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Some Asymptotic Properties of the Transition Densities of One-Dimensional Quasidiffusions

By

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§1. Introduction and Preliminary Results

Let $(X_t)_{t\geq 0}$ be a quasidiffusion on $[0,1)$ given by its speed measure m and scale p. Suppose the boundary point 0 is reflecting regular. If the boundary point 1 is regular we restrict ourselves to the case that (X_t) is killed as soon as it hits 1, although the arguments hold equally well for all other locally boundary conditions.

As m is not assumed to be strictly increasing besides the classical diffusions birth- and death-processes are also included.

In this introducing chapter we consider semigroups of contractions generated by (X_t) in $L_2(m)$ and in a Banach space B of continuous m-integrable functions. We construct a spectral expansion of the transition densities of (X_t) making use of M.G. Krein's results on the spectral functions of a string ([10]) and study some of their properties. The results given below generalize known facts for birth- and death-processes ([9]) and for some diffusion processes ([3], [8]). They can be proved by using standard methods of the theory of generalized differential operators ([2], [4], [10]) and therefore some of the proofs are omitted.

In Chapter 2 we will show that the spectra of the infinitesimal operators D_mD_p of (X_t) in $L_2(m)$ and B are identical assuming 1 is not a natural boundary. This is used for studying some boundedness and integrability properties of eigenfunctions of this operator.

The main results of this paper are contained in Chapters 3 and 4. In Chapter 3 we define a transformation of the pair (m,p) using an eigenfunction of D_mD_p. This transformation gives a far reaching duality between accessible

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and entrance boundaries and elucidates once more the connections between limit circle and limit point cases on the one hand and Feller's boundary classification on the other hand.

Chapter 4 is devoted to the investigation of asymptotic properties of the transition densities $p(t,x,y)$ of (X_t). Assuming 1 is an entrance or an accessible boundary we will show that $p(t,x,y)$ and $\frac{p(t,x,y)}{p(1)-p(y)}$ respectively have strictly positive limits for $y\to 1$ and/or $t\to\infty$ in a certain space- and time-weak sense. The asymptotic properties given below are not true in general if 1 is a natural boundary. In this case the spectral properties of D_mD_p and the asymptotic behaviour of $p(t,x,y)$ are of different type. This is known from the example of Brownian motion.

The results of this paper will be used in [13] to study parabolic functions of quasidiffusions. Let us mention that some asymptotic properties of transition densities of diffusions for a bounded time interval were investigated with somewhat other methods in [19]. In this paper we shall use the theory of the generalized differential operator D_mD_p and the terminology given in [10] (see also [2]). Some necessary notions and properties are summarized in an appendix.

We denote by R the set of real, by K the set of complex numbers and define $K_0:=K\setminus(-\infty,0]$. A nondecreasing function m from

$$J:=[0,1] \text{ into } [0,\infty] \text{ with}$$
$$0=m(0)<m(x)<m(1-0)=m(1) \qquad (x\in(0,1))$$

is called a *speed measure,* a (strictly) increasing continuous function p from J into $[0,\infty]$ is called a *scale*, and a pair (m,p) of such functions is said to be a *canonical pair*. With the same letter m and p we denote the measures generated by m and p. We put $I:=J\setminus\{1\}$, $F:=\operatorname{supp} m$ and $E:=F\setminus\{1\}$.

We will make use of Feller's boundary classification (see e.g. [7], [17]) i.e. with the notation $u(x):=\int_0^x m\,dp$ and $v(x):=\int_0^x p\,dm$ $(x\in J)$ we shall call the boundary 1 *regular* if $u(1)<\infty$ and $v(1)<\infty$, *entrance* if $u(1)=\infty$ and $v(1)<\infty$, *pure exit* if $u(1)<\infty$ and $v(1)=\infty$ and *natural* if $u(1)=\infty$ and $v(1)=\infty$. The natural case is divided in *pure natural boundary* $(p(1)=\infty)$ and *inaccessible exit boundary* $(p(1)<\infty)$ (shortly: *i.a. exit boundary*). Regular and pure exit boundaries are called *accessible,* the other *inaccessible.* Here the boundary 0 is always regular in an analogous classification.

Let (m,p) be a canonical pair and ϑ the set of all complex valued functions f on I such that there exists an m-locally integrable function $g=:D_mD_pf$ and two numbers $a,b\in K$ with

$$f(x) = a+bp(x)+\int_0^x (p(x)-p(s))g(s)m(ds) \qquad (x\in I). \tag{1}$$

For every $\lambda\in K$ there exist the *fundamental solutions*

$$\varphi(.,\lambda) \text{ and } \psi(.,\lambda) \text{ of } D_mD_pg-\lambda g = 0\ (g\in\vartheta) \text{ and}$$
$$D_p^-\varphi(0,\lambda) = \psi(0,\lambda) = 0,\ \varphi(0,\lambda) = D_p^-\psi(0,\lambda) = 1\,.$$

It is known that either both fundamental solutions belong to $L_2(m)$ for every $\lambda\in K$ or at least one of their linear combinations belongs to $L_2(m)$ for every $\lambda\in K_0$ depending on $\int_0^1 p^2dm<\infty$ or $\int_0^1 p^2dm=\infty$ respectively. In the first case we speak of the *limit circle case* (lcc) and in the second of the limit point case (lpc). In the lcc the boundary 1 has to be regular or entrance. If 1 is regular we have the lcc, if 1 is entrance both lcc or lpc are possible.

To formulate the first proposition let us define

$$\vartheta^*:= \{f\in\vartheta\cap L_2(m)\,|\,D_mD_pf\in L_2(m),\ D_p^-f(0) = 0\} \text{ and}$$

$$\Delta:=\begin{cases} \{f\in\vartheta^*\,|\,D_pf(1) = 0\} & \text{if } \int_0^1 p^2dm<\infty \text{ and 1 is entrance,}\\ \{f\in\vartheta^*\,|\,f(1) = 0\} & \text{if } \int_0^1 p^2dm<\infty \text{ and 1 is regular,}\\ \vartheta^* & \text{if } \int_0^1 p^2dm = \infty\,. \end{cases}$$

Proposition 1. *The restriction $\mathbf{D}_m\mathbf{D}_p$ of D_mD_p to Δ is a selfadjoint non-positive operator in $L_2(m)$ and its resolvent R_λ is given by*

$$R_\lambda f(x) = \int_0^1 r_\lambda(x,y)f(y)m(dy) \qquad (f\in L_2(m),\ \lambda\in K_0) \tag{2}$$

with

$$r_\lambda(x,y) = r_\lambda(y,x) = \varphi(x,\lambda)\chi(y,\lambda) \qquad (x,y\in I,\ x\leqq y)\,.$$

Here $\chi(.,\lambda)$ denotes the unique solution of $D_mD_pg-\lambda g=0$ belonging to $L_2(m)$ and (if $\int_0^1 p^2dm<\infty$) satisfying $D_p^-\chi(0,\lambda)=-1$ and $\chi(1,\lambda)=0$ if 1 is regular, $D_p\chi(1,\lambda)=0$ if 1 is entrance.

The proof is omitted, see e.g. [1], [2], [4], [15] for similar results.

Corollary. *If* $\int_0^1 p^2 dm = \infty$ *and* 1 *is not natural then all elements of* Δ *also satisfy certain boundary conditions. Precisely, for any* $f \in \Delta$ *it holds*

$$f(1) = 0 \text{ if } 1 \text{ is a pure exit boundary,}$$
$$D_p f(1) = 0 \text{ if } 1 \text{ is an entrance boundary.}$$

Proof. The first property follows from $m(1)=\infty$ in the pure exit case. To prove the second we assume 1 is entrance and fix $\lambda > 0$. Then every $f \in \Delta$ can be represented by $R_\lambda g$ with $g = \lambda f - D_m D_p f \in L_2(m)$. Thus

$$D_p f(x) = (D_p \chi(x,\lambda)) \int_0^x \varphi(.,\lambda) g dm + (D_p \varphi(x,\lambda)) \int_x^1 \chi(.,\lambda) g dm .$$

The boundary 1 is entrance, therefore we have $\lim_{x \uparrow 1} D_p \varphi(x,\lambda) < \infty$ (see [2], p. 166) and thus the second integral tends to zero for $x \uparrow 1$. Using (A3) it follows

$$D_p f(1) = \lim_{x \uparrow 1} (\varphi(x,\lambda))^{-1} \int_0^x \varphi(.,\lambda) g dm .$$

(The limit exists and is finite because $\varphi(.,\lambda)$ is increasing and $g \in L_2(m) \subset L_1(m)$ if 1 is entrance.) Thus we have

$$f(x) \sim p(x) D_p f(1) \quad \text{for } x \uparrow 1 .$$

Now $\int_0^1 p^2 dm = \infty$ and $f \in L_2(m)$ imply $D_p f(1) = 0$. Q.E.D.

Let (m,p) be a canonical pair and τ its main spectral function (see the appendix). We define

$$(3) \quad p(t,x,y) := \int_{-\infty}^0 e^{\lambda t} \varphi(x,\lambda) \varphi(y,\lambda) \tau(d\lambda) \qquad (t \in K, \ \mathrm{Re}\, t > 0;\ x,y \in I) .$$

By virtue of (A1–2) these integrals converge uniformly in $x,y \in [0,c]$ and $t \geqq t_0$ for every $c < 1$ and $t_0 > 0$ (uniformly in $x,y \in [0,1]$ and $t \geqq t_0$ for every $t_0 > 0$ if 1 is regular). Without proof we mention that $t \rightarrow p(t,x,y)$ is holomorphic in t with $Re\ t > 0$ for every $x,y \in I$ ($x,y \in J$ if 1 is regular).

Proposition 2. *The formula*

$$(4) \quad T_t f(x) := \int_0^1 p(t,x,y) f(y) m(dy) \quad (f \in L_2(m),\ x \in I, t \in K,\ \mathrm{Re}\, t > 0)$$

defines a holomorphic semigroup $\{T_t \mid t \in K, \mathrm{Re}\, t > 0\}$ *of selfadjoint contractions* T_t *on* $L_2(m)$ *being strongly continuous at* $t=0$ *with* $T_0 := I$ *and having* $D_m D_p$ *as its infinitesimal operator. Furthermore we have*

(5) $p(t,.,y)\in\Delta,\ \boldsymbol{D}_m\boldsymbol{D}_p p(t,x,y)=\frac{\partial}{\partial t}p(t,x,y)\quad (t\in K,\ \mathrm{Re}\,t>0;\ x,y\in I)$

and

(6) $p(t,x,y)=p(t,y,x)>0\quad (t>0,\ x,y\in I)\,.$

Proof. It is known (see e.g. [2], [10]) that the generalized Fourier transformation U from $L_2(m)$ given by

$$(Uf)(\lambda)=\int_0^1 f(x)\varphi(x,\lambda)m(dx)\quad (f\in L_2(m))$$

maps $L_2(m)$ isometrically onto $L_2(\tau)$ and that for the inverse mapping U^{-1} holds

$$(U^{-1}F)(x)=\int_{-\infty}^0 F(\lambda)\varphi(x,\lambda)d\tau(\lambda)\quad (F\in L_2(\tau))\,.$$

The mapping $Q:=U\boldsymbol{D}_m\boldsymbol{D}_pU^{-1}$ is the operator of multiplication acting in $L_2(\tau)$:

$$(QF)(\lambda)=\lambda F(\lambda)\quad (F\in\tilde{\Delta}:=\{F\in L_2(\tau)\,|\,QF\in L_2(\tau)\})\,.$$

By standard methods it can be shown that with the definition $T_t:=U^{-1}e^{Qt}U$ $(t\in K,\ \mathrm{Re}\,t>0)$ the first part of the proposition holds.

From (A1–2) it follows that for every $y\in I$ and t with $\mathrm{Re}\,t>0$ the function $\lambda\to e^{\lambda t}\varphi(y,\lambda)$ belongs to $\tilde{\Delta}$. Therefore $p(t,x,y)=U^{-1}(e^{\cdot t}\varphi(y,.))(x)$ as a function of x belongs to Δ and we get

$$\boldsymbol{D}_m\boldsymbol{D}_p p(t,x,y)=U^{-1}(Qe^{\cdot t}\varphi(y,.))(x)=\int_{-\infty}^0 \lambda e^{\lambda t}\varphi(x,\lambda)\varphi(y,\lambda)\tau(d\lambda)\,.$$

Thus from the uniform convergence of the last integral in (t,x,y) with $\mathrm{Re}\,t\geqq t_0$; $x,y\in[0,c]$ for every $t_0>0, c<1$ we obtain that this integral is equal to $\frac{\partial}{\partial t}p(t,x,y)$. Hence (5) is proved.

Now we shall show (6). The symmetry of $p(t,x,y)$ is trivial by definition. Let $\lambda>0$. If $f\in L_2(m)$, $f\geqq 0$ then $R_\lambda f\geqq 0$ by the positiveness of the kernel $r_\lambda(x,y)$. The Hille-Yosida-theorem (see e.g. [18]) implies that $T_tf\geqq 0$ for all $f\in L_2(m)$ with $f\geqq 0$ and every $t>0$. Using the continuity of $p(t,x,y)$ it is easy to see that $p(t,x,y)\geqq 0$ $(t>0,\ x,y\in I)$. Suppose $p(t_0,x_0,y_0)=0$ for a certain tripel (t_0,x_0,y_0) with $t_0>0$; $x_0,y_0\in I$. We can assume x_0,y_0 are points of increasing of m (because $p(t,x,.)$ and $p(t,.,y)$ depend linear in scale p on intervals where m is constant).

Now from the semigroup property of (T_t) we have for every $h\in(0,t_0)$

$$(7)\qquad \int_0^1 p(t_0-h, x_0, z)p(h, z, y_0)m(dz) = p(t_0, x_0, y_0) = 0\,.$$

Let $h' \in (0, t_0)$ be fixed. Using $T_{h'} \neq 0$ it follows $p(h', z', y_0) > 0$ for some $z' \in I$. By continuity we have $p(h, z, y_0) > 0$ in some neighbourhood of (h', z'). Hence (7) implies $p(t_0-h, x_0, z)=0$ for all (h, z) in some neighbourhood of (h', z'). Thus by the holomorphy of $p(., x_0, z')$ it follows that $p(s, x_0, z')=0$ for every $s>0$. Therefore $\int_0^\infty e^{-\lambda s}p(s, x_0, z')ds=0$ $(\lambda>0)$. But this integral is equal to $r_\lambda(x_0, z')$, because x_0 is an increasing point of m (see (A6)). This contradicts $r_\lambda(x, y)>0$ $(x, y \in I)$. Hence (6) holds. Q.E.D.

Remark. Without proof let us mention that $P_t(x, A) := \int_A p(t, x, y)m(dy)$ are the transition probabilities of a strongly Fellerian stochastically continuous Markov process with state space E. This process is reflected at the boundary 0 and killed as soon as it hits the boundary 1. It is called the quasidiffusion corresponding to (m, p) and the boundary conditions mentioned above (see e.g. [7], [16], [20]), or, because the boundary conditions are fixed here, shortly corresponding to (m, p). In this sense we call $p(t, x, y)$ the transition densities corresponding to (m, p).

To obtain further informations about the function $p(t, x, y)$ we shall study corresponding semigroups in spaces of continuous and m-integrable functions. To this purpose we take into consideration the Banach space $(C, ||.||)$ of continuous functions from J into K being linear in scale p on intervals where m is constant. The norm $||.||$ is given by $||f|| := \sup_{x \in J} ||f(x)||$ $(f \in C)$. Moreover let $C_0(C_1)$ be the subset of all $f \in C$ such that $f(1)=0$ ($\int_0^1 |f| dm < \infty$ resp.) Then $(C_0, ||.||)$ and $(C_1, ||.||_1)$ with $||f||_1 := \max(||f||, \int_0^1 |f| dm)$ $(f \in C_1)$ are also Banach spaces.

We will study the restrictions of $D_m D_p$ to $C_1 C_0$ and C_1 and we will show that some eigenfunctions of these restrictions are m-integrable. This is obvious if 1 is entrance or regular, because in this case $m(1)<\infty$ (i.e. $C \subset L_1(m)$) holds. If 1 is pure exit we have $m(1)=\infty$ and thus we must apply another methods to prove the m-integrability of the eigenfunctions. Let us define

$$(B, |.|) = \begin{cases} (C, ||\cdot||) \text{ if 1 is inaccessible,} \\ (C_0, ||\cdot||) \text{ if 1 is regular,} \\ (C_1, ||\cdot||_1) \text{ if 1 is pure exit.} \end{cases}$$

For every $\lambda > 0$ we define by

$$R_\lambda f(x) := \int_0^1 r_\lambda(x,y) f(y) m(dy) \tag{8}$$

a linear mapping R_λ on the set of functions f such that the integral in (8) exists.

Proposition 3. *The formula*

$$S_t f(x) := \int_0^1 p(t,x,y) f(y) m(dy), \; S_0 f := f \quad (f \in B,\, t > 0) \tag{9}$$

defines a strongly continuous semigroup $\{S_t \mid t \geqq 0\}$ *of contractions* S_t *on* $(B, \|.\|)$. *The infinitesimal operator of* (S_t) *is the restriction of* $D_m D_p$ *to* $\Delta_B : \{f \in \vartheta \cap B \mid D_m D_p f \in B\}$, *and for* $\lambda > 0$ *the corresponding resolvent operator* R_λ *on* B *is given by* (8).

The proof of this proposition can also be given by standard methods and is omitted here.

§2. Spectral Properties of $D_m D_p$

In the following theorem we consider the spectra of the restrictions of $D_m D_p$ to $\Delta \subset L_2(m)$ and to $\Delta_B \subset B$ which are denoted by σ and σ_B respectively.

Theorem 1. *Let* 1 *be an accessible or an entrance boundary. Then the spectra* σ *and* σ_B *coincide and consist of a strictly decreasing sequence* $(\lambda_n, n \geqq 0)$ *of nonpositive simple eigenvalues* λ_n $(n \geqq 0)$ *having no finite acccumulation points. The eigenfuntion corresponding to* λ_n is $\varphi(.,\lambda_n)$. *We have*

$$\begin{aligned} &\varphi(x,\lambda_0) > 0 \;\; (x \in I) \text{ and} \\ &\lambda_0 < 0 \text{ if 1 is accessible}, \\ &\lambda_0 = 0 \text{ if 1 is entrance}. \end{aligned} \tag{10}$$

Proof. Let $\lambda > 0$. It is easy to see that R_λ on $L_2(m)$ is a compact operator. Indeed under the assumptions of the theorem we have

$$\int_0^1 r_\lambda(x,x) m(dx) < \infty \,. \tag{11}$$

This follows from $r_\lambda(x,x) \leqq \chi(0,\lambda)\varphi(x,\lambda)$ and $D_p \varphi(1,\lambda) = \lambda \int_0^1 \varphi(x,\lambda) m(dx) < \infty$ if 1 is regular or entrance, $r_\lambda(x,x) \leqq \varphi(1,\lambda)\chi(x,\lambda)$ and $\varphi(1,\lambda)\int_0^1 \chi(x,\lambda) m(dx) < \infty$ if 1 is pure exit (see (A4) and [2], p. 166). Remarking $r_\lambda^2(x,y) \leqq r_\lambda(x,x) r_\lambda(y,y)$, (11) implies $\iint r_\lambda^2 dm dm < \infty$, i.e. R_λ is compact in $L_2(m)$.

The properties of its spectrum imply that σ consists of a sequence $(\lambda_n, n \geqq 0)$ of eigenvalues having no finite accumulation point. The corresponding eigenfunctions g_n satisfy the condition $D_p^- g_n(0)=0$. Thus $g_n=\varphi(., \lambda_n)$ which proves the simplicity of λ_n. From the nonpositivity of $D_m D_p$ it follows $\lambda_n \leqq 0$. Let (λ_n) be ordered: $0 \geqq \lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots$. To continue the proof we shall prove the following

Lemma 1. *For every $\lambda>0$ the operator R_λ is compact on $(B, |.|)$.*

Proof. If 1 is regular, then $r_\lambda(.,.)$ is continuous on $[0,1]^2$, and from $m(1)<\infty$ as usual (see e.g. [8]) it follows the compactness of R_λ in $(C, ||.||)$ and therefore in $(C_0, ||.||)$. If 1 is entrance we have from $m(1)<\infty$ and from the continuity of $r_\lambda(.,.)$ on $[0,1)^2$ that for every n the operator $R_\lambda^{(n)}$ generated by $r_\lambda^{(n)}(.,.)=\min(r_\lambda(.,.), n)$ is compact in $(C, ||.||)$. By the Lagrange identity it follows $D_p(\varphi(x,\lambda)\chi(x,\lambda))=2\chi(x,\lambda)D_p\varphi(x,\lambda)-1$. Because 1 is entrance we have $\chi(1,\lambda)>0$ and $D_p\varphi(1,\lambda)<\infty$ (see [2], p.166). Furthermore $\chi(1,\lambda)D_p\varphi(1,\lambda)=1$ holds (see (A3)). Thus $\lim_{x\uparrow 1} D_p(\varphi(x,\lambda)\chi(x,\lambda))=1$. Consequently the function $\varphi(.,\lambda)\chi(.,\lambda)$ increases near 1 and converges to ∞ for $x \uparrow 1$. Thus there exists a sequence $(x_n)\uparrow 1$ such that $\varphi(x_n,\lambda)\chi(x_n,\lambda)=n$. Hence $r_\lambda^{(n)}(x,y)=r_\lambda(x,y)$ if x or $y \leqq x_n$. Using $D_p\varphi(1,\lambda)<\infty$ it follows

$$||R_\lambda^{(n)}-R_\lambda|| = \sup_x \int_0^1 (r_\lambda(x,y)-r_\lambda^{(0)}(x,y))m(dy)$$

$$\leqq \sup_{x\geqq x_n}(\chi(x,\lambda)\int_{x_n}^x (\varphi(y,\lambda)-\varphi(x_n,\lambda))m(dy)+\varphi(x,\lambda)\int_x^1(\chi(y,\lambda)-\chi(1,\lambda))m(dy))$$

$$\leqq \frac{\chi(0,\lambda)}{\lambda}(D_p\varphi(1,\lambda)-D_p\varphi(x_n,\lambda))+\frac{(\chi(0,\lambda)-\chi(1,\lambda))}{\lambda}(D_p\varphi(1,\lambda)-D_p\varphi(x_n,\lambda))$$

$$\xrightarrow[n\to\infty]{} 0\,.$$

Therefore R_λ is compact in $(C, ||.||)$ if 1 is entrance.

Let 1 be pure exit. At first we will show that for $S:=\{f\in C \mid ||f||\leqq 1\}$ the set $R_\lambda S$ is totally bounded. This implies that R_λ is compact on $(C, ||.||)$. Of course $R_\lambda S$ is a bounded set. Using (A3) we have

$$(R_\lambda 1)(x) = \chi(x,\lambda)\int_0^x \varphi(.,\lambda)dm+\varphi(x,\lambda)\int_x^1 \chi(.,\lambda)dm$$

$$= \frac{1}{\lambda}(\chi(x,\lambda)D_p\varphi(x,\lambda)+\varphi(x,\lambda)\,(D_p\chi(1,\lambda)-D_p\chi(x,\lambda)))$$

$$=\frac{1}{\lambda}(1+\varphi(x,\lambda)D_p\chi(1,\lambda))=\frac{1}{\lambda}\left(1-\frac{\varphi(x,\lambda)}{\varphi(1,\lambda)}\right).$$

Thus for every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)$ such that $(R_\lambda 1)(x)<\varepsilon$ for every x with $x\geqq 1-\delta$. Therefore $|R_\lambda f(x)|<\varepsilon$ for every $f\in S$ and every $x\geqq 1-\delta$. Now it is easy to show that for every $\varepsilon>0$ there exists a finite set of points x_i with $0<x_1\leqq\cdots\leqq x_n<1-\delta(\varepsilon)$ such that $|R_\lambda f(x)-R_\lambda f(x_i)|<\varepsilon$ for every $x\in[x_{i-1},x_{i+1}]$ $(i=1,\cdots,n-1)$ with $x_0=0$ and $x_{n+1}=1-\delta(\varepsilon)$. Thus the total boundedness of R_λ is proved.

Now we have to show that R_λ is compact in $(C_1,||.||_1)$ if 1 is pure exit. Let (f_n) be a sequence in C_1 with $||f_n||_1\leqq 1$. Then by the compactness of R_λ in C there exists a subsequence (f_n') converging with respect to the norm of C. The elements of C_1 can be identified with elements of the dual $C^*: f\to F_f(g):=\int_0^1 f\cdot g dm$ $(f\in C_1, g\in C)$.

The dual operator R_λ^* of R_λ in C^* is also compact and from the symmetry of $r_\lambda(.,.)$ it follows that $R_\lambda^* F_f$ can be identified with $R_\lambda f$ if $f\in C_1$. Hence there exists a subsequence (f_n'') of (f_n') converging in the sense of L_1-norm and thus in the sense of the norm $||.||_1$. Summarizing we have proved that R_λ for $\lambda>0$ is a compact operator in $(B,\|.\|)$. In particular the spectrum σ_B is a sequence of eigenvalues having no finite accumulation point.

To continue the proof of Theorem 1 we have to show that σ_B and σ are identical. We remark that $B\subset L_2(m)$ and that B is dense in $L_2(m)$. Moreover there exists a constant $C>0$ such that

$$||f||_{L_2}\leqq C\|f\| \quad (f\in B). \tag{12}$$

Obviously every eigenvalue of $D_m D_p$ in B is an eigenvalue of $D_m D_p$ in $L_2(m)$, thus $\sigma_B\subseteqq\sigma$. If $\mu\in R_1$ not belongs to σ_B then R_μ is a bounded linear operator in B which has by (12) and by the symmetry property of $r_\lambda(x,y)$ a bounded extension R'_μ to $L_2(m)$ (see [8], [12]). Using the closeness of $D_m D_p$ it is easy to show that R'_μ is the inverse of $(\mu I-D_m D_p)$ i.e. $\mu\notin\sigma$ and therefore it holds $\sigma\subseteqq\sigma_B$.

Now we shall show the positiveness of $\varphi(.,\lambda_0)$. If $\lambda_0=0$ then $\varphi(.,\lambda_0)=1>0$. Let $\lambda_0<0$ and $\lambda>0$. Then $\frac{1}{\lambda-\lambda_0}>0$ is the greatest eigenvalue of the compact and positive linear operator R_λ in the Banach lattice $(B,\|.\|)$. Thus the eigenfunction $\varphi(.,\lambda_0)$ is nonnegative. Suppose $x_0<1$ is the first zero of $\varphi(.,\lambda_0)$.

Then $D_p\varphi(x_0, \lambda_0)=\lambda_0\int_0^{x_0}\varphi(s, \lambda_0)m(ds)<0$ and thus $\varphi(x, \lambda_0)=\int_{x_0}^{x} D_p\varphi(s, \lambda_0)p(ds)<0$ for some x in neighbourhood of x_0. This contradicts $\varphi(., \lambda_0)\geqq 0$. Thus $\varphi(x, \lambda_0)>0$ if $x\in I$. If 1 is accessible we have $\int_{-\infty}^{0}\frac{\tau(d\lambda)}{|\lambda|}=\sum_0^\infty \frac{\tau_k}{|\lambda_k|}=\Gamma(0)=p(1)<\infty$. Thus $\lambda_0\neq 0$, i.e. $\lambda_0<0$. (See the appendix for notations.) If 1 is entrance we have $\varphi(., 0)\equiv 1\in\Delta_B$ and therefore $\lambda_0=0$. Q.E.D.

Corollary. *We have*

$$\varphi(1, \lambda_n) = 0,\ |D_p\varphi(1, \lambda_n)| < \infty \text{ if 1 is accessible and}$$
$$|\varphi(1, \lambda_n)| < \infty,\ D_p\varphi(1, \lambda_n) = 0 \text{ if 1 is entrance.}$$

Proof. Theorem 1 implies $\varphi(., \lambda_n)\in L_1(m)$ if 1 is accessible or entrance. Therefore $|D_p\varphi(1, \lambda_n)|=|\lambda_n\int_0^1\varphi(x, \lambda_n)m(dx)|<\infty$ in both cases. The other properties follow from $\varphi(., \lambda_n)\in\Delta\cap\Delta_B$ and the corollary after Proposition 1. U.E.D.

Let 1 be accessible or entrance and (λ_k) the sequence of eigenvalues of D_mD_p. We know from (A1–2) that the series

$$p(t, x, y) = \sum_0^\infty e^{\lambda_k t}\varphi(x, \lambda_k)\varphi(y, \lambda_k)\tau_k \tag{13}$$

converge uniformly in (t, x, y) with $t\geqq t_0$; $x, y\leqq c$ for every $t_0>0$ and $c<1$. If 1 is accessible somewhat more holds. (See also the remark (ii) below.)

Proposition 4. *Assume* 1 *is accessible. Then the series* (13) *converge uniformly in* $t\geqq t_0$ *and* $x, y<1$ *for every* $t_0>0$.

Proof. Let $\lambda>0$. Then by $r_\lambda(x, .)\in L_2(m)$ for every $x\in I$ it follows

$$r_\lambda(x, .) = \sum_{k=0}^\infty \frac{\varphi(x, \lambda_k)\varphi(., \lambda_k)}{\lambda-\lambda_k}\tau_k \qquad (x\in I)$$

in the sense of $L_2(m)$-convergence. Thus for the operator $R_\lambda^{(n)}$ generated by the kernel

$$r_\lambda(x, y, n) = r_\lambda(x, y)-\sum_{k=0}^n \frac{\varphi(x, \lambda_k)\varphi(y, \lambda_k)}{\lambda-\lambda_k}\tau_k \qquad (x, y\in I)$$

we have

$$(R_\lambda^{(n)}f, f)\geqq 0 \qquad (f\in L_2(m)).$$

Therefore $r_\lambda(x,x,n)\geqq 0$ for every $x\in E$, i.e.

$$r_\lambda(x,x)\geqq \sum_{k=0}^{n}\frac{(\varphi(x,\lambda_k))^2}{\lambda-\lambda_k}\tau_k \qquad (x\in E)\,. \tag{14}$$

The accessibility of 1 implies that $r_\lambda(x,x)$ is bounded in x. Therefore the series on the right side converge uniformly in $x\in E$ and thus also in $x<1$. From

$$\sum_{n}^{m}\left|\frac{\varphi(x,\lambda_k)\varphi(y,\lambda_k)}{\lambda-\lambda_k}\right|\tau_k\leqq\left\{\sum_{n}^{m}\frac{|\varphi(x,\lambda_k)|^2}{\lambda-\lambda_k}\tau_k\sum_{n}^{m}\frac{|\varphi(y,\lambda_k)|^2}{\lambda-\lambda_k}\tau_k\right\}^{1/2} \tag{15}$$

and $e^{\lambda_k t_0}\leqq\frac{1}{\lambda-\lambda_k}$ for sufficiently large k the proposition follows. Q.E.D.

By the same conclusions as in the preceding proof it can be shown that (14) holds also if 1 is entrance. Thus from (11) it follows

Corollary. *If* 1 *is accessible or entrance then*

$$\sum_{1}^{\infty}\frac{1}{|\lambda_k|}<\infty\,. \tag{16}$$

Remarks. (i) It can be shown that the $(k+1)$-th eigenfunction $\varphi(.,\lambda_k)$ has exactly k zeros in $(0,1)$ if 1 is accessible or entrance. (See e.g. [3] for the method).

(ii) Let 1 be entrance and $\int_0^1 p^2dm<\infty$. Then the series (13) also converge uniformly in $x,y<1$ and $t\geqq t_0$ for every $t_0>0$. The proof is included in the proof of Theorem 3 below.

§3. A Transformation of the Canonical Pair

In this chapter we will show that there exists a far reaching connection between accessible and entrance boundaries. By a relatively simple transformation of (m,p) in a new canonical pair (m^*,p^*) the type of the boundary 1 changes, and under this transformation regular and entrance boundaries being in 1cc, pure exit and entrance boundaries being in 1pc correspond to another (see Propositions 5 and 5′ below).

Let (m,p) be a canonical pair, τ the corresponding main spectral function and $\lambda_0:=\sup\operatorname{supp}\tau$. We recall $\lambda_0\leqq 0$.For every $\mu\geqq\lambda_0$ we introduce a new canonical pair $(m^{(\mu)},p^{(\mu)})$ by

$$dm^{(\mu)}:=\varphi^2(.,\mu)dm,\quad dp^{(\mu)}:=\varphi^{-2}(.,\mu)dp\,. \qquad \text{(cf. [22])}$$

Obviously $dm^{(0)}=dm$ and $dp^{(0)}=dp$.

Let $\varphi^{(\mu)}$, $\psi^{(\mu)}$ and $\tau^{(\mu)}$ be the fundamental solutions and the main spectral function of $(m^{(\mu)}, p^{(\mu)})$ respectively. (More generally, all symbols connected with $(m^{(\mu)}, p^{(\mu)})$ get the superscript $.^{(\mu)}$.)

Lemma 2. *The following properties hold*:

(i) $\varphi^{(\mu)}(x,\lambda)=\dfrac{\varphi(x,\lambda+\mu)}{\varphi(x,\mu)}$, $\psi^{(\mu)}(x,\lambda)=\dfrac{\varphi(x,\lambda+\mu)}{\varphi(x,\mu)}$ $\quad (x\in I,\ \lambda\in K)$

(ii) $\tau^{(\mu)}(\lambda)=\tau(\lambda+\mu)$ $\quad (\lambda\in(-\infty,\infty))$

(iii) $((m^{(\mu)})^{(\eta)}, (p^{(\mu)})^{(\eta)})=(m^{(\mu+\eta)}, p^{(\mu+\eta)})$ $\quad (\mu,\ \mu+\eta\geqq\lambda_0)$.

Proof. Using [10], p. 661, (2.23) and an analogous formula for $\varphi(.,\mu)$ and $\psi(.,\lambda+\mu)$ it follows (i). From $\Gamma^{(\mu)}(\lambda)=\lim\limits_{x\uparrow 1}\dfrac{\psi^{(\mu)}(x,\lambda)}{\varphi^{(\mu)}(x,\lambda)}=\Gamma(\lambda+\mu)$ we have (ii). (iii) can be shown by an easy calculation. Q.E.D.

Now we study the character of the boundary point 1 with respect to $(m^{(\mu)}, p^{(\mu)})$. To this purpose we define

$$u^{(\mu)}(x):=\int_0^x m^{(\mu)}dp^{(\mu)}=\int_0^x\int_0^s\frac{\psi^2(t,\mu)}{\varphi^2(s,\mu)}\,dmdp \qquad \text{and}$$

$$v^{(\mu)}(x):=\int_0^x p^{(\mu)}dm^{(\mu)}=\int_0^x\psi(s,\mu)\varphi(s,\mu)dm \qquad (x\in J).$$

The following lemma is useful for studying the given transformation but is not needed in the sequel explicitely. Therefore we shall omit the proof.

Lemma 3. *If* $p(1)<\infty$, $\mu>\lambda_0$ *or* $p(1)=\infty$, $\mu=\lambda_0$ *then the character of the boundary* 1 *with respect to* $(m^{(\mu)}, p^{(\mu)})$ *is the same as to* (m,p).

Lemma 3 shows that a change of the character of 1 may be only if $p(1)<\infty$ and $\mu=\lambda_0$ or if $p(1)=\infty$ and $\mu>\lambda_0$.

In the first case the boundary 1 is an accessible or an i.a. exit boundary with respect to (m,p), in the second it is an entrance or a natural boundary with respect to (m,p).

Proposition 5. *Let* $p(1)<\infty$ *and* $\mu=\lambda_0$. *Then the boundary* 1 *is*

(i) $(m^{(\mu)}, p^{(\mu)})$*-entrance (lcc) if* 1 *is* (m,p)*-regular,*

(ii) $(m^{(\mu)}, p^{(\mu)})$*-entrance (lpc) if* 1 *is* (m,p)*-pure exit,*

(iii) $(m^{(\mu)}, p^{(\mu)})$*-natural, if* 1 *is* (m,p)*-i.a. exit.*

Proof. Suppose $p(1)<\infty$, $\mu=\lambda_0$ and let be at first $u(1)<\infty$. Thus λ_0 is a point of discontinuity of τ, and therefore

$$p^{(\mu)}(1)=\Gamma(\mu)=\Gamma(\lambda_0)=\int_{-\infty}^{0}\frac{d\tau(\lambda)}{\lambda_0-\lambda}=\infty \text{ holds.}$$

(i): If 1 is (m,p)-regular we have $\int_0^1 p^2 dm<\infty$. This implies

$$\psi(.,\lambda_0)\in L_2(m),\text{ i.e. }\int_0^1 (p^{(\mu)})^2 dm^{(\mu)}=\int_0^1(\psi(.,\mu))^2 dm<\infty .$$

Therefore 1 is $(m^{(\mu)},p^{(\mu)})$-nonregular with 1cc, this means $(m^{(\mu)},p^{(\mu)})$-entrance with 1cc. Thus (i) is proved.

(ii): If 1 is (m,p)-pure exit then $\mu=\lambda_0<0$ and thus $0<\psi(x,\mu)<p(x)<\infty$ for all $x\leqq 1$. It follows

$$v^{(\mu)}(1)=\int_0^1\psi(x,\mu)\varphi(x,\mu)m(dx)\leqq\psi(1,\mu)D_p\varphi(1,\mu)\frac{1}{\mu}<\infty .$$

From $p^{(\mu)}(1)=\infty$ we have $u^{(\mu)}(1)=\infty$. Thus 1 has to be $(m^{(\mu)},p^{(\mu)})$-entrance. Using the inequality

$$\int_0^1(p^{(\mu)})^2 dm^{(\mu)}=\int_0^1\psi^2(.,\mu)dm\geqq\psi^2(1,\mu)m(1)=\infty ,$$

it follows that in this case 1 is in the $(m^{(\mu)},p^{(\mu)})$-1pc. Therefore (ii) is proved.

(iii): Let 1 be an (m,p)-i.a. exit (in particular $u(1)=\infty$). Let us assume $\mu=\lambda_0<0$ (otherwise we have $\lambda_0=0$ and the conclusion is obvious). We have

$$\int_0^1 r_{-\mu}^{(\mu)}(x,x)m^{(\mu)}(dx)=\int_0^1\varphi(x,\lambda_0)\chi(x,\lambda_0)m(dx)=\int_0^1(p(1)-p(x))m(dx)$$
$$=u(1)=\infty .$$

Thus 1 cannot be $(m^{(\mu)},p^{(\mu)})$-accessible or -entrance because in this cases the trace of $R_\lambda^{(\mu)}$ is finite ((see (16)). Therefore (iii) holds. Q.E.D.

Remark. In (iii) of Pproposition 5 the boundary 1 need not be $(m^{(\mu)},p^{(\mu)})$-pure natural. For example let (m,p) be the canonical pair corresponding to the spectral function $\tau(\mu)=-\min(\mu^2,1)$ $(\mu\leqq 0)$, which exists by the solution of the inverse spectral problem (see [10]). Then $\lambda_0=0$, $p(1)=\int_{-\infty}^{0}\frac{d\tau}{\mu}=2<\infty$. Therefore 1 is an (m,p)- and $(m^{(\mu)},p^{(\mu)})$-i.a. exit.

The following proposition is the converse of Proposition 5 in a certain sense. We remark that $p(1)=\infty$ implies that 1 is (m,p)-entrance or -pure natural.

Proposition 5′. *Let $p(1)=\infty$ and $\mu>\lambda_0$. Then the boundary 1 is*

(i′) *$(m^{(\mu)}, p^{(\mu)})$-regular, if 1 is (m,p)-entrance (lcc),*

(ii′) *$(m^{(\mu)}, p^{(\mu)})$-pure exit, if 1 is (m,p)-entrance (lpc),*

(iii′) *$(m^{(\mu)}, p^{(\mu)})$-i.a. exit, if 1 is (m,p)-pure natural.*

Proof. If $p(1)=\infty$ then $\lambda_0=0$. From $\mu>\lambda_0=0$ it follows $p^{(\mu)}(1)=\Gamma(\mu)<\infty$, i.e. 1 is $(m^{(\mu)}, p^{(\mu)})$-accessible or-i.a. exit. Let $\tilde{m}:=m^{(\mu)}$, $\tilde{p}:=p^{(\mu)}$. Then $-\mu=\lambda_0:=\sup \operatorname{supp} \tau^{(\mu)}<0$. We apply Proposition 5 to $(\tilde{m},\tilde{p})$ and remark $\tilde{m}^{(-\mu)}=m$, $\tilde{p}^{(-\mu)}=p$ (see Lemma 1).

If 1 is (m,p)-entrance with 1cc then $p^{(\mu)}(1)<\infty$, i.e. 1 is (m,p)-accessible or (m,p)-i.a. exit. From Proposition 5 it follows that 1 is (m,p)-regular. Thus (i′) holds. (ii′) and (iii′) are proved analogously. Q.E.D.

§4. Asymptotic Properties of $p(t,x,y)$

Now we study the properties of $p(t,x,y)$ for $y\to 1$ or/and $t\to\infty$ under the assumption that 1 is accessible or entrance. If 1 is natural such properties as are proven below do not hold. For short formulation of the results we introduce the following notation: Let $h(t,x,y)$ and $h(t,x,1)$ $(t>0;\ x,y\in[0,1))$ be nonnegative measurable functions. We shall say that $h(t,x,y)$ converges for $y\to 1$ in *space-weak* (shortly: *s-weak*) *sense* to $h(t,x,1)$ if

$$\lim_{y\to 1}\int_0^1 h(t,x,y)f(x)m(dx)=\int_0^1 h(t,x,1)f(x)m(dx) \qquad (f\in C_c,\ t>0)$$

where $C_c:=\{f\in C\,|\,f(x)=0 \text{ for } x\in[a,1) \text{ and some } a<1\}$.

We shall say that $h(t,x,y)$ converges for $y\to 1$ in *time-weak* (shortly: *t*-weak) sense to $h(t,x,1)$ if

$$\lim_{y\to 1}\int_0^t h(s,x,y)ds=\int_0^t h(s,x,1)ds \qquad (x\in[0,1),\ t>0)\,.$$

If $h(t,x,y)$ converges to $h(t,x,1)$ in *s*- and *t*-weak sense, we write $\operatorname{st-lim}_{y\to 1} h(t,x,y)=h(t,x,1)$.

Now we can formulate the following

Theorem 2. (i): *Let 1 be entrance. Then*

$$\lim_{y\to 1}\frac{p(t,x,y)}{p(y)}=0 \qquad (x\in[0,1),\ t>0) \tag{19}$$

pointwise and there exists a strictly positive and continuous function $p(t,x,1)$ with

(20) $$\text{st-}\lim_{y\to 1} p(t,x,y) = p(t,x,1) \qquad (x\in[0,1),\ t>0)\,.$$

(ii): *Let* 1 *be accessible. Then*

(21) $$\lim_{y\to 1} p(t,x,y) = 0 \qquad (x\in[0,1),\ t>0)$$

pointwise and there exists a strictly negative and continuous function $D_p p(t,x,1)$ *with*

(22) $$\text{st-}\lim_{y\to 1} \frac{p(t,x,y)}{p(y)-p(1)} = D_p p(t,x,1) \qquad (x\in[0,1),\ t>0)\,.$$

Proof: From Proposition 2 and the corollary after Proposition 1 the properties (19) and (21) follow directly. Let 1 be entrance. By the theorem of Riesz for every $t>0$ there exists a measure on $[0,1]$ denoted by $P_t(1,dy)$ such that

$$S_t f(1) = \int_0^1 P_t(1,dy)f(y) \qquad (f\in C)\,.$$

From $S_t 1=1$ it follows $P_t(1,[0,1])=1$. The semigroup property of (S_t) implies

$$S_t f(1) = (S_h(S_{t-h}f))\,(1) = \int_0^1 f(z)\,(\int_0^1 P_h(1,dy)p(t-h,y,z))m(dz)$$

$(t>0,\ 0<h<t)$. Defining $p(t,x,1):=\int_0^1 P_h(1,dy)p(t-h,y,x)$ we have

$$\lim_{y\to 1} \int_0^1 f(x)p(t,x,y)m(dx) = \int_0^1 f(x)p(t,x,1)m(dx) \qquad (f\in C)$$

by continuity of $S_t f$ on $[0,1]$ for $f\in C$. From the continuity of $p(t-h,y,x)$ in (t,x) with $x\in[0,1)$ and $t>0$ it follows that $p(t,x,1)$ is continuous there.

Now we show that $p(t,x,1)$ is strictly positive. Suppose conversely $p(t_0,x_0,1)=0$ for some (t_0,x_0) with $t_0>0$, $x_0\in[0,1)$. Then $P_h(1,[0,1))=0$ for any $h\in(0,t_0)$ by definition and strong positiveness of the density $p(t-h,y,x)$. Thus $P_h(1,\{1\})=1$ and therefore $S_h f(1)=f(1)$ for every $h>0$. This implies $R_\lambda f(1)=\frac{f(1)}{\lambda}$ $(\lambda>0)$. But if 1 is entrance we have $0\leq\varphi(x,\lambda)\int_x^1 f\chi(.,\lambda)dm\leq\chi(x,\lambda)\int_x^1 f\varphi(.,\lambda)dm\underset{x\to 1}{\to}0$ in virtue of the monodromy of φ and χ and in virtue of $D_p\varphi(1,\lambda)=\frac{1}{\lambda}\int_0^1\varphi(.,\lambda)dm<\infty$ and $f\in C$. Thus $R_\lambda f(1)=\chi(1,\lambda)\int_0^1\varphi f dm$, which contradicts $R_\lambda f(1)=\frac{f(1)}{\lambda}$.

Hence we have proved the part (i) without the t-weak convergence in (20). The part (ii) of the theorem (without t-weak convergence in (22) we shall prove using (i) and the transformation studied in Chapter 3.

Let 1 be accessible. Then $\lambda_0 := \sup \operatorname{supp} \tau$ is an isolated eigenvalue of $\boldsymbol{D}_m\boldsymbol{D}_p$ (see theorem 1). We define a new canonical pair (m^*, p^*) by $dm^* = \varphi^2(., \lambda_0)dm$ and $dp^* = \varphi^{-2}(., \lambda_0)dp$. A simple calculation shows that

$$p^*(t,x,y) = e^{-\lambda_0 t}\frac{p(t,x,y)}{\varphi(x,\lambda_0)\varphi(y,\lambda_0)} \qquad (t>0;\ x,y\in[0,1))$$

is the transition density corresponding to (m^*, p^*).

By Proposition 5 the boundary 1 is (m^*, p^*)-entrance. Thus from (i) it follows the existence of a strictly positive continuous function $p^*(t,x,1)$ $(t>0, x\in[0,1))$ such that

$$\lim_{y\to 1}\int_0^1 p^*(t,x,y)f(x)m^*(dx) = \int_0^1 p^*(t,x,1)f(x)m^*(dx) \quad (f\in C,\ x\in[0,1),\ t>0)\,.$$

Thus we have

$$\lim_{y\to 1}\int_0^1\frac{p(t,x,y)}{\varphi(y,\lambda_0)}e^{-\lambda_0 t}f(x)\varphi(x,\lambda_0)m(dx) = \int_0^1 p^*(t,x,1)f(x)\varphi^2(x,\lambda_0)m(dx)\,.$$

Remarking that $\varphi(., \lambda_0)$ is continuous and strictly positive on $[0,1)$ (see Theorem 1) and $-\infty < D_p\varphi(1,\lambda_0) < 0$ (see Proposition 4) it follows for $g = f\cdot\varphi(., \lambda_0)$

$$\lim_{y\to 1}\frac{p(t,x,y)}{p(y)-p(1)}e^{-\lambda_0 t}g(x)m(dx) = D_p\varphi(1,\lambda_0)\int_0^1 p^*(t,x,1)\varphi(., \lambda_0)g(x)m(dx)$$

Defining $D_p p(t,x,1) := e^{\lambda_0 t}D_p\varphi(1,\lambda_0)\varphi(x,\lambda_0)p^*(t,x,1)$ for $x\in[0,1)$, $t>0$ and using that every $g\in C_c$ has the representation $g = f\cdot\varphi(., \lambda_0)$ for some $f\in C_c$ it follows

$$s-\lim_{y\to 1}\frac{p(t,x,y)}{p(y)-p(1)} = D_p p(t,x,1) < 0 \qquad (t>0,\ x\in[0,1))\,.$$

Thus (ii) without the t-weak convergence in (22) is proved.

Now we will show the t-weak convergence in (20) and (22). To this purpose we will use a lemma which was proven for diffusion processes in [19].

Lemma 4. *Let $p(1) < \infty$. Then we have*

$$\lim_{y\to 1}\int_0^t\frac{p(s,x,y)ds}{p(1)-p(y)} = 1 - \int_0^1 p(t,x,z)m(dz) \qquad (t>0) \tag{23}$$

uniformly in $x\in[0,c]$ for every $c<1$.

Proof. We assume that y is a point of increasing of m. Then (see (A6))

$$\int_0^t p(s,x,y)\,ds = r_0(x,y) - \int_t^\infty p(s,x,y) = (p(1)-p(y)) - \int_0^\infty p(s+t,x,y)ds$$
$$= (p(1)-p(y)) - \int_0^1 p(t,x,z)r_0(z,y)m(dz) \qquad (x \leqq y)$$

and it follows

$$\int_0^t \frac{p(s,x,y)}{p(1)-p(y)}\,ds = 1 - \int_0^y p(t,x,z)m(dz) - \int_y^1 p(t,x,z)\frac{p(1)-p(z)}{p(1)-p(y)}\,m(dz)\,.$$

Remarking that the two integrals on the right side converge for every $c<1$ uniformly in $x \leqq c$ if $y \to 1$ (the last integral to zero), (23) follows.

Let 1 be accessible. The t-weak convergence in (22) is proved if we can identify the limit in (23) with $\int_0^t D_p p(s,x,1)ds$ i.e. if for every $f \in C_c$

$$\text{(24)} \qquad \int_0^1\int_0^t D_p p(s,x,1)ds\, f(x)m(dx) = \int_0^1 (\lim_{y\to 1}\int_0^t \frac{p(s,x,y)}{p(y)-p(1)}\,ds) f(x)m(dx)$$

Using $f \in C_c$ the integrals on the left side of (24) can be changed. Furthermore we have

$$\int_0^1 \frac{p(s,x,y)}{p(y)-p(1)} f(x)m(dx) = \int_0^1 \frac{f(x)}{\varphi(x,\lambda_0)}\, p^*(s,x,y)m^*(dx)\,\frac{\varphi(y,\lambda_0)}{p(y)-p(1)}\,e^{\lambda_0 s}$$

with m^* and p^* as above. This integral is majorized by $\sup_x \left|\frac{f(x)}{\varphi(x,\lambda_0)}\right|$ $(|D_p\varphi(1,\lambda_0)|+\varepsilon)e^{\lambda_0 s}$ for y sufficiently near 1. Summarizing the above argument we have the s-weak convergence in (22)

$$\text{(25)} \qquad \lim_{y\to 1}\int_0^1\int_0^t \frac{p(s,x,y)}{p(y)-p(1)}\,ds\, f(x)m(dx) = \lim_{y\to 1}\int_0^t\int_0^1 \frac{p(s,x,y)}{p(y)-p(1)} f(x)m(dx)ds$$
$$= \int_0^t \lim_{y\to 1}\int_0^1 \frac{p(s,x,y)}{p(y)-p(1)} f(x)m(dx)ds = \int_0^t\int_0^1 D_p p(s,x,1)f(x)m(dx)ds$$
$$= \int_0^1\int_0^t D_p p(s,x,1)ds\, f(x)m(dx) \qquad (f \in C_c)\,.$$

The left hand side of (25) can be written as $\int_0^1 \lim \int_0^t \cdots$ because of the uniform convergence in Lemma 4 and $f \in C_c$. Thus (24) holds, i.e. the t-weak convergence in (22) is proved. Let 1 be entrance. We show the t-weak convergence in (20) by using (22) and the methods of the transformation studied in Chapter 3.

Assume $\mu>0$. Then 1 is $(m^{(\mu)}, p^{(\mu)})$-accessible and from (22) we have

$$\lim_{y\to 1}\int_0^t \frac{p^{(\mu)}(s,x,y)}{p^{(\mu)}(y)-p^{(\mu)}(1)}\,ds = \int_0^t D_{p^{(\mu)}}p^{(\mu)}(s,x,1)ds\,. \tag{26}$$

By definition and a short calculation it follows that the left hand side integrals are equal to

$$\int_0^t p(s,x,y)ds\cdot(\varphi(x,\mu)\chi(y,\mu))^{-1}\,.$$

Remarking that $\chi(1,\mu)>0$ and that the limit in (26) is uniformly in $x\leqq c<1$ (see Lemma 4), we have that $\lim_{y\to 1}\int_0^t p(s,x,y)ds$ exists uniformly in $x\leqq c$ for all $c<1$ and all $t>0$.

As in the case of accessibility of 1 we can show that for every $f\in C_c$ the equality $\lim_{y\to 1}\int_0^1(\int_0^t p(s,x,y)ds)f(x)m(dx) = \int_0^1\int_0^t p(s,x,1)dsf(x)$ holds. (The calculations are left to the reader.) Thus $\lim_{y\to 1}\int_0^t p(s,x,y)ds=\int_0^t p(s,x,1)ds$ as was to be shown. Therefore the proof of the Theorem 2 is finished.

If the boundary 1 is in the limit circle case the convergence character in Theorem 2 can be improved. Moreover in this case a spectral expansion of the limit functions holds. This is the contents of the following theorem. (We remark that in 1cc the boundary 1 has to be regular or entrance.)

Theorem 3. *Let* 1 *be entrance with* $\int_0^1 p^2dm<\infty$ *(let* 1 *be regular). Then*

$$\lim_{y\to 1}p(t,x,y) = p(t,x,1) \qquad \left(\lim_{y\to 1}\frac{p(t,x,y)}{p(y)-p(1)} = D_p p(t,x,1)\right)$$

uniformly in $x\leqq 1$ *and* $t\geqq t_0$ *for any* $t_0>0$.

Moreover the following formula holds:

$$p(t,x,1) = \sum_0^\infty e^{\lambda_k t}\varphi(x,\lambda_k)\varphi(1,\lambda_k)\tau_k \tag{27}$$

$$(D_p p(t,x,1) = \sum_0^\infty e^{\lambda_k t}\varphi(x,\lambda_k)D_p\varphi(1,\lambda_k)\tau_k) \tag{28}$$

uniformly in $x\in[0,1]$, $t\geqq t_0$ *for any* $t_0>0$.

Here (λ_k) *denote the eigenvalues of* $\boldsymbol{D}_m\boldsymbol{D}_p$ *see Theorem* 1).

Proof. Let 1 be entrance with $\int_0^1 p^2dm<\infty$. Then for every $\lambda>0$ we have $\varphi(.,\lambda)\in L_2(m)$ and therefore by the monotonity of $\varphi(.,\lambda)$ and $\chi(.,\lambda)$

$$r_{\lambda}^{(2)}(x,x):=\int_0^1 r_{\lambda}(x,z)r_{\lambda}(z,x)m(dz)=\chi^2(x,\lambda)\int_0^x \varphi^2(y,\lambda)m(dy)$$
$$+\varphi^2(x,\lambda)\int_x^1 \chi^2(y,\lambda)m(dy)\leqq 2\chi^2(0,\lambda)\|\varphi(.,\lambda)\|_{L_2(m)}^2 .$$

Thus the function $r_{\lambda}^{(2)}(x,x)$ is bounded in $x<1$.

From the Schwartz' inequality and the symmetry of $r_{\lambda}(x,y)$ it follows $r_{\lambda}^{(2)}(x,y)^2\leqq r_{\lambda}^{(2)}(x,x)r_{\lambda}^{(2)}(y,y)$. Hence $r_{\lambda}^{(2)}(x,y)$ is bounded and continuous on $[0,1]^2$.

We know that $\varphi(.,\lambda_k)$ is the eigenfunction of $\boldsymbol{D}_m\boldsymbol{D}_p$ corresponding to λ_k. Thus

$$(R_{\lambda}\varphi(.,\lambda_k))(x)=\frac{\varphi(x,\lambda_k)}{\lambda-\lambda_k}.$$

This and $r_{\lambda}(x,.)\in L_2(m)$ for every $x\in I$ imply the eigenfunction expansion

$$r_{\lambda}^{(2)}(x,.)=R_{\lambda}r_{\lambda}(x,.)=\sum_0^{\infty}\frac{\varphi(x,\lambda_k)\varphi(.,\lambda_k)}{(\lambda-\lambda_k)^2}\tau_k \tag{29}$$

for every $x\in I$ in $L_2(m)$-sense.

As in the proof of Proposition 4 it follows that $\sum_0^{\infty}\frac{\varphi^2(x,\lambda_k)}{(\lambda-\lambda_k)^2}\tau_k$ is bounded by $r_{\lambda}^{(2)}(x,x)$ and converges uniformly in $x\leqq 1$. From an analogous estimation as (14) and (15) it follows the absolute and uniform convergence of the series in (29). In particular the series

$$p(t,x,y)=\sum_0^{\infty}e^{\lambda_k t}\varphi(x,\lambda_k)\varphi(y,\lambda_k)\tau_k \tag{30}$$

are absolutely and uniformly convergent in $x,y\in I$, $t\geqq t_0$ for every $t_0>0$. We know from Theorem 1 that $\varphi(.,\lambda_k)\in C$. Thus (27) holds.

Let 1 be regular. Then 1 is $(m^{(\lambda_0)},p^{(\lambda_0)})$-entrance with 1cc. Thus we can apply the already proved part of the Theorem 3, and after simple calculations similarly to those in the proof of Theorem 2 we obtain (28). Q.E.D.

Now we study the properties of $p(t,x,y)$ if $t\to\infty$. To this purpose we prove the following

Lemma 5. *Let* 1 *be entrance and* P *the projection* $f\to\int_0^1\frac{f}{m(1)}dm$ *from* C *to the subspace* N *of constant functions. Then* $\lim_{t\to\infty}\|S_t-P\|=0$.

Proof. In virtue of Theorem 1 and the spectral mapping theorem the number 1 is a simple eigenvalue of $S_t(t>0)$ with the eigenspace N and with $||S_t-P||<1$. Therefore $S_t-P=S_t(I-P)$. This implies $(S_{nt}-P)=(S_t(I-P))^n$ $=(S_t-P)^n$. Thus the lemma is proved.

Theorem 4. *Let 1 be accessible or entrance. Then we have*

$$\text{(31)}\quad \lim_{t\to\infty} e^{-\lambda_0 t}\int_0^1 \frac{p(t,x,y)}{\varphi(x,\lambda_0)\varphi(y,\lambda_0)} f(y)m(dy) = (\int_0^1 \varphi^2(z,\lambda_0)m(dz))^{-1}\int_0^1 f(y)m(dy)$$

uniformly in $x\leqq 1$ *and* $f\in C_c$ *with* $||f||\leqq 1$. *If 1 is in the limit circle case then moreover*

$$\text{(32)}\qquad \lim_{t\to\infty} e^{-\lambda_0 t}\frac{p(t,x,y)}{\varphi(x,\lambda_0)\varphi(y,\lambda_0)} = (\int_0^1 \varphi^2(z,\lambda_0)m(dz))^{-1}$$

uniformly in $x,y\in[0,1]$.

Remark. If 1 is entrance then $\lambda_0=0$ and the formulas (31) and (32) simplify to

$$\text{(31')}\qquad \lim_{t\to\infty}\int_0^1 p(t,x,y)f(y)m(dy) = (m(1))^{-1}\int_0^1 f(y)m(dy)$$

uniformly in $x\leqq 1$ and $f\in C_c$ with $||f||\leqq 1$ and if 1 is moreover in the limit circle case, then it follows

$$\text{(32')}\qquad \lim_{t\to\infty} p(t,x,y) = (m(1))^{-1} \text{ uniformly in } x,y\in[0,1]\,.$$

Proof. Suppose 1 is entrance. From Lemma 5 it follows

$$\lim_{t\to\infty}\sup_{||f||\leqq 1}\sup_x |S_tf(x)-Pf| = 0\,.$$

Hence (31′) i.e. (31) for the entrance case is proved. If 1 is accessible, by transformation on (m,p) to (m^*,p^*) (see the proof of Theorem 2) and using (31′) it follows (31). The uniform and absolute convergence in (30) and τ_0^{-1} $=\int_0^1\varphi^2(z,\lambda_0)m(dz)$ imply (32). Q.E.D.

As mentioned in the first chapter the theorems above are used in [13] to study parabolic functions connected with quasidiffusions. To this aim we formulate the following

Corollary. *Let* f *be a continuous function on* $E\times(0,\infty)$ *with compact support. If 1 is accessible or entrance then*

$$(33)\qquad \lim_{t\to\infty}\int_0^\infty\int_0^1 \frac{p(t-s,y,x)}{e^{\lambda_0(t-s)}\varphi(y,\lambda_0)\varphi(x,\lambda_0)}f(x,s)m(dx)ds = \frac{\int_0^\infty\int_0^1 f(x,s)m(dx)ds}{\int_0^1 \varphi^2(z,\lambda_0)m(dz)}$$

uniformly in $y\leqq 1$.

Remark. If 1 is entrance we have $\lambda_0=0$ and thus (33) means

$$(33')\qquad \lim_{t\to\infty}\int_0^\infty\int_0^1 p(t-s,y,x)f(x,s)m(dx)ds = \frac{1}{m(1)}\int_0^\infty\int_0^1 f(x,s)m(dx)ds\,.$$

Proof. Apply (31) to the family $\{f(.,s)\}$ of continuous functions on E given by the f, use the compactness of supp f and integrate with respect to ds.

Q.E.D.

Appendix

Here we will summarize some notions and facts from the theory of the generalized differential operator $D_m D_p$ which we have used above. They can be found in [10] (see also [2], [7], [16]) or can be proved by standard methods of ordinary differential operators. We will follow the terminology of [10] in a slight changed manner.

Let (m,p) be a canonical pair and ϑ the set of all complex-valued functions f on $I=[0,1)$ such that there exist an m-locally integrable function $g=:D_m D_p f$ and two number $a, b\in K$ with

$$f(x) = a+bp(x)+\int_0^x (p(x)-p(s))g(s)m(ds) \qquad (x\in I)\,.$$

Every $f\in\vartheta$ is linear in p on intervals where m is constant. Put $D_p f(x):= b+\int_0^x D_m D_p f dm$ and $D_p^- f(0)=b$ $(f\in\vartheta,\ x\in I)$. If $m(0+)=0$ the function $D_m D_p f$ is uniquely determined (modulo m) by $f\in\vartheta$ and it holds $D_p f(0)=D_p^- f(0)$, if $m(0+)>0$ it is uniquely determined by $[b,f]$ with $b\in K, f\in\vartheta$. In this case $[b,f]$ is called an *extended function*. Especially we have

$$D_m D_p f(0) = \frac{D_p f(0)-D_p^- f(0)}{m(0+)}\,.$$

We shall speak about functions $f\in\vartheta$ and mean extended functions if necessary. If for a function f on I there exists the limit $\lim_{x\to 1} f(x)$ (finite or not) it is denoted by $f(1)$.

Let φ and ψ be the fundamental solutions of $D_m D_p g - \lambda g = 0$ (see Chapter 1). For any $\lambda \in K$ and $x \in I$ we have

$$\text{(A1)} \qquad |\varphi(x,\lambda)| \leqq \cosh (2p(x)m(x)|\lambda|)^{1/2}.$$

For every $\lambda \in K_0 := K \backslash (-\infty, 0]$ there exists the finite limit $\Gamma(\lambda) := \lim_{x \uparrow 1} \frac{\psi(x,\lambda)}{\varphi(x,\lambda)}$ and has a representation $\Gamma(\lambda) = \int_{-\infty}^{0} \frac{\tau(d\mu)}{\lambda - \mu}$, where τ is a uniquely determined nondecreasing function on $(-\infty, 0]$ with $\tau(x) = \frac{1}{2}(\tau(x+) + \tau(x-))$ $(x<0)$. It is called the *main spectral function* of (m,p). Obviously $\Gamma(.)$ has a holomorphic extension to $K \backslash \operatorname{supp} \tau$ and τ has the property

$$\text{(A2)} \qquad \int_{-\infty}^{0} \frac{\tau(d\mu)}{1+|\mu|} < \infty .$$

Put $\lambda_0 := \sup \operatorname{supp} \tau$. If λ is a point of discontinuity of τ, we have $\tau(\lambda+0) - \tau(\lambda-0) = (\int_0^1 \varphi^2(.,\lambda)dm)^{-1}$. By $\chi(x,\lambda) = \Gamma(\lambda)\varphi(x,\lambda) - \psi(x,\lambda)$ $(x \in I, \lambda \in K \backslash \operatorname{supp} \tau)$ a new solution of $D_m D_p g - \lambda g = 0$ is defined. We have $\chi(.,\lambda) \in L_2(m)$ for every $\lambda \in K \backslash \operatorname{supp} \tau$. For every $\lambda > 0$ the function $\varphi(.,\lambda)$ $(\chi(.,\lambda))$ is positive and strictly increasing (decreasing). For the behaviour of φ and χ near 1 see e.g. [2], p. 166. Moreover we have $\lim_{x \to 1} \varphi(x,\lambda) D_p \chi(x,\lambda) = -1$, and if 1 is not regular

$$\text{(A3)} \qquad \lim_{x \to 1} \chi(x,\lambda) D_p \varphi(x,\lambda) = 1 .$$

This can be proved similarly as was be done in [6] for diffusions. Let $\lambda > 0$. Then

$$\text{(A4)} \qquad 0 \geqq D_p \chi(x,\lambda) = -1 + \lambda \int_0^x \chi(s,\lambda) m(ds), \quad \text{i.e.} \quad \int_0^x \chi(s,\lambda) m(ds) \leqq \frac{1}{\lambda}.$$

If 1 is accessible then $\lambda_0 < 0$ (see Theorem 1) and

$$\text{(A5)} \qquad \begin{array}{ll} 0 < \varphi(x,\lambda_0) < 1, \; D_p \varphi(x,\lambda_0) < 0 & \\ 0 < \psi(x,\lambda_0) < p(x) \qquad (x \neq 0) & \end{array} \qquad (x \in I)$$

(This follows from $\varphi(x,\lambda_0) > 0$ $(x \in I)$ (see Theorem 1) and a comparison theorem as in the theory of ordinary differential operators, $\psi(x,\lambda) = \int_0^x \varphi^{-2}(y,\lambda) p(dy)$ $\varphi(x,\lambda)$ and

$$\psi(x,\lambda) = p(x) + \lambda \int_0^x (p(x) - p(y)) \psi(y,\lambda) m(dy)) .$$

From Proposition 2 it follows

$$\int_A r_\lambda(x,y)m(dy) = \int_A\int_0^\infty e^{-\lambda t}p(t,x,y)dtm(dy)$$

for every measurable subset A of I and every $x\in I$. If y is a point of increasing of m then it implies

$$r_\lambda(x,y) = \int_0^\infty e^{-\lambda t}p(t,x,y)dtm(dy)\,. \tag{A6}$$

The functions $p(t,\,.\,,y)$, φ and χ are linear in scale p on intervals where m is constant. Thus (A6) holds for all $x,y\in I$ such that x and y are not in the same interval where m is constant.

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