

An Equisingular Deformation Theory via Embedded Resolution of Singularities

By

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In this paper we shall give a definition of equisingular deformations of isolated singularities and prove some results about it such as the existence of semi-universal deformations.

Roughly speaking, equisingularity means in this paper the existence of a simultaneous *embedded* resolution. Of course, in the case of plane curve singularities the definition coincides with Zariski's classical one.

In Section 1 we shall define equisingular deformation and prove some elementary properties of it. In Section 2 we shall study the deformation of locally product type of the pair obtained by the embedded resolution of singularity. In Section 3 we shall prove the existence theorem of semi-universal deformations.

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§1. Definitions

Let (X, o) be a germ of reduced complex space which is non-singular outside of o . By abuse of language, we sometimes write only X instead of (X, o) and mix up the concepts of a germ of complex space and a representative of it. Since X is local, it admits a closed immersion $i: X \rightarrow A$ into a non-singular complex space A , where $\dim A = \dim m_{X,o}/m_{X,o}^2$ (A is also local). Denote $N = \dim A$ and $n = \dim X$. Note that $N - n$ is not necessarily 1. Let us consider the following commutative diagram (*):

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$$(*) \quad \begin{array}{ccc} \bar{X} & \xrightarrow{\bar{i}} & \bar{A} \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{i} & A, \end{array}$$

where f is a point modification of A at o obtained by succession of monoidal transformations with non-singular centers, such that $\bar{X} = \text{red } f^{-1}(X)$ is normal crossing in the sense of the following Definition 1. That is, $(*)$ is an *embedded resolution* of X in A . Denote $g = f|_{\bar{X}}$, and let $\bar{X} = Y \cup E$, where Y is the strict transform of X and $E = \text{red } f^{-1}(o)$ is the exceptional locus. In particular,

$$\begin{array}{ccc} Y - Y \cap E & \xrightarrow{\cong} & X - \{o\} \\ \downarrow & & \downarrow \\ A - E & \xrightarrow{\cong} & A - \{o\}. \end{array}$$

Definition 1. Let A be a complex manifold and X a closed analytic subset of A . X is called an analytic subset of *normal crossing* if the following conditions are satisfied:

- (1) $X = \bigcup_{i=1}^h X_i$, where the $X_i (1 \leq i \leq h)$ are complex submanifolds of A .
- (2) For each $p \in A$, there exists a neighborhood U of p and a system of local coordinates $\{z_1, \dots, z_n\}$ on U such that $X_i = \{z_{r_{i-1}+1} = \dots = z_{r_i} = 0\}$ for $1 \leq i \leq h$, where the r_i are integers such that $-1 \leq r_i \leq n$ and $r_i \leq r_j$ if $i \leq j$, and we understand that $z_0 = 1$.

We note that for any X there exists at least one such $(*)$ by [5].

Definition 2. An *equisingular deformation* (in this paper) of X of type $(*)$ consists of the following data:

- (1) complex spaces $\bar{\mathcal{X}}, \mathcal{X}, S, \mathcal{A}$ and a point s_0 of S ,
- (2) morphisms p, q, I, \bar{I}, F and G which make up the following commutative diagram

$$(**) \quad \begin{array}{ccc} \bar{\mathcal{X}} & \xrightarrow{\bar{I}} & \bar{\mathcal{A}} \\ G \downarrow & & \downarrow F \\ \mathcal{X} & \xrightarrow{I} & A \times S \\ q \searrow & & \swarrow p \\ & & S \end{array}$$

where p is the projection, q is flat, I , and \bar{I} are closed immersions and F is a

modification of $A \times S$ along $\{o\} \times S$. We put $\bar{p} = p \circ F$ and $\bar{q} = q \circ G$.

(3) an isomorphism $\bar{\psi}$ from $(*)$ to the diagram obtained by the base change of $((*)$) by $\{s_0\} \rightarrow S$,

(4) we claim that $((*)$) is *locally product* in the sense of the following (such a deformation will be called also to be *logarithmic* cf. [7]): for every $x \in \mathcal{A}$ there is a neighborhood U (resp. V) of x in \mathcal{A} (resp. of $y = \bar{p}(x)$ in S) and an isomorphism $\alpha: U \rightarrow (U \cap \bar{p}^{-1}(y)) \times \mathcal{V}$, which sends $U \cap \bar{\mathcal{X}}$ onto $(U \cap \bar{\mathcal{X}} \cap \bar{p}^{-1}(y)) \times V$ and $\bar{p} = \text{pr}_V \circ \alpha$.

Definition 3. A flat deformation (q, ψ) of X , where $q: \mathcal{X} \rightarrow S$ is a morphism of complex spaces and $\psi: X \rightarrow \mathcal{X} \times_S \{s_0\}$ is an isomorphism, is said to be *equisingular* (in this paper) if there is some embedded resolution $(*)$ of X and an equisingular deformation of type $(*)$ which induces (q, ψ) .

Remark 1. (1) In case $n=1$ and $N=2$, our definition coincides with Zariski's original one.

(2) Our definition is strictly stronger than the one defined by J. Wahl [15] (see Example 1).

Remark 2. For any geometric fiber X_s of q , $\dim m_{X_s, o_s} / m_{X_s, o_s}^2 = \dim A$. (This fact justifies the Definition 2.)

Proof. Since q is flat, the dimension of the Zariski tangent space is upper semi-continuous. Suppose that this is not constant. By a base change we obtain the following situation: (1) S is a smooth curve, (2) There is a point $s_0 \in S$ such that $\dim m_{X_s, o_s} / m_{X_s, o_s}^2$ is equal to N_1 if $s = s_0$ and to N_2 if otherwise, where $N_1 > N_2$ are integers.

Let L be the first infinitesimal neighborhood of the zero section $o \times S$ in $A \times S$, that is, the ideal of L is the square of that of $o \times S$. Put $\mathcal{X}_L = \mathcal{X} \cap L$. By assumption, $\mathcal{X}_L \times_S (S - \{s_0\})$ defines an $(S - \{s_0\})$ -valued point α of the Grassmanian variety $G(N_1, N_2)$. Since the latter is proper and separated, there is a unique S -valued point $\bar{\alpha}$ of $G(N_1, N_2)$ inducing α . $\bar{\alpha}$ defines a subspace V of L inducing $\mathcal{X}_L \times_S (S - \{s_0\})$ on $L \times_S (S - \{s_0\})$. Then $\mathcal{X}_L = V$, which contradicts the assumption. Q.E.D.

Proposition 1. *Let $q: \mathcal{X} \rightarrow S$ be an equisingular deformation of X where S is reduced. Then there exists a homeomorphism $h: A \times S \rightarrow A \times S$ which sends \mathcal{X} to $X \times S$ and $q = \text{pr}_S \circ h$.*

Proof. Using a partition of unity, we obtain a diffeomorphism $\bar{h}: \bar{\mathcal{A}} \rightarrow \bar{A} \times S$ which sends each component of $\bar{\mathcal{X}}$ to the corresponding component of $\bar{X} \times S$ and $\bar{q} = \text{pr}_S \circ \bar{h}$. \bar{h} is blown down to the desired h . Q.E.D.

Corollary 2. *When $N = n + 1$, i.e., in case of hypersurface singularities, any equisingular deformation preserves the Milnor number μ .*

Let \mathcal{H} be a general family of hypersurfaces in A parametrized by S . Let $H \subset A$ be the fiber of \mathcal{H} over s_0 . Let $\mathcal{X}_{\mathcal{H}}$ (resp. X_H) be the smallest closed analytic subset of $A \times S$ (resp. A) which induces $(\mathcal{X} \cap \mathcal{E}) - (o \times S)$ (resp. $(X \cap H) - \{o\}$) on $(A - \{o\}) \times S$ (resp. $A - \{o\}$). They are called the *strict general hyperplane section* of \mathcal{X} (resp. X) by \mathcal{H} (resp. H).

Proposition 3. *$\mathcal{X}_{\mathcal{H}} \rightarrow S$ is an equisingular deformation of X_H .*

Proof. Trivial.

Corollary 4. *When $N = n + 1$, any equisingular deformation preserves μ^* of Teissier, hence satisfies the Whitney condition (cf. [11] and [12]).*

Example 1. (Brianchon—Speder [12]) $N = 3$ and $n = 2$.

$A = \text{Specan } \mathbf{C}\{x, y, z\}$, $S = \text{Specan } \mathbf{C}\{t\}$ and $\mathcal{X} \subset A \times S$ is defined by the equation $x^5 + y^7z + z^9 + txy^6 = 0$.

Then $q: \mathcal{X} \rightarrow S$ is (0) flat, (1) equisingular in the sense of J. Wahl [15], (2) μ is constant, but (3) μ^* is *not* constant, hence (4) *not* equisingular in the sense of this paper. Note that (5) $\mathcal{X}_{\mathcal{H}}$ is *not* equisingular.

Remark 3. In case $N = n + 1$, we do not know whether our definition is stronger or weaker than the Z -equisingularity in the sense of J.-P. Speder [10].

For a fixed $X \subset A$, there can be various embedded resolutions. Let

$$(*)' \quad \begin{array}{ccc} \bar{X}' & \xrightarrow{\bar{i}} & \bar{A}' \\ g' \downarrow & & \downarrow f \\ X & \xrightarrow{i} & A \end{array}$$

be another embedded resolution of $X \subset A$. Suppose that there is a morphism $\alpha: A \rightarrow A'$ such that $f = f' \circ \alpha$. Then α induces $\beta: \bar{X} \rightarrow \bar{X}'$.

Proposition 5. *Let $\mathcal{X}, S, \bar{\mathcal{X}}, \bar{\mathcal{A}}, p, q$, etc. be an equisingular deformation of type $(*)$ of $X \subset A$. Then there exists a neighborhood S' of s_0 in S such that there exists an equisingular deformation $\mathcal{X}', S', \bar{\mathcal{X}}', \bar{\mathcal{A}}', p', q'$, etc. of type $(*)'$*

where $\mathfrak{X}' = \mathfrak{X} \times_S S'$ and $q' = q \times_S S'$.

Proof. We regard A as an open set of the projective space \mathbf{P}^N . The point modifications f and F can be seen to be done for \mathbf{P}^N and $\mathbf{P}^N \times S$, respectively. Since f is blown down to f' , F can also be blown down to some F' in some neighborhood S' of s_0 in S by Theorem 3 of [7]. It is easy to check that F' induces an equisingular deformation of type (*). Q.E.D.

Remark 4. There is no minimal embedded resolution even in case $N = 3$

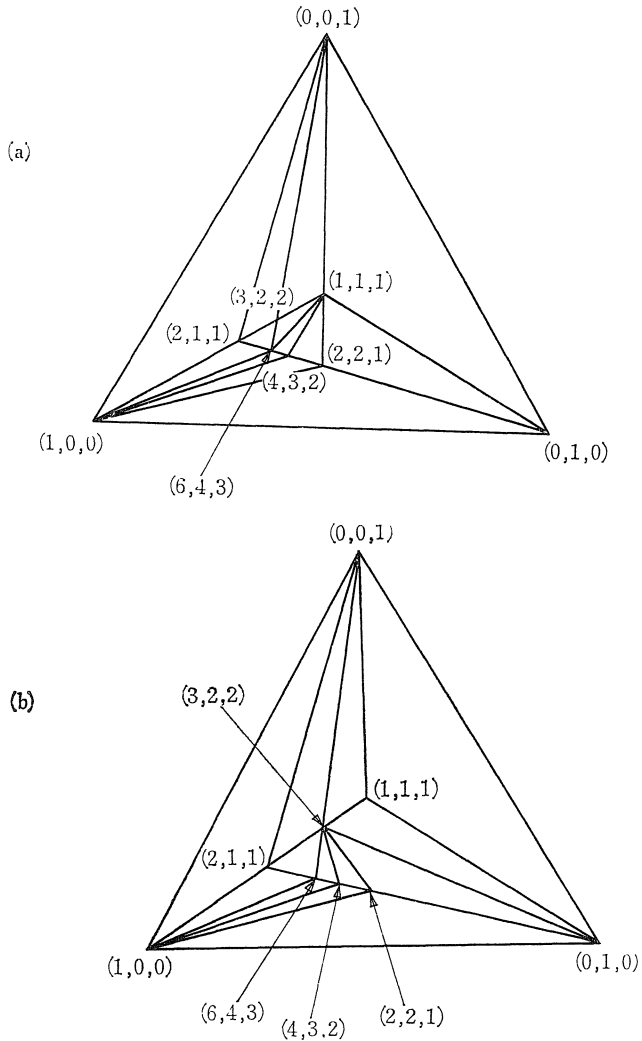


Figure 1

and $n=2$. For example, let X be defined by an equation $P=x_1^2+x_2^3+x_3^4$. Then there are two embedded resolutions corresponding to the two decompositions of positive cone of dimension 3 drawn in Figure 1. (See Example 2). We can easily check that there is no embedded resolution which factors through both.

Example 2. (A. N. Varchenko [13]). Let X be a hypersurface singularity defined by an equation $P=0$, and let $\Gamma(P)$ be the Newton polygon associated to P . P is said to be *non-degenerate* if for any facet $\gamma \in \Gamma(P)$, the polynomials $x_1 \frac{\partial P_\gamma}{\partial x_1}, \dots, x_N \frac{\partial P_\gamma}{\partial x_N}$ have no common zero in $\{x \in \mathbb{C}^N; x_1, \dots, x_N=0\}$. In this case Varchenko shows using the theory of toroidal embeddings that there is a modification $f: X(\Gamma) \rightarrow \mathbb{C}^N$ depending only on $\Gamma = \Gamma(P)$ such that $\text{red } f^{-1}(X)$ is normal crossing. Hence we know that any small deformation of P preserving the Newton polygon is equisingular with respect to f . However, for arbitrary complicated singularity which is usually degenerate, we should consider the geometric process of resolution, to which corresponds a sequence of Newton polygons.

§2. Logarithmic Deformations

Let us consider the situation (*) of Section 1. For some time we shall set aside the lower half of (*) and investigate the logarithmic deformation of $\bar{i}: \bar{X} \rightarrow A$, that is, the deformation which is locally product in the sense of Definition 2. We recall the definition of the logarithmic tangent sheaf (Definition 4 of [7]).

Definition 4. Let \bar{A} be a complex manifold and X a closed analytic subset of A . The *logarithmic tangent sheaf* $T_A(\log X)$ is the subsheaf of the tangent sheaf T_A consisting of derivations which sends the ideal sheaf of X in A into itself.

By the general theory of deformations of manifolds (Exp. III of [4]), the infinitesimal deformation of $\bar{\mathcal{A}} \subset \bar{\mathcal{A}}$ is described by the cohomology groups of $T_{\bar{A}}(\log \bar{X})$ as follows (cf. §1 of [7]): (1) $H^1(\bar{A}, T_{\bar{A}}(\log \bar{X}))$ is the set of isomorphism classes of all deformations over $S = \text{Specan } \mathbb{C}[x]/(x^2)$, (2) The obstruction of extending the deformation lies in $H^2(\bar{A}, T_{\bar{A}}(\log \bar{X}))$ (Note that since X is a germ, $H^i(\bar{A}, T_{\bar{A}}(\log \bar{X})) = R^i f_* T_{\bar{A}}(\log \bar{X})$ for $i > 0$.)

Let

$$\begin{array}{ccc} & I & \\ \overline{\mathcal{X}} & \longrightarrow & \overline{\mathcal{A}} \\ \overline{q} \searrow & & \swarrow \overline{p} \\ & S & \end{array}$$

be a logarithmic deformation of $\bar{i}: \overline{\mathcal{X}} \rightarrow A$. Replacing neighborhood of s_0 in S , if necessary, $\overline{\mathcal{A}}$ can be blown down to $A \times S$. Denote by $F: \overline{\mathcal{A}} \rightarrow A \times S$ the canonical morphism. F induces an isomorphism $\overline{\mathcal{A}} - \mathcal{E} \rightarrow (A - \{o\}) \times S$. Let $\mathcal{X} = \mathcal{X}_S$ be the smallest subspace of $A \times S$ inducing $\overline{\mathcal{X}} \cap (\overline{\mathcal{A}} - \mathcal{E})$ in $(A - \{o\}) \times S$. $\overline{\mathcal{X}}_S$ is called the *strict image* of $\overline{\mathcal{X}}$ over S .

Proposition 6. *Under the situation above, there exists a germ T of closed analytic subset of S at s_0 satisfying the following conditions: If Z is a closed subspace of S such that the strict image \mathcal{X}_Z of $\overline{\mathcal{X}} \times_S Z$ over Z is a flat deformation of X over Z , then the germ of Z at s_0 is contained in T .*

We need the following

Lemma. *\mathcal{X}_Z is a flat deformation of X over Z if and only if the geometric fiber $\mathcal{X}_Z \times_{\overline{Z}} \{s_0\}$ of \mathcal{X}_Z over s_0 is reduced, hence isomorphic to X (we shall say shortly that Z satisfies the property (H) in this case).*

Proof. Put $C = \mathcal{O}_{Z, s_0}$, $D = \mathcal{O}_{A \times Z}$, $E = \mathcal{O}_{\mathcal{X}_Z} = D/I$ and $m = m_{Z, s_0}$. There is an exact sequence

$$0 \rightarrow \text{Tor}_1^C(C/m, E) \rightarrow m \otimes E \rightarrow E.$$

Let a_1, \dots, a_h be elements of m which induces a basis of m/m^2 . Suppose that an element $f = \sum_{i=1}^h a_i \otimes [f_i]$ ($f_i \in D$) of $m \otimes E$ is mapped to zero in E . Considering the values of the functions on $\overline{\mathcal{A}}_Z$, we conclude that f_i is in (mD, I) . Therefore, replacing the representatives f_i of $[f_i]$ by suitable ones, we may suppose $f_i \in mD$, that is, $f_i = \sum_{j=1}^h a_j g_{ij}$ for some $g_{ij} \in D$. By a similar argument, we may suppose g_{ij} is in m , and so on. Hence f is in the image of $(\cap m^h) \otimes E$, and the latter is zero. Thus, $\text{Tor}_1^C(C/m, E) = 0$, which shows the lemma.

Q.E.D.

Proof of the Proposition 6. We have to show the following two claims:

- (1) If Z_1 and Z_2 satisfy (H), then $Z_1 \cup Z_2$ also satisfies (H).
- (2) If an ascending chain of subspaces $Z_1 \subset Z_2 \subset \dots$ satisfy (H) and Z is the smallest closed subset of S containing all the Z_i , then Z satisfies (H).

Proof of (1). As long as the germs are concerned, Z satisfies (H) if and only if every function f_0 in $\mathcal{O}_{\bar{A}}(-\bar{X})$ can be extended to some f in $\mathcal{O}_{\bar{A}_Z}(-\bar{X}_Z)$, where $\bar{A}_Z = \bar{A} \times_S Z$ and $\bar{X}_Z = \bar{X} \times_S Z$. Since $\mathcal{X}_{Z_1} \times_{s_1} \{s_0\}$ is reduced, $\mathcal{X}_{Z_1} \times_{s_1} (Z_1 \cap Z_2)$ has no function whose support is on the zero section. Therefore, every function f_{12} in $\mathcal{O}_{\bar{A}_{Z_1 \cap Z_2}}(-\bar{X}_{Z_1 \cap Z_2})$ can be extended to some f_1 in $\mathcal{O}_{\bar{A}_{Z_1}}(-\bar{X}_{Z_1})$. Now, starting from f_0 we find f_{12} over $Z_1 \cap Z_2$, and then f_1 and f_2 over Z_1 and Z_2 , respectively. They patch together to the desired f .

Proof of (2). Let I_j be the ideal of Z_j in Z for $j=1, 2, \dots$ and $m = m_{Z, s_0}$. Since for any $k \in \mathbb{Z}$ there is some j such that $m^k \supset I_j$, (2) is equivalent to the following

(2 bis) If \hat{Z} satisfies (H), then Z satisfies (H), where \hat{Z} is the completion of Z by the ideal m .

Proof of (2 bis). We have only to prove that the completion $\hat{\mathcal{X}}_Z = \text{proj-lim } \mathcal{X}_Z \times_Z Z/m^k$ of \mathcal{X}_Z with respect to the ideal $m\mathcal{O}_{\mathcal{X}_Z}$ has no function whose support lies on the zero section. For, if this is shown, then $\hat{\mathcal{X}}_Z$ is flat over Z by the assumption and hence $\hat{\mathcal{X}}_Z \times_{\hat{Z}} \{s_0\} = \mathcal{X}_Z \times_{\hat{Z}} \{s_0\}$ is reduced.

By assumption, for any general hyperplane defined by the equation $R=0$, the multiplication $R: \mathcal{O}_{\mathcal{X}_Z} \rightarrow \mathcal{O}_{\mathcal{X}_Z}$ is injective. Since the completion is a flat operation, $R: \mathcal{O}_{\hat{\mathcal{X}}_Z} \rightarrow \mathcal{O}_{\hat{\mathcal{X}}_Z}$ is also injective, which shows (2 bis). Thus the proof of Proposition 8 is completed. Q.E.D.

Proposition 7. *If X is normal, then the T in Proposition 6 is equal to S .*

Proof. By proposition 6, we may assume that $S = \text{Specan } \mathbb{C}[t]/(t^h)$ for some integer h . Let f_0 be any element of $\mathcal{O}_{\bar{A}}(-\bar{X})$. We shall denote by the same letter the trivial extension along S . Since $f_0 = 0$ on \bar{X} , there is some f_1 in $\mathcal{O}_{\bar{X}}$ such that $f_0 = tf_1$ on \bar{X} . Since X is normal, the restriction of f_1 to \bar{X} is also a function on X . Let g_1 be a representative of it in \mathcal{O}_A . Since $g_1 = f_1$ on \bar{X} , there is some f_2 in $\mathcal{O}_{\bar{X}}$ such that $f_1 = g_1 + tf_2$, hence $f_0 - tg_1 = t^2f_2$. Repeating the same process as above, we shall at last obtain elements g_1, \dots, g_{h-1} of \mathcal{O}_A (or $\mathcal{O}_{A \times_S}$) such that $f_0 - tg_1 - t^2g_2 - \dots - t^{h-1}g_{h-1} = 0$ on \bar{X} . The left hand side of the equality gives the desired extension of f_0 . Q.E.D.

Corollary 8. *When $N = n + 1$, $T = S$.*

Proof. When n is greater than one, X is normal. When $n = 1$, the Corollary will be proved in Section 3 (see proof of Corollary 10.)

Example 3. (F. Pham p.190 of [3]). $N=2$ and $n=1$. $A = \text{Specan } \mathbf{C}\{x_1, x_2\}$ and $X = \text{Specan } \mathbf{C}\{x_1, x_2\}/(x_1^3 + x_2^9)$. By a minimal embedded resolution of X , we obtain \bar{A} , \bar{X} , etc. Then, $\dim H^1(A, T_{\bar{A}}(\log \bar{X}))=2$. A semi-universal deformation is given by $S = \text{Specan } \mathbf{C}\{t_1, t_2\}$ and $\mathfrak{X} = \text{Specan } \mathbf{C}\{x_1, x_2, t_1, t_2\}/(x_1^3 + x_2^9 + t_1x_1x_2^6 + t_2x_1x_2^3)$ in $A \times S$. For each $t = (t_1, t_2)$ in S , $\dim H^1(\bar{A}_t, T_{\bar{A}_t}(\log \bar{X}_t))$ equals to 2 if $t_2=0$, and to 1 if $t_2 \neq 0$. On the other hand, F. Pham has shown that the singularity along the line $t_1=0$ is different from that along $t_1 \neq 0$ in connection with the versal flat deformations.

§3. Semi-Universal Deformations.

In this section we shall prove the following

Theorem 1. *Consider the situation (*). Then there exists a semi-universal object for all equisingular deformations of type (*), that is, there exists an equisingular deformation $\mathcal{X}, S, \bar{\mathcal{X}}, \bar{\mathcal{A}}, p, q$, etc. of type (*) such that for any equisingular deformation $\mathcal{X}', S', \bar{\mathcal{X}}', \bar{\mathcal{A}}', p', q'$, etc. of type (*), when we replace S' by a suitable neighborhood of s'_0 if necessary, there is a morphism $\alpha: S' \rightarrow S$ and $\mathcal{X}', \bar{\mathcal{X}}', p', q'$, etc. are obtained by the pull-back of $\mathcal{X}, \bar{\mathcal{X}}, p, q$, etc. by α , and moreover the tangent map $T_\alpha: T_{S', s'_0} \rightarrow T_{S, s_0}$ is uniquely determined.*

Proof. By Proposition 6, we have only to prove the existence of a semi-universal logarithmic deformation of $\bar{i}: \bar{X} \rightarrow \bar{A}$.

As in the proof of Proposition 5, embed \bar{A} in a point modification Z of a projective space \mathbf{P}^N . By Theorem 1 of [7], there exists a semi-universal logarithmic deformation

$$\begin{array}{ccc} \mathcal{E}^1 & \longrightarrow & \mathfrak{S}^1 \\ & \searrow & \swarrow \\ & S^1 & \end{array}$$

of $E \xrightarrow{\bar{i}} Z$. Over this we shall construct the desired deformation.

First, we shall prove the theorem under the condition that n is greater than one. Let $E = \bigcup_{i=1}^m E_i$ be the irreducible decomposition and $a = (a_1, \dots, a_m)$ a sequence of non-negative integers. Denote by E_a (resp. $Y_a, \mathcal{E}_a^1, \mathcal{Q}_a^1$) the infinitesimal neighborhood of E (resp. $Y \cap E, \mathcal{E}^1, \mathcal{Q}^1 \cap \mathcal{E}^1$) of order a_i along E_i (resp. $E_i, \mathcal{E}_i^1, \mathcal{E}_i^1$). Applying the relative Douady theorem (Theorem 2 of [9]) for S^1, \mathcal{E}_a^1 and $\mathcal{O}_{\mathcal{E}_a^1}$ instead of S, X and \mathcal{E} there, respectively, we obtain a universal family $\mathcal{Q}_a^2 \rightarrow_{S^1} \mathcal{E}_a^1 \times_{H_{S^1}(\mathcal{O}_{\mathcal{E}_a^1})} H_{S^1}(\mathcal{O}_{\mathcal{E}_a^1})$. We consider

a small enough neighborhood in $H_{S^1}(\mathcal{O}_{\mathcal{E}_a^1})$ of the point corresponding to $Y_a \cap E_a$. Then there is a closed analytic subset V_a of $H_{S^1}(\mathcal{O}_{\mathcal{E}_a^1})$ near that point which represents the flat families of locally product type in the sense of Definition 2. On the other hand, the tangent space of the functor of the deformation of locally product type is $H_a^1 = H^1(A, T_{\bar{A}}(\log E_a)(\log Y_a))$ by Exp. III of [4], where $T_{\bar{A}}(\log E_a)(\log Y_a)$ is the intersection of $T_{\bar{A}}(\log E_a)$ and $T_{\bar{A}}(\log Y_a)$ in $T_{\bar{A}}$. Therefore, there is a canonical surjection $T_{V_a} \rightarrow H_a^1$. Let M_a be the kernel and L_a some complementary space of it in T_{V_a} . We may suppose that V_a is embedded in a linear space $W_a = T_{V_a}$ and L_a is a linear subspace of W_a . Put $S_a = V_a \cap L_a$ and let $\mathcal{Q}_a \subset \mathcal{E}_a \subset \bar{\mathcal{A}}_a$ be the induced family over S_a , where $\mathcal{E}_a = \mathcal{E}_a^1 \times_{S^1} S_a$ and $\bar{\mathcal{A}}_a = \bar{\mathcal{A}}^1 \times_{S^1} S_a$.

Lemma. $\mathcal{Q}_a \subset \mathcal{E}_a \subset \bar{\mathcal{A}}_a$ over S_a gives a semi-universal deformation of locally product type of $Y_a \subset E_a \subset \bar{\mathcal{A}}$.

Proof. Let G be the automorphism group of an infinitesimal neighborhood of some order k of the origin of A . G acts on V_a if k is large enough. Let $\mathcal{Q}' \subset \mathcal{E}' \subset \bar{\mathcal{A}}'$ over B be any deformation of locally product type of $Y_a \subset E_a \subset A$. By the universality of the relative Douady space, there is a canonical morphism $\sigma: B \times G \rightarrow V_a \subseteq W_a$. If the order k is sufficiently large, $\sigma \times T_\sigma$ is sent onto M_a by T_σ . Applying the implicit function theorem to the composite morphism $B \times G \xrightarrow{\sigma} W_a \rightarrow W_a/L_a = M_a$, we obtain a morphism $s: B \rightarrow G$ such that $\sigma(b, s(b))$ falls into L_a for all b near the origin. This proves the Lemma. Q.E.D.

Note that S_a is an S^1 -complex space and if $a \leq b$, then there is a canonical morphism $S_a \rightarrow S_b$. By the exact sequence of sheaves:

$$0 \rightarrow T_{\bar{A}}(\log \bar{X}) \rightarrow T_{\bar{A}}(\log E_a)(\log Y_a) \rightarrow N_Y(-Y_a) \rightarrow 0,$$

where N_Y is the normal sheaf of Y in the sense of Proposition 1 of [7], we obtain the following exact sequence:

$$\begin{aligned} H^0(\bar{A}, T_{\bar{A}}(\log E_a)(\log Y_a)) &\rightarrow H^0(Y, N_Y(-Y_a)) \rightarrow H^1(A, T_{\bar{A}}(\log \bar{X})) \\ &\rightarrow H^1(\bar{A}, T_{\bar{A}}(\log E_a)(\log Y_a)) \rightarrow H^1(Y, N_Y(-Y_a)). \end{aligned}$$

We assume that $a = (a_1, \dots, a_M)$ is large enough and the a_i have some appropriate ratios each other. Then $H^1(Y, N_Y(-Y_a))$ vanishes by Serre's theorem. On the other hand, the image of the map $H^0(\bar{A}, T_{\bar{A}}(\log E_a)(\log Y_a)) \rightarrow H^0(Y, N_Y(-Y_a))$ coincides with the intersection of $H^0(Y, N_Y(-Y_a))$ and the image of the map $H^0(\bar{A}, T_{\bar{A}}(\log E)) \rightarrow H^0(T, N_Y)$. Thus the former is surjective

under the same kind of the conditions on the a_i as above. Therefore, under the assumptions above, $H^1(\bar{A}, T_{\bar{A}}(\log \bar{X}))$ is mapped bijectively onto $H^1(A, T_{\bar{A}}(\log E_a)(\log Y_a))$. This implies that the canonical map $S_a \rightarrow S_b$ is a closed immersion (near the origin) for such a and b , $a \geq b$. Since the local ring of the complex space is noetherian, we finally conclude that (the germ of) the spaces S_a are stationary for large a and appropriate ratios. Denote by S the space thus obtained. Put $\mathcal{E} = \mathcal{E}^1 \times_{S^1} S$ and $\bar{\mathcal{A}} = \bar{\mathcal{A}}^1 \times_{S^1} S$.

Summing up, we have obtained a complex space S over which a sequence $\{\mathcal{Q}_a \subset \mathcal{E}_a \subset \bar{\mathcal{A}}\}$ of semi-universal deformations stands. We have to construct *analytic* spaces from the data given above. Let \hat{A}_S be the completion of $A_S = A \times S$ along $o \times S$. Then, $\hat{\mathcal{A}} = \bar{\mathcal{A}} \times_{\hat{A}_S} \hat{A}_S$ is isomorphic to the completion of $\bar{\mathcal{A}}$ along \mathcal{E} . We have a subspace \mathcal{Q} of $\bar{\mathcal{A}}$ corresponding to the sequence $\{\mathcal{Q}_a\}$. Note that there is a canonical isomorphism from $\bar{\mathcal{A}} - \mathcal{E}$ to $\hat{A}_S - (o \times S)$. Let $\hat{\mathcal{X}}$ be the smallest subspace of \hat{A}_S which induces $\mathcal{Q} - \mathcal{E} \cap \mathcal{Q}_S$ in $\hat{A} - (o \times S)$. By Theorem 1.4 of [1] and using the technique of Chapter II of [2], for arbitrary number c we obtain a complex subspace \mathcal{X} of A_S which induces the same thing as $\hat{\mathcal{X}}$ on the c -th infinitesimal neighborhood of $o \times S$ in A_S . Then by the main theorem of [6], if c is large enough, the completion of \mathcal{X} along $o \times S$ is isomorphic to $\hat{\mathcal{X}}$. We shall show that the space \mathcal{X} thus obtained is the desired semi-universal deformation.

Let $\mathcal{X}' \rightarrow B$ (and $\bar{\mathcal{X}}'$, etc.) be any other equisingular deformation of type (*). For large c , the c -th infinitesimal neighborhood \mathcal{X}'_c is isomorphic to the pull back of \mathcal{X}_c by a morphism $\alpha_c: B \rightarrow S$. By [6] again, \mathcal{X}' is isomorphic to the pull back of \mathcal{X} by $\alpha = \alpha_c$. Thus, the proof of the Theorem 1 in case of $n \geq 2$ is completed.

In case $n = 1$, there is nothing difficult. The movements of the curve segments \mathcal{X}^1 in $\bar{\mathcal{A}}^1$ have no obstructions, since they are strictly local. We can make all at once the semiuniversal family instead of the step-by-step construction above. Q.E.D.

Corollary 9. *If $S = T$ in the notation of Proposition 6 and $H^2(\bar{A}, T_{\bar{A}}(\log \bar{X})) = 0$, then there exists a semi-universal equisingular deformation of type (*) whose base space is non-singular.*

Corollary 10. *If $N = 2$ and $n = 1$, then there exists a semi-universal equisingular deformation whose base space is non-singular.*

Proof. Let $(*)^m$ be the minimal embedded resolution of X . Since any

equisingular deformation is obtained as of type $(*)^m$ by Proposition 5, we may restrict our attention to the situation $(*)^m$. Since the point blowing ups have no obstructions, the base space S^m of the semi-universal logarithmic deformation of $\bar{X}^m \subset \bar{A}^m$ is non-singular. Then the space $\mathcal{X}^m = \mathcal{X}_{S^m}^m$ obtained by a blowing down is a Cartier divisor in $A \times S^m$. Therefore, the geometric fiber $\mathcal{X}^m \times_{S^m} \{s_0\}$ is also a Cartier divisor and hence reduced. Thus \mathcal{X}^m is flat over S^m . On the other hand, the consideration of dimensions shows that $H^2(\bar{A}, T_{\bar{A}}(\log \bar{X})) = 0$. Thus Corollary 10 is proved by Corollary 9. Q.E.D.

Proof of Corollary 8 continued. Since any \mathcal{X}_Z (in the notation of Proposition 6) is obtained from \mathcal{X}^m by a pulling back, \mathcal{X}_Z is flat. Q.E.D.

Remark 5. Corollary 10 was first obtained by J. Wahl [14].

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