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A Difference Scheme for Solving Two Phase Stefan Problem of Heat Equation

By

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§1. Introduction

We consider a one-dimensional two phase Stefan problem of heat equation with some specified temperature on the boundary. It is to seak a pair of unknown functions $(u(x, t), y(t))$ satisfying the following equations:

$$
(1.1)\qquad \begin{cases}
c_1\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} & (0<x<y(t),\ 0<t\leq T),\\
c_2\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} & (y(t)<x<1,\ 0<t\leq T),\\
u(0, t)=\psi_1(t)>0 & (0\leq t\leq T),\\
u(1, t)=\psi_2(t)<0 & (0\leq t\leq T),\\
u(y(t), t)=0 & (0\leq t\leq T),\\
u(x, 0)=\phi(x)\begin{cases}\geq 0\\ \leq 0\end{cases} & \begin{matrix}(0\leq x\leq l),\\ (l\leq x\leq 1),\end{matrix}\\
b\dot{y}(t)=\dfrac{\partial u}{\partial x}(y(t)+0, t)-\dfrac{\partial u}{\partial x}(y(t)-0, t) & (0<t\leq T),\\
y(0)=l.
\end{cases}
$$

This is, for example, a mathematical model of a water-ice system being homogeneous on each cross section perpendicular to the x-axis. Here u is temperature and y is width of the water region which, we assume, is left to the ice region. We call the last two relations of (1.1) Stefan's condition as usual.

We assume, by the physical reason, that the boundary and initial data in the water region are positive and those in the ice region are negative, and that c_1, c_2, b and l are positive constants, l being in the interval (0, 1).

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The same and similar problems were considered by several authors ([1]–[6]) and others ([7]–[11]). In the former references cited solutions of problems were constructed classically by using Green's functions of heat equations or by the method of retarding the argument (see [4] and [5]) only for fairly smooth data or small data. In the latter references weak solutions were constructed by some ways for more general cases including several dimensional case, and it was turned out that only for one dimensional cases weak solutions were classical ones even for bounded and pieceweise continuous data. (See especially [11].)

We also consider the one dimensional problem (1.1) with bounded and pieceweise data, and construct its solution directly by a finite difference method which we can call a 'semi implicit method'. The method is simple and useful for numerical computation.

One phase problem also can be solved by the method proposed in this paper more easily than the 'fully implicit method' of [12]. (See [13].) Boundary conditions of other types may be treated by the same way.

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§2. Difference Scheme

We use a net of rectangular meshes with a uniform space width h and variable time steps $\{k_n\}$ $(n=1, 2, 3,\dots)$. The time steps $\{k_n\}$ are assumed to be unknown a priori and to be determined in the process of computation by the rule that h/k_n might give gradient of a free boundary at each time $t=t_n$, so that the free boundary might cross each line of ordinate $x=x_j$ just at each corresponding mesh point.

Let us introduce discrete coordinates

$$x_j=jh \qquad (j=0, 1, 2,\dots, M; Mh=1),$$
$$t_n=\sum_{p=1}^{n} k_p \qquad (n=1, 2, 3,\dots)$$

and net functions y_n and u_j^n which correspond to $y(t_n)$ and $u(x_j, t_n)$ respectively. By the rule mentioned above we can put

$$y_n=J_n h \qquad (J_n\text{: integers, } n=0, 1, 2,\dots; J_0 h=l).$$

Then we introduce usual divided differences:

$$(u_j^n)_x = \frac{1}{h}(u_{j+1}^n - u_j^n), \quad (u_j^n)_{\bar{x}} = \frac{1}{h}(u_j^n - u_{j-1}^n),$$

$$(u_j^n)_{x\bar{x}} = \frac{1}{h^2}(u_{j+1}^n - 2u_j^n + u_{j+1}^n), \quad (u_j^n)_{\bar{t}} = \frac{1}{k_n}(u_j^n - u_j^{n-1}), \text{ etc.}$$

In our scheme heat equations are replaced by the pure implicit difference equations

$$c_i(u_j^{n+1})_{\bar{t}} = (u_j^{n+1})_{x\bar{x}} \qquad (i=1, 2) \tag{2.1}$$

and Stefan's condition is once replaced by an explicit formula

$$\pm b\frac{h}{k_{n+1}} = (u_{J_n}^n)_x - (u_{J_n}^n)_{\bar{x}} \qquad (=\text{the heat flow to the interface}), \tag{2.2}$$

where sign $+$ or $-$ corresponds to the case of positive heat flow to the interface or negative one at $t=t_n$ respectively. This equation is used for determining k_{n+1}. In the case of positive heat flow we admit for the interface to move to the right by one space mesh a time interval, while in the case of negative heat flow to the left. That is

$$\begin{aligned} &J_{n+1} = J_n + 1 \quad (y_{n+1} = y_n + h) \qquad \text{if} \quad (u_{J_n}^n)_x - (u_{J_n}^n)_{\bar{x}} > 0, \\ &J_{n+1} = J_n - 1 \quad (y_{n+1} = y_n - h) \qquad \text{if} \quad (u_{J_n}^n)_x - (u_{J_n}^n)_{\bar{x}} < 0. \end{aligned} \tag{2.3}$$

The boundary and initial conditions are put in the followings obviously;

$$u_0^{n+1} = \psi_1^{n+1} = \psi_1(t_{n+1}), \quad u_M^{n+1} = \psi_2^{n+1} = \psi_2(t_{n+1}), \quad u_{J_{n+1}}^{n+1} = 0, \tag{2.4}$$

$$u_j^0 = \phi_j = \phi(x_j), \quad y_0 = l. \tag{2.5}$$

In computation we start from the initial condition (2.5) and ask the first time step k_1 from (2.2), and J_1 from (2.3). Then we find $\{u_j^1\}$ from the difference equations (2.1) with the time step k_1 and the boundary conditions (2.4). Again from (2.2) and (2.3) we get k_2 and J_2, and further $\{u_j^2\}$, and so on.

Since this scheme is very simple, it has been used by many people. But, as far as we know, there were no proof of its convergence. In the followings we will give a revised scheme and prove its convergence.

The scheme mentioned above has a defect, which can be easily seen from (2.2) or the formula

$$k_{n+1} = \frac{bh}{|(u_{J_n}^n)_x - (u_{J_n}^n)_{\bar{x}}|}. \tag{2.2}'$$

In fact, if, for a fixed h, the denominator of the right hand side tend to zero, the time step k_{n+1} might increase infinitely. This feature might take place at some turning points of the free boundary. It is disadvantageous for numerical computation and also for the convergence of the scheme. Therefore we need a device of 'regularization' (or a 'zero decision') in order to avoid the 'singularity' of the algorithm. The device is the followings; if the heat flow to the interface at $t=t_n$ is less than a prescribed small quantity, which we take $\beta\sqrt{h}$ (β is a positive constant),

$$|(u^n_{J_n})_x-(u^n_{J_n})_{\bar{x}}|<\beta\sqrt{h},$$

then (2.2) is not used but, instead of it, the formula

$$k_{n+1}=b\sqrt{h}/\beta \tag{2.6}$$

is employed and the position of the free boundary is retained for the interval (t_n, t_{n+1}):

$$J_{n+1}=J_n \qquad (y_{n+1}=y_n).$$

By this rule we have, in general

$$k_{n+1}\leq b\sqrt{h}/\beta\,. \tag{2.7}$$

Hence $\{k_n\}$ are uniformly bounded and tend uniformly to zero as $h\to 0$. This means avoidance of singularity.

We need one more device of prohibiting a sudden change of direction of the free boundary in order to simplify proof of convergence. That is the followings; if, by the algorithm mentioned above, $J_{n+1}<J_n$ $(J_{n+1}>J_n)$ in addition to $J_n>J_{n-1}$ $(J_n<J_{n-1})$ hold, then k_{n+1} and J_{n+1} are revised so that

$$k_{n+1}=b\sqrt{h}/\beta, \quad J_{n+1}=J_n \qquad (y_{n+1}=y_n) \tag{2.8}$$

and the main routine of the algorithm is again repeated.

The complete description of the scheme and algorithm is the following:

1° $\qquad u^0_j=\phi_j \quad (1\leq j\leq M-1), \quad y_0=J_0h=l\,.$

For $n=0, 1, 2,\ldots$, successively

2.1° if $(u^n_{J_n})_x-(u^n_{J_n})_{\bar{x}}>\beta\sqrt{h}$, then $J_{n+1}=J_n+1$ and k_{n+1} is determined from (2.2)$'$,

2.2° if $(u^n_{J_n})_x-(u^n_{J_n})_{\bar{x}}<-\beta\sqrt{h}$, then $J_{n+1}=J_n-1$ and k_{n+1} is determined from (2.2)$'$,

2.3° if $|(u^n_{J_n})_x-(u^n_{J_n})_{\bar{x}}|\leq\beta\sqrt{h}$, then $J_{n+1}=J_n$ and $k_{n+1}=b\sqrt{h}/\beta$,

3° $\qquad u^{n+1}$ are found from (2.1) and (2.4), and

4° if $J_{n+1}<J_n>J_{n-1}$ ($J_{n+1}>J_n<J_{n-1}$), then J_{n+1} and k_{n+1} are revised like $J_{n+1}=J_n$ and $k_{n+1}=b\sqrt{h}/\beta$, and return again to the step 3°. (When $n=0$, $J_{-1}=J_0$.)

§3. Some Properties of the Solution of the Difference Scheme

We will show some a priori properties of the solution of our scheme under some stringent conditions of data, that are the followings: assume that $\psi_i(t)$ ($i=1, 2$) and $\phi(x)$ are bounded pieceweise continuous and

$$\psi_1(t)>0, \quad \psi_2(t)<0 \quad (0\le t\le T), \tag{3.1}$$

$$\phi(x)\ge 0 \quad (0\le x\le l), \qquad \phi(x)\le 0 \quad (l\le x\le 1), \tag{3.2}$$

and that there is a positive constant K such that

$$\psi_1(0)\le\phi(0), \quad \psi_2(0)\ge\phi(1), \quad \psi_{1\bar{t}}(t)<K, \quad \psi_{2\bar{t}}(t)>-K \tag{3.3}$$

and

$$\phi_{\bar{x}}(x)>-K, \quad \phi(l)=0. \tag{3.4}$$

First of all, it is easily shown that a maximum principle follows from boundedness of data:

Lemma 3.1.

$$\begin{aligned} &0\le u_j^n\le\max\{\max_{1\le j\le J_0}\phi_j,\ \max_{1\le p\le n}\psi_1^p\} && (0\le j\le J_n,\ t_n\le T),\\ &0\ge u_j^n\ge\min\{\min_{J_0\le j\le M-1}\phi_j,\ \min_{1\le p\le n}\psi_2^p\} && (J_n\le j\le M,\ t_n\le T). \end{aligned} \tag{3.5}$$

Next we have

Lemma 3.2.

$$d<y_n<1-d \qquad (0\le t_n\le T) \tag{3.6}$$

where $d=\min\{l/2, (1-l)/2, \varepsilon/2K_1\}$, $\varepsilon=\min_{0\le t\le T}\{\psi_1(t), -\psi_2(t)\}$ *and* K_1 *is a positive constant such that*

$$K_1>\max\left\{\frac{2}{1-l}\max_{0\le x\le l}\phi(x),\ \frac{2}{1+l}\max_{0\le t\le T}\psi_1(t),\right.$$
$$\left.\frac{2}{2-l}\max_{0\le t\le T}(-\psi_2(t)),\ \frac{2}{l}\max_{l\le x\le 1}(-\phi(x))\right\}.$$

Proof. First of all it is clear from the assumption (3.1) that such a positive constant ε exists.

Assume that y_n $(0\le n\le n_0+1)$ becomes most close to the boundary $x=1$ firstly at $n=n_0+1$. If $y_{n_0}<(1+l)/2$, $y_n<1-d$ $(0\le n\le n_0+1)$ follows directly. Consider the case of $y_{n_0}\ge(1+l)/2$. Introduce the function

$$w_{n_0}(x_j,\,t_n)=K_1(y_{n_0}-x_j)\qquad (0\le x_j\le y_n,\ 0\le t_n\le t_{n_0}).$$

It satisfies the difference equation $(w_{n_0})_{x\bar{x}}=(w_{n_0})_{\bar{t}}$ and the conditions

$$w_{n_0}(y_n,\,t_n)=K_1(y_{n_0}-y_n)\ge 0 \qquad (0\le n\le n_0),$$

$$w_{n_0}(0,\,t_n)=K_1 y_{n_0}\ge K_1\frac{(1+l)}{2}>\psi_1(t_n)\qquad (0\le n\le n_0)$$

and

$$w_{n_0}(x_j,\,0)=K_1(y_{n_0}-x_j)\ge K_1\Big(\frac{1+l}{2}-l\Big)>\phi(x_j)\qquad (0\le j\le J_0).$$

Hence we get $w_{n_0}(x_j,\,t_{n_0})\ge u_j^{n_0}$ $(0\le x_j\le y_{n_0})$ by the maximum principle, or especially for $j=J_{n_0}-1$

$$\frac{0-w_{n_0}(x_{J_{n_0}-1},\,t_{n_0})}{h}\le\frac{0-u_{J_{n_0}-1}^{n_0}}{h},$$

that is

$$-K_1\le(u_{J_{n_0}}^{n_0})_{\bar{x}}\qquad(\le 0). \tag{3.7}$$

Next we introduce the function

$$z_{n_0}(x_j,\,t_n)=-\varepsilon\frac{x_j-y_{n_0}}{1-y_{n_0}}\qquad (y_n\le x_j\le 1,\ 0\le t_n\le t_{n_0}).$$

It satisfies the same difference equation $(z_{n_0})_{x\bar{x}}=(z_{n_0})_{\bar{t}}$ and the conditions

$$z_{n_0}(y_n,\,t_n)=-\varepsilon\frac{y_n-y_{n_0}}{1-y_{n_0}}\ge 0\qquad (0\le n\le n_0),$$

$$z_{n_0}(1,\,t_n)=-\varepsilon\ge\psi_2(t_n)\qquad (0\le n\le n_0)$$

and

$$z_{n_0}(x_j,\,0)=-\varepsilon+\frac{\varepsilon}{1-y_n}(1-x_j)\qquad (J_0\le j\le M).$$

Now if $1-\dfrac{\varepsilon}{K_1}<y_{n_0}$ were to hold, we should have from the last equation

$$z_{n_0}(x_j,\,0)\ge-\varepsilon+K_1(1-x_j)\ge\phi(x_j).$$

Then by the maximum principle, $z_{n_0}(x_j,\,t_{n_0})\ge u_j^{n_0}$ $(y_{n_0}\le x_j\le 1)$ and especially for $j=J_{n_0}+1$, $-\varepsilon\dfrac{h}{1-y_{n_0}}\ge u_{J_{n_0}+1}^{n_0}$, and hence

$$(u_{J_{n_0}}^{n_0})_x \leq -\frac{\varepsilon}{1-y_{n_0}}. \tag{3.8}$$

Since $(u_{J_{n_0}}^{n_0})_x-(u_{J_{n_0}}^{n_0})_{\bar{x}}>0$ by the assumption of n_0, we should have from (3.7) and (3.8)

$$\frac{\varepsilon}{1-y_{n_0}}<K_1.$$

This would be a contradiction to the hypothesis of $1-\frac{\varepsilon}{K_1}<y_{n_0}$. Therefore the inequality

$$y_{n_0}\leq 1-\frac{\varepsilon}{K_1}$$

and also

$$y_n\leq y_{n_0+1}\leq 1-\frac{\varepsilon}{K_1}+h<1-d \qquad (0\leq n\leq n_0+1)$$

must hold for sufficiently small h. Since n_0 is arbitrary among ones compatible with the above definition, we have in conclusion

$$y_n<1-d \qquad \text{for all} \quad n.$$

We can prove the other inequality $y_n>d$ by an analogous way. We assume that $\{y_n\}$ $(0\leq n\leq n_1+1)$ becomes most close to the boundary $x=0$ firstly at $n=n_1+1$. In the case of $y_{n_1}>l/2$, we have no problems. In the opposite case, we introduce the auxiliary functions

$$w_{n_1}(x_j, t_n)=K_1(y_{n_1}-x_j) \qquad (y_n\leq x_j\leq 1,\ 0\leq t_n\leq t_{n_1})$$

and

$$z'_{n_1}(x_j, t_n)=\varepsilon\left(1-\frac{x_j}{y_{n_1}}\right) \qquad (0\leq x_j\leq y_n,\ 0\leq t_n\leq t_{n_1}),$$

and then we get by using these function for comparison similarly

$$y_n>d \qquad \text{for all} \quad n.$$

Lemma 3.3.

$$(u_0^n)_x \quad \textit{and} \quad (u_M^n)_{\bar{x}}\geq -L \qquad (0<t_n\leq T), \tag{3.9}$$

where L is a positive constant such that

$$L>\max\{cK, \max_{0\leq t\leq T} 2\psi_1(t)/d, \max_{0\leq t\leq T}(-2\psi_2(t))/(1-d)\}$$
$$(c=\max\{2, c_1, c_2\}).$$

Proof. Taking a number n_0 ($t_{n_0}\leq T$) arbitrarily, we consider the functions

$$\zeta_{n_0}(x_j, t_n)=\psi_1(t_{n_0})-K(t_{n_0}-t_n)-Lx_j+\frac{c_1}{2}Kx_j^2$$
$$(0\leq j\leq J_n,\ 0\leq t_n\leq t_{n_0})$$

and

$$\tilde{\zeta}_{n_0}(x_j, t_n)=\psi_2(t_{n_0})+K(t_{n_0}-t_n)+L(1-x_j)-\frac{c_2}{2}K(1-x_j)^2$$
$$(J_n\leq j\leq M,\ 0\leq t_n\leq t_{n_0}).$$

The function ζ_{n_0} satisfies the difference equation $(\zeta_{n_0})_{x\bar{x}}=c_1(\zeta_{n_0})_{\bar{t}}$ and the conditions

$$\zeta_{n_0}(y_n, t_n)=\psi_1(t_{n_0})-K(t_{n_0}-t_n)-Ly_n+\frac{c_1}{2}Ky_n^2$$
$$<\psi_1(t_n)-Ly_n\left(1-\frac{c_1K}{2L}y_n\right)<\psi_1(t_n)-\frac{K_2}{2}d<0,$$
$$\zeta_{n_0}(0, t_n)=\psi_1(t_{n_0})-K(t_{n_0}-t_n)\leq\psi_1(t_n)$$

and

$$\zeta_{n_0}(x_j, 0)=\psi_1(t_{n_0})-Kt_{n_0}-Lx_j+\frac{c_1}{2}Kx_j^2$$
$$<\phi(0)-Lx_j\left(1-\frac{c_1K}{2L}x_j\right)<\phi(0)-Kx_j\leq\phi(x_j).$$

Hence by the maximum principle we have

$$\zeta_{n_0}(x_j, t_{n_0})\leq u(x_j, t_{n_0})\qquad(0\leq j\leq J_{n_0})$$

and especially for $j=1$

$$\frac{\zeta_{n_0}(x_1, t_{n_0})-\psi_1(t_{n_0})}{h}\leq\frac{u_1^{n_0}-\psi_1^{n_0}}{h},$$

that is

$$(u_0^{n_0})_x\geq -L+\frac{c_1}{2}kh>-L.$$

By using the function $\tilde{\zeta}_{n_0}$ we can get the inequality about $(u_M^{n_0})_{\bar{x}}$ similarly. Since n_0 is arbitrary, we get (3.9) in conclusion.

We go to the next lemma being essential to our discussion.

Lemma 3.4.

$$(u_j^n)_{\bar{x}}\geq -L\qquad(1\leq j\leq M,\ 0<t_n\leq T),\tag{3.10}$$

$$0\geq(u_{J_n}^n)_{\bar{x}}\geq -L,\quad 0\geq(u_{J_n}^n)_x\geq -L\qquad(0<t_n\leq T),\tag{3.11}$$

where L is the constant appeared in Lemma 3.3.

Proof. i) Assume first that the sequence $\{y_n\}$ is strictly monotone increasing for some interval $0\leq t_n\leq t_{n_0}$:

$$y_n<y_{n'} \qquad \text{for} \quad t_n<t_{n'}\leq t_{n_0}.$$

Then by the same way used for (3.7) we get

$$0\leq -(u^n_{J_n})_{\bar{x}}\leq L \qquad (0<t_n\leq t_{n_0}).$$

Hence we have from the assumption of monotonicity of $\{y_n\}$ and the algorithm (see the step 4° in § 2)

$$0\leq -(u^n_{J_n})_x< -(u^n_{J_n})_{\bar{x}}\leq L \qquad (0\leq t_n\leq t_{n_0}). \tag{3.12}$$

The function $(u^n_j)_x\equiv\eta^n_j$ satisfies the same difference equations $(\eta^n_j)_{x\bar{x}}=c_1(\eta^n_j)_{\bar{t}}$ $(0<j<J_n-1,\ 0<t_n\leq t_{n_0})$ and $(\eta^n_j)_{x\bar{x}}=c_2(\eta^n_j)_{\bar{t}}$ $(J_n<j<M-1,\ 0<t_n\leq t_{n_0})$ and the conditions

$$-\eta^n_0, \quad -\eta^n_{J_n-1}, \quad -\eta^n_{J_n}, \quad -\eta^n_{M-1} \quad \text{and} \quad -\eta^0_j\leq L$$
$$(0<t_n\leq t_{n_0},\ 0\leq j\leq M-1)$$

(see (3.9), (3.12) and (3.4)). Hence by the maximum principle we have

$$-(u^n_j)_x\leq L \qquad (0\leq j\leq M-1,\ 0<t_n\leq t_{n_0}). \tag{3.13}$$

ii) Assume next that $\{y_n\}$ is strictly monotone decreasing for some interval $0\leq t_n\leq t_{n_0}$:

$$y_n>y_{n'} \qquad \text{for} \quad t_n<t_{n'}\leq t_{n_0}.$$

Then we get as above

$$0\leq -(u^n_{J_n})_{\bar{x}}< -(u^n_{J_n})_x\leq L \qquad (0<t_n\leq t_{n_0})$$

and

$$-(u^n_j)_x\leq L \qquad (0\leq j\leq M-1,\ 0<t_n\leq t_{n_0}).$$

iii) If, in addition to the assumption of i) or ii), $J_{n_0+1}=J_{n_0}$ hold, we should have, by applying the same discussion as that for (3.7) in the both right and left regions for the interval (t_{n_0}, t_{n_0+1}),

$$0\leq -(u^{n_0+1}_{J_{n_0+1}})_{\bar{x}}\leq L \quad \text{and} \quad 0\leq -(u^{n_0+1}_{J_{n_0+1}})_x\leq L,$$

and hence again by the maximum principle

$$-(u^{n_0+1}_j)_x\leq L \qquad (0\leq j\leq M-1).$$

iv) It is not expected in general that $\{y_n\}$ is monotone. For the general cases, however, we can repeat the above discussion for each time interval of monotonicity. Thus we get (3.10) and (3.11) in conclusion.

Directly from the last lemma we can obtain the following lemma;

Lemma 3.5.

$$|(u^n_{J_n})_x-(u^n_{J_n})_{\bar{x}}|\leq L \qquad (0<t_n<T), \tag{3.14}$$

$$\frac{h}{k_n}\leq\frac{L}{b} \qquad (0<t_n<T), \tag{3.15}$$

$$|(u^n_{J_n})_t| \quad and \quad |(u^n_{J_n})_{\bar{t}}|\leq\frac{L^2}{b} \qquad (0<t_n<T) \tag{3.16}$$

and

$$\begin{aligned} &|(u^n_{J_n-1})_{x\bar{x}}|\leq\frac{L^2}{b}c_1 \qquad if \quad J_n>J_{n-1} \quad or \quad J_{n+1}<J_n, \\ &|(u^n_{J_n+1})_{x\bar{x}}|\leq\frac{L^2}{b}c_2 \qquad if \quad J_n<J_{n-1} \quad or \quad J_{n+1}<J_n \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0<t_n<T). \end{aligned} \tag{3.17}$$

Proof. The inequality (3.14) is obvious from (3.11). If $J_{n+1}\neq J_n$, by (3.14) and (2.2)′ we get $h/k_{n+1}\leq L/b$. If $J_{n+1}=J_n$, by (2.6) and (2.8) we have $h/k_{n+1}\leq\sqrt{h}/b\beta<L/b$ for small h. Thus we obtain (3.15) for all n.

By using the relations $u^n_{J_n}=0$ (for all n), we find that

$$(u^{n-1}_{J_{n-1}})_t=-\frac{h}{k_n}(u^n_{J_n})_{\bar{x}}, \quad (u^n_{J_n})_{\bar{t}}=-\frac{h}{k_n}(u^{n-1}_{J_{n-1}})_x \qquad \text{if} \quad J_n>J_{n-1},$$

$$(u^n_{J_n})_{\bar{t}}=\frac{h}{k_n}(u^{n-1}_{J_{n-1}})_{\bar{x}}, \quad (u^{n-1}_{J_{n-1}})_t=\frac{h}{k_n}(u^n_{J_n})_x \qquad \text{if} \quad J_n<J_{n-1}$$

and

$$(u^n_{J_n})_{\bar{t}}=0 \qquad \text{if} \quad J_n=J_{n-1}.$$

Hence, by (3.11) and (3.15) we get (3.16). By applying (3.16) to the difference equations $(u^n_{J_n\pm1})_{x\bar{x}}=c_i(u^n_{J_n\pm1})_{\bar{t}}$ ($i=1, 2$) we obtain (3.17), too.

Remark. (3.15) produces the Lipshitz type inequality

$$|y_{n_1}-y_{n_2}|\leq\frac{L}{b}|t_{n_1}-t_{n_2}| \qquad (0\leq t_{n_1}, t_{n_2}\leq T). \tag{3.18}$$

§4. Convergence of the Difference Scheme

Here we will show convergence of our scheme under the conditions of data given in Section 3.

We take a sequence $\{h_\alpha\}$ ($\alpha\to\infty$) tending to zero. Then the corresponding sequences $\{k_{n\alpha}\}$ ($n=1, 2, 3,\dots$) also tend to zero uniformly (see (2.7)). We define a pieceweise linear function $y_\alpha(t)$ as follows:

$$y_\alpha(t)=[(t-t_n)y_{n+1}+(t_{n+1}-t)y_n]/k_{n+1} \qquad \text{for} \quad t_n\le t\le t_{n+1} \quad (n=0, 1, 2,\dots).$$

Then we have from (3.6)

$$d<y_\alpha(t)<1-d \qquad (0\le t\le T)$$

and from (3.18)

$$|y_\alpha(t^2)-y_\alpha(t^1)|\le\frac{L}{b}|t^2-t^1| \qquad (0\le t^2, t^1\le T).$$

They mean that the functions $\{y_\alpha(t)\}$ are uniformly bounded and equicontinuous in $0\le t\le T$. Therefore there is a subsequence (which we denote again by $\{y_\alpha(t)\}$) which converges to a continuous function $y(t)$ uniformly in $0\le t\le T$. Clearly the limit function $y(t)$ itself satisfies

$$d\le y(t)\le 1-d \qquad (0\le t\le T), \tag{4.1}$$

$$|y(t^2)-y(t^1)|\le\frac{L}{b}|t^2-t^1| \qquad (0\le t^2, t^1\le T). \tag{4.2}$$

Let u_α be the solution of the difference scheme corresponding to h_α, and u be the solution of the auxiliarly problems

$$\begin{cases} c_1\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} & (0<x<y(t),\ 0<t\le T),\\ u(0, t)=\psi_1(t) & (0<t\le T),\\ u(y(t), t)=0 & (0<t\le T),\\ u(x, 0)=\phi(x) & (0\le x\le l)\end{cases}$$

and

$$\begin{cases} c_2\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} & (y(t)<x<1,\ 0<t\le T),\\ u(y(t), t)=0 & (0<t\le T),\\ u(1, t)=\psi_2(t) & (0<t\le T),\\ u(x, 0)=\phi(x) & (l\le x\le 1)\end{cases}$$

with the prescribed boundary $x=y(t)$, $y(t)$ being specified above.

We mention the following lemma;

Lemma 4.1. *Uniform boundedness of the family* $\{u_\alpha\}$ *in* $\Omega\{0<x<1,$

$0<t\leq T\}$ *implies uniform boundedness of each family of any divided difference of the first or higher order constructed from* $\{u_\alpha\}$ *in every region* Ω^* *whose closure is contained in the boundary of* Ω *and the moving boundary* $x=y(t)$ $(0<t\leq T)$.

From Lemmas 3.1, 4.1 and uniqueness of the solution (see [7] for example), it is seen that the sequence $\{u_\alpha(x_j, t_n)\}$, $\alpha\to\infty$ (or strictly speaking, $\{u_\alpha(x, t)\}$ defined by extending each net function for all (x, t) in Ω conveniently) converges to $u(x, t)$ in Ω and uniformly except some neighbourhoods of discontinuous points of the boundary data. For the details and the proof of the last lemma see the textbook [14] by I. G. Petrowsky.

What remains is to show that $y(t)$ and $u(x, t)$ satisfies Stefan's condition. Combining Petrowsky's technique and Lemmas 3.4 and 3.5 we also get

Lemma 4.2. *In every region* Ω^* *whose closure is contained in only the boundary of* Ω, *the families*

$$\{u_{\alpha\bar{x}}\}, \quad \{u_{\alpha\bar{t}}\} \quad \textit{and} \quad \{u_{\alpha x\bar{x}}\},$$

only $\{u_{\alpha x\bar{x}}(y_n, t_n)\}$, $n=1, 2,\ldots$, *being excluded from the families, are uniformly bounded.*

Now we get from uniform boundedness of $\{u_{\alpha x\bar{x}}\}$ mentioned above

$$|(u_\alpha(x, t))_x-(u_\alpha(x', t))_x|\leq C(\delta, \sigma)|x-x'|$$

for any (x, t) and (x', t) in any region $\{0<\delta<x,\ x'<y_\alpha(t),\ 0<\sigma<t\leq T\}$ or $\{y_\alpha(t)<x,\ x'<1-\delta<1,\ 0<\sigma<t\leq T\}$, where δ and σ are arbitrary small constants and $C(\delta, \sigma)$ is some constant depending on δ and σ, but not on α. Hence we also have

$$\left|\frac{\partial u}{\partial x}(x, t)-\frac{\partial u}{\partial x}(x', t)\right|\leq C(\delta, \sigma)|x-x'|$$

for $\{\delta<x,\ x'<y(t),\ \sigma<t<T\}$ or $\{y(t)<x,\ x'<1-\delta,\ \sigma<t<T\}$. It follows from these inequalities that there exist limits

$$\lim_{x\to y_\alpha(t)\pm0} u_x(x, t)=v_{\alpha\pm}(t) \tag{4.3}$$

and

$$\lim_{x\to y(t)\pm0} \frac{\partial u}{\partial x}(x, t)=v_\pm(t) \tag{4.4}$$

and convergence is uniform in any $\sigma\leq t\leq T\,(\sigma>0)$. Clearly the limit functions are continuous in $0<t\leq T$, and also bounded by Lemma 4.2. From convergence of $\{y_\alpha(t)\}$ and $\{u_{\alpha x}(x, t)\}$, (4.3) and (4.4) we find that

$$\lim_{\alpha\to\infty} v_{\alpha\pm}(t)=v_{\pm}(t)$$

uniformly in any $\sigma\leq t\leq T\,(\sigma>0)$.

Taking arbitrary τ and t $(t_m\leq\tau<t_{m+1}<t_n\leq t<t_{n+1})$, we have

$$\begin{aligned} y_\alpha(t)&=y_\alpha(\tau)+\int_\tau^t \dot{y}_\alpha(\theta)\,d\theta \\ &=y_\alpha(\tau)+\sum_{p=m+1}^{n-1}\ \underset{t_p\leq\theta<t_{p+1}}{\mathrm{sign}}\ (\dot{y}_\alpha(\theta))k_{p+1}\frac{h}{k_{p+1}} \\ &\quad+\int_\tau^{t_{m+1}}\dot{y}_\alpha(\theta)\,d\theta+\int_{t_n}^t \dot{y}_\alpha(\theta)\,d\theta\,. \end{aligned}$$

This can be put in the form

$$\begin{aligned} y_\alpha(t)&=y_\alpha(\tau)+\frac{1}{b}\sum_{p=m+1}^{n-1}{}' k_{p+1}[v_{\alpha+}(t_p+0)-v_{\alpha-}(t_p+0)] \\ &\quad+\int_\tau^{t_{m+1}}\dot{y}_\alpha(\theta)\,d\theta+\int_{t_n}^t \dot{y}_\alpha(\theta)\,d\theta\,, \end{aligned} \tag{4.5}$$

where $\sum'$ means summation except for the number p's such that $\dot{y}_\alpha(t_p+0)=0$, which occur at those times when $|v_{\alpha+}(t_p+0)-v_{\alpha-}(t_p+0)|\leq\beta\sqrt{h}$ hold or

$$v_{\alpha+}(t_{p-1}+0)-v_{\alpha-}(t_{p-1}+0)\gtrless\pm\beta\sqrt{h} \quad \text{and} \quad v_+(t_p+0)-v_-(t_p+0)\lessgtr\mp\beta\sqrt{h}$$

hold successively. By taking account of uniform convergence of $v_{\alpha\pm}(t)$ and uniform continuity of $v_\pm(t)$ in $\tau\leq t\leq T$, we find that (4.5) can be written in the form

$$y_\alpha(t)=y_\alpha(\tau)+\frac{1}{b}\sum_{p=m+1}^{n-1} k_{p+1}[v_{\alpha+}(t_p+0)-v_{\alpha-}(t_p+0)]+o(1) \tag{4.6}$$

where the term $o(1)$ tends to zero as $\alpha\to\infty$ $(h\to 0)$. Taking $\alpha\to\infty$ in the last formula, we obtain

$$y(t)=y(\tau)+\frac{1}{b}\int_\tau^t [v_+(\theta)-v_-(\theta)]\,d\theta$$

for any τ and t $(0<\tau<t\leq T)$. This means further that $y(t)$ is differentiable and

$$\dot{y}(t)=\frac{1}{b}[v_+(t)-v_-(t)] \qquad (0<t\leq T)$$

which is not but Stefan's condition. Thus we have found that the pair of functions $\{y(t), u(x, t)\}$ is a solution of the problem (1.1), that is, the selected subsequence $\{y_\alpha(t), u_\alpha(x, t)\}$ converges to the desired solution. However, since it is unique as well known, the full sequence itself converges. Thus we have proved

Theorem 4.1. *For bounded pieceweise continuous data* $\psi_i(t)$ $(i=1, 2)$ *and* $\phi(x)$ *satisfying* (3.1)$\sim$(3.4), *the solution of the difference scheme mentioned in Section* 2 *converges to the solution of* (1.1).

Remark. The convergence order all over the scheme might be subject to the slowest term $o(1)$ in (4.6). However, from the well known fact that $y(t)$ is infinitely times differentiable in $0<t\leq T$ ([3]), we find that $v_+(t)-v_-(t)$ also is infinitely times differentiable and hence $o(1)$ can be replaced by $O(\max k_n)$ $=O(\sqrt{h})$ at least.

§5. Existence Theorem

In the previous section we also proved the existence of the solution under the slightly stringent conditions (3.1)$\sim$(3.4) on the way proving Theorem 4.1. In this section we will show that the part of the conditions (3.3), (3.4) could be dropped for the unique existence, that is,

Theorem 5.1. *Suppose that* $\psi_i(t)$ $(i=1, 2)$ *and* $\phi(x)$ *are bounded, pieceweise continuous and*

$$\psi_1(t)>0, \quad \psi_2(t)<0 \qquad (0\leq t\leq T), \tag{5.1}$$

$$\phi(x)\geq 0 \quad (0\leq x\leq l), \quad \phi(x)\leq 0 \qquad (l\leq x\leq 1). \tag{5.2}$$

Then there is one and only one solution of (1.1).

Since the uniqueness is well known, we consider only the existence.

Before the proof, we will prepare some more facts under the stringent conditions.

Lemma 5.1. *Assume, in addition to* (3.1)$\sim$(3.4), *that* $\phi(x)$ *is continuously differentiable in some small intervals* $[l-\varepsilon, l]$ *and* $[l, l+\varepsilon]$. *Then* $v_\pm(t)$ $\equiv \lim\limits_{x\to y(t)\pm 0} \dfrac{\partial u}{\partial x}(x, t)$ *are continuous also at* $t=0$, *and*

$$v_\pm(t)=\phi'(l\pm 0)+O(L^2\sqrt{t}) \quad as \quad t\to 0. \tag{5.3}$$

Proof. Introduce the following Green's functions:

$$g_1(x, t; \xi, \tau)=U_1(x-\xi, t-\tau)-U_1(x+\xi, t-\tau),$$

$$G_1(x, t; \xi, \tau)=U_1(x-\xi, t-\tau)+U_1(x+\xi, t-\tau),$$

$$g_2(x, t; \xi, \tau)=U_2(x-\xi, t-\tau)-U_2(x+\xi-2, t-\tau)$$

and

$$G_2(x, t; \xi, \tau)=U_2(x-\xi, t-\tau)+U_2(x+\xi-2, t-\tau),$$

where

$$U_i(x, t)=\frac{1}{2}\sqrt{\frac{c_i}{\pi t}}e^{-\frac{c_i x^2}{4t}} \qquad (i=1, 2)$$

are the fundamental solutions of the differential operators $L_{c_i}\equiv\frac{\partial^2}{\partial x^2}-c_i\frac{\partial}{\partial t}$ $(i=1, 2)$. g_1 and G_1 are Green's functions of the first and second kind of boundary value problems of L_1 in the half plane $x>0$ respectively. g_2 and G_2 are those of L_2 in the half plane $x<1$. By using Green's functions we can represent the solution u of the problem (1.1) as follows;

$$\begin{aligned} u(x, t)=&\int_0^l g_1(x, t; \xi, 0)\phi(\xi)d\xi+\int_0^t g_{1\xi}(x, t; 0, \sigma)\psi_1(\sigma)d\sigma \\ &+\int_0^t g_1(x, t; y(\sigma), \sigma)v_-(\sigma)d\sigma \qquad (0<x<y(t), t>0), \\ u(x, t)=&\int_l^1 g_2(x, t; \xi, 0)\phi(\xi)d\xi-\int_0^t g_{2\xi}(x, t; 1, \sigma)\psi_2(\sigma)d\sigma \\ &-\int_0^t g_2(x, t; y(\sigma), \sigma)v_+(\sigma)d\sigma \qquad (y(t)<x<1, t>0) \end{aligned}$$

in the respective regions. (Here $(\ )_\xi$ means differential in ξ.) Hence, by differentiating both equations in x and taking limits $x\to y(t)\pm 0$, we obtain the well known formula

$$\begin{aligned} v_-(t)=&-2\int_0^l G_{1\xi}(y(t), t; \xi, 0)\phi(\xi)d\xi+2\int_0^t G_{1\sigma}(y(t), t; 0, \sigma)\psi_1(\sigma)d\sigma \\ &-2\int_0^t G_{1\xi}(y(t), t; y(\sigma), \sigma)v_-(\sigma)d\sigma \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} v_+(t)=&-2\int_l^1 G_{2\xi}(y(t), t; \xi, 0)\phi(\xi)d\xi-2\int_0^t G_{2\sigma}(y(t), t; 1, \sigma)\psi_2(\sigma)d\sigma \\ &+2\int_0^t G_{2\xi}(y(t), t; y(\sigma), \sigma)v_+(\sigma)d\sigma \end{aligned} \tag{5.5}$$

(for derivation, see [2]).

We consider first $v_-(t)$. Denote the three terms appearing in the right hand side of (5.4) by I_i $(i=1, 2, 3)$ respectively. Since $v_-(t)$ is bounded and continuous in $0<t\le T$ $(|v_-(t)|\le L$; see the previous section), I_3 is estimated as follows:

$$|I_3| \leq \left| -2\int_0^t G_{1\xi}(y(t), t; y(\sigma), \sigma)v_-(\sigma)d\sigma \right|$$
$$\leq \frac{c_1^{3/2}}{2\sqrt{\pi}} L\left[\int_0^t \frac{|y(t)-y(\sigma)|}{(t-\sigma)^{3/2}} e^{-\frac{c_1(y(t)-y(\sigma))^2}{4(t-\sigma)}} d\sigma\right.$$
$$\left.+\int_0^t \frac{y(t)+y(\sigma)}{(t-\sigma)^{3/2}} e^{-\frac{c_1(y(t)+y(\sigma))^2}{4(t-\sigma)}} d\sigma\right]$$
$$< \frac{c_1^{3/2}}{2\sqrt{\pi}} L\left[\frac{2L}{b}\sqrt{t} + 2\int_0^t \frac{1}{(t-\sigma)^{3/2}} e^{-\frac{c_1 d^2}{(t-\sigma)}} d\sigma\right]$$
(by (4.1) and (4.2)).

The last integral is

$$\int_0^t \frac{1}{(t-\sigma)^{3/2}} e^{-\frac{c_1 d^2}{(t-\sigma)}} d\sigma = \frac{2}{\sqrt{c_1}d}\int_0^\infty e^{-\left(\theta+\frac{c_1 d}{\sqrt{t}}\right)^2} d\theta$$
$$< \frac{2}{\sqrt{c_1}d} e^{-\frac{c_1 d^2}{t}} \int_0^\infty e^{-\theta^2} d\theta = \frac{\sqrt{\pi}}{\sqrt{c_1}d} e^{-\frac{c_1 d^2}{t}},$$

which decreases exponentially as $t\to 0$. Hence we find $|I_3| < O(L^2\sqrt{t})$ $(t\to 0)$.

Next, denoting a bound of $\psi_1(t)$ by K_0, we have

$$|I_2| = \left| 2\int_0^t G_{1\sigma}(y(t), t; 0, \sigma)\psi_1(\sigma)d\sigma \right|$$
$$\leq K_0\sqrt{\frac{c_1}{\pi}}\left[1+\frac{c_1}{2}y(t)^2\right]\int_0^t \frac{1}{(t-\sigma)^{3/2}} e^{-\frac{c_1 y(t)^2}{4(t-\sigma)}} d\sigma,$$

which also decreases exponentially as above.

Consider I_1.

$$I_1 = -2\int_0^l G_{1\xi}(y(t), t; \xi, 0)\phi(\xi)d\xi$$
$$= -\frac{c_1^{3/2}}{2\sqrt{\pi}t^{3/2}}\left[\int_0^l (y(t)-\xi)e^{-\frac{c_1(y(t)-\xi)^2}{4t}}\phi(\xi)d\xi\right.$$
$$\left.-\int_0^l (y(t)+\xi)e^{-\frac{c_1(y(t)+\xi)^2}{4t}}\phi(\xi)d\xi\right].$$

Clearly the last integral decreases exponentially as $t\to 0$. Consider the former integral. Taking a number $\alpha\left(0<\alpha<\frac{1}{2}\right)$, we assume t so small that $t^\alpha<\varepsilon$. Then we devide the interval of the integral into two parts: $(0, l-t^\alpha)$ and $(l-t^\alpha, l)$. By the mean value theorem, we have

$$\int_0^{l-t^\alpha} (y(t)-\xi)e^{-\frac{c_1(y(t)-\xi)^2}{4t}}\phi(\xi)d\xi$$
$$= \int_0^{l-t^\alpha} (y(t)-\xi)e^{-\frac{c_1(l-\xi)^2}{4t}} e^{-\frac{c_1(y(t)-l)(y(t)+l-2\xi)}{4t}}\phi(\xi)d\xi$$

$$=(y(t)-\bar{\xi})e^{-\frac{c_1(y(t)-l)(y(t)+l-2\bar{\xi})}{4t}}\int_0^{l-t^\alpha} e^{-\frac{c_1(l-\xi)^2}{4t}}\phi(\xi)d\xi$$

for a $\bar{\xi}$ $(0<\bar{\xi}<l-t^\alpha)$. By using $l-\xi\geq t^\alpha$, $y(t)+l-2\bar{\xi}>0$ (for sufficiently small t) and $|y(t)-l|\leq\frac{L}{b}t$, we find that the last integral decreases as $O(e^{-\frac{c_1}{4t^{1-2\alpha}}})$ as $t\to 0$. On the other hand,

$$\int_{l-t^\alpha}^{l}(y(t)-\xi)e^{-\frac{c_1(y(t)-\xi)^2}{4t}}\phi(\xi)d\xi$$

$$=\frac{2t}{c_1}\int_{l-t^\alpha}^{l}\frac{\partial}{\partial\xi}e^{-\frac{c_1(y(t)-\xi)^2}{4t}}\phi(\xi)d\xi$$

$$=-\frac{2t}{c_1}\phi(l-t^\alpha)e^{-\frac{c_1(y(t)-l+t^\alpha)^2}{4t}}-\frac{2t}{c_1}\int_{l-t^\alpha}^{l}e^{-\frac{c_1(y(t)-\xi)^2}{4t}}\phi'(\xi)d\xi$$

$$=-\frac{2t}{c_1}\phi(l-t^\alpha)e^{-\frac{c_1(y(t)-l+t^\alpha)^2}{4t}}$$

$$-\frac{2t}{c_1}e^{-\frac{(y(t)-l)(y(t)+l-2\tilde{\xi})}{4t}}\phi'(\tilde{\xi})\int_{l-t^\alpha}^{l}e^{-\frac{c_1(l-\xi)^2}{4t}}d\xi$$

$$(l-t^\alpha<\tilde{\xi}<l).$$

The first term on the right hand side decreases as fast as $O(t\cdot e^{-\frac{c_1}{4t^{1-2\alpha}}})$ $(t\to 0)$. Consider the second term. We find that

$$\left|\frac{(y(t)-l)(y(t)+l-2\tilde{\xi})}{4t}\right|=\frac{(y(t)-l)^2+2|y(t)-l|(l-\tilde{\xi})}{4t}$$

$$\leq\left(\frac{L}{2}\right)^2 t+\frac{L}{2b}t^\alpha\to 0$$

and

$$\int_{l-t^\alpha}^{l}e^{-\frac{c_1(l-\xi)^2}{4t}}d\xi=2\sqrt{\frac{t}{c_1}}\int_0^{\frac{\sqrt{c_1}}{2t^{1/2-\alpha}}}e^{-\theta^2}d\theta,$$

and the last integral converges to $\frac{\sqrt{\pi}}{2}$ as $t\to 0$. Thus we get

$$I_1=\phi'(l-0)+O(t^{-\frac{3}{2}}e^{-\frac{c_1}{4t^{1-2\alpha}}}).$$

Hence, from the estimates of I_2 and I_3 mentioned above, we obtain in conclusion

$$v_-(t)=\phi'(l-0)+O(L^2\sqrt{t}),$$
$$\lim_{t\to 0}v_-(t)=\phi'(l-0).$$

By the similar way, we get also

$$v_+(t)=\phi'(l+0)+O(L^2\sqrt{t}),$$
$$\lim_{t\to 0} v_+(t)=\phi'(l+0).$$

Thus we have proved Lemma 5.1.

Directly from Lemma 5.1, we find

Lemma 5.2. *Under the same assumption as in Lemma* 5.1, *if*

$$\phi'(+0)\gtrless\phi'(-0),$$

then

$$v_+(t)\gtrless v_-(t) \quad \text{and} \quad \dot{y}(t)\gtrless 0$$

for sufficiently small $t>0$.

The next lemma is a so-called principle of monotonicity which holds even for the general data as stated in Theorem 5.1.

Lemma 5.3. *Denote the solutions of* (1.1) *corresponding to the two sets of data* $(l_i, \psi_{1i}, \psi_{2i}, \phi_i)$ $(i=1, 2)$ *by* (y_i, u_i) $(i=1, 2)$ *respectively. If* $l_1<l_2$, $\psi_{11}\leq\psi_{12}$, $\psi_{21}\leq\psi_{22}$ *and* $\phi_1\leq\phi_2$, *then*

$$y_1(t)<y_2(t) \qquad (t\geq 0) \tag{5.6}$$

holds.

Proof. We will prove it by showing a contradiction derived from the opposite case. Suppose that there were a time $t=\lambda$ where

$$y_1(\lambda)=y_2(\lambda)$$

hold firstly. Then, since $y_1(t)<y_2(t)$ for $0\leq t<\lambda$,

$$\dot{y}_1(\lambda)\geq\dot{y}_2(\lambda) \tag{5.7}$$

should hold. However, from the assumption of data, we get by the well known strong maximum principle (see, for example, [2])

$$u_1(x, t)-u_2(x, t)<0$$

in $0<x<y_2(t)$, $0<t\leq\lambda$ and $y_2(t)<x<1$, $0<t\leq\lambda$. Hence and from $u_1(y_2(\lambda), \lambda)-u_2(y_2(\lambda), \lambda)=0$, we obtain by Friedman's lemma ([2])

$$\frac{\partial u_1}{\partial x}(y_2(\lambda)-0, \lambda)-\frac{\partial u_2}{\partial x}(y_2(\lambda)-0, \lambda)>0,$$
$$\frac{\partial u_1}{\partial x}(y_2(\lambda)+0, \lambda)-\frac{\partial u_2}{\partial x}(y_2(\lambda)+0, \lambda)<0.$$

This means $\dot{y}_1(\lambda)<\dot{y}_2(\lambda)$, which contradicts to (5.7). Thus we get (5.6).

In order to construct a desired solution for general data mentioned in Theorem 5.1, we use a sequence of solutions (y^τ, u^τ) $(\tau\to 0)$ of the following auxiliary problem depending on the approximating parameter τ:

$$
(5.8)\qquad \begin{cases}
c_1\dfrac{\partial u^\tau}{\partial t}=\dfrac{\partial^2 u^\tau}{\partial x^2} & (0<x<y^\tau(t),\ 0<t\le T),\\[2mm]
c_2\dfrac{\partial u^\tau}{\partial t}=\dfrac{\partial^2 u^\tau}{\partial x^2} & (y^\tau(t)<x<1,\ 0<t\le T),\\
u^\tau(0,t)=\psi_1(t) & (0<t\le T),\\
u^\tau(1,t)=\psi_2(t) & (0<t\le T),\\
u^\tau(y^\tau(t),t)=0 & (0<t\le T),\\
u^\tau(x,0)=\phi^\tau(x) & (0\le x\le 1),\\
y^\tau(t)=l & (0\le t\le \tau),\\
b\dot{y}^\tau(t)=\dfrac{\partial u^\tau}{\partial x}(y^\tau(t)+0,t)-\dfrac{\partial u^\tau}{\partial x}(y^\tau(t)-0,t) & (\tau<t\le T),
\end{cases}
$$

where

$$
(5.9)\qquad \phi^\tau(x)=\begin{cases}
\phi(x) & (0\le x\le l-\tau^\alpha,\ l+\tau^\alpha\le x\le 1),\\[1mm]
\dfrac{\bar{\phi}^\tau}{\tau^\alpha}(l-x) & (l-\tau^\alpha\le x\le l).\\[2mm]
\dfrac{\underline{\phi}^\tau}{\tau^\alpha}(x-l) & (l\le x\le l+\tau^\alpha),
\end{cases}
$$

$$
0<\alpha<\frac{1}{2},\quad \bar{\phi}^\tau=\inf_{l-\tau^\alpha\le x\le l}\phi(x),\quad \underline{\phi}^\tau=\sup_{l\le x\le l+\tau^\alpha}\phi(x)
$$
$$
(\text{if}\quad \phi(l+0)+\phi(l-0)\neq 0),
$$
$$
\bar{\phi}^\tau=\inf_{l-\tau^\alpha\le x\le l}\phi(x)+\beta_1\tau^{\alpha/2},\quad \underline{\phi}^\tau=\sup_{l\le x\le l+\tau^\alpha}\phi(x)-\beta_2\tau^{\alpha/2}
$$
$$
(\text{if}\quad \phi(l+0)+\phi(l-0)=0),
$$

and β_1, β_2 are arbitrarily fixed constants $(\beta_1>\beta_2>0)$. Here we notice that, in general,

$$
(5.10)\qquad (\phi^\tau)'(l+0)\neq 0,\quad (\phi^\tau)'(l-0)\neq 0\quad\text{and}\quad (\phi^\tau)'(l+0)\neq(\phi^\tau)'(l-0).
$$

The problem (5.6) differs from the original problem (1.1) only in the small time interval $[0, \tau]$, in which two ordinary initial and boundary value problems with the initial data $\phi^\tau(x)$ are assigned for both regions $(0, l)$ and $(l, 1)$.

Lemma 5.4. *Under the same conditions stated in Theorem* 5.1, *there is one and only one solution* (y^τ, u^τ) *of the problem* (5.8).

We will mention only an outline of its proof briefly. For construction of a solution we use again the difference scheme in Section 2, but we have to change it slightly in $(0, \tau)$. In fact we take the time steps $\{k_n\}$ appearing in $(0, \tau)$ arbitrarily but, at most, as much as $O(\sqrt{h})$ and so as for both τ and $\tau/2$ to be equal to some discrete times respectively: $\tau = t_{n_0}$, $\tau/2 = t_{n_1}$. Using the given time steps we solve the difference scheme (2.1), (2.4) and (2.5) (ϕ being replaced by ϕ^τ) in two fixed region $(0, l)$ and $(l, 1)$ up to the time $t = \tau$ $(n \leq n_0)$. After that, for $t > \tau$ $(n > n_0)$ we use the same algorithm mentioned in Section 2. We denote the solution obtained in such away by $\{(y^\tau)^n, (u^\tau)_j^n\}$. It is clear that Lemmas 3.1, 3.2 and 4.1 hold also in this case. Hence it follows that $\{(u^\tau)_j^n\}$ and their divided differences of any high order converge uniformly in any fixed closed region strictly away from the fixed boundaries and the moving boundary and hence the limit function $u^\tau(x, t) = \lim_{h \to 0} (u^\tau)_j^n$ is infinitely times differentiable in such interior region and satisfies the heat equations.

From the fact stated above, we have especially for $j = j_1$ and j_2 such as $j_1 h = d/2$ and $j_2 h = 1 - d/2$ (if necessary, by adjusting d so that both equalities may hold)

$$|(u^\tau)_{j_1 \bar{t}}^n|, \quad |(u^\tau)_{j_2 \bar{t}}^n| < L_d^\tau \qquad \left(\frac{\tau}{2} \leq t_n \leq T\right) \tag{5.11}$$

where L_d^τ is a positive constant depending on τ and d, but not on h.

Next we will show that

$$0 \leq -(u^\tau)_{J_0 \bar{x}}^n \quad \text{and} \quad -(u^\tau)_{J_0 x}^n \leq \frac{K_2}{\tau} \qquad \left(\frac{\tau}{2} \leq t_n < \tau\right) \tag{5.12}$$

where K_2 and τ are positive numbers such that

$$K_2 > \max\{c_1 l M_0, \quad c_2(1-l) M_0\}, \quad M_0 = \sup|\phi(x)|,$$
$$\tau \leq \max\left\{\frac{K_2}{2 \sup \psi_1}, \quad \frac{K_2(1-l)}{2 \sup(-\psi_2)}\right\}.$$

For the purpose we first introduce the function

$$\eta^\tau(x_j, t_n) = \frac{M_0}{\tau}(\tau - t_n) + \frac{K_2}{\tau}(l - x_j) - \frac{M_0 c_1}{2\tau}(l - x_j)^2$$
$$(0 \leq x_j \leq l, \ 0 \leq t_n \leq \tau).$$

It satisfies the difference equation $(\eta^\tau)_{x\bar{x}} = c_1 (\eta^\tau)_{\bar{t}}$ and the conditions

$$\eta^\tau(0, t_n) = \frac{M_0}{\tau}(\tau - t_n) + \frac{K_2 l}{\tau}\left(1 - \frac{M_0 c_1 l}{2K_2}\right) > \frac{K_2 l}{2\tau} > \psi_1(t_n) \qquad (t_n \leq \tau),$$

$$\eta^\tau(l, t_n) = \frac{M_0}{\tau}(\tau - t_n) \geq 0 \qquad (t_n \leq \tau)$$

and

$$\eta^\tau(x_j, 0) = M_0 + \frac{K_2}{\tau}(l - x_j)\left\{1 - \frac{M_0 c_1}{2K_2}(l - x_j)\right\} \geq \phi(x_j) \qquad (0 \leq x_j \leq l).$$

Hence by the maximum principle we get

$$\eta^\tau(x_j, t_n) \geq (u^\tau)_j^n \qquad (0 \leq t_n \leq \tau),$$

and especially for $n = n_0$, $j = J_0 - 1$

$$(0 \leq) - (u^\tau)_{J_0 \bar{x}}^{n_0} \leq \frac{K_2}{\tau}\left(1 - \frac{M_0 c_1}{2K_2}\right) \leq \frac{K_2}{2}$$

as far as $h < K_2/M_0 c_1$ holds. By repeating above discussion for each interval $[0, \sigma]$ $\left(\frac{\tau}{2} \leq \sigma \leq \tau\right)$ instead of the interval $[0, \tau]$ considered above, we obtain

$$0 \leq -(u^\tau)_{J_0 \bar{x}}^n \leq \frac{K_2}{\tau} \qquad \text{for any} \quad t_n, \ \frac{\tau}{2} \leq t_n \leq \tau.$$

By making the similar argument for $(u^\tau)_{J_0 x}^n$, we get (5.12) in conclusion.

From (5.12) and the uniform boundedness of $\{(u^\tau)_j^n\}$, we can derive

$$|(u^\tau)_{j\bar{x}}^{n_0}| < \frac{K_3}{\tau} \qquad \left(\frac{d}{2} \leq x_j \leq 1 - \frac{d}{2}\right) \tag{5.13}$$

with another constant K_3 not depending on h and τ, by the modified method from Petrowsky's one for deriving uniform boundedness of the high order difference quotients.

If the solution of the difference scheme is considered only in a restricted region

$$\Omega_{d,\tau} = \left\{\frac{d}{2} \leq x_j \leq 1 - \frac{d}{2}, \quad \tau \leq t_n \leq T\right\},$$

then the function $\{(u^\tau)_j^n\}$ are subject to the conditions (5.11) and (5.13) on the boundaries $x = d/2$, $1 - d/2$ and the initial line $t = \tau$, and $(u^\tau)_{J_0}^{n_0} = 0$ as we have shown above. Therefore the conditions corresponding to (3.2)–(3.4) are satisfied by $(u^\tau)_j^n$ in $\Omega_{d,\tau}$. We remember that strict positiveness or negativeness contained in (3.1) was used for making the free boundary strictly away from the fixed boundary. In $\Omega_{d,\tau}$ the corresponding condition is not necessary because y_n is always contained in $(d, 1-d)$ and it is clearly away from the boundary of $\Omega_{d,\tau}$. Then we find that $\{y_n^\tau\}$ converges to a Lipshitz continuous function $y^\tau(t)$ and $\{y^\tau(t), u^\tau(x, t)\}$ satisfies Stefan's condition for $t > \tau$ (by the same way as in Section 4). Thus the proof of Lemma 5.4 is completed.

Remark. Directly from (5.11) and (5.13), the inequalities

$$\left|\left(u^{\tau}\left(\frac{d}{2},\, t\right)\right)_{\bar{t}}\right|,\quad \left|\left(u^{\tau}\left(1-\frac{d}{2},\, t\right)\right)_{\bar{t}}\right| < L_d \qquad (\tau \le t \le T), \tag{5.14}$$

$$|(u^{\tau}(x,\, \tau))_{\bar{x}}| < \frac{K_3}{\tau} \qquad \left(\frac{d}{2} \le x \le 1-\frac{d}{2}\right) \tag{5.15}$$

and $u^{\tau}(l, \tau)=0$ hold. In reality, the corresponding inequalities hold for differentiation instead of differenciation. Such facts admit us to apply Lemma 5.2 to the problem (5.8) in the restricted region $\Omega_{d,\tau}$. In fact we shall use it for proving the next lemma.

Lemma 5.5. *For an arbitrarily fixed and sufficiently small λ, the function $y^{\tau}(t)$ is strictly monotone increasing or decreasing in $[\tau, \lambda]$.*

Proof. Applying (5.4) and (5.5) to the present problem, we have

$$\begin{aligned} v_{-}^{\tau}(\tau) = & -2\int_0^l G_{1\xi}(l,\, \tau;\, \xi,\, 0)\phi^{\tau}(\xi)\, d\xi \\ & +2\int_0^{\tau} G_{1\sigma}(l,\, \tau;\, 0,\, \sigma)\psi_1(\sigma)d\sigma - 2\int_0^{\tau} G_{1\xi}(l,\, \tau;\, l,\, \sigma)v_{-}^{\tau}(\sigma)d\sigma, \\ v_{+}^{\tau}(\tau) = & -2\int_l^1 G_{2\xi}(l,\, \tau;\, \xi,\, 0)\phi^{\tau}(\xi)\, d\xi \\ & -2\int_0^{\tau} G_{2\sigma}(l,\, \tau;\, 1,\, \sigma)\psi_2(\sigma)d\sigma + 2\int_0^{\tau} G_{2\xi}(l,\, \tau;\, l,\, \sigma)v_{+}^{\tau}(\sigma)d\sigma. \end{aligned}$$

Just as in the proof of Lemma 5.1, we get the following asymptotical expression

$$v_{-}^{\tau}(\tau) = -\frac{2}{\sqrt{\pi \tau^{\alpha}}}\bar{\phi}^{\tau}\int_0^{\frac{\sqrt{c_1}}{2\tau^{1/2-\alpha}}} e^{-\theta^2}d\theta + R_{-},$$

$$v_{+}^{\tau}(\tau) = \frac{2}{\sqrt{\pi \tau^{\alpha}}}\underline{\phi}^{\tau}\int_0^{\frac{\sqrt{c_2}}{2\tau^{1/2-\alpha}}} e^{-\theta^2}d\theta + R_{+},$$

where $R_{\pm}$ tend to zero exponentially and

$$\int_0^{\frac{\sqrt{c_i}}{2\tau^{1/2-\alpha}}} e^{-\theta^2}d\theta \to \frac{\sqrt{\pi}}{2}$$

as $\tau \to 0$. Thus we find that, for sufficiently small λ, the sign of $v_{+}^{\tau}(\tau) - v_{-}^{\tau}(\tau)$ $(\tau < \lambda)$ is the same as that of $\underline{\phi}^{\tau} + \bar{\phi}^{\tau}$ or $\phi(l+0)+\phi(l-0)$ (or, if it is zero, the sign of $\beta_1 - \beta_2$ which is positive as we defined).

Now, as we mentioned in the above remark, we can apply Lemma 5.2 in the restricted region $\Omega_{d,\tau}$. Then we find that for the small interval $[\tau, \lambda]$ $\dot{y}^{\tau}(t)$ is positive or negative according to that $\phi(l+0)+\phi(l-0)\geq 0$ or <0. Thus we have proved Lemma 5.5.

Proof of Theorem 5.1. Suppose that τ tends to zero through a sequence $\{\tau_i\}$ $(i=1, 2, 3,\dots)$.

We find from Lemma 5.5 that

$$l=y^{\tau_i}(\tau_i)<y^{\tau_{i+1}}(\tau_i) \qquad (i=1, 2, 3,\dots)$$

or

$$l=y^{\tau_i}(\tau_i)>y^{\tau_{i+1}}(\tau_i) \qquad (i=1, 2, 3,\dots)$$

according to $\phi(l+0)+\phi(l-0)\geq 0$ or <0. We consider only the former case, for the latter case can be treated similarly. From the assumption of data, we have

$$u^{\tau_i}(x, \tau_i)\leq u^{\tau_{i+1}}(x, \tau_i) \qquad (0\leq x\leq 1).$$

Then, by applying Lemma 5.3 to the solutions (y^{τ_i}, u^{τ_i}) and $(y^{\tau_{i+1}}, u^{\tau_{i+1}})$ for $t\geq\tau_i$, we get

$$y^{\tau_i}(t)<y^{\tau_{i+1}}(t) \qquad \text{for} \quad t\geq\tau_i,$$

and hence

$$y^{\tau_i}(t)\leq y^{\tau_{i+1}}(t) \qquad \text{for all} \quad t\geq 0.$$

Therefore the sequence of the functions $y^{\tau_i}(t)$ is monotone increasing and bounded above:

$$y^{\tau_1}(t)\leq y^{\tau_2}(t)\leq\cdots\leq y^{\tau_i}(t)\leq\cdots\leq 1-d \qquad \text{for all} \quad t\geq 0,$$

hence we find existence of the limit

$$\lim_{i\to\infty} y^{\tau_i}(t)=y(t).$$

It can be easily seen from (5.14) and (5.15) that

$$|y(t^1)-y(t^2)|\leq\frac{K_4}{b\tau}|t^1-t^2| \qquad \text{for any} \quad t^1, t^2\in[\tau, T],$$

that is, $y(t)$ is Lipshitz continuous in any interval $[\tau, T]$ away from the origin $t=0$.

Next we will show the continuity of $y(t)$ at $t=0$. For comparison we consider the following one-phase problem:

$$(5.16)\qquad \begin{cases} c_1\dfrac{\partial z}{\partial t}=\dfrac{\partial^2 z}{\partial x^2} & (0<x<\zeta(t),\ t>0),\\ z(0,t)=M_0 & (t>0),\\ z(\zeta(t),t)=0 & (t>0),\\ z(x,0)=M_0 & (0<x<l),\\ b\dot{\zeta}(t)=-\dfrac{\partial z}{\partial x}(\zeta(t)-0,t) & (t>0),\\ z(x,t)=0 & (x>\zeta(t),\ t>0), \end{cases}$$

where M_0 is an upper bound of $\psi_1(t)$ $(0<t\leq T)$ and $\phi(x)$ $(0\leq x\leq l)$. The problem (5.16), as well known, has one and only one solution $(\zeta(t), z(x, t))$ and

$$\zeta(t)=l+A\sqrt{t} \qquad (A \text{ is a constant}).$$

(See, for example, [2].) Clearly we get

$$y^{\tau_1}(t)\leq y^{\tau_i}(t)\leq\zeta(t) \qquad (0\leq t\leq T)$$

and hence

$$y^{\tau_1}(t)\leq y(t)\leq\zeta(t) \qquad (0<t\leq T).$$

Since $y^{\tau_1}(t)$ and $\zeta(t)$ are continuous at $t=0$, we find that $y(t)$ is also continuous there.

Thus we have shown that $\{y^{\tau_i}(t)\}$ converges monotoneously to the continuous function $y(t)$ on the closed interval $[0, T]$. Therefore, by Dini's theorem, we find also that $\{y^{\tau_i}(t)\}$ converges uniformly to $y(t)$ on $[0, T]$.

Now, by using $y(t)$ defined above, we define $u(x, t)$ as the solution of the following problems with the prescribed boundaries:

$$\begin{cases} c_1\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} & (0<x<y(t),\ 0<t\leq T),\\ u(0,t)=\psi_1(t) & (0<t\leq T),\\ u(y(t),t)=0 & (0<t\leq T),\\ u(x,0)=\phi(x) & (0\leq x\leq l), \end{cases}$$

$$\begin{cases} c_2\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2} & (y(t)<x<1,\ 0<t\leq T),\\ u(y(t),t)=0 & (0<t\leq T),\\ u(1,t)=\psi_2(t) & (0<t\leq T),\\ u(x,0)=\phi(x) & (l\leq x\leq 1). \end{cases}$$

Then we can see that $\{u^{\tau_i}(x, t)\}$ converges uniformly to $u(x, t)$ in any time interval $[\lambda, T]$ $(\lambda>0)$. In fact, it is clear that $u^{\tau_i}(x, \lambda)\leq u^{\tau_{i+1}}(x, \lambda)(\lambda\geq\tau_i>$

τ_{i+1}) and hence by the Dini's theorem $\{u^{\tau_i}(x, \lambda)\}$ converges uniformly. Thus we get

$$(5.17) \qquad |u(x, \lambda)-u^{\tau_i}(x, \lambda)|<\varepsilon \qquad (0\leq x\leq 1)$$

for sufficiently small τ_i. And from uniform convergence of $\{y^{\tau_i}(t)\}$ above mentioned and (5.15) we have for sufficiently small τ_i

$$(5.18) \qquad -\frac{\varepsilon}{2}<u^{\tau_i}(x, t)\leq 0 \qquad (y^{\tau_i}(t)\leq x\leq y(t),\ \lambda\leq t\leq T)$$

and also from uniform continuity of $u(x, t)$ in any vicinity of the curve $x=y(t)$ for $\lambda\leq t\leq T$,

$$(5.19) \qquad 0<u(x, t)<\frac{\varepsilon}{2} \qquad (y^{\tau_i}(t)\leq x\leq y(t),\ \lambda\leq t\leq T).$$

From (5.17) and (5.18), we have by the maximum principle

$$(5.20) \qquad |u(x, t)-u^{\tau_i}(x, t)|<\varepsilon$$

for $y(t)\leq x\leq 1$, $\lambda\leq t\leq T$, and from (5.17) and (5.19) we get again (5.20) for $0\leq x\leq y^{\tau_i}(t)$, $\lambda\leq t\leq T$, and further from (5.18) and (5.19) also for $y^{\tau_i}(t)\leq x \leq y(t)$, $\lambda\leq t\leq T$. Thus we have proved that $\{u^{\tau_i}(x, t)\}$ converges to $u(x, t)$ uniformly in any time interval $[\lambda, T]$.

Up to now, we showed that the sequence of the solutions (y^{τ_i}, u^{τ_i}) converges to (y, u) which satisfies all the conditions of (1.1) except Stefan's condition. Finally we will show that (y, u) itself satisfies it. Here we mention that, since $y(t)$ is Lipshitz continuous for $0<t\leq T$, $\frac{\partial u}{\partial x}(y(t)\pm 0, t)$ exist and are continuous in $0<t\leq T$. We use the following lemma (which was used also in [11]):

Lemma 5.6. *If $y(t)$ is Lipshitz continuous and $\frac{\partial u}{\partial x}(y(t)\pm 0, t)$ are continuous in $0<t\leq T$, then Stefan's condition is equivalent to that the following relation holds for any λ and t $(0<\lambda<t\leq T)$ and any positive number δ $(0<\delta<d)$:*

$$\begin{aligned}(5.21) \quad y(t)=y(\lambda)&-\frac{c_1}{b}\left[\int_0^{y(t)} u(x, t)dx-\int_0^{y(\lambda)} u(x, \lambda)dx\right]\\ &-\frac{c_2}{b}\left[\int_{y(t)}^1 u(x, t)dx-\int_{y(\lambda)}^1 u(x, \lambda)dx\right]\\ &+\frac{c_2}{b\delta}\int_0^{\delta}(\delta-x)[u(x, t)-u(x, \lambda)]dx\\ &+\frac{c_2}{b\delta}\int_{1-\delta}^1(\delta+x-1)[u(x, t)-u(x, \lambda)]dx\end{aligned}$$

$$+\frac{1}{b\delta}\int_{\lambda}^{t}[\psi_1(\sigma)-u(\delta,\sigma)]d\sigma+\frac{1}{b\delta}\int_{\lambda}^{t}[\psi_2(\sigma)-u(1-\delta,\sigma)]d\sigma.$$

Leaving its proof until later, we go to apply it for our last purpose. It follows from Lemma 5.6 that the solutions of the problems (5.8) (y^{τ_i}, u^{τ_i}) $(\tau^i<\lambda)$ satisfy the relations

$$\begin{aligned}
y^{\tau_i}(t)=y^{\tau_i}(\lambda)&-\frac{c_1}{b}\left[\int_0^{y^{\tau_i}(t)}u^{\tau_i}(x,t)dx-\int_0^{y^{\tau_i}(\lambda)}u^{\tau_i}(x,\lambda)dx\right]\\
&-\frac{c_2}{b}\left[\int_{y^{\tau_i}(t)}^{1}u^{\tau_i}(x,t)dx-\int_{y^{\tau_i}(\lambda)}^{1}u^{\tau_i}(x,\lambda)dx\right]\\
&+\frac{c_1}{b\delta}\int_0^{\delta}(\delta-x)[u^{\tau_i}(x,t)-u^{\tau_i}(x,\lambda)]dx\\
&+\frac{c_2}{b\delta}\int_{1-\delta}^{1}(\delta+x-1)[u^{\tau_i}(x,t)-u^{\tau_i}(x,\lambda)]dx\\
&+\frac{1}{b\delta}\int_{\lambda}^{t}[\psi_1(\sigma)-u^{\tau_i}(\delta,\sigma)]d\sigma+\frac{1}{b\delta}\int_{\lambda}^{t}[\psi_2(\sigma)-u^{\tau_i}(\delta,\sigma)]d\sigma.
\end{aligned}$$

Here we take $\tau_i\to 0$, then we get the formula (5.21) directly by uniform convergence of $\{y^{\tau_i}(t)\}$ in $0\le t\le T$ and that of $\{u^{\tau_i}(x,t)\}$ in $\delta\le x\le 1-\delta$, $\lambda\le t\le T$. Therefore we find again by Lemma 5.6 that (y, u) satisfies Stefan's condition, and hence that it is surely the solution of the original problem (1.1). Thus we have proved Theorem 5.1.

Proof of Lemma 5.6. We devide the region $\{0<x<1, \lambda<\sigma<t\}$ into four regions

$$D_1=\{0<x<\delta,\ \lambda<\sigma<t\},\quad D_2=\{\delta<x<y(t),\ \lambda<\sigma<t\},$$
$$D_3=\{y(t)<x<1-\delta,\ \lambda<\sigma<t\}\quad\text{and}\quad D_4=\{1-\delta<x<1,\ \lambda<\sigma<t\}.$$

In each region we consider Green's formula

$$\iint_D\left[v\left(\frac{\partial^2 u}{\partial x^2}-c_i\frac{\partial u}{\partial t}\right)-u\left(\frac{\partial^2 v}{\partial x^2}+c_i\frac{\partial v}{\partial t}\right)\right]dxdt$$
$$=\int_D\left(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}\right)dt+c_iuvdx.$$

We take the solution of (1.1) for $y(t)$ appearing in the above definition of D_2, D_3 and $u(x, t)$ in the formula, and take v and c_i like as

$$v=x,\ c_i=c_1\ \text{in}\ D_1,\quad v=1,\ c_i=c_1\ \text{in}\ D_2,$$
$$v=1,\ c_i=c_2\ \text{in}\ D_3\quad\text{and}\quad v=1-x,\ c_i=c_2\ \text{in}\ D_4.$$

Then we get the following formula:

$$\int_\lambda^t u_x(\delta,\sigma)d\sigma=\frac{1}{\delta}\int_\lambda^t u(\delta,\sigma)d\sigma-\frac{1}{\delta}\int_\lambda^t \psi_1(\sigma)d\sigma$$
$$+\frac{c_1}{\delta}\int_0^\delta xu(x,t)dx-\frac{c_1}{\delta}\int_0^\delta xu(x,t)dx\,,$$
$$\int_\lambda^t u_x(y(\sigma)-0,\sigma)d\sigma=\int_\lambda^t u_x(\delta,\sigma)d\sigma$$
$$+c_1\int_\delta^{y(t)}u(x,t)dx-c_1\int_\delta^{y(\lambda)}u(x,\lambda)dx\,,$$
$$\int_\lambda^t u_x(y(\sigma)+0,\sigma)d\sigma=\int_\lambda^t u_x(1-\delta,\sigma)d\sigma-c_2\int_{y(t)}^{1-\delta}u(x,t)dx$$
$$+c_2\int_{y(\lambda)}^{1-\delta}u(x,\lambda)dx\,,$$
$$\int_\lambda^t u_x(1-\delta,\sigma)d\sigma=-\frac{1}{\delta}\int_\lambda^t u(1-\delta,\sigma)d\sigma+\frac{1}{\delta}\int_\lambda^t \psi_2(\sigma)d\sigma$$
$$-\frac{c_2}{\delta}\int_{1-\delta}^1(1-x)u(x,t)dx+\frac{c_2}{\delta}\int_{1-\delta}^1(1-x)u(x,\lambda)dx\,.$$

Eliminating $\int_\lambda^t u_x(\delta,\sigma)d\sigma$ and $\int_\lambda^t u_x(1-\delta,\sigma)d\sigma$ from these formula, seaking an expressions of $\int_\lambda^t u_x(y(\sigma)\pm 0,\sigma)d\sigma$ and using the formula

$$y(t)-y(\lambda)=\frac{1}{b}\int_\lambda^t [u_x(y(\sigma)+0,\sigma)-u_x(y(\sigma)-0,\sigma)]d\sigma$$

derived by integrating Stefan's condition, we get (5.21) immediately.

Conversely we assume that (5.21) hold. Differentiating it by t and using $u(y(t),t)=0$, we obtain

$$\dot y(t)=-\frac{c_1}{b}\int_0^\delta u_t(x,t)dx-\frac{c_2}{b}\int_{y(t)}^1 u_t(x,t)dx$$
$$+\frac{c_1}{b\delta}\int_0^\delta(\delta-x)u_t(x,t)dx+\frac{c_2}{b\delta}\int_{1-\delta}^1(\delta+x-1)u_t(x,t)dx$$
$$+\frac{1}{b\delta}[\psi_1(t)-u(\delta,t)]+\frac{1}{b\delta}[\psi_2(t)-u(1-\delta,t)].$$

Here we use the equations $c_iu_t=u_{xx}$, then we get

$$\dot y(t)=-\frac{1}{b}[u_x(y(t)-0,t)-u_x(+0,t)]-\frac{1}{b}[u_x(1-0,t)-u_x(y(t)+0,t)]$$
$$+\frac{1}{b\delta}\left[(\delta-x)u_x\Big|_0^\delta+\int_0^\delta u_x(x,t)dx\right]+\frac{1}{b\delta}\left[(\delta+x-1)u_x\Big|_{1-\delta}^1\right.$$
$$\left.-\int_{1-\delta}^1 u_x(x,t)dx\right]+\frac{1}{b\delta}[\psi_1(t)-u(\delta,t)]+\frac{1}{b\delta}[\psi_2(t)-u(1-\delta,t)]$$
$$=\frac{1}{b}[u_x(y(t)+0,t)-u_x(y(t)-0,t)].$$

Thus we have obtained Stefan's condition from the formula (5.21). Q. E. D.

§6. Numerical Examples

Here we show some numerical examples. The data are the followings:

$$c_1=1, \quad c_2=1/2,$$
$$\psi_1(t)=t, \quad \psi_2(t)=(t-1)/2 \qquad (0\leq t\leq 1),$$
$$\phi(x)=\begin{cases} 0 & (0\leq x\leq 1/2), \\ 1/2-x & (1/2\leq x\leq 1). \end{cases}$$

The values of the parameters h and β are taken like as

$$h=0.005, \quad \beta=0.5, \quad 0.1, \quad 0.01, \quad 0.001.$$

Figure 1 shows the result for the case of $\beta=0.01$; the change of the free boundary and the profile of u at the time $t=0.6$. As expected from the given data, the ice region first grew and then the water region recovered and grew. Even for the case of $\beta=0.001$, the position of the free boundary was scarcely exposed to change in the figure. When β was taken larger, it got slight change,

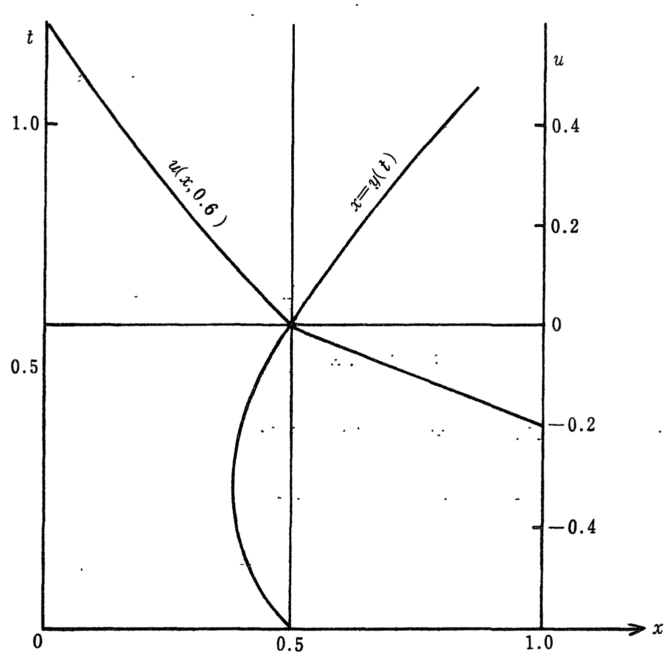

Figure 1. The change of the free boundary $x=y(t)$ and the profile of $u(x, 0.6)$ at the time $t=0.6$

the convexity of the curve to the left became smaller and the accuracy of the solution did worse.

The computation time needed was 1/38 of that needed in the computation using Kamenomostskaya's explicit scheme [7].

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