-1} \oint e^{iy\lambda} (P^+(\xi, \lambda))^{-1} d\lambda \cdot \phi(\xi)$$

where $\phi(\xi)$ is a $C_0^\infty(W)$ function. It is obvious that $G(\xi, y)$ satisfies the equations (5.1) and (5.2). Put

$$u(x, y) = (2\pi)^{-n} \int G(\xi, y) \exp (ix\xi) d\xi .$$

It is obvious that $u(x, y)$ belongs to $\mathcal{S}(\boldsymbol{R}_+^{n+1})$ and satisfies the equations (1.1) and (1.2), since $G(\xi, y)$ satisfies the equations (5.1) and (5.2). Next, we suppose that the assumption (A-1) is fulfilled but the assumption (A-2) is not fulfilled for some W. Then $p \geqq \tilde{b}$ and $L_{W,\sigma}^+(\xi)$ stated in Section 1 vanish identically for all $\sigma = \{\sigma_1, \cdots \sigma_{\tilde{b}}\} \subset \{1, \cdots, p\}$ in W. Thus it follows from the same reason as the first case that there exists a solution $u(x, y)$ of the equations (1.1) and (1.2) which belongs to $\mathcal{S}(\boldsymbol{R}_+^{n+1})$. This completes the proof.

Remark. When we put $N=1$ and $\Gamma=\boldsymbol{R}^{n+1}$, for any system $\{P(D), B_j(D), j=1, \cdots, p\}$ which satisfies the assumptions (A-1) and (A-2) we have that if $u\in \boldsymbol{L}^2_{\text{loc}}(\overline{\boldsymbol{R}^{n+1}_+})\cap C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n))$ is a solution of the equations (1.1) and (1.2) with

$$\lim_{R\to\infty} R^{-1} \int_{\substack{R<|(x,y)|<2R \\ y\geq 0}} |u(x, y)|^2 dxdy = 0\,,$$

then $u=0$.

Example 1 (Rellich [12] or Agmon [1]). We consider a solution $u\in C^\infty([0, \delta)\mathcal{S}'(\boldsymbol{R}^n))\cap \boldsymbol{L}^2_{\text{loc}}(\boldsymbol{R}^{n+1}_+)$ of the equations:

$$\Delta u+ku = 0 \quad \text{in} \quad \boldsymbol{R}^{n+1}_+, \tag{5.3}$$

$$u|_{y=0} = 0 \quad \text{in} \quad \boldsymbol{R}^n\,, \tag{5.4}$$

where $\Delta = -\sum_{j=1}^n D_j^2 - D_y^2$ and $k>0$. Put $A^+ = \{\xi\in\Xi^n;\ |\xi|^2>k\}$, $A^0_+ = \{\xi\in\Xi^n\,|\,|\xi|^2<k, \xi_1>0\}$, $A^0_- = \{\xi\in\Xi^n;\ |\xi|^2<k, \xi_1<0\}$. When $\xi\in A^+$, we can denote by $\lambda^\pm(\xi)=\pm i\sqrt{|\xi|^2-k}$ the roots of the equation $\lambda^2+|\xi|^2-k=0$. It follows from Lemma 2.5 that

$$\langle (D_y-\lambda^+(\xi))\hat{u}(\xi, y), v(\xi, y)\rangle = 0\,,$$

for any $v(\xi, y)\in\mathcal{S}_0(A^+\times R^1_+)$. Since $u|_{y=0}=0$, we have

$$\begin{aligned}
&\langle (D_y-\lambda^+(\xi))\hat{u}(\xi, y), v(\xi, y)\rangle \\
&\quad = i\langle \hat{u}(\xi, 0), v(\xi, 0)\rangle + \langle \hat{u}(\xi, y), (-D_y-\lambda^+(\xi))v(\xi, y)\rangle \\
&\quad = \langle \hat{u}(\xi, y), (-D_y-\lambda^+(\xi))v(\xi, y)\rangle\,.
\end{aligned}$$

Then it follows from Lemma 2.6 that the support of $\hat{u}(\xi, y)$ is contained in $\{\xi\in\Xi^n;\ |\xi|^2\leqq k\}$. Put $\lambda^0_\pm(\xi)=\pm\sqrt{k-|\xi|^2}$, $\xi\in\{\xi\in\Xi^n;\ |\xi|^2<k\}$, Let u satisfy the condition:

$$\lim_{R\to\infty} R^{-1}\int_{\Gamma_R} |u(x, y)|^2 dxdy = 0$$

where Γ is an open cone which contains a normal of $M=\{(x, y)\in\boldsymbol{R}^{n+1};\ |x|^2+y^2=k,\ |x|^2\leqq k$ and $x_1\geqq 0\}$ for every $(x, y)\in M$ and $\Gamma_R=\{(x, y)\Gamma;\ y\geqq 0, R<|(x, y)|<2R\}$. When $\xi\in A^0_+$, Γ contains an outer normal of $\{(\xi, \lambda^0_+(\xi));\ \xi\in A^0_+\}$ at $\xi\in A^0_+$. It follows from Lemma 2.7 that

$$\langle (D_y-\lambda^0_-(\xi))\hat{u}(\xi, 0), \chi(\xi)\rangle = 0\,,$$

for any $\chi(\xi)\in C_0^\infty(A_+^0)$. Since $u|_{y=0}=0$, we have that

$$\langle D_y^j\hat{u}(\xi, 0), \chi(\xi)\rangle = 0\,, \quad j = 0, 1\,,$$

for any $\chi(\xi)\in C_0^\infty(A_+^0)$. Thus we have that the support of $\hat{u}(\xi, y)$ is contained in $[\{\xi\in\Xi^n;\ |\xi|^2<k,\ \xi_1\leqq 0\}\cup\{\xi\in\Xi^n;\ |\xi|^2=k\}]\times\overline{R_+^1}$. When $\xi\in A_-^0$, Γ contains an inner normal of $\{(\xi, \lambda_-^0(\xi));\ \xi\in A_+^0\}$. It follows from Lemma 2.7 that

$$\langle (D_y-\lambda_+^0(\xi))u(\xi, 0), \chi(\xi)\rangle = 0\,,$$

for any $\chi(\xi)\in C_0^\infty(A_-^0)$. So we have that the support of $\hat{u}(\xi, y)$ is contained in $[\{\xi\in\Xi^n;\ |\xi|^2<k,\ \xi_1=0\}\cup\{\xi\in\Xi^n;\ |\xi|^2=k\}]\times\overline{R_+^1}$. Let ϕ be any $C_0^\infty(\{\xi\in\Xi^n;\ |\xi|^2<k\})$ function. The support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $\{\xi\in\Xi^n;\ |\xi|^2<k,\ \xi_1=0\}\times\overline{R_+^1}$ and $\{\xi\in\Xi^n;\ |\xi|^2<k,\ \xi_1=0\}$ is a real analytic set of codimension 1. Γ contains some normal of $\{(\xi\cdot\lambda_+^0(\xi));\ \xi_1=0,\ |\xi|^2<k\}$. $\phi(\xi)\hat{u}(\xi, y)$ has the form:

$$\phi(\xi)\hat{u}(\xi, y) = \textstyle\sum_{|\alpha|\leqq s} v_\alpha(\xi', y)\otimes D_{\xi_1}\delta(\xi_1)\,,$$

where $\xi'=(\xi_2, \cdots, \xi_n)$. It follows from Lemma 2.7 that

$$\langle (D_y^2+|\xi'|^2-k)v_\alpha(\xi', y), w(\xi', y)\rangle = 0\,,$$
$$\langle v_\alpha(\xi', y)|_{y=0}, \chi(\xi')\rangle = 0\,,$$
$$\langle (D_y-\lambda_-^0(0, \xi')v_\alpha(\xi', y)|_{y=0}, \chi(\xi')\rangle = 0\,, \quad |\alpha| = s$$

for any $w(\xi', y)\in\mathcal{S}_0(\{\xi\in\Xi_{\xi'}^{n-1};\ |\xi'|^2<k\}\times\overline{R_+^1})$ and any $\chi(\xi')\in C_0^\infty(\{\xi\in\Xi_{\xi'}^{n-1};\ |\xi'|^2<k\})$. Then we have that $\phi(\xi)\hat{u}(\xi, y)$ has the form:

$$\phi(\xi)\hat{u}(\xi, y) = \textstyle\sum_{|\alpha|\leqq s-1} v_\alpha(\xi', y)\otimes D_{\xi_1}\delta(\xi_1)\,.$$

Repeating this reasoning on v_α ($|\alpha|\leqq s-1$), we obtain that $\phi(\xi)\hat{u}(\xi, y)=0$. So we have that the support of $\hat{u}(\xi, y)$ is contained in $\{\xi\in\Xi^n;\ |\xi|^2=k\}$. Let U be any open set contained in $\{\xi\in\Xi^n;\ \xi_1>0,\ \xi_2^2+\cdots+\xi_n^2<k\}$ and let ϕ be any $C_0^\infty(U)$ function. The support of $\phi(\xi)\hat{u}(\xi, y)$ is contained in $A_1\times\overline{R_+^1}$ where $A_1=\{(\sqrt{k-|\xi'|^2}, \xi')\in\Xi^n;\ \xi'=(\xi_2, \cdots, \xi_n),\ |\xi'|^2<k\}$ which is real analytic manifold of codimension 1 in Ξ^n. When $\xi\in A_1$, the roots of the equation $P(\xi, \lambda)=\lambda^2+|\xi|^2-k=0$ in λ is zero with multiplicity 2, that is, $P(\xi, \lambda)=\lambda^2$, $\xi\in A_1$. $(0, \cdots, 0, 1)$ is normal of real analytic manifold $\{(\xi, 0);\ \xi\in A_1\}$ where $P(\xi, \lambda)=0$ and Γ contains $(0, \cdots, 0, 1)$. Let v be the composition of $\phi(\xi)\hat{u}(\xi, y)$ and the map $\xi\mapsto(\xi_1+\sqrt{k-|\xi'|^2}, \xi')$. $v(\xi, y)$ has the form:

$$v(\xi, y) = \textstyle\sum_{|\alpha|\leqq s} v_\alpha(\xi', y)\otimes D_{\xi_1}\delta(\xi_1)\,,$$

for the support of $v(\xi, y)$ is contained in the plane $\xi_1=0$. It follows from Lemma 2.7 that

$$\langle D_y^2 v_\alpha(\xi', y), w(\xi', y)\rangle = 0\,,$$
$$\langle v_\alpha(\xi', y)|_{y=0}, \chi(\xi')\rangle = 0\,,$$
$$\langle D_y v_\alpha(\xi', y)|_{y=0}, \chi(\xi')\rangle = 0\,,$$

for any $w(\xi', y)\in \mathcal{S}_0(\{\xi'\in \Xi^{n-1};\ |\xi'|^2<k\}\times \overline{\boldsymbol{R}_+^1})$ and $\chi(\xi')\in C_0^\infty(\{\xi'\in \Xi^{n-1};\ |\xi'|^2<k\})$. In the same way, we have that the support of $\hat{u}(\xi, y)$ is contained in $\{\xi\in \Xi^n;\ \xi_1=0,\ \xi_2^2+\cdots+\xi_n^2=k\}$. Repeating the argument we conclude that $u=0$. Summing up, we have proved.

Theorem (Rellich [12] or Agmon [1]). *Let Γ be an open cone in $\boldsymbol{R}^{n+1}$ which contains a normal of $M=\{(x, y)\in \boldsymbol{R}^{n+1};\ x_1\geqq 0,\ |x|^2+y^2=k,\ |x|^2\leqq k\}$ at every $(x, y)\in M$. Set*

$$\Gamma_R = \{(x, y)\in \Gamma;\ y\geqq 0,\ R<|(x, y)|<2R\}\ .$$

If $u\in C^\infty([0, \delta);\ \mathcal{S}'(\boldsymbol{R}^n))\cap \boldsymbol{L}^2_{\mathrm{loc}}(\overline{\boldsymbol{R}_+^{n+1}})$ is a solution of the equations (5.3) *and* (5.4) *with*

$$\varliminf_{R\to\infty} R^{-1}\int_{\Gamma_R} |u(x, y)|^2 dxdy = 0\,,$$

then $u=0$.

Remark. Let $\phi(\xi)$ be a $C_0^\infty(\{\xi\in \Xi^n;\ |\xi|^2<k,\ \xi_1>0\})$. Put

$$v(x, y) = \int e^{ix\cdot\xi}\{e^{i\lambda_+^0(\xi)y}-e^{i\lambda_-^0(\xi)y}\}\phi(\xi)d\xi\ .$$

Then $v(x, y)$ is a solution of the equations (5.3) and (5.4) with

$$C_1 R\leqq \int_{\Gamma_R} |v(x, y)|^2 dxdy\leqq C_2 R$$

for some positive constants C_1 and C_2.

Example 2. We consider a solution $u\in C^\infty([0, \delta);\ \mathcal{S}'(\boldsymbol{R}^n))\cap \boldsymbol{L}^2_{\mathrm{loc}}(\overline{\boldsymbol{R}_+^{n+1}})$ of the equations:

(5.5) $$P(D)u = (D_y^2+D_1^2-k)u = 0\,, \quad \text{in } \boldsymbol{R}_+^{n+1}\,,$$

(5.6) $$B_1(D)u|_{y=0} = (D_y-i(k-1/2))u|_{y=0} = 0\,, \quad \text{in } \boldsymbol{R}^n\,,$$

(5.7) $$B_2(D)u|_{y=0} = (D_y-iD_1+i)u|_{y=0} = 0\,, \quad \text{in } \boldsymbol{R}^n\,,$$

where k satisfies $((k+1)/2)^2>k>1$. When $\xi_1^2>k$, we can denote by $\lambda^{\pm}(\xi)=\pm i\sqrt{\xi_1^2-k}$ the roots of the equations $P(\xi,\lambda)=0$ in λ. When $\xi_1^2>k$ and $\xi_1=(k+1)/2$, we have

$$(2\pi i)^{-1}\oint B_j(\xi,\lambda)(\lambda-\lambda^+(\xi))^{-1}d\lambda=0\,,\quad j=1,2\,.$$

When $\xi_1^2>k$ and $\xi_1\neq(k+1)/2$, we have

$$(2\pi i)^{-1}\oint B_j(\xi,\lambda)(\lambda-\lambda^+(\xi))^{-1}d\lambda\neq 0\,,\quad j=1,2\,.$$

Where $\xi_1^2<k$, the roots of the equation $P(\xi,\lambda)=0$ are all real and we have

$$\begin{aligned}&\det\left((2\pi i)^{-1}\oint B_j(\xi,\lambda)\lambda^{k-1}(P(\xi,\lambda))^{-1}d\lambda\right)_{j,k=1,2}\\&=\det\begin{bmatrix}1, & -i(k-1)/2\\ 1, & -i\xi_1+i\end{bmatrix}=-i(\xi_1-(k+1)/2)\neq 0\,.\end{aligned}$$

In the same way, when $\xi_1^2=k$, we have

$$\det\left((2\pi i)^{-1}\oint B_j(\xi,\lambda)\lambda^{k-1}(P(\xi,\lambda))^{-1}d\lambda\right)_{j,k=1,2}\neq 0\,.$$

Then the support of $\hat{u}(\xi,y)$ is contained in the plane $\{\xi\in\Xi^n;\ \xi_1=(k+1)/2\}\times\overline{R^1_+}$. This is real analytic manifold of codimension 1 in Ξ^n and $(1,0,\cdots,0)$ is its normal. Let Γ be an open conic neighborhood of $(1,0,\cdots,0)$ in R^{n+1} and set

$$\Gamma_R=\{(x,y)\in\Gamma;\ y\geqq 0,\ R<|(x,y)|<2R\}\,.$$

If u satisfies the condition:

$$\varliminf_{R\to\infty}R^{-1}\int_{\Gamma_R}|u(x,y)|^2dxdy=0\,,$$

then $u=0$.

Remark. Let ψ be a $C_0^\infty(\Xi^{n-1})$ function. Put

$$w(x,y)=\int\exp\{i((k+1)/2\cdot x_1+x''\cdot\xi'')-(k-1)/2\cdot y\}\cdot\psi(\xi'')d\xi'',$$

where $\xi''=(\xi_2,\cdots,\xi_n)$. $w(x,y)$ satisfies the equations (5.5)–(5.7) and there exist positive constants C_1 and C_2 such that

$$C_1R\leqq\int_{\substack{|(x,y)/R-(1,0,\cdots,0)|<\varepsilon\\ y\geqq 0}}|w(x,y)|^2dxdy\leqq C_2R\,,$$

for small positive ε.

Example 3. Let us consider a solution $u \in C^\infty([0, \delta); \mathcal{S}'(\boldsymbol{R}^n)) \cap \boldsymbol{L}^2_{loc}(\boldsymbol{R}^{n+1}_+)$ of the equations (1.1) and (1.2) for a system:

$$P(D) = (D_y - D_1)^4 (D_y - D_1 - 1)^2, \quad B_1(D) = 1, \quad B_2(D) = D_y,$$
$$B_3(D) = D_y^2, \quad B_4(D) = D_y^2 + \sum_{j=1}^n D_j D_y, \quad B_5(D) = D_y^4 - (4D_1 + 2) D_y^3.$$

When $\xi_1 \neq 1/6$, we have

$$\det ((2\pi i)^{-1} \oint B_j(\xi, \lambda) \lambda^{k-1} (\lambda - \xi_1)^{-3} (\lambda - \xi_1 - 1)^{-2} d\lambda)_{j,k=1,\cdots,5} \equiv 0,$$

$$\det ((2\pi i)^{-1} \oint B_j(\xi, \lambda) \lambda^{k-1} (\lambda - \xi_1)^{-4} (\lambda - \xi_1 - 1)^{-1} d\lambda)_{j,k=1,\cdots,5} \equiv 0,$$

$$\det ((2\pi i)^{-1} \oint B_{\sigma_j}(\xi, \lambda) \lambda^{k-1} (\lambda - \xi_1)^{-4} d\lambda)_{j,k=1,\cdots,4} \neq 0,$$

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with $\sigma_4 = 5$,

$$\det ((2\pi i)^{-1} \oint B_{\sigma_j}(\xi, \lambda) \lambda^{k-1} (\lambda - \xi_1)^{-3} (\lambda - \xi_1 - 1)^{-1} d\lambda)_{j,k=1,\cdots,4} \neq 0,$$

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ with $\sigma_4 = 5$,

$$\det ((2\pi i)^{-1} \oint B_{\sigma_j}(\xi, \lambda) \lambda^{k-1} (\lambda - \xi_1)^{-2} (\lambda - \xi_1 - 1)^{-2} d\lambda)_{j,k=1,\cdots,4} \equiv 0 \quad \text{for all } \sigma.$$

Since the normal of $\{(\xi, \lambda); \lambda = \xi_1, \xi_1 \neq 1/6\}$ and $\{(\xi, \lambda); \lambda = \xi_1 + 1, \xi_1 \neq 1/6\}$ is $(-1, 0, \cdots, 0, 1)$, it follows from Lemma 2.17 that if u satisfies the condition:

$$\varliminf_{R \to \infty} R^{-3} \int_{\substack{|(x,y)/R - (-1,0,\cdots,0,1)| < \varepsilon \\ y \geq 0}} |u(x, y)|^2 dx dy = 0,$$

then the support of $\hat{u}(\xi, y)$ is contained in $\{\xi \in \Xi^n; \xi_1 = 1/6\} \times \overline{\boldsymbol{R}^1_+}$. On the other hand, there exist $C^\infty(\{\xi \in \Xi^n; \xi_1 \neq 1/6\})$ functions $C_{j,s}(\xi)$ such that

$$v(x, y) = \int [\sum_{s=0}^2 C_{1,s}(\xi)(iy)^s e^{i(x\cdot\xi + y\cdot\xi_1)} + \sum_{s=0}^1 C_{2,s}(\xi)(iy)^s e^{ix\cdot\xi + y\cdot\xi_1}] \phi(\xi) d\xi$$

is non-trivial solution of the equations (1.1) and (1.2) where $C_{1,2}(\xi) = 1$ and $\phi(\xi) \in C_0^\infty(\{\xi \in \Xi^n; \xi_1 \neq 1/6\})$. Further, we have

$$C_1 R^5 \leqq \int_{\substack{|(x,y)/R - (-1,0,\cdots,0,1)| < \varepsilon \\ y \geq 0}} |v(x, y)|^2 dx dy \leqq C_2 R^5$$

for some positive constants C_1 and C_2. This is an example for what we could not take $N_j=\alpha_j-\delta_j$ in Proposition 2.18.

§4. Appendix

Here for the convenience of readers we state some results due to Hörmander [6] and Agmon-Hörmander [3] which is used in our paper without proof.

Theorem A_p-1 ([6]). *If u is a smooth density with compact support on a C^∞ submanifold M of $\boldsymbol{R}^n$ of codimension k, then*

$$(A_p\text{-}1)\qquad \int_{|\xi|<R} |\hat{u}(\xi)|^2 d\xi \leqq CR^k, \quad R>0.$$

If Γ is a closed cone in $\boldsymbol{R}^n$ which contains no element $\neq 0$ which is normal to M at a point in supp u,

$$(A_p\text{-}2)\qquad |\hat{u}(\xi)|<C_N(1+|\xi|)^{-N}, \quad \xi\in\Gamma,$$

for every integer N. Here $\hat{u}(\xi)$ denotes the Fourier transform of u.

Theorem A_p-2 ([6]). *Let $u\in\mathcal{S}'(\boldsymbol{R}^n)$, $\hat{u}\in \boldsymbol{L}^2_{\mathrm{loc}}$, $\theta\in\boldsymbol{R}^n$ and $\varepsilon>0$. If $\chi\in C_0^\infty(\boldsymbol{R}^n)$ and $v=\chi u$, it follows that for every $k\in\boldsymbol{R}$*

$$(A_p\text{-}3)\qquad \varlimsup_{R\to\infty} R^{-k} \int_{|\xi/R-\theta|<\varepsilon} |\hat{v}(\xi)|^2 d\xi \leqq C \varlimsup_{R\to\infty} R^{-k} \int_{|\xi/R-\theta|<2\varepsilon} |\hat{u}(\xi)|^2 d\xi,$$

where $C=(2\pi)^{-n}\int|\chi|\,d\xi$.

Theorem A_p-3 ([6]). *Let $u\in\mathcal{S}'(\boldsymbol{R}^n)$ be supported by a real analytic set A of codimension $k>0$, and assume that $\hat{u}\in\boldsymbol{L}^2_{\mathrm{loc}}$. Set*

$$\Gamma_R=\{\xi\in\Gamma;\ R<|\xi|<2R\},$$

where Γ is an open cone in $\boldsymbol{R}^n$ which for every analytic manifold $M\subset A$ and $x_0\in M$ contains some normal of M at x_0. If

$$\varliminf_{R\to\infty} R^{-k}\int_{\Gamma_R}|\hat{u}(\xi)|^2 d\xi=0,$$

it follows that $u=0$.

Theorem A_p-4 ([3]). *Let ϕ be a continuous function with compact support in $\boldsymbol{R}^n$. If $u\in\mathcal{S}'$ and $\hat{u}=\hat{u}_0 dS$ is an $\boldsymbol{L}^2$ density with compact support on a C^1*

manifold $M\subset \boldsymbol{R}^n$ of codimension k, then

$$\lim_{R\to\infty}\int|u(x)|^2\phi(x/R)dx/R^k = (2\pi)^{-k-n}\int_M|u_0(\xi)|^2\Big(\int_{N\xi}\phi(x)d\sigma(x)\Big)dS(\xi)\,,$$

where dS is the Euclidean surface element on M and dσ is the Euclidean integration element in the normal plane N_ξ of M at ξ, passing through 0.

Modifying the proof of Theorem A_p-4 slightly, we have the following.

Corollary A_p-5. *Let U be an open set in $\Xi^{n-k}_{\xi'}$, let $\mu_j(\xi')$, $j=1, 2$, be $C^\infty(U)$ functions such that $\mu_1(\xi')\neq\mu_2(\xi')$ when $\xi'\in U$ and let χ be a $C_0^\infty(U)$ function. Put*

$$F_j(x) = \int \exp\{i(x'\cdot\xi'+x''\cdot\mu_j(\xi'))\}\chi(\xi')d\xi'\,,\quad j = 1, 2\,.$$

Let φ be a $C_0^\infty(\boldsymbol{R}^n_x)$ function. Then it follows that

$$\lim_{R\to\infty}\int F_1(x)F_2(x)\phi(x/R)dx/R^k = 0\,.$$

Here $\mu_1(\xi')\neq\mu_2(\xi')$ means that $|\mu_1(\xi')-\mu_2(\xi')|>0$ when $\xi'\in U$, $x'=(x_1,\cdots, x_{n-k})$, $x''=(x_{n-k+1},\cdots,x_n)$ and $\mu_j(\xi')=(\mu^j_{n-k+1}(\xi'),\cdots,\mu^j_n(\xi'))$ $(j=1, 2)$.

Proof. Let $\phi(x)=\hat{\Phi}(-x)$, thus $\Phi\in\mathcal{S}$. The Fourier transform of $R^{n-k}\Phi(R\xi)$ and $F_j(x)$ are $R^{-k}\phi(-x/R)$ and $\chi(\xi')\otimes\delta(\xi''-\mu_j(\xi'))$, respectively, so the Fourier transform of $F_1(x)\phi(x/R)R^{-k}$ is

$$(2\pi)^{-n}R^{n-k}\int\Phi(R(\xi'-\eta'),\ R(\xi''-\mu_1(\eta'))\chi(\eta')d\eta'\,.$$

Hence it follows from Parseval's equality that

$$\begin{aligned}&\int F_1(x)F_2(x)\phi(x/R)dx/R^k\\ &\quad= (2\pi)^{-n}R^{n-k}\int\Phi(R(\xi'-\eta'),\ R(\mu_2(\xi')-\mu_1(\eta'))\chi(\xi')\chi(\eta')d\xi'd\eta'\\ &\quad= (2\pi)^{-n}\int\Phi(\eta',\ R(\mu_2(\xi')-\mu_1(\xi'-\eta'/R)))\chi(\xi'-\eta'/R)\chi(\xi')d\eta'd\xi'\,.\end{aligned}$$

Since

$$|\chi(\xi'-\eta'/R)|<C\,,$$
$$|\Phi(\eta',\ R(\mu_2(\xi')-\mu_1(\xi'-\eta'/R)))|<C_N(1+|\eta'|)^{-N}\qquad\text{for any}\quad N\,,$$

for any $\varepsilon>0$ there exists a large number K such that

$$\left|\iint_{|\eta'|\geqq K} \chi(\xi')\chi(\xi'-\eta'/R)\Phi(\eta', R(\mu_2(\xi')-\mu_1(\xi'-\eta'/R)))d\xi'd\eta'\right|$$
$$\leqq C'\int |\chi(\xi')|d\xi'\cdot K^{-1}<\varepsilon ,$$

where C' is independent of K. On the other hand, we have

$$R|\mu_2(\xi')-\mu_1(\xi'-\eta'/R)|>R|\mu_2(\xi')-\mu_1(\xi')|-|\mu_1'(\xi')||\eta'|-CR^{-1},$$

when $|\eta'|\leqq K$ and $\xi'\in\operatorname{supp}\chi$, where C is some constant independent of R. Since $|\mu_2(\xi')-\mu_1(\xi')|\geqq C'$ when $\xi'\in\operatorname{supp}\chi$, we have that

$$|\Phi(\eta', R(\mu_2(\xi')-\mu_1(\xi'-\eta'/R)))|\leqq C/(1+C'R),\ \xi'\in\operatorname{supp}\chi \text{ and } |\eta'|\leqq K.$$

Therefore we have

$$\left|\lim_{R\to\infty}\iint \chi(\xi')\chi(\xi'-\eta'/R)\Phi(\eta', R(\mu_2(\xi')-\mu_1(\xi'-\eta'/R)))d\xi'd\eta'\right|<\varepsilon ,$$

for any $\varepsilon>0$, that is,

$$\lim_{R\to\infty}\iint \chi(\xi')\chi(\xi'-\eta'/R)\Phi(\eta', R(\mu_2(\xi')-\mu_1(\xi'-\eta'/R)))d\xi'd\eta' = 0 .$$

Q.E.D.

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