Publ. RIMS, Kyoto Univ.
16 (1980), 269–287

On Parabolic Functions of One-Dimensional Quasidiffusions

By

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§1. Introduction and Definitions

In this paper we will deal with quasidiffusions $X=(X_t)_{t\geqq 0}$ on [0, 1) assuming that 0 is a reflecting regular boundary, 1 is an accessible or entrance boundary and X is killed as soon as it hits the boundary 1. We shall give a Martin representation of space-time-excessive functions for X. This includes in particular a representation of all parabolic functions $f(x, t)$ satisfying a certain integrability condition by minimal ones (see Theorem 2 below).

We shall show that the set of minimal points of space-time Martin boundary is homeomorphic to $(0, \infty[$; in particular the minimal parabolic functions form a one-parameter family (k_t) $(t\in(0, \infty])$. If $t<\infty$, the function $k_t(x, s)$ is the limit (in a weak sense) of the transition density $p(t-s, x, y)$ or its derivative with respect to y, where y converges to the boundary 1. In the limit circle case we will give an uniformly convergent expansion of k_t $(t\in(0, \infty])$ in eigenfunctions of the infinitesimal operator of X.

Using these results we consider the problem which minimal parabolic functions factorize (i.e. have the form $k_t(x, s)=\phi_t(x)\psi_t(s)$) and which factorizing parabolic functions are minimal. (For some Markov chains and diffusion processes this problem was studied in [9], [11].)

As another application we shall give a necessary and sufficient condition in order that for a parabolic function f the process $\{f(X_t, s+t), t\in[a, b]\}$ is a martingale. Parabolic functions which are martingales on the trajectories of X are used to determine the probabilities that X ever hits some time-varying boundaries (see e.g. [12]) and were studied e.g. in [2], [8].

The assumption that the boundary 1 is accessible or entrance is essential.

Communicated by K. Itô, March 17, 1978.
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If 1 is natural then the transition densities and their derivatives have a complete other asymptotic behaviour and therefore the minimal parabolic functions are of other type. This case will be studied in a further note (see [6]).

In the following we will use the terminology and the results of [5] without further explanation.

Let (m, p) be a canonical pair (see [5]) and $X:=(X_t)_{t\geqq 0}$ the corresponding quasidiffusion on $E:=\operatorname{supp} m\backslash\{1\}\subseteqq[0, 1)$ with the condition that 0 is reflecting regular and (X_t) is killed as soon as it hits 1. The transition probabilities $P_t(x, A)$ of X have densities $p(t, x, y)$ with respect to dm:

$$P_t(x, A) = \int_A p(t, x, y)m(dy)\,.$$

Let $\tilde{X}:=(\tilde{X}_t)_{t\geqq 0}$ be the corresponding to X *space-time process*, i.e. a Markov standard process on $\tilde{E}:=E\times(0, \infty)$ with the transition probabilities $\tilde{P}_t$ generated by

$$\tilde{P}_t((x, s), A\times\Gamma) = P_t(x, A)\cdot\chi_\Gamma(s+t) \qquad ((x, s)\in\tilde{E},\ t>0,\ A\in\mathfrak{B}_E,\ \Gamma\in\mathfrak{B}_{(0,\infty)})\,,$$

where χ_Γ denotes the indicator function of Γ and where $\mathfrak{B}_G$ is the trace of the σ-algebra of Borel subsets of R_1 on G $(G\subseteqq R_1)$. Furthermore let $\tilde{\tilde{X}}$ be the Markov standard process on $\tilde{E}$ with the transition probabilities $\tilde{\tilde{P}}_t$ generated by

$$\tilde{\tilde{P}}_t((x, s), A\times\Gamma) = P_t(x, A)\chi_\Gamma(s-t)$$
$$((x, s)\in\tilde{E},\ t>0,\ A\in\mathfrak{B}_E,\ \Gamma\in\mathfrak{B}_{(,0\infty)})\,.$$

$\tilde{\tilde{X}}$ is called the *coprocess of* $\tilde{X}$. The operators $\tilde{\tilde{T}}_t$ and $\tilde{T}_t$ acting on positive (and on bounded) measurable functions are defined by

$$\tilde{T}_t f(x) := \int f(y, s+t)P(t, x, dy) \qquad (\tilde{x} = (x, s)\in\tilde{E})$$

and

$$\tilde{\tilde{T}}_t f(x) := \int f(y, s-t)P(t, x, dy) \quad (t<s),\ = 0 \quad (t\geqq s) \quad (\tilde{x} = (x, s)\in\tilde{E})\,.$$

In the following the elements of $\tilde{E}$ always are denoted by $\tilde{x}, \tilde{y}$ etc. with $\tilde{x}=(x, s)$, $\tilde{y}=(y, t)$ etc.

The resolvent kernels

$$\tilde{R}_\lambda(\tilde{x}, \tilde{A}) = \int_0^\infty e^{-\lambda u}\tilde{P}_u(\tilde{x}, \tilde{A})du$$

and

$$\tilde{\tilde{R}}_\lambda(\tilde{x}, \tilde{A}) = \int_0^\infty e^{-\lambda u} \tilde{\tilde{P}}_u(\tilde{x}, \tilde{A}) du \qquad (\tilde{x} \in \tilde{E},\ \tilde{A} \text{ Borel subset of } \tilde{E},\ \lambda > 0)$$

of $\tilde{X}$ and $\tilde{\tilde{X}}$ respectively are absolutely continuous with respect to $d\tilde{m} := dmdt$ on $\tilde{E}$ and have the densities

$$\tilde{r}_\lambda(\tilde{x}, \tilde{y}) = e^{-\lambda(t-s)} p(t-s, x, y) \chi_{(0,\infty)}(t-s)$$

and

$$\tilde{\tilde{r}}_\lambda(\tilde{x}, \tilde{y}) = \tilde{r}_\lambda(\tilde{y}, \tilde{x}) \qquad (\tilde{x}, \tilde{y} \in \tilde{E},\ \lambda > 0)$$

respectively. A nonnegative nearly Borel function f on $\tilde{E}$ is called *excessive with respect to* $\tilde{X}$ (shortly said *excessive*) if

$$\lambda \tilde{R}_\lambda f \leqq f \qquad (\lambda > 0),\ \lim_{\lambda \to \infty} \lambda \tilde{R}_\lambda f = f$$

pointwise. An excessive function f with $\lim_{\lambda \to 0} \lambda \tilde{R}_\lambda f = 0$ is called *purely excessive.*

Let f be excessive. If for every $\tilde{x} = (x, s) \in \tilde{E}$ and every open subset $\tilde{A}$ of $\tilde{E}$ with compact closure in $\tilde{E}$ we have

(1) $$f(\tilde{x}) = E_x f(X_{\tau_{\tilde{A}}}, s + \tau_{\tilde{A}}) \qquad (\tau_{\tilde{A}} := \inf \{t > 0 \mid \tilde{X}_t \notin \tilde{A}\})$$

then f is said to be a *harmonic function for* $\tilde{X}$ or a *parabolic function for* X (shortly said a *parabolic function*). The excessive (harmonic etc.) functions for $\tilde{\tilde{X}}$ will be called *coexcessive* (*coharmonic* etc.) *for* $\tilde{X}$ or shortly *coexcessive* (*coharmonic etc.*).

One can show (see e.g. [8], [13]) that a continuous function h on $\tilde{E}$ with continuous derivatives $\frac{\partial}{\partial t} h$ and $D_m D_p h$ satisfying $\frac{\partial}{\partial t} h + D_m D_p h = 0$ $\left(\frac{\partial}{\partial t} h - D_m D_p h = 0\right)$ on $\tilde{E}$ is parabolic (coparabolic). In particular if $\mu \geqq \lambda_0$ the function

$$\tilde{f}_\mu(\tilde{x}) := e^{-\mu s} \varphi(x, \mu) \qquad (\tilde{x} \in \tilde{E})$$

is parabolic and the function

$$\tilde{\tilde{f}}_\mu(x) := e^{\mu s} \varphi(x, \mu) \qquad (\tilde{x} \in \tilde{E})$$

is coparabolic.

The process $\tilde{\tilde{X}}$ starting in $\tilde{x} = (x, s)$ is killed at time s. This implies

Proposition 1. *Every coexcessive function is purely coexcessive.*

Proof. If f is coexcessive, we have for $\lambda > 0$

$$\lambda\tilde{\tilde{R}}_\lambda f(\tilde{x}) = \lambda \int_0^s \int_0^1 e^{-\lambda(s-t)} p(s-t, y, x) f(y, t) m(dy) dt$$
$$= \lambda \int_0^s e^{-\lambda(s-t)} (T_{s-t} f(\cdot, t))(x) dt \leqq f(\tilde{x}) \qquad (\tilde{x} \in \tilde{E})$$

by definition. Thus the integral $\int_0^s (P_{s-t} f(\cdot, t))(x) dt$ exists and hence we have $\lambda\tilde{\tilde{R}}_\lambda f(\tilde{x}) \to 0$ for $\lambda \to 0$.

Especially $\tilde{\tilde{f}}_\mu$ is purely coexcessive for every $\mu \geqq \lambda_0$. But generally $\tilde{f}_\mu$ is not purely excessive e.g. if 1 is entrance then $\tilde{f}_{\lambda_0} \equiv 1$ and the identity $\lambda\tilde{R}_\lambda \tilde{f}_{\lambda_0} = \tilde{f}_{\lambda_0}$ holds.

The application of the theory of Martin boundaries to X requires at first to check that X satisfies some basic assumptions of this theory. These are the so called conditions KW (see [10]) which can be formulated in our case as follows:

(i) $\tilde{R}_0(\cdot, \tilde{K})$ is bounded for every compact subset $\tilde{K}$ of $\tilde{E}$,

(ii) $\lim_{\lambda \to \infty} \lambda\tilde{\tilde{R}}_\lambda f(\tilde{x}) = f(\tilde{x})$ $(\tilde{x} \in \tilde{E})$ holds for every $f \in C_c(\tilde{E}) := \{f \mid f$ real-valued continuous function on $\tilde{E}$ having compact support$\}$,

(iii) $\tilde{\tilde{R}}_\lambda f$ is continuous and bounded for every bounded real-valued Borel-function f on $\tilde{E}$ having compact support and for every $\lambda \geqq 0$,

(iv) $\int_{\tilde{E}} f \tilde{R}_\lambda g \, d\tilde{m} = \int_{\tilde{E}} g \tilde{\tilde{R}}_\lambda f \, d\tilde{m}$ ($\lambda \geqq 0$; f, g positive Borel-functions on $\tilde{E}$).

Proposition 2. *The space-time process $\tilde{X}$ satisfies the condition KW.*

Proof. (i): If $\tilde{K}$ is compact there exist constants l, u, v with $0 < l < 1$; $0 < u < v < \infty$ such that $\tilde{K} \subset [0, l] \times [u, v]$. Thus

$$\tilde{R}_0(\tilde{x}, \tilde{K}) \leqq v - u \qquad (\tilde{x} \in \tilde{E}).$$

(ii): Let $f \in C_c(\tilde{E})$. Using the uniform continuity of f it follows that for every $\varepsilon > 0$ there exists *a* $\delta > 0$ such that

$$|f(x, t) - f(x, s)| < \varepsilon \qquad (x \in E,\ t \in (s-\delta, s),\ s > 0).$$

Because $(T_h)_{h \geqq 0}$ is a strongly continuous semigroup, for every $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that

$$|(T_h f(\cdot, s))(x) - f(x, s)| < \varepsilon \qquad (x \in E,\ s > 0,\ h < \delta_1).$$

Summarizing these remarks it follows

$$|(T_h f(\cdot, t))(x) - f(x, s)| < 2\varepsilon \qquad (x \in E,\ s > \delta,\ t \in (s-\delta, s),\ h < \delta_1),$$

and thus we have

$$\lim_{\lambda\to\infty}\lambda\tilde{\tilde{R}}_\lambda f(\tilde{x})=\lim_{\lambda\to\infty}\lambda\int_0^{s-\delta_1}e^{-\lambda(s-t)}(T_{s-t}f(\cdot,t))(x)dt$$
$$+\lim_{\lambda\to\infty}\lambda\int_{s-\delta_1}^{s}e^{-\lambda(s-t)}(T_{s-t}f(\cdot,t))(x)dt$$
$$=\lim_{\lambda\to\infty}\lambda\int_{s-\delta_1}^{s}e^{-\lambda(s-t)}(T_{s-t}f(\cdot,t))(x)dt=f(x)\,.$$

(iii): It suffices to prove (iii) for $f=\chi_{\tilde{A}\cap\tilde{E}}$, where $\tilde{A}=[0,l]\times[u,v]$ is a compact rectangle of $I\times(0,\infty)$. Let f be of such type and suppose (x_0,s_0), $(x_1,s_1)\in\tilde{E}$. Then

$$|\tilde{\tilde{R}}_\lambda f(x_0,s_0)-\tilde{\tilde{R}}_\lambda f(x_1,s_1)|\leqq\iint_{\tilde{A}}|p(s_0-t,x_0,y)-p(s_1-t,x_1,y)|m(dy)dt\,.$$

If $v<s_0$ the right side tends to zero for $(x_1,s_1)\to(x_0,s_0)$ because $p(h,x,y)$ is uniformly continuous on $[h_0,h_1]\times[0,l]^2$ $(0<h_0<h_1<\infty,\ l<1)$.

If $v<s_0$ the right side of the inequality can be written as $\int_u^{s_0-\eta}\int_0^l\cdots$ $\cdots m(dy)dt+\int_{s_0-\eta}^{s_0}\int_0^l\cdots m(dy)dt$ for every η with $0<\eta<s_0$. The first integral tends to zero for $(x_1,s_1)\to(x_0,s_0)$ as we have shown above. The second is bounded by 2η. Thus (iii) is proved.

(iv): Using the definition of $\tilde{R}_\lambda$ and $\tilde{\tilde{R}}_\lambda$ the proof follows by an easy calculation.

§2. A Martin Boundary for the Space-Time Process

At first let us define Martin boundaries for the space-time process X following the general line of Meyer [9]. The boundary very depends on the so called Martin kernel used for the construction, i.e. on a renormed resolvent kernel. To construct a Martin boundary for X which describes sufficiently many excessive functions, e.g. the bounded ones, it is necessary to renorm the resolvent kernels by a suitable purely coexessive function. Therefore we define

$$n(\tilde{y}):=\tilde{\tilde{f}}_{\lambda_0}(\tilde{y})=e^{\lambda_0 t}\varphi(y,\lambda_0)\,,\qquad d\zeta(y):=n(\tilde{y})m(d\tilde{y})\,,$$
$$k_\lambda(\tilde{x},\tilde{y}):=\frac{\tilde{g}_\lambda(\tilde{x},\tilde{y})}{n(\tilde{y})}\,,$$
$$\check{K}_\lambda(\tilde{x},d\tilde{y}):=k_\lambda(\tilde{x},\tilde{y})\zeta(d\tilde{y})=\tilde{R}_\lambda(\tilde{x},d\tilde{y})\,,$$
$$\tilde{\tilde{K}}_\lambda(\tilde{x},d\tilde{y}):=k_\lambda(\tilde{y},\tilde{x})\zeta(d\tilde{y})=\frac{n(\tilde{y})}{n(\tilde{x})}\tilde{\tilde{R}}_\lambda(\tilde{x},d\tilde{y})\qquad(\lambda\geqq0,\ \tilde{x},\tilde{y}\in\tilde{E})\,.$$

We remark $n(\tilde{y})\equiv 1$ if the boundary 1 is entrance.

Let d_0 be a metric on $\tilde{E}$ generating the usual topology on $\tilde{E}$.

Definition ([9]). A compact metric space $(\tilde{F}, \tilde{d})$ is called a *Martin compact for* $\tilde{X}$ if

(a): $\tilde{E}$ is a dense subset of $\tilde{F}$ such that the injection from $(\tilde{E}, d_0)$ into $(\tilde{F}, \tilde{d})$ is continuous,

(b): for every $f\in C_c(\tilde{E})$ the function $\tilde{\tilde{K}}_0 f$ can be extended to a continuous function on $\tilde{F}$, again denoted by $\tilde{\tilde{K}}_0 f$.

The set $\tilde{F}':=\tilde{F}\backslash\tilde{E}$ is called a *Martin boundary of* $\tilde{X}$.

For any $\tilde{y}\in\tilde{F}$ by $f\to\tilde{\tilde{K}}_0 f(\tilde{y})$ $(f\in C_c(\tilde{E}))$ a Radon measure on $\tilde{E}$ is given which is absolutely continuous with respect to $d\zeta$. The corresponding density for $\tilde{y}\in\tilde{E}$ coincides with $k_0(\cdot, \tilde{y})=: k_{\tilde{y}}(\cdot)$ and is denoted for $\tilde{y}\in\tilde{F}'$ also by $k_{\tilde{y}}(\cdot)$. For every $\tilde{y}\in\tilde{F}$ the function $k_{\tilde{y}}(\cdot)$ is excessive ([10]). Let d_1 be a metric on $\tilde{E}$ such that a sequence $(\tilde{x}_n)$ of points $\tilde{x}_n\in\tilde{E}$ is a d_1-Cauchy sequence if and only if either $|x_n-x_m|+|s_n-s_m|\to 0$ $(n, m\to\infty)$ or $s_n\to\infty$ $(n\to\infty)$ independent of x_n. It is clear that the completion of the metric space $(\tilde{E}, d_1)$ is equal (up to isomorphism) to $\tilde{F}:=(F\times[0, \infty))\cup\{(1, \infty)\}$ with the extension $\tilde{d}_1$ of d_1 to $\tilde{F}$.

Theorem 1. *Let* 1 *be an entrance or an accessible boundary. Then* (F, d_1) *is a Martin compact for the space-time process* X.

Proof. It is clear that $(\tilde{F}, \tilde{d}_1)$ is compact and that the point (a) of the definition above is satisfied. Thus we shall prove only (b).

Let $f\in C_c(\tilde{E})$. Thus there exist numbers l, u, v with $0<l<1$, $0<u<v<\infty$ such that f vanishes outside of $[0, l]\times[u, v]$. We will prove that

$$\tilde{\tilde{K}}_0 f(\tilde{y}) = \int_0^1\int_0^t p(t-s, x, y)\frac{\varphi(x, \lambda_0)}{\varphi(y, \lambda_0)}e^{-\lambda_0(t-s)}f(x, s)m(dx)ds \qquad (\tilde{y}\in\tilde{E})$$

has a continuous extension to $\tilde{F}$ and, moreover, we shall determine the densities $k_{\tilde{y}_0}(\cdot)$ for $\tilde{y}_0\in\tilde{F}'$. To this purpose we consider three cases.

First Case: Let $\tilde{y}_0=(y_0, 0)$ for some $y_0\in F$. Then $\tilde{\tilde{K}}_0 f(\tilde{y})=\tilde{\tilde{K}}_0 f(y, t)=0$ in neighbourhood of $\tilde{y}_0$ (namely if $t<u$). Thus $\tilde{\tilde{K}}_0 f$ can be continuous extended to $\tilde{y}_0$ and it holds

$$k_{\tilde{y}_0}(\tilde{x}) = 0 \qquad (\tilde{y}_0 = (y_0, 0),\ y_0\in F;\ \tilde{x}\in\tilde{E})\,. \tag{2}$$

Second Case: Let $\tilde{y}_0=(1, t_0)$ for some $t_0\in(0, \infty)$. Firstly we suppose 1 is entrance. Then $\lambda_0=0$ and $\varphi(\cdot, \lambda_0)=1$. From [5] Theorem 1 follows the identity

$$(3)\qquad \lim_{y\to 1}\int_0^1 p(t_0-s,x,y)f(x,s)m(dx)=\int_0^1 p(t_0-s,x,1)f(x,s)m(dx)$$

for every $s<t_0$, where $p(h,x,1)$ is an uniquely determined strictly positive continuous function of $(x,h)\in\tilde{E}$.

The integrals on the left side of (3) are dominated by $C:=\max|f(\tilde{z})|<\infty$. Integrating (3) with respect to s and using Fubini's theorem and Lebesgue's dominated convergence theorem it follows for $t>0$

$$\lim_{y\to 1}\tilde{\tilde{K}}_0 f(y,t)=\int_0^1\int_0^t p(t-s,x,1)f(x,s)m(dx)ds\,.$$

Now we shall estimate $\tilde{\tilde{K}}_0 f(y,t)-\tilde{\tilde{K}}_0 f(y,t_0)$. We can assume $t<t_0$. Then this difference is equal to

$$\int_0^1\int_0^t (p(t-s,x,y)-p(t_0-s,x,y))f(x,s)dsm(dx)$$
$$-\int_0^1\int_t^{t_0} p(t_0-s,x,y)f(x,s)dsm(dx)\,.$$

The absolute value of the first integral can be estimated by $\int_0^t |(T_{t_0-t}f(\circ,s))(y)-f(y,s)|ds$. The strong continuity of (T_h) implies $(T_{t_0-t}f(\circ,s))(y)-f(y,s)\xrightarrow[t\to t_0]{}0$ uniformly in y for every s. Thus by Lebesgue's dominated convergence theorem the first integral tends to zero for $t\to t_0$.

The second integral vanishes for $t—t_0$ uniformly in y because $|\int_0^1 p(t_0-s,x,y)f(x,s)m(dx)|\leqq C<\infty$ uniformly in $x\in E$ and $s\leqq t_0$.

Summarizing we obtain under the assumption that 1 is entrance

$$\lim_{\tilde{y}\to\tilde{y}_0}\tilde{\tilde{K}}_0 f(\tilde{y})=\int_0^1\int_0^{t_0} p(t_0-s,x,1)f(x,s)m(dx)ds$$

and

$$(4)\qquad k_{\tilde{y}_0}(x)=p(t_0-s,x,1)\chi_{(0,\infty)}(t_0-s))$$
$$(\tilde{x}=(x,s)\in\tilde{E},\ y_0=(1,t_0),\ t_0\in(0,\infty))\,.$$

Now let 1 be accessible. Then by $dm^*=\varphi^2(\circ,\lambda_0)dm$ and $dp^*=\varphi^{-2}(\circ,\lambda_0)dp$ a new speed measure m^* and a new scale p^* on [0, 1) are given. The boundary 1 is entrance for (m^*,p^*) (see [5]) and the corresponding transition density is

$$p^*(h,x,y)=\frac{e^{-\lambda_0 h}p(h,x,y)}{\varphi(x,\lambda_0)\varphi(y,\lambda_0)}\qquad (x,y\in E,\ h>0)\,.$$

Applying the preceding results to (m^*, p^*) it follows

$$\lim_{y\to 1}\int_0^1 p(t-s, x, y)e^{-\lambda_0(t-s)}\frac{\varphi(x, \lambda_0)}{\varphi(y, \lambda_0)}f(x, s)m(dx)$$

$$(5)\qquad = \int_0^1 p^*(t-s, x, 1)\varphi^2(x, \lambda_0)f(x, s)m(dx) \qquad (0<s<t).$$

We know from [5] that $D_p\varphi(1, \lambda_0)=\lim_{x\to 1}\frac{\varphi(x, \lambda_0)}{p(x)-p(1)}$ exists and is finite and negative. Defining

$$D_p p(h, x, 1):= p^*(h, x, 1)e^{\lambda_0 h}\varphi(x, \lambda_0)D_p\varphi(1, \lambda_0) \qquad (h>0,\ x\in E),$$

the right side of (5) can be written as

$$\int_0^1 e^{-\lambda_0(t-s)}\varphi(x, \lambda_0)D_p p(t-s, x, 1)f(x, s)m(dx)\cdot(D_p\varphi(1, \lambda_0))^{-1},$$

and on an analogous way as above it can be shown

$$\lim_{\tilde{y}\to\tilde{y}_0}\tilde{\tilde{K}}_0 f(y) = (D_p\varphi(1, \lambda_0))^{-1}\int_0^1\int_0^{t_0} e^{-\lambda_0 t_0}D_p p(t_0-s, x, 1)d\zeta(\tilde{x})$$

and

$$(6)\qquad k_{\tilde{y}_0}(\tilde{x}) = e^{-\lambda_0 t_0}\frac{D_p p(t_0-s, x, 1)}{D_p\varphi(1, \lambda_0)}\chi_{(0\ \infty)}(t_0-s)$$

$$(\tilde{x} = (x, s)\in\tilde{E},\ \tilde{y}_0 = (1, t_0),\ t_0\in(0, \infty))$$

under the assumption that 1 is accessible.

Third Case: $\tilde{y}_0=(1, \infty)$. If 1 is entrance, from [5], Theorem 3 it follows the identity

$$\lim_{t\to\infty}\int_0^1\int_0^t p(t-s, x, y)f(x, s)dsm(dx) = (m(1))^{-1}\int_0^1\int_0^\infty f(x, s)dsm(dx).$$

This implies

$$(7)\qquad k_{\tilde{y}_0}(\tilde{x}) = (m(1))^{-1} \qquad (\tilde{x}\in\tilde{E}).$$

If 1 is accessible by the same transformation and the same method as in the second case it follows

$$\lim_{t\to\infty}\tilde{\tilde{K}}_0 f(\tilde{y}) = \left(\int_0^1 \varphi^2(x, \lambda_0)m(dx)\right)^{-1}\int_0^1\int_0^\infty \varphi(x, \lambda_0)f(x, s)e^{-\lambda_0 s}d\zeta(\tilde{x})$$

and

$$(8)\qquad k_{\tilde{y}_0}(x) = \left(\int_0^1 \varphi^2(x, \lambda_0)m(dx)\right)^{-1}e^{-\lambda_0 s}\varphi(x, \lambda_0) \qquad (\tilde{x}\in\tilde{E}).$$

Thus Theorem 1 is proved.

Corollary. *Let* $\int_0^1 p^2 dm < \infty$, *i.e.* (m, p) *is in the limit circle case (see e.g.* [4], [5]). *Then we have an uniformly converging spectral expansion for* $p(h, x, 1)$ (*if* 1 *is intrance*) *and for* $D_p p(h, x, 1)$ (*if* 1 *is accessible*) (*see* [5], *Theorem* 2).

Hence we can give such an expansion for k_{y_0}:

$$k_{\tilde{y}_0}(\tilde{x}) = \left(\frac{1}{m(1)} + \sum_{k=1}^{\infty} e^{\lambda_k(t_0-s)}\varphi(x, \lambda_k)\varphi(1, \lambda_k)\tau_k\right)\chi_{(0,\infty)}(t_0-s) \tag{9}$$
$$(\tilde{x}\in\tilde{E},\ \tilde{y}_0 = (1, t_0),\ t_0\in(0, \infty))$$

if 1 *is entrance with* $\int_0^1 p^2 dm < \infty$ *and*

$$\begin{aligned} k_{\tilde{y}_0}(\tilde{x}) = (e^{-\lambda_0 s}\varphi(x, \lambda_0)\tau_0 \\ + \sum_{k=1}^{\infty} e^{\lambda_k(t_0-s)}\varphi(x, \lambda_k)\frac{D_p\varphi(1, \lambda_k)}{D_p\varphi(1, \lambda_0)}\cdot e^{-\lambda_0 t_0}\tau_k)\chi_{(0,\infty)}(t_0-s) \end{aligned} \tag{10}$$

$(\tilde{x}\in\tilde{E}, \tilde{y}_0=(1, t_0), t_0\in(0, \infty))$ *if* 1 *is accessible with* $\int_0^1 p^2 dm < \infty$ (*i.e. regular*), *where the series in* (9) *and* (10) *are uniformly converging in* $t \geqq t_0 > 0$ *and* $x\in[0, 1)$.

Theorem 2 below gives an integral representation of $\tilde{X}$-excessive and -parabolic functions. To formulate it we need some more notations and propositions from the theory of Martin boundaries which will now be given (see [10], also [7]).

Let f be excessive. From the mentioned theory it is known that the limit

$$L(f) := \lim_{\lambda\to\infty} \lambda \int_{\tilde{E}} (1-\tilde{\tilde{K}}_\lambda 1)(\tilde{y})f(\tilde{y})\zeta(d\tilde{y})$$

exists (possibly infinite). A short calculation using the definition of $\tilde{\tilde{K}}_\lambda$ and ζ in our case implies

$$L(f) = \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda s}\left(\int_0^1 f(x, s)\varphi(x, \lambda_0)m(dx)\right)ds\,.$$

Because f is excessive we have

$$(T_h f(\cdot, t+h))(x) \leqq f(x, t) \qquad (h, t>0,\ x\in E)\,.$$

Thus

$$\int_0^1 e^{h\lambda_0} f(y, t+h)\varphi(y, \lambda_0)m(dy) = \int_0^1 f(y, t+h)(T_h\varphi(\cdot, \lambda_0))(y)m(dy)$$
$$= \int_0^1 (T_h f(\cdot, t+h))(x)\varphi(x, \lambda_0)m(dx) \leqq \int_0^1 f(x, t)\varphi(x, \lambda_0)m(dx) \qquad (h, t>0)\,.$$

Therefore the function $h \to \int_0^1 e^{\lambda_0 h} f(y, t+h)\varphi(y, \lambda_0)m(dy)$ decreases if t is fixed and h increases. Hence the function $h \to e^{-\lambda_0 h} \int_0^1 f(y, t-h)\varphi(y, \lambda_0)m(dy)$ increases if h increases from 0 to t for every $t > 0$. Thus the limit $\lim_{h \downarrow 0} \int_0^1 f(y, h)\varphi(y, \lambda_0)m(dy)$ exists and by properties of the Laplace-transformation we have

$$L(f) = \lim_{h \downarrow 0} \int_0^1 f(y, h)\varphi(y, \lambda_0)m(dy) \tag{11}$$

with $L(f) \in (0, \infty]$ if f is excessive and non identical zero. In the theory of Martin boundaries every excessive function f satisfying a certain condition (e.g. $L(f) < \infty$ in our case) is represented by minimal excessive functions (see below). An essential point is to choose the purely coexcessive function n in the definition of $\tilde{K}_\lambda$ in such a way that as many as possible excessive functions f satisfy the mentioned condition. Here we have taken $n(y, t) = \varphi(y, \lambda_0)e^{+\lambda_0 t}$. This implies e.g. that the bounded excessive functions and the functions $\tilde{f}_\mu(x, s) = e^{-\mu s}\varphi(x, \mu)$ $(\mu \geqq \lambda_0)$ are included if 1 is entrance or accessible because $\varphi(\cdot, \lambda_0)$ is bounded and m-integrable and $\varphi(\cdot, \lambda_0)$ is bounded (if 1 is accessible) or m-integrable (if 1 is entrance) (see [5]).

An excessive function $f \neq 0$ is called *extreme* if for every two excessive functions f_1, f_2 with $f = \alpha f_1 + (1-\alpha)f_2$ for some $\alpha \in (0, 1)$ it follows $f = f_1 = f_2$.

We say that the point $\tilde{y} \in \tilde{F}$ is a *minimal point* if $\tilde{y} \in \tilde{E}$ or $\tilde{y} \in \tilde{F}'_m := \{\tilde{z} \in \tilde{F}' \mid L(k_{\tilde{z}}) = 1,\ k_{\tilde{z}} \text{ is extreme and parabolic}\}$. The set of minimal points of $\tilde{F}$ is a Borel set and we denote it by $\tilde{F}_m$. (Using the fact that for the process $\tilde{X}$ considered here the function $k_{\tilde{y}}$ for no $\tilde{y} \in \tilde{E}$ is harmonic (the proof is not difficult and omitted here) one can show that the definition given above coincides with the definition of minimal points given in [10]. See also [7].)

The functions $k_{\tilde{y}} (\tilde{y} \in \tilde{F}_m)$ are also called *minimal.*

As already said, from the theory of Martin boundaries it follows that for any excessive function f with $L(f) = 1$ there exists a Radon measure μ_f on $\tilde{F}_1 := \{\tilde{y} \in \tilde{F} \mid L(k_{\tilde{y}}) = 1\}$ with $\mu_f(\tilde{F}_1) = 1$ such that

$$f(\tilde{x}) = \int_{\tilde{F}} k_{\tilde{y}}(\tilde{x})\mu_f(d\tilde{y}) \qquad (\tilde{x} \in \tilde{E}) \tag{12}$$

(see [10]).

If the functions $\tilde{\tilde{K}}_0 g$ $(g \in C_c(\tilde{E}))$ separate the points of $\tilde{F}_m$, then there exists an uniquely determined measure μ_f supported on $\tilde{F}_m$ such that (12) holds.

Now we are ready to formulate and prove a representation theorem for quasidiffusion-space-time excessive functions f satisfying $L(f)<\infty$.

Theorem 2. *Let* 1 *be an entrance or an accessible boundary.*

Then we have:

(i): *The set* $\tilde{F}_m$ *of minimal points of* $\tilde{F}$ *is equal to*

$$\tilde{E} \cup \{(1, t_0) \mid t_0 \in (0, \infty]\},$$

(ii): *For every excessive function* f *with* $L(f)=1$ *there exists an uniquely determined measure* μ_f *on* $\tilde{F}_m$ *such that*

$$\mu_f(\tilde{F}_m) = 1 \quad \text{and} \quad f(\tilde{x}) = \int_{\tilde{E}} k_{\tilde{y}}(\tilde{x})\mu_f(d\tilde{y}) \qquad (\tilde{x} \in \tilde{E}),$$

(iii): *For every parabolic function* f *with* $L(f)=1$ *there exists an uniquely determined measure* μ_f *on* $(0, \infty]$ *such that*

$$\mu_f((0, \infty]) = 1, \quad f(\tilde{x}) = \int_{0+}^{\infty+} k_{(1,t)}(\tilde{x})\mu_f(dt) \qquad (\tilde{x} \in \tilde{E}) \tag{13}$$

and

$$\mu_f((0, s]) = -D_p\varphi(1, \lambda_0) \lim_{x\to 1} \int_0^s e^{\lambda_0 t} f(x, t)dt \qquad (s \in (0, \infty)) \tag{14}$$

if 1 *is accessible,*

$$\mu_f((0, s]) = \lim_{x\to 1} (p(x))^{-1} \int_0^x f(x, t)dt \qquad (s \in (0, \infty)) \tag{15}$$

if 1 *is entrance.*

Proof. At first we remark that the points $(y_0, 0)$ not belong to $\tilde{F}_m$ because $k_{(y_0,0)} \equiv 0$ (see also (2)). Thus we have the part $\subseteq$ of (i). To prove the $\supseteq$-part we have to show that every point $\tilde{y}_0=(1, t_0)$ $(0<t_0 \leqq \infty)$ belongs to $\tilde{F}'_m$. At first suppose $t_0 \in (0, \infty)$ and let $\tilde{y}_0=(1, t_0)$ be fixed. Then $L(k_{\tilde{y}_0}) = \lim_{s\downarrow 0} \int_0^1 p(t_0-s, y, 1)m(dy) = 1$ if 1 is entrance and $L(k_{\tilde{y}_0}) = \lim_{s\downarrow 0} \frac{e^{-\lambda_0 t_0}}{D_p\varphi(1, \lambda_0)} \int_0^1 D_p p(t_0 - s, x, 1)\varphi(x, \lambda_0)m(dx) = \lim_{s\downarrow 0} e^{-\lambda_0 s} p^*(t_0-s, x, 1)m^*(dx) = 1$ if 1 is accessible. (For notations see the proof of Theorem 1.)

Now we shall show that $k_{\tilde{y}_0}$ is parabolic. We have already mentioned that $k_{\tilde{y}_0}$ is excessive. Thus only (1) is to prove. If $\tilde{x}=(x, s) \in \tilde{E}$ with $s \geqq t_0$ we have by definition $k_{\tilde{y}_0}(\tilde{x})=0$ and therefore $k_{\tilde{y}_0}(X_t, s+t)=0$ a.e. with respect to P_x for every $t>0$. Thus (1) holds for such $\tilde{x}$.

Let $\tilde{x}=(x, s)$ with $s<t_0$ be fixed. To show (1) we can restrict ourselves to

rectangles of the form

$$\tilde{A}_{\delta,h} := \{(z,u) \mid |x-z| < \delta,\ x \pm \delta \in E,\ 0 < u-s < h\}$$
$$(\max(\delta, 1-\delta) \quad 1;\ \delta, h > 0).$$

Let 1 be entrance. Then $k_{\tilde{y}_0}(\tilde{x}) = p(t_0-s, x, 1) = \int_0^1 P_h(1, dz) p(t_0-s-h, x, z)$ $(s+h<t_0)$, by definition (see [5]), where $P_h(1, dz)$ is the transition function of X at the point 1. Suppose at first that $\tilde{A} := \tilde{A}_{\delta,h}$ is a rectangle of the mentioned form with $h<t_0-s$. Then we have

$$E_x k_{\tilde{y}_0}(X_{\tau_{\tilde{A}}}, s+\tau_{\tilde{A}}) = \int_{\partial\tilde{A}} p(t_0-u, z, 1) P_x((X_{\tau_{\tilde{A}}}, \tau_{\tilde{A}}+s) \in d\tilde{z})$$
$$= \int_0^1 P_{t_0-s-h}(1, dy) \int_{\partial\tilde{A}} p(s+h-u, z, y) P_x((X_{\tau_{\tilde{A}}}, \tau_{\tilde{A}}+s) \in d\tilde{z})$$
$$(\tilde{z} = (z, u)).$$

Remarking that for every $(y, s+h)$ the function $k_{(y,s+h)}(z, u) := p(s+h-u, z, y)$ is parabolic in (z, u) with $u<s+h$ by continuity and the Kolmogorov equations, it follows that the inner integral is equal to $p(s+h-s, x, y) = p(h, x, y)$. Thus

$$E_x k_{\tilde{y}_0}(X_{\tau_{\tilde{A}}}, s+\tau_{\tilde{A}}) = \int_0^1 P_{t_0-s-h}(1, dy) p(h, x, y) = p(t_0-s, x, 1)$$
$$= k_{\tilde{y}_0}(x, s) \qquad ((x, s) \in \tilde{E}).$$

Now let $\tilde{A} := A_{\delta,h}$ be a rectangle as above with $h \geq t_0-s$. Fixing h' with $0<h'<t_0-s$ and using the preceding step it follows with the notation $\tilde{A}' = \tilde{A}_{\delta,h'}$

$$p(t_0-s, x, 1) = \int_{\partial\tilde{A}'} p(t_0-u, z, 1) P_x(X_{\tau_{\tilde{A}'}}, \tau_{\tilde{A}}+s) \in dz)\,. \tag{16}$$

The right hand side splits into three parts, namely the integral about the lines $(x+\delta, s)\cdots(x+\delta, s+h')$, $(x-\delta, s)\cdots(x-\delta, s+h')$ and $(x-\delta, s+h')\cdots(x+\delta, s+h')$. The integral along the third line tends to zero if $s+h' \to t_0$:

$$\lim_{h' \to t_0-s} \int_{x_0-\delta}^{x_0+\delta} p(t_0-s-h', z, 1) P_x(\tau_{\tilde{A}} > h', X_{h'} \in dz) = 0\,. \tag{17}$$

This is proved as follows. The integrals in (17) are less than $\int_{x-\delta}^{x+\delta} p(t_0-s-h', z, 1) p(h', x, z) m(dz) = \int_{x-\delta}^{x+\delta} p(h', x, z) P_{t_0-s-h'}(1, dz)$ and this term converges to zero for $h' \to t_0-s$ by continuity of the semigroup $(T_h)_{h\geq 0}$. Therefore from (16) and (17) it follows

$$p(t_0-s, x, 1) = \int_s^{t_0} p(t_0-u, x+\delta, 1)P_x(\tau_{\tilde{A}}+s\in du, X_{\tau_{\tilde{A}}}\geqq x+\delta)$$

$$+\int_s^{t_0} p(t_0-u, x-\delta, 1)P_x(\tau_{\tilde{A}}+s\in du, X_{\tau_{\tilde{A}}}\leqq x-\delta)$$

$$=\int_{\partial\tilde{A}} p(t_0-u, z, 1)P_x((X_{\tau_{\tilde{A}}}, \tau_{\tilde{A}}+s)\in d\tilde{z}) .$$

Thus we have proved that $k_{\tilde{y}_0}(\cdot)$ is parabolic if 1 is entrance. If 1 is accessible, the proof follows by transformation of (m, p) as in the proof of Theorem 1 and applying the preceding result for the entrance boundary.

The proof that $k_{\tilde{y}_0}$ is extreme depends on the following lemmata.

Lemma 1. *We have*

$$\lim_{x\to 1}\int_0^s k_{(1,t_0)}(x, u)du = e^{-\lambda_0 t_0}(-D_p\varphi(1, \lambda_0))^{-1}\chi_{[0,s]}(t_0) \tag{18}$$

if 1 *is accessible and*

$$\lim_{x\to 1}(p(x))^{-1}\int_0^s k_{(1,t_0)}(x, u)du = \chi_{[0,s]}(t_0) \qquad (s\in(0, \infty)) \tag{19}$$

if 1 *is entrance.*

Proof of Lemma 1. Let 1 be accessible. Then by Theorem 2 and Lemma 4 of [5] it follows

$$-D_p\varphi(1, \lambda_0)e^{\lambda_0 t_0}\int_0^s k_{(1,t_0)}(x, u)du = \lim_{y\to 1}\int_0^{t_0\wedge s}\frac{p(t_0-u, x, y)}{p(1)-p(y)}$$

$$= \begin{cases} 1-\int_0^1 p(t_0, x, z)m(dz) & \text{if } s\geqq t_0 , \\ \int_0^1 p(t_0-s, x, z)m(dz)-\int_0^1 p(t_0, x, z)m(dz) & \text{if } s<t_0 . \end{cases} \tag{20}$$

Also from the accessibility of 1 it follows $T_h 1(x)\to 0$ for every $h>0$ if $x\to 1$. Thus we have (18).

Let 1 be entrance. We choose $\lambda>0$ and consider the new speed measure $dm^{(\lambda)}=\varphi^2(\cdot, \lambda)dm$ and the new scale $dp^{(\lambda)}=\varphi^{-2}(\cdot, \lambda)dp$. One can show that 1 is accessible with respect to $(m^{(\lambda)}, p^{(\lambda)})$ and that the corresponding to $(m^{(\lambda)}, p^{(\lambda)})$ transition density is

$$p^{(\lambda)}(h, x, y) = \frac{p(h, x, y)e^{-h}}{\varphi(x, \lambda)\varphi(y, \lambda)} \qquad (x, y\in E,\ h>0)$$

(see [5]). Moreover, we have

$$k^{(\lambda)}_{(1,t_0)}(x, s) = e^{\lambda t_0} \frac{D_{p^{(\lambda)}} p^{(\lambda)}(t_0 - s_1 x_1 1)}{D_{p^{(\lambda)}} \varphi^{(\lambda)}(1, -\lambda)} \chi_{(0,\infty)}(t_0 - s) \qquad ((x, s) \in \tilde{E}) .$$

Using (18) it follows

$$\lim_{x \to 1} (-1) \int_0^s D_{p^{(\lambda)}} p^{(\lambda)}(t_0 - u, x, 1) du = \chi_{[0,u]}(t_0) .$$

From Theorem 2 of [5] we have

$$-\int_0^s D_{p^{(\lambda)}} p^{(\lambda)}(t_0 - u, x, 1) du = \lim_{y \to 1} \int_0^s \frac{p^{(\lambda)}(t_0 - u, x, y)}{p^{(\lambda)}(1) - p^{(\lambda)}(y)} du$$

$$= \lim_{y \to 1} \int_0^s \frac{e^{-\lambda(t_0 - u)} p(t_0 - u, x, y)}{\varphi(x, \lambda)(\Gamma(\lambda)\varphi(y, \lambda) - \psi(y, \lambda))} du .$$

where $\psi(\cdot, \lambda)$ is the solution of $D_m D_p g = \lambda g$ satisfying $\psi(0, \lambda) = 0$, $D_p^- \psi(0, \lambda) = 1$, and $\Gamma(\lambda) = p^{(\lambda)}(1) = \lim_{x \uparrow 1} \frac{\psi(x, \lambda)}{\varphi(x, \lambda)}$. Using known properties of the function $\chi(\cdot, \lambda) := \Gamma(\lambda)\varphi(\cdot, \lambda) - \psi(\cdot, \lambda)$ (see also [5]) it follows that the last integral is equal to

$$e^{-\lambda t_0} (\varphi(x, \lambda)\chi(1, \lambda))^{-1} \int_0^s e^{\lambda u} p(t_0 - u, x, 1) du .$$

Thus by using $D_p \varphi(1, \lambda) \cdot \chi(1, \lambda) = 1$ (this follows from [5] remarking $D_p \varphi(1, \lambda) < \infty$, $\chi(1, \lambda) > 0$) we have

$$e^{-\lambda t_0} \lim (p(x))^{-1} \int_0^s e^{\lambda u} p(t_0 - u, x, 1) du = -1$$

$$= -\lim_{x \to 1} \int_0^s D_{p^{(\lambda)}} p^{(\lambda)}(t_0 - u, x, 1) du = \chi_{[0,s]}(t_0) \qquad \text{for every } \lambda > 0 .$$

Hence (19) follows by letting $\lambda \to 0$. Thus Lemma 1 is proved.

Lemma 2. *Let g be a parabolic function with $L(g) = 1$ and $g(\tilde{x}) = \int_{0+}^{\infty+} k_{(1,t)}(\tilde{x}) \times \mu(dt)$ for some finite measure μ on $(0, \infty]$. Then*

$$-D_p \varphi(1, \lambda_0) \lim_{x \to 1} \int_0^s e^{\lambda_0 u} g(x, u) du = \mu((0, s]) \qquad (s < \infty) \tag{21}$$

if 1 is accessible and

$$\lim_{x \to 1} (p(x))^{-1} \int_0^s g(x, u) du = \mu((0, s]) \qquad (s < \infty) \tag{22}$$

if 1 is entrance.

Proof of Lemma 2. Let 1 be accessible. Then by Fubini's theorem $\int_0^s g(x, u)du = \int_0^\infty \int_0^s k_{(1,t)}(x, u)du\mu(dt)$. If x converges to 1, the limitation and integration can be changed. This is seen as follows. From (20) we have

$$\int_0^s k_{(1,t)}(x, u)du \leqq e^{-\lambda_0 t}(-D_p\varphi(1, \lambda_0))^{-1} \int_0^1 p(t-s, x, z)m(dz) \qquad (t>s,\ x\in E).$$

We show that the continuous function $h \to e^{-\lambda_0 h}\, p(h, x, z)m(dz)$ is bounded for large h. This justifies the change in virtue of the Lebesgue's dominated convergence theorem. To this purpose we calculate the Laplace-transform (see [5]):

$$\lambda \int_0^\infty e^{-\lambda h} e^{-\lambda_0 h} dh \int_0^1 p(h, x, z)m(dz) = \frac{\lambda}{\lambda+\lambda_0}\left(1 - \frac{\varphi(x, \lambda+\lambda_0)}{\varphi(1, \lambda+\lambda_0)}\right) \qquad (\lambda>0)$$

and let λ tend to zero. Then the right hand side converges to $-\frac{\varphi(x, \lambda_0)}{\lambda_0} \times \left(\frac{\partial\varphi(1, \lambda_0+\lambda)}{\partial\lambda}\Big|_{\lambda=0}\right)^{-1}$. For every $x<1$ the function $\lambda \to \varphi(x, \lambda)$ is entire and has the representation

$$\varphi(x, \lambda) = \prod_{k=0}^\infty \left(1 - \frac{\lambda}{\lambda_k(x)}\right) \tag{23}$$

where $(\lambda_k(x))$ are the zeros of $\varphi(x, \cdot)$ (see [4]).

If x tends to 1, the numbers $\lambda_k(x)$ converge to the points λ_k of increasing of the main spectral function τ. Because $\sum_{k=1}^\infty \frac{1}{|\lambda_k|} < \infty$ the right hand side of (23) tends to the entire function $\prod_{k=1}^\infty \left(1 - \frac{\lambda}{\lambda_k}\right)$ and this limit is equal to $\varphi(1, \lambda)$. Therefore $\frac{\partial\varphi(1, \lambda_0+\lambda)}{\partial\lambda}\Big|_{\lambda=0} = -\prod_{k=1}^\infty \left(1 - \frac{\lambda_0}{\lambda_k}\right)$ is finite and nonzero. Hence by $\varphi(x, \lambda_0)>0$ $(x\in E)$ it follows that $e^{-\lambda_0 h}\int_0^1 p(h, x, z)m(dz)$ converges to a finite limit if $h\to\infty$ Summarizing and using Lemma 1 we obtain

$$\lim_{x\to 1} \int_0^s g(x, u)du = -\int_0^\infty (D_p\varphi(1, \lambda_0))^{-1} \cdot e^{-\lambda_0 t}\chi_{(0,s]}(t)\,\mu(dt)$$

and hence (21) follows.

If 1 is entrance we can transform (m, p) as in the proof of Lemma 1 into a new pair $(m^{(\lambda)}, p^{(\lambda)})$ such that 1 is $(m^{(\lambda)}, p^{(\lambda)})$-accessible. Applying (21) and transforming back we obtain (22).

Now we shall prove that $k_{\tilde{y}_0}$ is extreme. To this aim we remark that every parabolic function g with $L(g)=1$ has a representation $g=\int_{\tilde{F}_1} k_{\tilde{y}}\mu_g(dy)$ where the Radon-measure μ_g is supported on F_1' (see above). (That μ_g is supported on F_1' follows from the fact that no $k_{\tilde{y}}$ for $\tilde{y}\in\tilde{E}$ is parabolic (see [10]). The last point has to be shown, but this step is easy to see and omitted here.)

Suppose g_1, g_2 are two excessive functions with $\alpha g_1+(1-\alpha)g_2=k_{\tilde{y}_0}$ for some $\alpha\in(0, 1)$. Then g_1 and g_2 have to be parabolic because k_{y_0} it is, and by the preceding remark they have the representation

$$g_i = \int_0^\infty k_{(1,t)}\mu_i(dt)$$

for some (finite) measure μ_i on $(0, \infty]$ $(i=1, 2)$.

Using Lemmata 1 and 2 it follows that $k_{\tilde{y}_0}$ is extreme. Thus we have proved that every $\tilde{y}_0=(1, t_0)$ with $0<t_0<\infty$ is a minimal point.

At least we consider $k_{(1,\infty)}(x)=e^{-\lambda_0 s}\varphi(x, \lambda_0)$ $(\tilde{x}\in\tilde{E})$. This function is parabolic and satisfies $L(k_{(1,\infty)})=1$ which is easy to see. Moreover $k_{(1,\infty)}$ is extreme, because $k_{(1,\infty)}(1, s)=0$ and thus $\mu_g((0, s])=0$ for every parabolic function g with $g\leqq k_{(1,\infty)}$ and every $s<\infty$. Hence $(1, \infty)\in\tilde{F}_m'$. Thus (i) is proved.

If we show that the functions $\tilde{\tilde{K}}_0 f$ $(f\in C_c(\tilde{E}))$ separate the points of $\tilde{F}_m$ then (ii) follows from [10] and (iii) is a consequence of (ii) and the preceding conclusions. Let $\tilde{y}_0=(y_0, t_0)$ and $\tilde{y}_1=(y_1, t_1)\in\tilde{F}_m$ and suppose at first $t_0<t_1$. Let $f\in C_c(\tilde{E})$, $f>0$ with support is bounded below by $\frac{t_0+t_1}{2}$ and above by t_1. Then $\tilde{\tilde{K}}_0 f(\tilde{y}_1)>0$ and $\tilde{\tilde{K}}_0 f(\tilde{y}_0)=0$. Suppose $t_0=t_1$, $y_0\neq y_1$ and $\tilde{\tilde{K}}_0 f(\tilde{y}_0)=\tilde{\tilde{K}}_0 f(\tilde{y}_1)$ for every $f\in C_c(\tilde{E})$. Then $k_{\tilde{y}_0}=k_{\tilde{y}_1}$. Let e.g. 1 be entrance. Then for $\delta<\frac{|y_0-y_1|}{2}$ we have

$$\int_{|x-y_0|<\delta} k_{\tilde{y}_i}(\tilde{x})m(dx) = \int_{|x-y_0|<\delta} p(t_0-s, x, y_i)m(dx) \xrightarrow[s\uparrow t_0]{} 1 \qquad \text{or } 0$$

if $i=0$ or 1 respectively by the stochastical continuity of X. This contradicts $k_{\tilde{y}_0}=k_{\tilde{y}_1}$. If 1 is accessible by using the new pair (m^*, p^*) (see the proof of Theorem 1) on the same way it can be shown that $k_{\tilde{y}_0}\neq k_{\tilde{y}_1}$ for $y_0\neq y_1$. Thus $\tilde{\tilde{K}}_0 f(f\in C_c(\tilde{E}))$ separate the points of $\tilde{F}_m$. Hence the theorem is proved.

§3. Applications

We give two applications of the preceding results. Several authors have studied the problem if every extreme parabolic function factorizes, i.e. has a representation of the form $\phi(x)\psi(t)$, and if every factorizing parabolic function is extreme (see e.g. [9], [11]).

For quasidiffusions we have the following corollary from the results above.

Corollary 1. ——*If* 1 *is entrance* (*accessible*) *the only factorizing minimal parabolic function is*

$$k_{(1,\infty)}(x) = (m(1))^{-1} \qquad (= e^{-\lambda_0 s}\varphi(x, \lambda_0)) .$$

——*For every* $\mu > \lambda_0$ *the factorizing parabolic function* $\tilde{h}_\mu(\tilde{x}) = e^{-\mu s}\varphi(x, \mu)$ *has representation*

$$e^{-\mu s}\varphi(x, \mu) = -D_p\varphi(1, \lambda) \int_s^\infty e^{-\mu u} p(u-s, x, 1) du$$

if 1 *is entrance and*

$$e^{-\mu s}\varphi(x, \mu) = \varphi(1, \mu) \int_s^\infty e^{-\mu u} D_p p(u-s, x, 1) du$$

if 1 *is accessible.*

Remark. The last but one formula can be proved elementary by using $\int_0^\infty e^{-\mu h} p(h, x, 1) dh = r_\mu(x, 1) = \varphi(x, \mu)\chi(1, \mu) = \varphi(x, \mu)\, (D_p\varphi(1, \mu))^{-1}$ where $r_\mu(x, y)$ denotes the resolvent kernel density of the corresponding to (m, p) semigroup (S_t) in $C(E)$.

In [2], [8] and other papers were studied conditions under which for a given function $h(x, t)$ and a given Markov process the composition $(h(X_t, s+t))_{t\geq 0}$ is a martingale. In this connection we can formulate the following

Corollary 2. *Let* 1 *be entrance or accessible and* f *a parabolic function with* $L(f) < \infty$ *and*

$$f(\tilde{x}) = \int_0^\infty k_{(1,t)}(\tilde{x}) \mu_f(dt) \qquad (\tilde{x} \in \tilde{E}) .$$

Then

$$E_x(f(X_t, t+s) \mid \mathfrak{F}_u) = f(X_u, u+s) - \int_{s+u}^{s+t} k_{(1,v)}(X_u, s+u) \mu_f(dv) \qquad (0 < u < t) ,$$

where $\mathfrak{F}_u$ *denotes the* σ*-algebra generated by* $\{X_r, r \leqq u\}$. *In particular* $(f(X_t, t+s))_{t\in[a,b]}$ *is a martingale with respect to every* $P_x (x \in E)$ *if and only if*

$\mu_f([a+s, b+s])=0.$

Proof: By the Markov property it follows

$$E_x(f(X_t, t+s)\,|\,\mathcal{F}_u) = E_{X_u}(f(X_{t-u}, t+s)) = \int_0^1 f(y, t+s)p(t-u, X_u, y)m(dy)$$

$$= \int_0^1 \int_{t+s}^{\infty} k_{(1,v)}(y, t+s)\mu_f(dv)p(t-u, X_u, y)m(dy)$$

$$= \int_{t+s}^{\infty} \int_0^1 k_{(1,v)}(y, t+s)p(t-u, X_u, y)m(dy)\mu_f(dv)$$

$$= \int_{t+s}^{\infty} k_{(1,v)}(X_u, s+u)\mu_f(dv)$$

$$= f(X_u, s+u) - \int_{s+u}^{s+t} k_{(1,v)}(X_u, s+u)\mu_f(dv)\,.$$

Thereby we have used

$$\int_0^1 k_{(1,v)}(y, t)p(h, z, y)m(dy) = k_{(1,v)}(z, t-h) \qquad (h<t)$$

which follows from [5] Theorem 2 and the Chapman-Kolmogorov equation.

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