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# On a Limit Theorem for Branching One-Dimensional Diffusion Processes

*Dedicated to Professor K. Itô on his 60th birthday*

By

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## §1. Introduction

Let $M$ be a simply connected complete Riemannian manifold with constant sectional curvature, and consider a branching Brownian motion $\boldsymbol{Y}=(y_t, \boldsymbol{P}_y)$ having $M$ as the underlying state space ([5]). In the case of $M=S^d$, the $d$-dimensional sphere, one can apply the results of Watanabe [16], [17] and Asmussen-Hering [1] to obtain a limit theorem on the number of particles in a domain for the process $\boldsymbol{Y}$. If $M=\boldsymbol{R}^d$, then, although $M$ is not compact, the process $\boldsymbol{Y}$ belongs to the class considered by Watanabe [17], and his argument works well. But, if $M$ has constant negative sectional curvature $-k$, then $\boldsymbol{Y}$ is not necessarily contained in the scheme of [17] and a new phenomenon appears. In this case, the Laplace-Beltrami operator is given by

$$\Delta=\partial^2/\partial r^2+\sqrt{k}\,(d-1)\coth\sqrt{k}\,r\partial/\partial r+\Delta'$$

in polar coordinates, where $\Delta'$ is the Laplace-Beltrami operator on the sphere $S_r=\{y\in M\colon \text{distance}(y, 0)=r\}$ ([4] p. 445). Hence the radial part process $\boldsymbol{X}$ of the branching Brownian motion $\boldsymbol{Y}$ on $M$ is reduced to a branching diffusion process on the underlying state space $S=[0, \infty)$, whose nonbranching part diffusion has the generator

$$L=d^2/2dr^2+\sqrt{k}\,(d-1)/2\coth\sqrt{k}\,rd/dr. \tag{1.1}$$

The spectrum of $L$ is the interval $(-\infty, -k(d-1)^2/8]$ and there exist many

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bounded nonnegative martingales for $\boldsymbol{X}$ (Section 5). This enables us to show that bounded domains may come to contain no particle in some cases, even when the total number of particles diverges to infinity as time elapses.

Having the above situation in the background, we shall study in this paper the branching diffusion process on the underlying state space $S=[0, \infty)$, whose nonbranching part diffusion has the generator

$$L=d^2/2dx^2+b(x)d/dx. \tag{1.2}$$

Precise conditions on $b(x)$ as well as boundary conditions will be given in Section 2. The radial part process $\boldsymbol{X}$ mentioned above is contained in our class. Our results are fully described in Section 2. We shall prepare some lemmas on ordinary differential equations in Section 3 and some comparison theorems for stochastic differential equations in Section 4. Sections 5 and 6 are devoted to the proof of Theorems.

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## §2. Notations and Results

Throughout this paper, we assume that $b(x)$ is a function satisfying the following conditions:

(2.1) $b(x)$ *is defined on* $S=[0, \infty)$ *and* $b(0)=0$,

(2.2) $b(x)$ *is continuous and nonincreasing on* $(0, \infty)$,

(2.3) $b(x)\geqq b_0>0$ *on* $(0, \infty)$,

(2.4) $\int_R^\infty x(b(x)-b_0)dx<\infty$ *for every* $R>0$.

Let

$$s(x)=\int_1^x e^{-B(y)}dy, \quad x\in S, \quad m(E)=2\int_E e^{B(x)}dx, \quad E\in\mathscr{B}(S),$$

where $B(x)=2\int_1^x b(y)dy$ and $\mathscr{B}(S)$ is the topological Borel field of $S$. Then there exists a unique conservative diffusion process $X=(x_t, P_x)$ on $[0, \infty)$ with scale $s(x)$ and speed measure $m(dx)$ satisfying

$$P_0(\sigma_{0+}=0)=1^{1)} \tag{2.5}$$

1) $\sigma_x$ is the first hitting time of $x_t$ for the point $x$, and $\sigma_{0+}=\lim_{x\downarrow 0}\sigma_x$.

([8]). The boundary $\infty$ is natural in Feller's sense (ibid. p. 108). Further, since $X$ is conservative and $m(\{0\})=0$, (2.3) and (2.5) ensure that the boundary 0 is reflecting if

$$\text{(R)} \qquad e^{-B(x)}\int_x^1 2e^{B(y)}dy \in L^1(0, 1)\,.$$

The boundary 0 is entrance non-exit if

$$\text{(E)} \qquad e^{-B(x)}\int_x^1 2e^{B(y)}dy \notin L^1(0, 1)$$

(ibid.). The generator of $X$ coincides with $L$ of (1.2) with the domain

$$\text{(2.6)} \qquad D(L)=\{u\in \boldsymbol{C}(S): Lu\in \boldsymbol{C}(S), \lim_{x\downarrow 0} e^{B(x)}u'(x)=0\}$$

(ibid.). Note that the last relation in the braces of (2.6) is automatically satisfied in the case of (E). We denote the transition density for the diffusion $X$ with respect to $m(dx)$ by $p(t, x, y)$, and the semigroup by $T_t$.

Following [5], let $S^n$ be the $n$-fold symmetric product of $S$ ($S^0=\partial$, $\partial$ is an extra point), and set $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n$. Thus an element $\boldsymbol{x}\in \boldsymbol{S}$ belongs to some $S^n$, and, if $n\geqq 1$, we have a coordinate expression $\boldsymbol{x}=[x^1, x^2,\dots, x^n]$. Define a stochastic kernel $\pi(x, \boldsymbol{E})$ ($x\in S$, $\boldsymbol{E}\in \mathscr{B}(\boldsymbol{S})$) by $\pi(x, \boldsymbol{E})=1$ if $[x, x]\in \boldsymbol{E}$, and $=0$ otherwise. Then, for a positive constant $c$, there exists a unique branching diffusion process (BDP) $\boldsymbol{X}=(\boldsymbol{x}_t, \boldsymbol{P}_x)$ ($x\in S$) on $\boldsymbol{S}$ corresponding to the fundamental system $(X, c, \pi)$. $X$ is called the nonbranching part, $c$ the branching rate and $\pi$ the branching law.

Let $\boldsymbol{B}$ be the set of all bounded Borel measurable functions on $S$. For each $g\in \boldsymbol{B}$, set

$$\hat{g}(\boldsymbol{x})=\begin{cases} g(x^1)g(x^2)\cdots g(x^n)\,, & \boldsymbol{x}=[x^1, x^2,\dots, x^n]\in \boldsymbol{S}\setminus\{\partial\}\,, \\ 1\,, & \boldsymbol{x}=\partial, \end{cases}$$

$$\check{g}(\boldsymbol{x})=\begin{cases} g(x^1)+g(x^2)+\cdots+g(x^n), & \boldsymbol{x}=[x^1, x^2,\dots, x^n]\in \boldsymbol{S}\setminus\{\partial\}\,, \\ 0\,, & \boldsymbol{x}=\partial\,. \end{cases}$$

Then, for each continuous $g\in \boldsymbol{B}$ with $\|g\|\leqq 1$, the function $u(t, x)=\boldsymbol{E}_x[\hat{g}(\boldsymbol{x}_t)]$ satisfies the so-called $S$-equation

$$\text{(2.7)} \qquad \frac{\partial u}{\partial t}=Lu+c(u^2-u), \quad u(0+, x)=g(x)^{2)}$$

(cf. [5]). Let, for $\alpha$, $l\in \boldsymbol{R}$,

2) More precisely, $u(t, \cdot)\in D(L)$ and (2. 7) is satisfied. In writing similar equations, the condition that $u$ belongs to $D(L)$ is meant implicitly.

$$\boldsymbol{B}_i(\alpha, l)=\{g\in\boldsymbol{B}\colon \lim_{x\to\infty} e^{\alpha x}g(x)/x^i=l\},$$
$$\boldsymbol{B}_i(\alpha)=\bigcup_{l\in\boldsymbol{R}}\boldsymbol{B}_i(\alpha, l), \qquad i=0, 1.$$

Thus, each $g\in\boldsymbol{B}_i(\alpha)$ belongs to $\boldsymbol{B}_i(\alpha, l)$ for some $l\in\boldsymbol{R}$, which we denote by $l(g)$.

Finally, set

$$\lambda_0=b_0^2/2, \quad \alpha_\lambda=b_0-\sqrt{b_0^2-2\lambda}, \quad \alpha_{\lambda_0}=b_0.$$

Now we are ready to state our results.

**Theorem 1.** 1) *For each* $0\leqq\lambda<\lambda_0$, *there is a nonnegative integrable random variable* $W_\lambda$ *such that for each* $g\in\boldsymbol{B}_0(\alpha_\lambda, l)$

(2.8) $$\lim_{t\to\infty} e^{(\lambda-c)t}\check{g}(\boldsymbol{x}_t)=lW_\lambda, \qquad a.s.$$

2) *There is a nonnegative integrable random variable* $W=W_{\lambda_0}$ *such that for each* $g\in\boldsymbol{B}_1(\alpha_{\lambda_0}, l)$

(2.9) $$\lim_{t\to\infty} e^{(\lambda_0-c)t}\check{g}(\boldsymbol{x}_t)=lW, \qquad \text{in probability}.$$

**Theorem 2.** *If* $0\leqq\lambda\leqq\lambda_0$ *and* $\alpha_\lambda^2\leqq 2c$, *then*

(2.10) $$\boldsymbol{P}_x(0<W_\lambda<\infty)=1, \qquad x>0.$$

**Remark 1.** Consider the case $0<c<\lambda_0$ $[c=\lambda_0]$. Theorem 1 says that

$$\lim_{x\to\infty} e^{\alpha_c x}g(x)=l \quad [\text{resp. } \lim_{x\to\infty} e^{b_0 x}g(x)/x=l]$$

implies

$$\lim_{t\to\infty}\check{g}(\boldsymbol{x}_t)=lW_\lambda, \quad \text{a.s.} \quad [=lW, \text{ in probability}].$$

Further, Theorems 1 and 2 say that

$$\lim_{x\to\infty} e^{\alpha_c x}g(x)=\infty \qquad [\text{resp. } \lim_{x\to\infty} e^{b_0 x}g(x)/x=\infty]$$

implies

$$\lim_{t\to\infty}\check{g}(\boldsymbol{x}_t)=\infty, \qquad \text{a.s. [resp. in probability]}.$$

Indeed, for $0\leqq\lambda<\lambda_0$ $[\lambda=\lambda_0]$, $\alpha_\lambda$ is the smaller [resp. unique] solution of

(2.11) $$t^2-2b_0t+2\lambda=0.$$

Hence $\alpha_\lambda^2\leqq 2\lambda$, and (2.10) holds for $\lambda=c$.

## §3. Lemmas on Ordinary Differential Equations

**Lemma 1.** 1) *For each $\lambda$, there is a unique solution $\varphi_\lambda(x)$ of the equation*

$$L\varphi_\lambda(x) = -\lambda\varphi_\lambda(x), \qquad \varphi_\lambda(0)=1. \tag{3.1}$$

2) *If $0<\lambda\leqq\lambda_0$, then $\varphi_\lambda(x)$ is positive and decreasing. $\varphi_0(x)\equiv 1$. Further,*

$$\varphi_\lambda \in \boldsymbol{B}_0(\alpha_\lambda), \qquad 0\leqq\lambda<\lambda_0, \tag{3.2}$$

$$\varphi_{\lambda_0} \in \boldsymbol{B}_1(b_0), \qquad \lambda=\lambda_0 \tag{3.3}$$

*with positive $l(\varphi_\lambda)$.*

3) *If $\lambda>\lambda_0$, then $\varphi_\lambda(x)<0$ for some $x\in(0,\infty)$.*

*Proof.* 1) First we shall show the uniqueness. Let $\varphi_\lambda(x)$ be a solution of (3.1), and put $\xi(x)=\varphi_\lambda(x)$ and $\eta(x)=\varphi'_\lambda(x)$[3]. Then (3.1) is equivalent to

$$\begin{cases} \xi'(x)=\eta(x), \\ \eta'(x)=-2\lambda\xi(x)-2b(x)\eta(x), \qquad \xi(0)=1, \end{cases} \tag{3.4}$$

and

$$\lim_{x\downarrow 0} e^{B(x)}\eta(x)=0. \tag{3.5}$$

Solving the second equation of (3.4), we have

$$\eta(x)=-2\lambda\int_{x_1}^{x}\xi(y)e^{-2\int_y^x b(z)dz}dy+e^{-B(x)}e^{B(x_1)}\eta(x_1),$$

for $x$, $x_1\in(0,\infty)$. Hence, by letting $x_1\downarrow 0$, we get

$$\eta(x)=-2\lambda\int_0^x \xi(y)e^{-2\int_y^x b(z)dz}dy. \tag{3.6}$$

This and (3.4) give

$$\xi(x)=1-2\lambda\int_0^x\int_0^y \xi(z)e^{-2\int_z^y b(u)du}dzdy. \tag{3.7}$$

Hence the solution $\xi(x)$ of (3.7) is unique. This means the uniqueness of the solution $\varphi_\lambda(x)$ of (3.1).

The existence of the solution $\varphi_\lambda(x)$ of (3.1) is obvious, because it is easy to see that (3.7) has a solution $\xi(x)$.

2) Clearly, $\varphi_0(x)\equiv 1$. We divide the proof for the case $0<\lambda\leqq\lambda_0$ into two steps.

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3) $\varphi'_\lambda(x)=d\varphi_\lambda(x)/dx$.

*Step* 1. We shall show that $\varphi_\lambda(x)$ is positive and decreasing. It is enough to see that $\xi(x)=\varphi_\lambda(x)>0$ and $\eta(x)=\varphi'_\lambda(x)<0$.

By (3.6) and (3.7),

$$\xi(x)>0\,,\quad \eta(x)<0\,,\qquad 0<x<\delta\,, \tag{3.8}$$

for some $\delta>0$. Set $T=\inf\{x>0\colon \xi(x)=0\}$[4] and $\zeta(x)=-\eta(x)/\xi(x)$, $x\in(0, T)$. Then $\zeta(0+)=0$ and $\zeta(x)>0$ for $0<x<\delta$. Further, by (3.4),

$$\zeta'(x)=\zeta(x)^2-2b(x)\zeta(x)+2\lambda\,,\qquad x\in(0\,,\,T)\,. \tag{3.9}$$

This implies

$$\zeta(x)>0\,,\qquad x\in(0\,,\,T)\,. \tag{3.10}$$

Indeed, if otherwise, contradiction occurs at the first zero point $x_0\in(0, T)$ of $\zeta(x)$, because $\zeta'(x_0)\leqq 0$ whereas $\zeta'(x_0)=2\lambda>0$.

Now (3.9) and (3.10) yield

$$\zeta'(x)\leqq\zeta(x)^2-2b_0\zeta(x)+2\lambda\,,\qquad x\in(0\,,\,T)\,. \tag{3.11}$$

Define $\alpha_+$ and $\alpha_-$ by

$$\alpha_\pm\equiv\alpha_\pm(\lambda)=b_0\pm\sqrt{b_0^2-2\lambda}\,.$$

Note that $\alpha_-=\alpha_\lambda$ and that (3.11) implies

$$\zeta'(x)\leqq(\zeta(x)-\alpha_+)(\zeta(x)-\alpha_-)\,,\qquad x\in(0,\,T)\,. \tag{3.12}$$

If $0<\lambda<\lambda_0$, then this yields

$$0<\zeta(x)<\alpha_-\,,\qquad x\in(0,\,T)\,. \tag{3.13}$$

If $\lambda=\lambda_0$, then (3.12) is written as $\zeta'(x)\leqq(\zeta(x)-b_0)^2$. On the other hand, there is a $\delta_0\in(0, T)$ such that $0<\zeta(x)<b_0$ for $x\in(0, \delta_0]$. Take any $x_1\in(\delta_0, T)$ and fix it. If $\zeta(x)<b_0$ for $\delta_0<x<x_1$, then

$$\frac{1}{b_0-\zeta(x)}\leqq x-\delta_0+\frac{1}{b_0-\zeta(\delta_0)}\,.$$

This excludes the possibility that $\zeta(x)\uparrow b_0$ as $x\uparrow x_1$. Hence (3.13) holds in this case also.

Now we shall show $T=\infty$. Suppose $T<\infty$. Then $\eta(T)=0$ by (3.13). Since the solution of (3.4) for $x\geqq T$ with the initial condition $\xi(T)=\eta(T)=0$ is unique, we have $\xi(x)=\eta(x)=0$ for $x\geqq T$. Then, by (3.7),

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4) We always consider that the infimum of the empty set is $\infty$.

$$0=\xi(x)=\xi(T)-2\lambda\int_T^x\int_0^y \xi(z)e^{-2\int_z^y b(u)du}dz\,dy<0\,,\qquad x>T\,.$$

This is a contradiction.

*Step* 2. We shall show the second assertion. By (3.4),

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}=A\begin{pmatrix}\xi\\ \eta\end{pmatrix}-\begin{pmatrix}0\\ 2\tilde{b}\eta\end{pmatrix},$$

where $A=\begin{pmatrix}0 & 1\\ -2\lambda & -2b_0\end{pmatrix}$ and $\tilde{b}(x)=b(x)-b_0$. It follows that, for each $x_0>0$,

$$(3.14)\quad \begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix}=-\int_{x_0}^x e^{(x-y)A}\begin{pmatrix}0\\ 2\tilde{b}(y)\eta(y)\end{pmatrix}dy+e^{(x-x_0)A}\begin{pmatrix}\xi(x_0)\\ \eta(x_0)\end{pmatrix}.$$

Now suppose that $0<\lambda<\lambda_0$. Then

$$(3.15)\qquad e^{xA}=\frac{1}{\alpha_+-\alpha_-}\begin{pmatrix}\alpha_+e^{-\alpha_-x}-\alpha_-e^{-\alpha_+x} & e^{-\alpha_-x}-e^{-\alpha_+x}\\ 2\lambda(e^{-\alpha_+x}-e^{-\alpha_-x}) & \alpha_+e^{-\alpha_+x}-\alpha_-e^{-\alpha_-x}\end{pmatrix}.$$

Hence, noting $\xi(0)=1$ and $\eta(0+)=0$ (by (3.6)), we have

$$(3.16)\qquad \xi(x)=\frac{1}{\alpha_+-\alpha_-}\Big(\alpha_+e^{-\alpha_-x}-\alpha_-e^{-\alpha_+x}-\int_0^x(e^{-\alpha_-(x-y)}-e^{-\alpha_+(x-y)})2\tilde{b}(y)\eta(y)dy\Big),$$

as well as convergence of the integral. On the other hand, (3.14) and (3.15) imply

$$e^{\alpha_-x}|\eta(x)|\leqq K_1\int_{x_0}^x \tilde{b}(y)e^{\alpha_-y}|\eta(y)|dy+K_2$$

for some $K_1$, $K_2>0$. Hence, by Gronwall's inequality and (2.4), we have

$$e^{\alpha_-x}|\eta(x)|\leqq K_2\exp\Big\{K_1\int_{x_0}^x \tilde{b}(y)dy\Big\}\leqq K_3<\infty,\ x>x_0\,,$$

for some $K_3>0$. Now, using this and (2.4) again, we obtain (3.2) with $l(\varphi_\lambda)>0$ from (3.16).

Turning to the case of $\lambda=\lambda_0$, we have

$$(3.17)\qquad e^{xA}=e^{-b_0x}\begin{pmatrix}1+b_0x & x\\ -4b_0^2x & 1-b_0x\end{pmatrix}.$$

Hence (3.14) gives

$$\xi(x)=xe^{-b_0x}\left(b_0-\int_0^x e^{b_0y}\left(1-\frac{y}{x}\right)2\tilde{b}(y)\eta(y)dy\right)+e^{-b_0x}, \tag{3.18}$$

$$e^{b_0x}|\eta(x)|/x\leqq K_5\int_{x_0}^x y\tilde{b}(y)\frac{e^{b_0y}|\eta(y)|}{y}dy+K_6. \tag{3.19}$$

By Gronwall's inequality, it follows from (3.19) and (2.4) that

$$e^{b_0x}|\eta(x)|/x\leqq K_6\exp\left\{K_5\int_{x_0}^x y\tilde{b}(y)dy\right\}\leqq K_7<\infty.$$

Hence we have (3.3) with $l(\varphi_{\lambda_0})>0$ from (3.18) and (2.4).

3) Let $\lambda>\lambda_0$. Let $T$ and $\zeta(x)$ be as in 2) Step 1. First we shall show $T<\infty$. Suppose $T=\infty$, and choose an $\varepsilon>0$ such that $2\lambda-(b_0+\varepsilon)^2=a^2$ for some $a>0$. Then there is an $x_0>0$ satisfying

$$b(x)\leqq b_0+\varepsilon, \qquad x\geqq x_0.$$

We note that (3.9) and (3.10) are valid also in this case. Hence,

$$\zeta'(x)\geqq(\zeta(x)-b_0-\varepsilon)^2+a^2, \qquad x\geqq x_0.$$

Solving this, we have

$$\mathrm{Tan}^{-1}[(\zeta(x)-b_0-\varepsilon)/a]-\mathrm{Tan}^{-1}[(\zeta(x_0)-b_0-\varepsilon)/a]\geqq a(x-x_0), \quad x\geqq x_0,$$

which is absurd. Hence $T<\infty$. If $\eta(T)=0$, then $\xi(T)=\eta(T)=0$. This is impossible by the last argument in 2) Step 1. Thus $\eta(T)<0$ since $\zeta(x)>0$ on $(0, T)$. This implies $\varphi_\lambda(x)=\xi(x)<0$ for some $x>T$. q.e.d.

**Remark 2.** *The spectrum of L on the space $L^2(m(dx))$ is contained in $(-\infty, -\lambda_0]$. Further, $-\lambda_0$ belongs to the spectrum of L.*

*Proof.* Lemma 1 implies

$$T_tg(x)=O(e^{-\lambda_0t}), \qquad t\to\infty,$$

for each $g\in\boldsymbol{B}$ with compact support. This assures the first assertion.

If $-\lambda_0$ does not belong to the spectrum of $L$, then the spectrum is contained in $(-\infty, -\lambda_0-\delta]$ for some $\delta>0$, because it is closed. Hence $p(t, x, y)=O(e^{-(\lambda_0+\delta)t})$ as $t\to\infty$, and the Green kernel

$$G_\lambda(x, y)=\int_0^\infty e^{\lambda t}p(t, x, y)dt$$

is convergent for $\lambda_0<\lambda<\lambda_0+\delta/2$. This, combined with Dynkin's formula, yields $E_{0+}[e^{\lambda\sigma_x}]<\infty$. Hence $\tilde{\varphi}_\lambda(x)\equiv1/E_{0+}[e^{\lambda\sigma_x}]$ is positive. But $\tilde{\varphi}_\lambda(x)$ solves (3.1) (cf. [8] 4.6), which contradicts Lemma 1.

**Lemma 2.** 1) *Let* $0<c\leqq\lambda_0$. *Then, for each* $0\leqq v_0\leqq 1$, *there is a unique solution of the equation*

$$\text{(3.20)}\qquad \begin{cases} Lv+c(v^2-v)=0\,, \\ v(0)=v_0\,, \quad 0\leqq v\leqq 1\,. \end{cases}$$

*Further, if* $0<v_0<1$,

$$\text{(3.21)}\qquad 1-v(x)\in \boldsymbol{B}_0(\alpha_c)\,, \qquad 0<c<\lambda_0\,,$$

$$\text{(3.22)}\qquad 1-v(x)\in \boldsymbol{B}_1(b_0)\,, \qquad c=\lambda_0$$

*with positive* $l(1-v)$.

2) *If* $c>\lambda_0$ *and* $0<v_0<1$, *then* (3.20) *admits no solution.*

*Proof.* The outline of the proof is similar to that of Lemma 1.

1) *Step* 1 (*Uniqueness*). Set $\xi(x)=1-v(x)$, $\eta(x)=-v'(x)$, and

$$f(\xi)=\begin{cases} 2c\xi(1-\xi)\,, & 0\leqq\xi\leqq 1, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then (3.20) is rewritten as

$$\text{(3.23)}\qquad \begin{cases} \xi'(x)=\eta(x)\,, \\ \eta'(x)=-f(\xi(x))-2b(x)\eta(x)\,, \\ \xi(0)=1-v_0\,, \quad 0\leqq\xi(x)\leqq 1 \end{cases}$$

with (3.5). Then, as in the proof of Lemma 1, we have

$$\text{(3.24)}\qquad \eta(x)=-\int_0^x f(\xi(y))e^{-2\int_y^x b(z)dz}dy\,,$$

$$\text{(3.25)}\qquad \xi(x)=1-v_0-\int_0^x\int_0^y f(\xi(z))e^{-2\int_z^y b(u)du}dzdy.$$

Now the uniqueness follows, since $f(\xi)$ is Lipschitz continuous.

*Step* 2 (*Existence*). Since there is a trivial solution $v(x)\equiv 0$ (resp. $\equiv 1$) in the case of $v_0=0$ (resp. $=1$), we assume $0<v_0<1$. Further, (3.25) has a unique solution $\xi(x)$. So it is enough to show

$$\text{(3.26)}\qquad 0<\xi(x)<1, \qquad x>0.$$

By (3.24) and (3.25), there exists a positive $\delta$ such that

$$\text{(3.27)}\qquad 0<\xi(x)<1, \quad \eta(x)<0\,, \qquad 0<x<\delta.$$

Set $T=\inf\{x>0\colon \xi(x)(1-\xi(x))=0\}$ and $\zeta(x)=-\eta(x)/\xi(x)$ for $x\in(0,T)$. Then (3.23) gives

$$\zeta'(x)=\zeta(x)^2-2b(x)\zeta(x)+2c(1-\xi(x)), \qquad x\in(0, T). \tag{3.28}$$

It follows from this and (3.27) that $\zeta(x)>0$ for $x\in(0, T)$. Hence,

$$\zeta'(x)\leqq(\zeta(x)-\alpha_+)(\zeta(x)-\alpha_-), \qquad x\in(0, T), \tag{3.29}$$

where

$$\alpha_\pm=\alpha_\pm(c)=b_0\pm\sqrt{b_0^2-2c}.$$

Now, as in the proof of Lemma 1, we have

$$0<\zeta(x)<\alpha_-, \qquad x\in(0, T). \tag{3.30}$$

Suppose that $T<\infty$. Then, since $\xi'(x)=\eta(x)<0$ for $x\in(0, T)$, it must hold that $\xi(T)=\eta(T)=0$. But this leads to contradiction by the same argument as in the proof of Lemma 1. Hence $T=\infty$.

*Step* 3 (*Proof of the second assertion*). First we shall show

$$\lim_{x\to\infty}\xi(x)=0. \tag{3.31}$$

Since $\xi(x)$ is decreasing, it tends to a limit $\xi_*$. If $\xi_*\neq 0$, then, by (3.24),

$$\overline{\lim_{x\to\infty}}\,\eta(x)\leqq -e^{-2\delta b_0}\lim_{x\to\infty}\int_{x-\delta}^{x} f(\xi(y))e^{-2\int_y^x \tilde{b}(z)dz}dy=-\delta e^{-2\delta b_0}f(\xi_*)<0.$$

This contradicts the fact $\xi(x)\geqq 0$, $x>0$. Hence, $\xi_*=0$.

Now, since $\eta(0+)=0$, (3.23) is written as

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}=A\begin{pmatrix}\xi\\ \eta\end{pmatrix}+\begin{pmatrix}0\\ \gamma\end{pmatrix}, \quad \begin{pmatrix}\xi(0+)\\ \eta(0+)\end{pmatrix}=\begin{pmatrix}1-v_0\\ 0\end{pmatrix}, \tag{3.32}$$

where $A=\begin{pmatrix}0 & 1\\ -2c & -2b_0\end{pmatrix}$ and

$$\gamma=\gamma(x)=2c\xi(x)^2-2\tilde{b}(x)\eta(x). \tag{3.33}$$

Let $0<c<\lambda_0$. Then as in the proof of Lemma 1, we have

$$\xi(x)=\frac{1}{\alpha_+-\alpha_-}\left[\int_0^x (e^{-\alpha_-(x-y)}-e^{-\alpha_+(x-y)})\gamma(y)dy\right.$$
$$\left.+(\alpha_+e^{-\alpha_-x}-\alpha_-e^{-\alpha_+x})(1-v_0)\right] \tag{3.34}$$

and, for each $x_0>0$,

$$|\eta(x)|\leqq K_1\int_{x_0}^{x} e^{-\alpha_-(x-y)}\gamma(y)|\eta(y)|dy+K_2e^{-\alpha_-x}. \tag{3.35}$$

Let $\mu(x)=\xi(x)+|\eta(x)|$. Then (3.33)–(3.35) imply

$$e^{\alpha_- x}\mu(x)\leqq K_3\int_{x_0}^{x}(\xi(y)+\tilde{b}(y))e^{\alpha_- y}\mu(y)dy+K_4,\quad x\geqq x_0.$$

Hence, by Gronwall's inequality,

$$e^{\alpha_- x}\mu(x)\leqq K_4\exp\left\{K_3\int_{x_0}^{x}(\tilde{b}(y)+\xi(y))dy\right\}. \tag{3.36}$$

By (3.31) and by $\lim_{x\to\infty}\tilde{b}(x)=0$, this gives

$$\mu(x)=O(e^{-(\alpha_- -\delta)x}),\quad x\to\infty \tag{3.37}$$

for each $\delta>0$. Since $\xi(x)\leqq\mu(x)$, (3.36) combined with (3.37) and (2.4) assures boundedness of $e^{-\alpha_- x}\mu(x)$. Hence (3.21) and the inequality $l(1-v)>0$ follow from (3.34), (3.33) and (2.4).

In case $c=\lambda_0$, we have

$$\xi(x)=\int_0^x e^{-b_0(x-y)}(x-y)\gamma(y)dy+(1+b_0x)e^{-b_0x}(1-v_0), \tag{3.38}$$

$$|\eta(x)|\leqq K_5\int_{x_0}^{x}e^{-b_0(x-y)}(x-y)\gamma(y)dy+K_6xe^{-b_0x} \tag{3.39}$$

in place of (3.34) and (3.35). Hence, as above,

$$\frac{e^{b_0x}\mu(x)}{x}\leqq K_7\int_{x_0}^{x}(y\xi(y)+y\tilde{b}(y))\frac{e^{b_0y}\mu(y)}{y}dy+K_8,\quad x\geqq x_0. \tag{3.40}$$

This implies (3.37) first, and then boundedness of $e^{b_0x}\mu(x)/x$. Now (3.22) and the inequality $l(1-v)>0$ follow from (3.38), (3.33) and (2.4).

2) Suppose that $c>\lambda_0$, $0<v_0<1$, and $v(x)$ satisfies (3.20) except for $0\leqq v(x)\leqq 1$. Notice that the argument in 1) Step 1 is valid also in this case. Thus (3.27) holds. Use $T$ and $\zeta(x)$ in 1) Step 2. Then (3.28) follows. Now we shall show $T<\infty$. Suppose that $T=\infty$. Then we have (3.31) by the previous argument. Choose an $\varepsilon>0$ satisfying $2c(1-\varepsilon)-(b_0+\varepsilon)^2>0$. We can find an $x_0>0$ such that $\xi(x)<\varepsilon$ and $b(x)\leqq b_0+\varepsilon$ for $x\geqq x_0$. This and (3.28) lead to a contradiction in the same way as in the proof of Lemma 1. Hence $T<\infty$. Since $\eta(x)<0$ on $(0, T)$, $\xi(T)$ must be 0. Thus it is enough to show $\eta(T)<0$. But this is clear if we repeat the consideration in the proof of Lemma 1. q.e.d.

## §4. Comparison Theorem for Stochastic Differential Equations

A comparison theorem for one-dimensional stochastic differential equations is found in [6] and [18]. Here we shall reformulate it in a form convenient for our use.

Let $(\beta(t), \mathscr{F}_t, P)$[5] be a one-dimensional Brownian motion and consider a stochastic integral equation

$$x(t)=x_0+\beta(t)+\int_0^t b(x(s))ds+\psi(t), \qquad x_0>0. \tag{4.1}$$

We say that $(x(t), \psi(t))$ is a solution of (4.1), if $x(t)$ and $\psi(t)$ are continuous and $\mathscr{F}_t$-adapted, $x(t)$ is nonnegative, $\psi(t)$ is nondecreasing, (4.1) holds and

$$\int_0^t I_{\{0\}}(x(s))d\psi(s)=\psi(t)^{6)}. \tag{4.2}$$

Later we shall see that (E) implies $\psi(t)\equiv 0$. It is well known that if (4.1) has a unique solution $(x(t), \psi(t))$, then $(x(t), P)$ is a diffusion process corresponding to the generator $L$ of (1.2) with the domain $D(L)$ of (2.6).

The next lemma is a variation of [13] and [19].

**Lemma 3.** *There is a solution* $(x(t), \psi(t))$ *of* (4.1), *which is pathwise unique.*

*Proof. Step* 1 (*Uniqueness*). Suppose that (4.1) has two solutions $(x^{(1)}(t), \psi^{(1)}(t))$ and $(x^{(2)}(t), \psi^{(2)}(t))$. Choose a sequence $\{x_n\}$ decreasing to 0 and continuous nonnegative functions $g_n(x)$, $n=1, 2,\dots,$ such that support $[g_n(x)] \subset (x_n, x_{n-1})$ and $\int_0^\infty g_n(x)dx=1$. Let $f_n(x)=\int_0^{|x|}\int_0^y g_n(z)dzdy$. Then we have

$$\begin{aligned}
&E[f_n(x^{(1)}(t)-x^{(2)}(t))]\\
&=E\left[\int_0^t f_n'(x^{(1)}(s)-x^{(2)}(s))(b(x^{(1)}(s))-b(x^{(2)}(s)))\,ds\right]\\
&\quad+E\left[\int_0^t f_n'(x^{(1)}(s)-x^{(2)}(s))d(\psi^{(1)}(s)-\psi^{(2)}(s))\right]
\end{aligned}$$

(e.g. [7]). Now since $b(x)$ is nonincreasing and (4.2) holds, we see that

$$x^{(1)}(s)>x^{(2)}(s)\geqq 0$$

implies

$$\begin{aligned}
&f_n'(x^{(1)}(s)-x^{(2)}(s))\geqq 0,\\
&b(x^{(1)}(s))-b(x^{(2)}(s))\leqq 0,\\
&(\psi^{(1)}-\psi^{(2)})(s+\varepsilon)-(\psi^{(1)}-\psi^{(2)})(s-\varepsilon)=-(\psi^{(2)}(s+\varepsilon)-\psi^{(2)}(s-\varepsilon))\leqq 0
\end{aligned}$$

5) We assume that $\mathscr{F}_t$ contains all $P$-null sets, and the assertions in this section should be read to hold almost surely $(P)$.

6) $I_A(x)$ is the indicator function of a set $A$.

for some $\varepsilon>0$. Similarly, $0\leqq x^{(1)}(s)<x^{(2)}(s)$ implies the opposite inequalities. Hence $E[f_n(x^{(1)}(t)-x^{(2)}(t))]\leqq 0$. In view of $f_n(x)\uparrow |x|$ as $n\uparrow\infty$, we obtain $E[|x^{(1)}(t)-x^{(2)}(t)|]=0$. Noting that $x^{(i)}(t)$ is continuous, we have $x^{(1)}(t)\equiv x^{(2)}(t)$ as well as $\psi^{(1)}(t)\equiv\psi^{(2)}(t)$.

*Step* 2 (*Existence in the case of* (E)). Let $\{x_n\}$ be a sequence decreasing to 0 and let $b^{(n)}(x)=b(x\vee x_n)$[7] for $x\in(-\infty,\infty)$. Since $b^{(n)}(x)$ is bounded and continuous, the equation

$$x^{(n)}(t)=x_0+\beta(t)+\int_0^t b^{(n)}(x^{(n)}(s))ds$$

has a solution ([14] pp. 76–77). This is unique by the same reason as in Step 1. Now let $\sigma_x^{(n)}=\inf\{t>0: x^{(n)}(t)\leqq x\}$. Then, by the uniqueness

$$x^{(m)}(t)=x^{(n)}(t), \qquad t\leqq\sigma_{x_n}^{(n)}, \quad m\geqq n\,.$$

Hence $\sigma_{x_n}^{(n)}=\sigma_{x_n}^{(m)}\leqq\sigma_{x_m}^{(m)}$. Letting $\sigma=\lim_{n\to\infty}\sigma_{x_n}^{(n)}$, we can define the inductive limit $\{x(t): t<\sigma\}$ of the processes $\{x^{(n)}(t): t<\sigma_{x_n}^{(n)}\}$. But $\sigma=\infty$, since $g(x_0)\equiv E[e^{-\mu\sigma}]$ $(\mu>0)$ is a solution of

$$\frac{1}{2}g''(x)+b(x)g'(x)=\mu g(x), \qquad 0\leqq g(x)\leqq 1\,,$$

and such a solution identically vanishes by virtue of (E). The process $x(t)$ is a solution of (4.1) with $\psi(t)\equiv 0$.

*Step* 3 (*Existence in the case of* (R)). Let $\{x_n\}$ be a sequence decreasing to 0 and let $\gamma^{(n)}(x)$, $n=1, 2,\dots$, be nonnegative continuous functions such that $\gamma^{(n)}(x)=1/x$ $(0<x\leqq x_{n+1})$, $\leqq 1/x$ $(x_{n+1}\leqq x\leqq x_n)$, $=0$ $(x_n\leqq x)$ and $b(x)+\gamma^{(n)}(x)$ is nonincreasing. Then, since the function $b(x)+\gamma^{(n)}(x)$ satisfies (2.1)–(2.4) as well as (E), the equation

$$x^{(n)}(t)=x_0+\beta(t)+\int_0^t b(x^{(n)}(s))ds+\int_0^t \gamma^{(n)}(x^{(n)}(s))ds \tag{4.3}$$

admits a unique solution $x^{(n)}(t)$ by Step 2. Further, noting $b(x)+\gamma^{(n)}(x)\geqq b(x)+\gamma^{(n+1)}(x)$, we see that $x^{(n)}(t)\geqq x^{(n+1)}(t)$, $t\geqq 0$, by the standard comparison theorem ([6], [18]). Hence there exists a nonnegative limit $x(t)=\lim_{n\to\infty}x^{(n)}(t)$. Since $b(x)$ is nonincreasing and $\int_0^t b(x^{(n)}(s))ds\leqq x^{(n)}(t)-x_0-\beta(t)$ by (4.3), it follows that

$$\lim_{n\to\infty}\int_0^t b(x^{(n)}(s))ds=\int_0^t b(x(s))ds, \qquad t\geqq 0\,, \tag{4.4}$$

7) $a\vee b=\max\{a, b\}$ and $a\wedge b=\min\{a, b\}$.

the both sides being finite. Now (4.3) and (4.4) imply existence of a finite limit

$$\psi(t)=\lim_{n\to\infty}\int_0^t \gamma^{(n)}(x^{(n)}(s))ds, \qquad t\geqq 0.$$

Hence $(x(t), \psi(t))$ satisfies (4.1).

It is clear that $x(t)\geqq 0$, and $x(t)$ and $\psi(t)$ are $\mathscr{F}_t$-adapted. We shall show they are continuous. Since $x(t)$ is nonnegative and upper semicontinuous, it is continuous at the time $t$ for which $x(t)=0$. Let $x(t_0)>0$. Take an $N>0$ such that $x(t_0)\geqq x_N$. We can find a $\delta>0$ such that

$$x^{(n)}(t)\geqq x_{N+1}, \qquad t\in(t_0-\delta, t_0+\delta)\cap[0, \infty), \quad n\geqq N+1. \tag{4.5}$$

If otherwise, then, by the continuity of $x^{(n)}(t)$, there is a sequence $\{t_n\}$ such that $t_n\to t_0$, $t_n\geqq 0$, $x^{(n)}(t_n)=x_{N+1}$ and $x^{(n)}(t)\geqq x_{N+1}$ for all $t\in[t_0\wedge t_n, t_0\vee t_n]$. But (4.3) gives

$$\begin{aligned}0<x_N-x_{N+1}&\leqq x^{(n)}(t_0)-x^{(n)}(t_n)\\ &\leqq|\beta(t_0)-\beta(t_n)|+b(x_{N+1})|t_0-t_n|, \qquad n\geqq N+1,\end{aligned}$$

which is impossible. Hence, there exists a $\delta$ satisfying (4.5). Also, equicontinuity of $\{x^{(n)}(t)\colon n\geqq N+1\}$ on $(t_0-\delta, t_0+\delta)\cap[0, \infty)$ follows from (4.3) and (4.5). Hence $x(t)$ is continuous at $t_0$. Therefore, $x(t)$ is continuous on $[0, \infty)$. By (4.1), this assures the continuity of $\psi(t)$.

Finally we shall show (4.2). By (4.5) it follows that

$$\int_{(t_0-\delta)\vee 0}^{t_0+\delta}\gamma^{(n)}(x^{(n)}(s))ds=0, \qquad n\geqq N+1.$$

Hence $\psi(t_0+\delta)-\psi((t_0-\delta)\vee 0)=0$. Therefore, for each $N$, $\int_0^t I_{[x_N,\infty)}(x(s))d\psi(s)=0$. Letting $N\to\infty$, we have (4.2). q.e.d.

**Lemma 4.** 1) *Suppose that, for* $i=1, 2$, $b_i(x)$ *satisfies* (2.1)–(2.4), *and* $(x_i(t), \psi_i(t))$ *is the solution of* (4.1) *with* $b(x)$ *and* $x_0$ *replaced by* $b_i(x)$ *and* $x_0^{(i)}$. *If* $x_0^{(1)}\leqq x_0^{(2)}$ *and* $b_1(x)\leqq b_2(x)$, *then* $x_1(t)\leqq x_2(t)$, $t\geqq 0$.

2) *Let* $(x(t), \psi(t))$ *be the solution of* (4.1) *and* $(\bar{x}(t), \bar{\psi}(t))$ *be that of* (4.1) *with* $\bar{b}(x)=b(x)-h$ *and* $\bar{x}_0$ *in place of* $b(x)$ *and* $x_0$. *If* $0\leqq h<b_0$ *and* $\bar{x}_0\leqq x_0$, *then*

$$x(t)-ht-x_0\leqq\bar{x}(t)-\bar{x}_0, \qquad t\geqq 0. \tag{4.6}$$

*Proof.* 1) *Case* 1. When both $b_1(x)$ and $b_2(x)$ satisfy (E), the assertion is none other than the standard comparison theorem ([6], [18]).

*Case* 2. Suppose that $b_1(x)$ satisfies (R) and $b_2(x)$ satisfies (E). Let $\{x_n\}$ be a sequence decreasing to 0 and let $\tilde{\gamma}^{(n)}(x)$, $n=1, 2,\dots$, be nonnegative continuous functions on $(0, \infty)$ such that $\tilde{\gamma}^{(n)}(x)=b_2(x)-b_1(x)$ $(0<x\leqq x_{n+1})$, $\leqq b_2(x)-b_1(x)$ $(x_{n+1}<x<x_n)$, $=0$ $(x_n\leqq x)$ and $b_1(x)+\tilde{\gamma}^{(n)}(x)$ is nonincreasing. Then the function $b_1(x)+\tilde{\gamma}^{(n)}(x)$ can play the role of $b(x)+\gamma^{(n)}(x)$ in Step 3 of the proof of Lemma 3. The corresponding solution $x_1^{(n)}(t)$ satisfies $x_1^{(n)}(t)\leqq x_2(t)$, $t\geqq 0$, by the result of Case 1. Since $x_1^{(n)}(t)$ tends to $x_1(t)$, this gives $x_1(t)\leqq x_2(t)$, $t\geqq 0$.

*Case* 3. When both $b_1(x)$ and $b_2(x)$ satisfy (R), we use the $\gamma^{(n)}(x)$ in the proof of Lemma 3. By the result of Case 1, the corresponding solutions satisfy $x_1^{(n)}(t)\leqq x_2^{(n)}(t)$, $t\geqq 0$. This gives $x_1(t)\leqq x_2(t)$.

2) Suppose that $b(x)$ satisfies (E). Then $\psi(t)\equiv 0$. Since $\bar{x}(t)\leqq x(t)$, $t\geqq 0$, and $b(x)$ is nonincreasing,

$$\begin{aligned} x(t)-ht-x_0&=\beta(t)+\int_0^t (b(x(s))-h)ds \\ &\leqq \beta(t)+\int_0^t (b(\bar{x}(s))-h)ds=\bar{x}(t)-\bar{x}_0\,. \end{aligned}$$

In the case that $b(x)$ satisfies (R), let $x^{(n)}(t)$ be the solution of (4.3) and $\bar{x}^{(n)}(t)$ be that of (4.3) corresponding to $\bar{b}(x)$ and $\bar{x}_0$. Then $\bar{x}^{(n)}(t)\leqq x^{(n)}(t)$, and

$$\begin{aligned} &x^{(n)}(t)-ht-x_0 \\ &\quad=\beta(t)+\int_0^t (b(x^{(n)}(s))-h)ds+\int_0^t \gamma^{(n)}(x^{(n)}(s))ds \\ &\quad\leqq \beta(t)+\int_0^t (\bar{b}(\bar{x}^{(n)}(s))+\gamma^{(n)}(\bar{x}^{(n)}(s)))ds=\bar{x}^{(n)}(t)-\bar{x}_0\,. \end{aligned}$$

We get the conclusion by letting $n\to\infty$.

## §5. Proof of Theorem 1

Using the $\varphi_\lambda(x)$ given in Lemma 1, we define

$$W^{(\lambda)}(t)=e^{(\lambda-c)t}\check{\varphi}_\lambda(\boldsymbol{x}_t)\,.$$

Then this is a martingale ([17]). Further, if $0\leqq\lambda\leqq\lambda_0$, then this is a nonnegative martingale, so that there is a nonnegative integrable limit

$$W^{(\lambda)}=\lim_{t\to\infty} W^{(\lambda)}(t)\,,\qquad \text{a.s.} \tag{5.1}$$

**Lemma 5.** *Let $g\in\boldsymbol{B}$ have a compact support. Then,* (5.2) $$\lim_{t\to\infty} e^{(\lambda-c)t}\,\check{g}(\boldsymbol{x}_t)=0\,,\quad a.s.\qquad for\quad \lambda<\lambda_0\,,$$

(5.3) $$\lim_{t\to\infty} e^{(\lambda_0-c)t}\,\check{g}(\boldsymbol{x}_t)=0\,,\qquad in\quad L^1(\Omega)\,.$$

*Proof.* Suppose $\lambda<\lambda_0$. Using Lemma 1, we can easily show that there is a positive constant $K_1$ such that $|g(x)|\leqq K_1\varphi_{\lambda_0}(x)$. Hence,

$$\overline{\lim_{t\to\infty}}\, e^{(\lambda-c)t}|\check{g}(\boldsymbol{x}_t)|\leqq K_1\lim_{t\to\infty} e^{(\lambda-\lambda_0)t}e^{(\lambda_0-c)t}\check{\varphi}_{\lambda_0}(\boldsymbol{x}_t)=0\,,\quad \text{a.s.,}$$

which completes the proof of (5.2).

We shall show (5.3). Without loss of generality, we may assume that $g(x)$ is nonnegative and nonincreasing. By [17],

$$\boldsymbol{E}_x[e^{(\lambda_0-c)t}\,\check{g}(\boldsymbol{x}_t)]=e^{\lambda_0 t}\,E_x[g(x_t)]\,.$$

Let $X^*=(x_t^*,\,P_x^*)$ be a diffusion process given in Section 2 with $b(x)\equiv b_0$, and denote the corresponding objects by $p^*(t,x,y)$, $T_t^*$ etc. Then we have $E_x[g(x_t)]\leqq E_x^*[g(x_t^*)]$ by Lemma 4. Hence, it is enough to show that

(5.4) $$\lim_{t\to\infty} e^{\lambda_0 t}\,T_t^* g(x)=0.$$

It is easily seen that the transition density $p^*(t,x,y)$ of $X^*$ is represented as

$$p^*(t,x,y)=\int_{\lambda_0}^{\infty} e^{-\lambda t}\varphi_\lambda^*(x)\varphi_\lambda^*(y)\sigma^*(d\lambda),\qquad x,y\in S,\ t>0,$$

where

$$\varphi_\lambda^*(x)=e^{-b_0 x}\left\{\cos\sqrt{2\lambda-b_0^2}\,x+\frac{b_0}{\sqrt{2\lambda-b_0^2}}\sin\sqrt{2\lambda-b_0^2}\,x\right\},$$

$$\sigma^*(d\lambda)=\frac{\sqrt{\lambda-\lambda_0}}{\sqrt{2}\,\pi\lambda}\,d\lambda.$$

Hence, using Fubini's theorem, we obtain

$$e^{\lambda_0 t}T_t^*g(x)=\int_{\lambda_0}^{\infty} e^{(\lambda_0-\lambda)t}\varphi_\lambda^*(x)\mathscr{F}g(\lambda)\sigma^*(d\lambda),$$

where

$$\mathscr{F}g(\lambda)=\int_0^{\infty} g(x)\varphi_\lambda^*(x)m^*(dx)\,.$$

Now for $t\geqq t_0>0$ and $\lambda\geqq\lambda_0$,

$$|e^{(\lambda_0-\lambda)t}\varphi_\lambda^*(x)\mathscr{F}g(x)|\leqq e^{-(\lambda-\lambda_0)t_0}\mathscr{F}g(\lambda_0),$$

and

$$\int_{\lambda_0}^{\infty} e^{-\lambda t_0}\,\sigma^*(d\lambda)=p^*(t_0,0,0)<\infty.$$

Therefore, using Lebesgue's dominated convergence theorem, we have (5.4).

*Proof of Theorem* 1. 1) Take a $g\in \boldsymbol{B}_0(\alpha_\lambda, l)$, and let

$$g_1(x: R)=(g(x)-\varphi_\lambda(x)l/l(\varphi_\lambda))I_{[0,R)}(x),$$

$$g_2(x: R)=(g(x)-\varphi_\lambda(x)l/l(\varphi_\lambda))I_{[R,\infty)}(x),$$

for each $R>0$. Then we have

$$g(x)=\varphi_\lambda(x)l/l(\varphi_\lambda)+g_1(x: R)+g_2(x: R). \tag{5.5}$$

Using (3.2), we can easily show that for each $\varepsilon_n>0$ there exists a positive $R_n$ such that $|g_2(x: R_n)|\leqq\varepsilon_n\varphi_\lambda(x)$. Hence,

$$e^{(\lambda-c)t}|\check{g}_2(\boldsymbol{x}_t: R_n)|\leqq\varepsilon_n W^{(\lambda)}(t).$$

Further, since $g_1(x: R_n)$ has a compact support,

$$\lim_{t\to\infty} e^{(\lambda-c)t}\check{g}_1(\boldsymbol{x}_t: R_n)=0, \qquad \text{a.s.},$$

by Lemma 5. Hence (5.5) implies

$$\overline{\lim_{t\to\infty}}\,|e^{(\lambda-c)t}\check{g}(\boldsymbol{x}_t)-W^{(\lambda)}l/l(\varphi_\lambda)|\leqq\overline{\lim_{t\to\infty}}\,|(W^{(\lambda)}(t)-W^{(\lambda)})l/l(\varphi_\lambda)|$$

$$+\varepsilon_n\lim_{t\to\infty}W^{(\lambda)}(t)=\varepsilon_n W^{(\lambda)},\ n=1, 2,\dots, \qquad \text{a.s.}$$

Now let $\varepsilon_n\downarrow 0$ and set $W_\lambda=W^{(\lambda)}/l(\varphi_\lambda)$. Then (2.8) follows.

2) (2.9) will be proved by a similar way. But the conclusion holds in probability since the assertion in (5.3) holds only in $L^1(\Omega)$. q.e.d.

## §6. Proof of Theorem 2

For the $W^{(\lambda)}(t)$ and $W^{(\lambda)}$ in the previous section, we define

$$\psi_t(x, a)=\boldsymbol{E}_x[\exp\{-aW^{(\lambda)}(t)\}],$$

$$\psi(x, a)=\boldsymbol{E}_x[\exp\{-aW^{(\lambda)}\}], \qquad a\geqq 0.$$

Denote $T_t^0=e^{-ct}T_t$.

**Lemma 6.** *Let* $0\leqq\lambda\leqq\lambda_0$. *Then*

$$\psi_t(x, a)=T_t^0 f_t(\cdot, a)(x)+c\int_0^t T_s^0\{\psi_{t-s}(\cdot, a\,e^{(\lambda-c)s})^2\}(x)ds, \tag{6.1}$$

*where* $f_t(x, a)=\exp\{-ae^{(\lambda-c)t}\varphi_\lambda(x)\}$. *Further, the function* $v(x)\equiv\boldsymbol{P}_x(W^{(\lambda)}$

$=0$) *satisfies* (3.20) *with some* $v_0$.

*Proof.* For a fixed $t_0 \geqq 0$, $\psi_t(x, a: t_0) \equiv \boldsymbol{E}_x[\hat{f}_{t_0}(\boldsymbol{x}_t, a)]$ satisfies the $S$-equation (2.7) with $g(x)=f_{t_0}(x, a)$. Hence

$$\psi_t(x, a: t_0)=T_t^0 f_{t_0}(\cdot, a)(x)+c\int_0^t T_s^0\{\boldsymbol{E}.[\hat{f}_{t_0}(\boldsymbol{x}_{t-s}, a)]^2\}(x)ds.$$

Since $\psi_t(x, a)=\psi_t(x, a: t)$ and $\boldsymbol{E}_x[\hat{f}_t(\boldsymbol{x}_{t-s}, a)]=\psi_{t-s}(x, ae^{(\lambda-c)s})$, we obtain (6.1).

To prove the second assertion, note that $\lim_{t\to\infty}\psi_t(x, a)=\psi(x, a)$ and

(6.2) $$\|T_t^0\| \leqq e^{-ct}, \quad \|f_t(\cdot, a)\| \leqq 1, \quad \|\psi_s(\cdot, a)\| \leqq 1,$$

where $\|f(\cdot)\|=\sup_{x\in S}|f(x)|$ and $\|T\|=\sup_{\|f\|=1}\|Tf\|$. Then by letting $t\to\infty$ in (6.1), we have

$$\psi(x, a)=c\int_0^\infty T_s^0\{\psi(\cdot, ae^{(\lambda-c)s})^2\}(x)ds.$$

Let $a\to\infty$. It follows that $v(x)\equiv\lim_{a\to\infty}\psi(x, a)$ satisfies

$$v(x)=c\int_0^\infty T_s^0\{v(\cdot)^2\}(x)ds.$$

This completes the proof.

**Lemma 7.** *If* $0<c\leqq\lambda_0$, *then*

(6.3) $$\boldsymbol{P}_x(W_c=0)=0, \qquad x\in S.$$

*Proof.* Fix $0<v_0<1$, and let $v(x)$ be the solution of (3.20). Then $u(t, x)$ $\equiv v(x)$ satisfies

(6.4) $$u(t, x)=T_t^0 v(x)+c\int_0^t T_s^0\{u(t-s, \cdot)^2\}(x)ds.$$

On the other hand, setting $\lambda=c$ in (6.1), we have

(6.5) $$\psi_t(x, a)=T_t^0 f(\cdot, a)(x)+c\int_0^t T_s^0\{\psi_{t-s}(\cdot, a)^2\}(x)ds,$$

where $f(x, a)=\exp\{-a\varphi_c(x)\}$. But Lemmas 1 and 2 ensure that, for any sufficiently large $a$,

$$v(x)\geqq f(x, a), \qquad x\in S.$$

Hence, it follows from (6.4) and (6.5) that

$$\psi_t(x, a)\leqq u(t, x)=v(x), \qquad x\in S, \quad t\geqq 0.$$

Now let $t\to\infty$ and then $a\to\infty$. We obtain $\psi(x)\equiv\lim_{a\to\infty}\psi(x, a)\leqq v(x)$, $x\in S$. Hence, using Lemmas 2 and 6, we complete the proof, because $0<v_0<1$ is arbitrary.

Now let us make a convenient realization of the BDP $\boldsymbol{X}$. Let $\beta_i(t)$, $i=1, 2,\ldots$, be one-dimensional Brownian motions, and $\tau_i$, $i=1, 2,\ldots$, be random variables with the exponential distribution of mean $1/c$ on a common probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$. Further suppose that $\{\beta_i(t), \tau_j: i, j=1, 2,\ldots\}$ are mutually independent. Let also $x_i(t: x_0)$ be the solution of (4.1) with $\beta_i(t)$ in place of $\beta(t)$. We shall define $\boldsymbol{x}^e(t)=(x^1(t), x^2(t),\ldots, x^{\xi_t}(t))$ inductively. First, define

$$\xi_t=1, \quad x^1(t)=x_1(t: x), \qquad 0\leqq t<T_1, \tag{6.6}$$

where $T_1=\tau_1$. Then define

$$\begin{aligned}&\xi_t=2,\\ &x^1(t)=x_2(t-T_1: x^1(T_1-)), \quad x^2(t)=x_3(t-T_1: x^1(T_1-)), \quad T_1\leqq t<T_2{}^{8)},\end{aligned}$$

where $T_2=\min\{\tau_2+T_1, \tau_3+T_1\}$. Suppose that we have obtained

$$\begin{aligned}&\xi_t=n,\\ &x^i(t)=x_{k_i}(t-T_{n_i}: x^{l_i}(T_{n_i}-)), \qquad 1\leqq i\leqq n, \quad T_{n-1}\leqq t<T_n,\end{aligned}$$

where $k_i$, $n_i$, and $l_i$ are some integers satisfying $1\leqq n_i\leqq n-1$, $1\leqq l_i\leqq n_i$, $T_n=\min\{\tau_{k_i}+T_{n_i}: 1\leqq i\leqq n\}$, and $k_i\neq k_j$ for $i\neq j$. If $T_n=\tau_{k_{i_0}}+T_{n_{i_0}}{}^{9)}$, then we define, as the next step,

$$\begin{aligned}&\xi_t=n+1,\\ &x^i(t)=x_{k_i}(t-T_{n_i}: x^{l_i}(T_{n_i}-)), \qquad i\neq i_0, \quad 1\leqq i\leqq n,\\ &x^{i_0}(t)=x_{2n}(t-T_n: x^{i_0}(T_n-)),\\ &x^{n+1}(t)=x_{2n+1}(t-T_n: x^{i_0}(T_n-)), \qquad T_n\leqq t<T_{n+1},\end{aligned}\tag{6.7}$$

where $T_{n+1}=\min\{\tau_{k_i}+T_{n_i}: i\neq i_0, 1\leqq i\leqq n\}\wedge(\tau_{2n}+T_n)\wedge(\tau_{2n+1}+T_n)$. Since $T_n$ is increasing in $n$ and $\boldsymbol{P}(T_n\leqq t)=(1-e^{-ct})^n$, $\lim_{n\to\infty}T_n=\infty$ a.s. Thus we can define $\boldsymbol{x}^e(t)$ for all $t\geqq 0$. Finally denote by $\boldsymbol{x}_t$ the equivalent class containing $\boldsymbol{x}^e(t)$ in the $n$-fold ($n=\xi_t$) symmetric product space $S^n$ of $S$. Then the process $(\boldsymbol{x}_t, \boldsymbol{P})$ is a realization of the BDP $\boldsymbol{X}$ ([5], [12]). $\{T_n\}$ is the sequence of splitting times and $\xi_t$ is the number of particles at time $t$.

Let $\bar{b}(x)=b(x)-h$ $(0\leqq h<b_0)$, and $\bar{x}_i(t: x_0)$ be the solution of (4.1) with

8) The probability that $x^i(T_j-)=0$ is equal to zero. So we can exclude such a case.

9) The probability that more than two $\tau_{k_i}+T_{n_i}$ attain the minimum simultaneously is equal to zero. Hence $i_0$ is well-defined.

$\beta_i(t)$ and $\bar{b}(x)$ in place of $\beta(t)$ and $b(x)$, respectively. Then, repeating the above procedure, we can construct on the same probability space a process $(\bar{\boldsymbol{x}}_t, \boldsymbol{P})$, a realization of the BDP $\bar{\boldsymbol{X}}$ corresponding to the fundamental system $(\bar{X}, c, \pi)$, where $\bar{X}$ is a diffusion process with the generator $\bar{L}=d^2/2dx^2+\bar{b}(x)d/dx$. Note that the splitting times $T_n$ are common to $(\boldsymbol{x}_t, \boldsymbol{P})$ and $(\bar{\boldsymbol{x}}_t, \boldsymbol{P})$, since the common $\{\tau_i\}$ is used.

**Lemma 8.** *For each* $0\leqq h<b_0$,

$$x^i(t)-ht\leqq \bar{x}^i(t)\leqq x^i(t)\,, \quad t\geqq 0\,, \quad i=1, 2,\dots, \xi_t\,, \quad a.s. \tag{6.8}$$

*Proof.* If $0\leqq t<T_1$, then the assertion is clear from (6.6) and Lemma 4. Suppose that the assertion is valid for all $t<T_n$. Then

$$x^{l_i}(T_{n_i}-)-hT_{n_i}\leqq \bar{x}^{l_i}(T_{n_i}-)\leqq x^{l_i}(T_{n_i}-) \tag{6.9}$$

for all $l_i$ and $n_i$ in (6.7). Let $t\in[T_n, T_{n+1}]$. By Lemma 4, it follows that

$$\begin{aligned}
&\bar{x}_{k_i}(t-T_{n_i}: \bar{x}^{l_i}(T_{n_i}-))\leqq x_{k_i}(t-T_{n_i}: x^{l_i}(T_{n_i}-))\,,\\
&x_{k_i}(t-T_{n_i}: x^{l_i}(T_{n_i}-))-h(t-T_{n_i})-x^{l_i}(T_{n_i}-)\\
&\quad\leqq \bar{x}_{k_i}(t-T_{n_i}: \bar{x}^{l_i}(T_{n_i}-))-\bar{x}^{l_i}(T_{n_i}-)\,, \qquad i\neq i_0\,, \quad 1\leqq i\leqq n\,.
\end{aligned}$$

Hence (6.7) and (6.9) give

$$x^i(t)-ht\leqq \bar{x}^i(t)\leqq x^i(t)\,, \qquad i\neq i_0\,, \quad 1\leqq i\leqq n\,.$$

Similar observation applies to $i=i_0$ and $n+1$. The proof is complete by induction.

Now we are ready to prove Theorem 2.

*Proof of Theorem* 2. *Case* 1. $0<\alpha_\lambda^2/2\leqq c\leqq\lambda<\lambda_0$. We may choose $g_\lambda(x)\equiv\exp\{-\alpha_\lambda x\}$ as $g$ in Theorem 1. Let $h=(\lambda-c)/\alpha_\lambda\geqq 0$. Then $b_0-h=\alpha_\lambda/2+c/\alpha_\lambda>0$, since $\alpha_\lambda$ solves (2.11). Now let $\bar{\boldsymbol{X}}$ be the BDP for this $h$ and use the realizations $(\boldsymbol{x}_t, \boldsymbol{P})$ and $(\bar{\boldsymbol{x}}_t, \boldsymbol{P})$ of $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$, respectively. Then, by Lemma 8,

$$e^{(\lambda-c)t}\,\check{g}_\lambda(\boldsymbol{x}_t)=\sum_{i=1}^{\xi_t} e^{-\alpha_\lambda(x^i(t)-ht)}\geqq \check{g}_\lambda(\bar{\boldsymbol{x}}_t)\,, \quad t\geqq 0\,, \quad \text{a.s.} \tag{6.10}$$

Denote the quantities for $\bar{\boldsymbol{X}}$ by putting a bar such as $\bar{\lambda}_0$, $\bar{b}_0$, $\bar{\alpha}_\lambda$, and $\overline{W}_\lambda$. Then, using (2.11) and the inequality $\alpha_\lambda^2\leqq 2c$, we have

$$c\leqq\bar{\lambda}_0\,, \qquad \alpha_\lambda=\bar{\alpha}_c\,.$$

Lemma 7 applied to the BDP $\bar{\boldsymbol{X}}$ says that $\boldsymbol{P}(\overline{W}_c=0)=0$. Hence, noting (6.10), we get (2.10).

*Case* 2. $0<\lambda<\lambda_0\wedge c$. Since $\lambda>\alpha_\lambda^2/2$ by Remark 1, $\sqrt{2c}\,\alpha_\lambda-\alpha_\lambda^2/2>\alpha_\lambda^2/2$. Hence we can choose a $\mu$ satisfying

$$\alpha_\lambda^2/2<\mu<(\sqrt{2c}\,\alpha_\lambda-\alpha_\lambda^2/2)\wedge\lambda\,. \tag{6.11}$$

For this $\mu$, set $h=(\lambda-\mu)/\alpha_\lambda>0$ and let $\bar{X}$ be the corresponding BDP. Then

$$\bar{b}_0\equiv b_0-h>0\,,\quad \mu<\bar{\lambda}_0\,,\quad \alpha_\lambda=\bar{\alpha}_\mu\,. \tag{6.12}$$

Now take the above realizations $(x_t, P)$ and $(\bar{x}_t, P)$ of $X$ and $\bar{X}$, respectively. Then, by Lemma 8,

$$e^{(\lambda-c)t}\check{g}_\lambda(x_t)\geqq e^{(\mu-c)}\check{g}_\lambda(\bar{x}_t)\,,\qquad \text{a.s.}$$

Hence it is enough to see $\bar{v}(x)\equiv P(\overline{W}^{(\mu)}=0)=0$.

First note that, in the present case,

$$c>\bar{\lambda}_0\,. \tag{6.13}$$

By Lemma 6 applied to the BDP $\bar{X}$, the function $\bar{v}(x)$ solves (3.20) with $L$ replaced by $\bar{L}$. It follows from Lemma 2 and (6.13) that $\bar{v}(x)\equiv 0$ or $\equiv 1$. So we have only to show

$$\bar{P}_x(\overline{W}^{(\mu)}=0)<1. \tag{6.14}$$

By [17],

$$\begin{aligned}\bar{E}_x[\overline{W}^{(\mu)}(t)^2]&=e^{2(\mu-c)t}\bar{E}_x[\bar{\varphi}_\mu(\bar{x}_t)^2]\\&=e^{2(\mu-c)t}\overline{M}_t\bar{\varphi}_\mu^2(x)+2c\int_0^t e^{2(\mu-c)s}\overline{M}_s\bar{\varphi}_\mu^2(x)\,ds,\end{aligned}$$

where $\overline{M}_t=e^{ct}\,\overline{T}_t$. By Lemma 1 applied to $\bar{L}$, we have $\bar{\varphi}_\mu(x)^2\leqq K\bar{\varphi}_\mu(x)$ for some $K>0$. Hence, noting $\overline{M}_t\bar{\varphi}_\mu(x)=e^{(c-\mu)t}\bar{\varphi}_\mu(x)$, we have

$$\bar{E}_x[\overline{W}^{(\mu)}(t)^2]\leqq K\left\{e^{(\mu-c)t}+2c\int_0^t e^{(\mu-c)s}\,ds\right\}\bar{\varphi}_\mu(x).$$

But $\mu<c$ by (6.12) and (6.13). Thus $\overline{W}^{(\mu)}(t)$ is an $L^2$-bounded martingale. So

$$\bar{E}_x[\overline{W}^{(\mu)}]=\lim_{t\to\infty}\bar{E}_x[\overline{W}^{(\mu)}(t)]=\bar{\varphi}_\mu(x)>0.$$

This assures (6.14).

*Case* 3. $\lambda=\lambda_0$. Since $\alpha_{\lambda_0}^2/2=\lambda_0$, it holds that $c\geqq\lambda_0$ by the assumption. If $c=\lambda_0$, then (2.10) is proved by Lemma 7. If $c>\lambda_0$, then the above argument ensures that $W^{(\lambda_0)}(t)$ is an $L^2$-bounded martingale and the function $v(x)=P_x(W^{(\lambda_0)}=0)$ must be $\equiv 0$ or $\equiv 1$. Hence, (2.10) follows as above.

*Case* 4. $\lambda=0$. In this case, we may take the constant function 1 as $g$ in

Theorem 1. Then $e^{-ct}\check{1}(\boldsymbol{x}_t)=e^{-ct}\xi_t$ is a martingale for the simple branching process. Hence the assertion is obvious ([3] pp. 109–110).

## References


[1] Asmussen, S. and Hering, H., Strong limit theorems for general supercritical branching processes with application to branching diffusions, *Z. Wahrscheinlichkeitstheorie*, **36** (1976), 195–212.

[2] Dynkin, E. B., Brownian motion in certain symmetric spaces and nonnegative eigenfunctions of the Laplace-Beltrami operator, *A.M.S. Translations*, Series 2, **72** (1968), 203–228. (Translation).

[3] Harris, T. E., *The Theory of Branching Processes*, Springer, 1963.

[4] Helgason, S., *Differential Geometry and Symmetric Spaces*, Academic Press, 1962.

[5] Ikeda, N., Nagasawa, M. and Watanabe, S., Branching Markov processes, I, II, III, *J. Math. Kyoto Univ.*, **8** (1968), 233–278: **8** (1968), 365–410: **9** (1969), 95–160.

[6] Ikeda, N. and Watanabe, S., A comparison theorem for solutions of stochastic differential equations and its application, *Osaka Math. J.*, **14** (1977), 619–633.

[7] Itô, K., Stochastic differentials, *Appl. Math. Optimization*, **1** (1975), 374–381.

[8] Itô, K. and McKean Jr., H. P., *Diffusion processes and their sample paths*, Springer, 1965.

[9] Kesten, H., Branching Brownian motions with absorption, *Stochastic Processes Appl.*, **7** (1978), 9–47.

[10] McKean, H. P., An upper bound to the spectrum of $\Delta$ on a manifold of negative curvature, *J. Diff. Geometry*, **4** (1970), 359–366.

[11] ———, Application of Brownian motion to the equation of Kolmogorov-Petrovsky-Piscunov, *Comm. Pure Appl. Math.*, **XXVIII** (1975), 323–331.

[12] Nagasawa, M., Branching property of Markov processes, *Lecture notes in Math.* **258**, *Séminaire de Probabilités VI Université de Strasbourg*, Springer, 1973, 177–198.

[13] Tanaka, H., A construction of one-dimensional Markov processes with reflecting barrier by stochastic differential equations, *RIMS Kokyuroku* **112**, Kyoto Univ., 1971, 120–133 (in Japanese).

[14] Tanaka, H., Tsuchiya, M. and Watanabe, S., Perturbation of drift-type for Lévy processes, *J. Math. Kyoto Univ.*, **14** (1974), 73–92.

[15] Titchmarsh, E. C., *Eigenfunction expansions*, Part one, Second ed., Oxford, 1962.

[16] Watanabe, S., On the branching process for Brownian particles with absorbing boundary, *J. Math. Kyoto Univ.*, **4** (1965), 385–398.

[17] ———, Limit theorem for a class of branching processes, *Markov processes and potential theory*, ed. Chover, Wiley, 1967, 205–232.

[18] Yamada, T., On a comparison theorem for solutions of stochastic differential equations and its application, *J. Math. Kyoto Univ.*, **13** (1973), 497–512.

[19] ———, On the uniqueness of solutions of stochastic differential equations with reflecting barrier conditions, *Lecture notes in Math.* **511**, *Séminaire de Probabilités X Université de Strasbourg*, Springer, 1976, 240–244.